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INFINITE PRODUCTS AND PARACONTRACTING MATRICES *

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Abstract. In [*Linear Algebra Appl.*, 161:227–263, 1992] the LCP-property of a finite set Σ of square complex matrices was introduced and studied. A set Σ is an LCP-set if all left infinite products formed from matrices in Σ are convergent. It was shown earlier in [*Linear Algebra Appl.*, 130:65–82, 1990] that a set Σ paracontracting with respect to a fixed norm is an LCP-set. Here a converse statement is proved: If Σ is an LCP-set with a continuous limit function then there exists a norm such that all matrices in Σ are paracontracting with respect to this norm. In addition the stronger property of ℓ-paracontractivity is introduced. It is shown that common ℓ-paracontractivity of a set of matrices has a simple characterization. It turns out that in the above mentioned converse statement the norm can be chosen such that all matrices are ℓ-paracontracting. It is shown that for Σ consisting of two projectors the LCP-property is equivalent to ℓ-paracontractivity, even without requiring continuity.

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1. Introduction.

In the investigation of chaotic iteration procedures for consistent linear systems, matrices which are paracontracting with respect to some vector norm play an important role. It was shown in [3], that if $A_1, \ldots, A_m$ are finitely many $k \times k$ complex matrices which are paracontracting with respect to the same norm, then for any sequence d_i, $1 \le d_i \le m$, $i = 1, 2, \ldots$ and any x_0 the sequence

$$x_i = A_{d_i} x_{i-1} \quad i = 1, 2, \ldots \tag{1}$$

is convergent. In particular $A^{(d)} = \lim_{i \to \infty} A_{d_i} \ldots A_{d_1}$ exists for all sequences $\{d_i\}_{i=1}^{\infty} = d$. Hence those sets are examples of sets of matrices all infinite products of which converge. Such sets have been studied in [2]. Following [2], we call them LCP–sets.

In this note we investigate the question of necessity. As our main result we show that under the additional assumption that the mapping

$$d = \{d_i\}_{i=1}^{\infty} \to A^{(d)} = \lim_{i \to \infty} A_{d_i} A_{d_{i-1}} \ldots A_{d_1} \tag{2}$$

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is continuous (which is equivalent to the set of fixed points of A_i being the same for all $1 \le i \le m$), an LCP–set is necessarily paracontracting with respect to some norm. In this sense paracontractivity is equivalent to the LCP–property. We show, in addition, that continuity implies even the stronger property of ℓ-paracontractiveness.

In the final section, we consider the case $m = 2$. We show that for Σ consisting of two projectors the LCP-property is equivalent to ℓ-paracontractivity, even without continuity.

2. Notations and known results. Let $|| \; ||$ denote a vector norm in $\mathbb{C}^k$. A $k \times k$ matrix P is *paracontracting* with respect to $|| \; ||$, if for all x

$$Px \neq x \Leftrightarrow ||Px|| < ||x||.$$

We denote by $\mathcal{N}(|| \; ||)$ the set of all $k \times k$ matrices paracontracting w.r.t. $|| \; ||$. We call P *ℓ-paracontracting* w.r.t. $|| \; ||$, if there exists $\gamma > 0$ such that

$$||Px|| \le ||x|| - \gamma ||Px - x||$$

holds for all $x \in \mathbb{C}^k$ and denote this set of matrices by $\mathcal{N}_\gamma(|| \; ||)$. Obviously

$$\mathcal{N}_\gamma(|| \; ||) \subset \mathcal{N}(|| \; ||). \tag{3}$$

The example of an orthogonal projection $P, P \neq I, P \neq 0$ which is paracontracting w.r.t. the Euclidean vector norm but never ℓ-paracontracting shows that in (3) equality does not hold in general.

