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SPECTRUM PRESERVING LOWER TRIANGULAR COMPLETIONS—THE NONNEGATIVE NILPOTENT CASE*

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Abstract. Nonnegative nilpotent lower triangular completions of a nonnegative nilpotent matrix are studied. It is shown that for every natural number between the index of the matrix and its order, there exists a completion that has this number as its index. A similar result is obtained for the rank. However, unlike the case of complex completions of complex matrices, it is proved that for every nonincreasing sequence of nonnegative integers whose sum is n, there exists an $n \times n$ nonnegative nilpotent matrix A such that for every nonnegative nilpotent lower triangular completion, B, of A, $B \neq A$, $ind(B) > ind(A)$.

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1. Introduction. In many problems related to the spectral properties of a matrix, an important role is played by an additive perturbation of some part of the matrix, e.g. the main diagonal, the off-diagonal part, or a strictly triangular part of the matrix.

The perturbed matrices of a given matrix can be considered as completions of a partial matrix, i.e., a matrix in which the entries that may be perturbed are considered as free independent variables. Many completion problems were solved for matrices over different algebraic structures. Properties of nonnegative additive perturbations of nonnegative partially specified matrices are used, for example, in the calculation of risk-free interest rates and some derivative securities of financial markets; see, e.g., [12].

In the present paper we are interested in investigating some properties of nonnegative matrices which preserve nilpotency under strictly lower triangular perturbations.

Since similar problems were solved for matrices with complex elements, we will use the term *completion* instead of *perturbation* in order to show that in comparison with the corresponding completion problem for complex matrices (instead of nonnegative matrices) some results still hold, while some are necessarily more limited.

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A *triangular completion* of a square matrix A is a matrix $B = A + T$, where T is a strictly lower triangular matrix. Triangular completions were intensively studied in various directions; see, e.g., [1], [7], [8], [9], [10], [19], [23], [24]. One of the directions is the study of triangular completions that preserve the spectrum of a matrix (see, e.g., [14], [15]) and in particular nilpotent completions; see, e.g., [16], [17], [18].

As it was already mentioned, we are interested here in the nonnegative case. Observe that a triangular completion of a complex matrix can be viewed as an additive perturbation with a strictly lower triangular matrix T. For example, if B is a triangular completion of A, namely, $B = A + T$, then it is clear that $A = B + (-T)$ is a triangular completion of B. In the nonnegative case the last statement is, of course, not true. Thus nonnegative completions may be defined in two ways:

1. A and T (and of course the completion $B = A + T$) are nonnegative.
2. A and B, but not necessary T, are nonnegative.

Let $\tilde{A}$ be an upper triangular matrix which has the same entries as A on and above the main diagonal. The set of completions of the second type of A, coincide with the set of completions of the first type of the matrix $\tilde{A}$. In the present paper we only study the case when **both** A and T are nonnegative, but using the remark above one may also interpret the results in the context of 2.

The structure of the paper is as follows. In Section 2 we give the notation and the necessary preliminaries on matrices and their graphs. In Section 3 we show that some results on triangular nilpotent completions that are known for general matrices, hold in the nonnegative case and some can not be reproved in this case. The paper is concluded with a discussion on the results obtained, and some open questions.

2. Notation and Preliminaries. Let $\mathbb{C}^{n,n}$ denote the set of all n by n complex matrices, $\mathcal{N}^n$ the subset of nilpotent matrices in $\mathbb{C}^{n,n}$, and $\mathcal{N}_+^n$ the subset of the nonnegative matrices in $\mathcal{N}^n$.
For $A \in \mathbb{C}^{n,n}$ –

The index of A, $ind(A)$, is the smallest integer k, such that $rank(A^k) = rank(A^{k+1})$ (it is equal to zero iff A is nonsingular),

$NTC(A)$ denotes the set of nilpotent triangular completions of A, and

$\underline{r}(A)$ denotes the minimum of the ranks of all matrices in $NTC(A)$.

For a nonnegative nilpotent matrix A –

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For a nilpotent matrix A –

$J(A) = \{J_1, ..., J_s\}$ denotes the Jordan structure of A, i.e. set of sizes of Jordan blocks of A in nonincreasing order. In this case $ind(A)$ is the size of the largest block J_1.