For a bounded set $\Sigma = \Sigma_1$ of complex $k \times k$ - matrices define $\Sigma_0 = \{I\}$ and for $n \ge 1$, $\Sigma_n = \{M_1 \; M_2 \dots M_n : M_i \in \Sigma\}$, the set of all products of matrices in Σ of length n. Let $\Sigma = \{A_1, \dots, A_m\}$ be finite. For $d = (d_1, d_2, \dots) \in \{1, \dots, m\}^{\mathbb{N}}$, i.e. $1 \le d_i \le m$ for $i \in \mathbb{N}$ define $A^{(d)} = \lim_{n \to \infty} A_{d_n} A_{d_{n-1}} \dots A_{d_1}$, if the limit exists. The set Σ is an LCP–set (left–convergent–product), if for all $d \in \{1, \dots, m\}^{\mathbb{N}}$ the limit $A^{(d)}$ exists. The function $d \to A^{(d)}$ mapping $\{1, \dots, m\}^{\mathbb{N}}$ into the space of $k \times k$ - matrices is called the *limit function.*

We note in passing that in [2] also the right–convergent–product property (RCP) was introduced. For convenience we restrict our considerations to the left convergence case.

Introducing in $\{1, \dots, m\}^{\mathbb{N}}$ the metric

$$\mathrm{dist}(d, d') = m^{-r} \qquad r \text{ smallest index such that } d_r \neq d'_r,$$

we define the concept of a *continuous limit function* in the standard way.

The set Σ is *product bounded,* if there exists $\Delta > 0$ such that

$$||A|| \le \Delta \quad \text{for all} \quad A \in \Sigma_n, \; n = 1, 2, \dots$$

Here $|| \; ||$ denotes any matrix norm. Obviously this concept is independent of the norm. G. Schechtman has proved that LCP–sets are product bounded (see [1, Theorem I]). We have the following result.

LEMMA 2.1. *For a set Σ of $k \times k$ - matrices the following are equivalent.*
(i) The set Σ is product bounded.
(ii) There exists a vector norm $|| \; ||$ such that $||Ax|| \leq ||x||$ for all $A \in \Sigma,\ x \in \mathbb{C}^k$.
(iii) There exists a multiplicative matrix norm $|| \; ||$ such that $||A|| \leq 1$ for all $A \in \Sigma$.

Proof. As $(ii) \Longrightarrow (iii)$ (the operator norm is multiplicative) and $(iii) \Longrightarrow (i)$ are obvious, only $(i) \Longrightarrow (ii)$ has to be shown.
For some vector norm ν define the norm

$$||x|| = \sup_{n \geq 0} \{ \sup_{A \in \Sigma_n} \nu(Ax) \}$$

which is finite by (i). Then $||Ax|| \leq ||x||$ for all $A \in \Sigma$. $\square$

We remark that this result could also be derived from [5]. For a given matrix norm $|| \; ||$ and bounded Σ let $\widehat{\rho}_n = \widehat{\rho}_n(\Sigma) = \max\{||A||, A \in \Sigma_n\}$ and let $\widehat{\rho} = \widehat{\rho}(\Sigma) = \lim_{n \to \infty} \widehat{\rho}_n^{1/n}$. The quantity $\widehat{\rho}$ is called the joint spectral radius of Σ. It was introduced in [5] for general bounded sets in a normed algebra. In [5] and in [2] the limit is replaced by lim sup, however, it is implicitly shown in [2] (see there (3.12)), that the limit exists.

We give here a characterization of $\widehat{\rho}(\Sigma)$, which can be found essentially in [5]. Hence the proof, which is also an easy consequence of the previous Lemma, is omitted.

LEMMA 2.2. *For any bounded set Σ of $k \times k$ - matrices*

$$\widehat{\rho}(\Sigma) = \inf_{\nu \ operator \ norm} \sup_{A \in \Sigma} \nu(A). \tag{4}$$

The following result is just a restatement of the Theorem in [3].

THEOREM 2.3. *Let $\Sigma \subset \mathcal{N}(|| \; ||)$ for some vector norm $|| \; ||$, Σ finite. Then Σ has the LCP–property.*

We finish this section by pointing out that if in addition $\Sigma \subset \mathcal{N}_\gamma(|| \; ||)$ for some positive γ, then the proof of Theorem 2.3 is very simple. This is outlined below. It is a consequence of the following characterization of ℓ-paracontractivity of the set Σ.

Let $\Sigma = \{A_i\}_{i \in I}$ be a set of matrices, not necessarily finite. Let $d = (d_1, \ldots, d_r) \in I^r$, ν a vector norm. Define

$$\nu_d(x) = \nu(x_r) + \sum_{k=1}^{r} \nu(x_k - x_{k-1}) \tag{5}$$

where the vectors x_i are defined as in (1) and $x = x_0$. Then obviously, for any $i \in I$ and $d' = (i, d_1, \dots, d_r)$

$$\nu_d(A_i x) = \nu_{d'}(x) - \nu(A_i x - x). \tag{6}$$

We define now

$$\nu_*(x) = \sup\{\nu_d(x) : d \text{ finite}\}. \tag{7}$$

This is a vector norm provided that $\nu_*(x) < \infty$ for all x.

THEOREM 2.4. *For a set of $k \times k$ - matrices $\{A_i\}_{i \in I}$ the following are equivalent.*
(i) There exists a norm ν and a positive γ such that

$$A_i \in \mathcal{N}_\gamma(\nu) \qquad \textit{for all} \qquad i \in I.$$

(ii) There exists a vector norm μ such that

$$\mu_*(x) < \infty \qquad \textit{for all} \qquad x \in \mathbb{C}^k$$

(iii) For all vector norms μ

$$\mu_*(x) < \infty \qquad \textit{for all} \qquad x \in \mathbb{C}^k$$

Proof. We show $(i) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$.
Assume that (i) holds. Then from

$$\nu(A_i x - x) \le \gamma^{-1}\{\nu(x) - \nu(A_i x)\} \qquad \forall i \in I, \forall x \tag{8}$$

we have, using the notation in (5), and assuming (w.l.o.g.) that $\gamma \le 1$,

$$\begin{aligned} \nu_d(x) &\le \nu(x_r) + \gamma^{-1} \sum_{k=1}^{r} (\nu(x_{k-1}) - \nu(x_k)) \\ &= \nu(x_r) + \gamma^{-1}\{\nu(x) - \nu(x_r)\} \le \gamma^{-1}\nu(x). \end{aligned} \tag{9}$$

If μ is a fixed vector norm, then due to the compatibility of any two norms we have a constant κ such that $\mu(x) \le \kappa\nu(x)$ and hence also $\mu_d(x) \le \kappa\nu_d(x)$. The inequality (9) gives that $\mu_*(x)$ exists, hence we have (iii).
Obviously (iii) implies (ii).
Now we assume (ii). From (6) we have

$$\mu_*(A_i x) \le \mu_*(x) - \mu(A_i x - x) \le \mu_*(x) - \gamma\mu_*(A_i x - x) \tag{10}$$

where we have chosen γ such that $\mu(\xi) \ge \gamma\mu_*(\xi)$ for all ξ. Hence (i) holds with $\nu = \mu_*$. □

We indicate now the easy proof of the fact that a finite set $\Sigma = \{A_1, \dots, A_m\} \subset \mathcal{N}_\gamma(\nu)$ has the LCP-property. It suffices to show that for any x_0 and any $d = (d_1, d_2, \dots) \in \{1, ..., m\}^{\mathbb{N}}$ the sequence $\{x_i\}_{i=1}^{\infty}$ defined by (1) is convergent. By Theorem 2.4 we have $\nu_*(x_0) < \infty$, hence the sequence $\sum_{i=1}^{\infty} \nu(x_i - x_{i-1})$ is convergent. This implies that the sequence of the $x_i's$ is a Cauchy sequence.

3. Main result. It is tempting to conjecture that the converse statement of Theorem 2.3 also holds, namely that if Σ is an LCP-set, then there exists a vector norm $||\ ||$ such that $\Sigma \subset \mathcal{N}(||\ ||)$. We were unable to decide this question in general. However, the converse is true if Σ is an LCP-set with a continuous limit function. More precisely, the following theorem holds.

THEOREM 3.1. *Let $\Sigma = \{A_1, \ldots, A_m\}$ be a finite set of $k \times k$ - matrices and let the subspaces $M_i = N(I - A_i),\ i = 1, \ldots, m$. Then the following are equivalent.*
(i) The set Σ has the LCP–property and $M_i = M_j$ for $i, j = 1, \ldots, m$.
(ii) The set Σ has the LCP–property with continuous limit function.
(iii) There exists a vector norm $||\ ||$ in $\mathbb{C}^k$ and a positive γ such that $\Sigma \subset \mathcal{N}_\gamma(||\ ||)$ and $M_i = M_j$ for $i, j = 1, \ldots, m$.
(iv) There exists a vector norm $||\ ||$ in $\mathbb{C}^k$ such that $\Sigma \subset \mathcal{N}(||\ ||)$ and $M_i = M_j$ for $i, j = 1, \ldots, m$.