Let $\Lambda = \{\lambda_1, ..., \lambda_p\}$ and $\mathcal{M} = \{\mu_1, ..., \mu_q\}$ be two nonincreasing sequences of nonnegative integers. We say that Λ majorizes $\mathcal{M}$ if

$$\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i, \quad 1 \leq k \leq min(p,q), \text{ and } \sum_{i=1}^{p} \lambda_i = \sum_{i=1}^{q} \mu_i.$$

Let $A \in \mathbb{C}^{n,n}$. The digraph $D(A)$ corresponding to A is a directed graph with vertices $\{1, \ldots, n\}$ such that there is an arc $i \rightarrow j$ if and only if $a_{i,j} \neq 0$.

A *path* in $D(A)$ is any ordered sequence of (not necessarily disjoint) vertices $\{i_1, \ldots, i_k\}$ such that there is an arc from i_p to i_{p+1} $(p = 1, \ldots, k-1)$; the number k is called the *length* of the path $\{i_1, \ldots, i_k\}$. Every one vertex sequence $\{i\}$ (with or without an arc from i to itself) will be also called a *path* (of length one). If there is an arc $i_k \rightarrow i_1$ in a path $\{i_1, \ldots, i_k\}$, then the path is called a *cycle*; a one vertex path $\{i\}$ is called a *cycle* if there is an arc from i to itself.

It is well known (see e.g. [3]), that if $D(A)$ has no paths of length $p > 1$, then $A^{p-1} = 0$ and if A is nonnegative, then the converse is also true, i.e. $A^{p-1} = 0$, $p > 1$ implies that $D(A)$ has no paths of length p. In particular, if A is nonnegative, nilpotency means that $D(A)$ has no cycles (if cycles exist one can make paths of any length). In this case the index of A is equal to the length of the largest path in $D(A)$; see, e.g., [20].

We call vertex v an *initial vertex* if there are no arcs in $D(A)$ entering v. It will be called *k-initial vertex* if the length of the maximal path starting with this vertex equals k. Note that path of maximal length always starts with an initial vertex. A vertex w is called *final vertex* if there no arcs exiting from w.

There is an extensive literature on the relation between the Jordan structure of a matrix and its graph, see, e.g., [6], [11], [21], [22]. In particular, for nilpotent generic matrices, i.e., matrices whose nonzero entries are algebraically independent, the Jordan structure is completely determined by their digraphs. This can be seen by the following theorem proved in [6], [21].

THEOREM 2.1. *Let A be a nilpotent generic matrix with digraph D, and let $p_k(D)$ be the maximal number of vertices which can be covered by k disjoint paths. Then $J_k(A) = p_k(D) - p_{k-1}(D)$, $1 \leq k \leq s$, where s is the number of Jordan blocks of A.*

3. Results. The following result was conjectured in [18] and proved in [16].

THEOREM 3.1. *Let J be the Jordan structure of a nilpotent matrix $A \in \mathbb{C}^{n,n}$. Then for every nonincreasing sequence K of nonnegative integers, that majorizes J, there exists a matrix $B \in NTC(A)$ with Jordan structure K.*

Unfortunately, as the following example shows, a result similar to Theorem 3.1 does not hold in the nonnegative case.

EXAMPLE 3.2. Let

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $J(A) = \{2,1,1\}$. The sequence $\{2,2\}$ majorizes J and it is not difficult to check that there exists no $A+T \in NNTC(A)$ with $J(A+T) = \{2,2\}$. This, of course, does not contradict Theorem 3.1, since for the (not nonnegative) matrix

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$J(A+S) = \{2,2\}$ and $A + S \in NTC(A)$.

There are, however, two corollaries of Theorem 3.1, which do have analogues in the nonnegative case.

COROLLARY 3.3. *Let $A \in \mathcal{N}^n$. Then for every integer r , such that $\underline{r}(A) \le r \le n-1$ there exists a lower triangular completion $B \in NTC(A)$ of $rank(B) = r$.*

COROLLARY 3.4. *Let $A \in \mathcal{N}^n$, and let $\underline{m}(A)$ be the minimum of the indices of matrices in $NTC(A)$. Then for every integer m , such that $\underline{m}(A) \le m \le n$ there exists a lower triangular completion $B \in NTC(A)$ of $ind(B) = m$.*

The analogous results for nonnegative matrices are given in the following theorem.