Proof. We will show $(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (iv) \Longrightarrow (i)$.
To prove $(i) \Longrightarrow (ii)$, we are going to show that

$$||A^{(d)} - A^{(d')}|| \leq (2 + \Delta)||A_{(r)} - A^{(d)}||, \tag{11}$$

where $||\ ||$ is a fixed operator norm, $(d), (d') \in \{1, \ldots, m\}^{\mathbb{N}}$, $d_i = d'_i$ for $i \leq r$, and Δ is the bound in the definition of product boundedness. Here we use the fact that by [1], Σ is product bounded. Also we use the notation

$$A_{(r)} = A_{d_r} A_{d_{r-1}} \ldots A_{d_1},\ A'_{(s)} = A_{d'_s} \ldots A_{d'_1}.$$

Let $M_0 = N(I - A_i),\ i = 1, \ldots, m$ the common pointwise invariant subspace of the matrices A_i. If $i \in \{1, \ldots, m\}$ occurs infinitely often in the sequence $d_1, d_2, \ldots$, then by the usual reasoning $A_i A^{(d)} = A^{(d)}$, and hence all columns of $A^{(d)}$ are in M_0. Hence $A_j A^{(d)} = A^{(d)}$ for all $A_j \in \Sigma$. This implies the relation

$$A'_{(r+s)} - A_{(r)} = (A_{d'_{r+s}} \ldots A_{d'_{r+1}} - I)(A_{(r)} - A^{(d)}) \qquad s > 0$$

and hence $||A'_{(r+s)} - A_{(r)}|| \leq (1 + \Delta)||A_{(r)} - A^{(d)}||$. Taking $s \to \infty$, we get

$$||A^{(d')} - A_{(r)}|| \leq (1 + \Delta)||A_{(r)} - A^{(d)}||,$$

from which (11) follows. This implies continuity: Given $\epsilon > 0$, as $A_{(r)} \to A^{(d)}$, there exists r_0 such that

$$||A_{(r_0)} - A^{(d)}|| \leq (2 + \Delta)^{-1}\epsilon.$$

Now, if (d') is such that $\mathrm{dist}(d, d') \leq m^{-r_0-1}$, then $d_i = d'_i$ for $i \leq r_0$ and hence by (11)

$$||A^{(d')} - A^{(d)}|| \leq (2 + \Delta)||A_{(r_0)} - A^{(d)}|| \leq \epsilon.$$

We remark that although this step is not directly contained in [2], we have used tools and ideas from that paper.
Finally, we show $(ii) \Longrightarrow (iii)$. Assume that (ii) holds. By Theorem 4.2 in [2] the subspaces M_i are the same for $i = 1, \ldots, m$. By a similarity transformation, i.e.

$$\Sigma \to S^{-1}\Sigma S = \{S^{-1}A_iS : \ i = 1, \ldots, m\}$$

which does not change the properties involved, we can assume that M_i is spanned by the first r unit vectors $e_1, \ldots, e_r$, so that for $i = 1, \ldots, m$,

$$A_i = \begin{pmatrix} I_r & C_i \\ 0 & \widetilde{A}_i \end{pmatrix}.$$

Obviously $\widetilde{\Sigma} = \{\widetilde{A}_1, \ldots, \widetilde{A}_m\}$ has the LCP–property also and its limit function is identically zero. Otherwise if $\widetilde{A}^{(d)} \neq 0$, for some $d \in \{1, \ldots, m\}^{\mathbb{N}}$ we would have $\widetilde{A}_r\widetilde{A}^{(d)} = \widetilde{A}^{(d)}$ for at least one r and $\widetilde{A}_r$ would have 1 as an eigenvalue. This contradicts our assumptions. But then, from Theorem 4.1 in [2], it follows that $\widehat{\rho}(\widetilde{\Sigma}) < 1$. We select some q in $(\widehat{\rho}(\widetilde{\Sigma}), 1)$. By Lemma 2.2 we find a norm $|| \ ||$ on $\mathbb{C}^{k-r}$ such that

$$||\widetilde{A}_i x|| \leq q||x|| \quad \text{for all} \quad x \in \mathbb{C}^{k-r} \quad \text{and all} \quad i = 1, \ldots, m. \tag{12}$$