THEOREM 3.5. *Let $A \in \mathcal{N}_+^n$. Then*
(a) For every integer m, $ind(A) \le m \le n$, there exists $B \in NNTC(A)$, such that $ind(B) = m$.
(b) For every integer r, $\underline{r}^+(A) \le r \le n-1$, there exists $C \in NNTC(A)$, such that $rank(C) = r$.

Proof. (a) First we show that if $ind(A) = k < n$, then there exists $B \in NNTC(A)$, such that $ind(B) = k+1$. We use induction on n. If $n = 2$, the only matrix A with $ind(A) < 2$ that satisfies the conditions of the theorem is the zero matrix and in this case B can be chosen as $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Thus let us assume that the theorem is correct for matrices of order less than n.
Consider the digraph $D(A)$ of A. We want to show that there are pairs of vertices $i_s \to j_s$, $i_s > j_s$, so that we can add arcs from i_s to j_s without creating cycles, and thus the length of the largest path increases to $k+1$.
Let v be the smallest k-initial vertex in $D(A)$. If there exists an initial vertex w, such that $w > v$, then we add the arc $w \to v$. If no such initial vertex

exists, it means, that v is the only k-initial vertex in $D(A)$. In this case we consider A_1 - the principal submatrix of A, obtained by deleting the (zero) column and the row, that correspond to v. Being a principal submatrix of A, the matrix $A_1 \in \mathcal{N}_+^{n-1}$ and since v is the only k-initial vertex in $D(A)$, then $ind(A_1) = k-1$. Thus A_1 satisfies the conditions of the theorem. By the induction hypothesis there exists $B_1 \in NNTC(A_1)$, such that $ind(B_1) = k$. Let $T_1 = B_1 - A_1$; $S = 0 \oplus T_1$ and $F = A + S$. Observe that $F \in NNTC(A)$. Look now at the k-initial vertices in $D(B_1)$. If $D(B_1)$ has a k- initial vertex V, such that $v \to V$ is an arc in $D(A)$, then v is a $(k+1)$-initial vertex in $D(F)$ and $ind(F) = k+1$, so taking $B = F$ completes the proof. If there is no arc from v to a k-initial vertex in $D(B_1)$, then $D(A)$ has at least two k-initial vertices, namely v and u – an initial vertex of a maximal (of length k) path in the subdigraph $D(B_1)$. The completion by an arc $max(u,v) \to min(u,v)$ yields a matrix $B \in NNTC(A)$ with index $k+1$. A repeated use of this construction completes the proof of (a).
(b) First we increase, if necessary, entries of A to obtain a matrix $A_0 \in NNTC(A)$, of rank $\underline{r}^+(A)$. Then we construct, using repeatedly the construction in the proof of (a), a sequence of matrices $A_0, A_1, ..., A_s$, such that $A_s \in NNTC(A)$, $ind(A_s) = n$ and each A_i is obtained from A_{i-1} by replacing a zero entry below the main diagonal with a positive number. To show that all the matrices in the sequence are nilpotent, we observe that they are nonnegative and $A_{i-1} \le A_i$. By the Perron-Frobenius Theorem (see, e.g., Corollary 1.5 in [2]), the spectral radius $\rho(A_{i-1}) \le \rho(A_i)$. Since A_0 and A_s are both nilpotent, $\rho(A_0) = \rho(A_s) = 0$ so this holds for all matrices in the sequence. We thus obtained a sequence of nonnegative nilpotent matrices, all in $NNTC(A)$. Each A_i is a one-dimensional perturbation of the previous one, so their ranks differ by at most one. The rank of A_0 is $\underline{r}^+(A)$ and the rank of A_s is $n-1$ ($ind(B) = n$). This implies that every integer r between $\underline{r}^+(A)$ and $n-1$ appears as the rank of one of the matrices $A_0, ..., A_s$. □

Now we return to Example 3.2, and show that the situation in this counterexample is not exceptional.

THEOREM 3.6. *For every nonincreasing sequence of nonnegative integers* $J = \{J_1, ..., J_s\}$*with* $\sum_{i=1}^{s} J_i = n$ *there exists* $A \in \mathcal{N}_+^n$ *with Jordan structure* $J(A) = J$*, such that for every* $B \in NNTC(A)$, $B \neq A$, $ind(B) > ind(A)$.

Proof. Denote by M the $n \times n$ matrix in the Jordan canonical form with a Jordan structure J. Observe, that the digraph $D(M)$ consists of simple disjoint paths of lengths $\{J_1, ..., J_s\}$. Define an integer-valued function f on the vertices of $D(M)$ by the following rules:

1. The value assigned to the endpoints of the paths is 1.
2. If there is an arc $v \to w$ in $D(M)$ then $f(v) = f(w) + 1$.

Now we add arcs to $D(M)$ by connecting all vertices with value s to all vertices with a smaller value for all $2 \le s \le ind(M)$. The digraph D obtained in this way has the following properties:
(a) The number of vertices of value s does not exceed the number of vertices

of value $s-1$; $2 \leq s \leq ind(M)$.
(b) Every path from v to w is contained in a path

$$v = v_0 \to v_1 \to \ldots \to v_{f(w)-f(v)} = w, \quad \text{where } f(v_i) = f(v_{i-1}) - 1.$$

Since $p_k(D)$ is the maximal number of vertices covered by k disjoint paths in D, it follows from (a) and (b) that $p_k = \sum_{i=1}^{k} J_k$, and so $p_i(D) = p_i(D(M))$. Now we construct an upper triangular matrix A from the digraph D. First, we renumerate the vertices of the digraph D such that $v > w$ if $f(v) > f(w)$. Then, we assign algebraically independent numbers to the arcs. By Theorem 2.1, the Jordan structure of A is J. Since A is an upper triangular matrix, adding arcs to its graph is the only type of completion allowed here. The proof is completed by observing that adding of an arc $v \to w$ to $D(A)$ either creates a cycle if $f(v) \neq f(w)$ or increases the length of the maximal path if $f(v) = f(w)$. ☐

4. Discussion. We conclude the paper with some remarks and open problems on nonnegative completions.
It is clear that a triangular completion of a matrix A may have a smaller rank than A does. For this reason it was of interest to characterize the minimum of ranks of all possible completions. This was done in [24]. A spectrum preserving triangular completion of a matrix A may also have a smaller rank than $rank(A)$.

The following example shows that there can be a matrix $A \in \mathcal{N}_+^n$ and a matrix $B \in NNTC(A)$ such that $rank(B) < rank(A)$.

Example 4.1. Let

$$A(k) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & k & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then for $0 \leq k < 1$, $\quad rank(A(1)) < rank(A(k))$.

Open Question 4.2. *What is the minimal rank of a spectrum preserving triangular completion of a given nonnegative matrix A? In particular, what is $\underline{r}^+$ for $A \in \mathcal{N}_+^n$?*

For a discussion of questions concerning minimal ranks of completions of different types, the reader is referred to [4], [5], [13], [19], [25].

In Theorem 3.5 we showed that the index and the rank of a matrix $A \in \mathcal{N}_+^n$ can be increased by one. It is not claimed (and it is not true in general) that this can be done by a one-dimensional perturbation.

Example 4.3. Let

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then the only matrices in $NNTC(A)$ with a greater rank or index are of the form

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & + & 0 & 0 \\ 0 & 1 & + & 0 \end{pmatrix}.$$

It is also not claimed that for *any* pair (r,i) such that $r \geq \underline{r}^+$, $i \geq ind(A)$, there exists $B \in NNTC(A)$ of rank r and index i, even if the necessary condition $r \geq i-1$ holds. This is shown (for $r=4$ and $i=3$) by Example 4.5 that relates to the following question.

Open Question 4.4. *What are the possible Jordan structures of the nonnegative nilpotent triangular completions of a given nonnegative nilpotent matrix A?*

It is tempting to hope that for every nonincreasing sequence of nonnegative integers K, such that K majorizes $J(A)$ and $K_1 > ind(A)$, there exists $B \in NNTC(A)$ with a Jordan structure K. However, the following example shows this is not the case.

Example 4.5. Let

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $J(A) = \{2,1,1,1,1,1\}$ and there is no nonnegative nilpotent completion with Jordan sequence $\{3,3\}$.

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