Denoting by $|| \ ||_2$ the Euclidean norm in $\mathbb{C}^r$, we introduce for any positive ϵ the following vector norm in $\mathbb{C}^k$:

$$\mu_\epsilon(x) = \mu_\epsilon\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \epsilon||x_1||_2 + ||x_2||.$$

Then we observe that

$$\begin{aligned} \mu_\epsilon(A_ix) &= \mu_\epsilon\begin{pmatrix} x_1 + C_ix_2 \\ \widetilde{A}_ix_2 \end{pmatrix} \\ &= \epsilon||x_1 + C_ix_2||_2 + ||\widetilde{A}_ix_2|| \\ &\leq \epsilon||x_1||_2 + (\epsilon||C_i|| + q)||x_2||, \end{aligned} \tag{13}$$

where $||C_i|| = \max\left\{\frac{||C_ix||_2}{||x||}, \ x \in \mathbb{C}^{k-r}\right\}$. Choose $\epsilon > 0$ such that $\tilde{q} = \max_i(\epsilon||C_i|| + q) < 1$ and let $\gamma = (1 - \tilde{q})/(1 + \tilde{q})$. Then we get after some manipulations using (12) and (13) the inequality

$$\mu_\epsilon(A_ix) \leq \mu_\epsilon(x) - \gamma\mu_\epsilon(A_ix - x).$$

Hence $\Sigma \subset \mathcal{N}_\gamma(\mu_\epsilon)$ and (iii) is proved.
$(iii) \Longrightarrow (iv)$ is trivial, while $(iv) \Longrightarrow (i)$ is Theorem 2.3. $\square$

4. Final remarks. The conjecture at the beginning of the previous section remains unsolved even in the case $m = 2$. The following related result was proved in [6].

THEOREM 4.1. *For $\Sigma = \{A_1, A_2\}$ the following are equivalent.*
(i) Σ is an LCP-set.
(ii) (a) there exist a vector norm $||\ ||$ such that

$$||A_i x|| \le ||x||, \qquad i = 1,2 \qquad \textit{for all} \qquad x \in \mathbb{C}^k,$$
$$||A_1 A_2 x|| = ||x|| \Longrightarrow A_1 x = A_2 x = x$$

(b) For $i = 1,2$ if λ is an eigenvalue of A_i, $|\lambda| = 1$, then $\lambda = 1$.
Notice that here we have finitely many conditions characterizing the LCP-property. Nevertheless *(ii)* seems not to imply paracontractivity of Σ.

In the case of two projectors $P_i, i = 1,2$, not necessarily orthogonal, the conjecture can be proved.

THEOREM 4.2. *Let P_i, $i = 1,2$ be projectors, i.e. $P_i^2 = P_i$, $i = 1,2$. Then the following are equivalent.*
(i) $\{P_1, P_2\}$ is an LCP-set.
(ii) There exists a vector norm $||\ ||$ and a positive γ such that

$$\{P_1, P_2\} \subset \mathcal{N}_\gamma(||\ ||).$$

The proof is given after the following auxiliary result.

LEMMA 4.3. *Let A, B be complex $k \times k$ -matrices such that*
(i) B is convergent, i.e. the powers of B converge, and
(ii) $\lim_{n\to\infty} AB^n = 0$.
Then there exists $\alpha \in (0,1)$ such that for any norm $||\ ||$

$$||AB^n|| \le C\alpha^n \qquad \textit{for all} \qquad n \in \mathbb{N}.$$

with $C > 0$ a constant depending on the norm.

Proof. By eventually changing the basis accordingly, we have by (i) that B is of the form

$$B = \begin{pmatrix} I_r & 0 \\ 0 & B_0 \end{pmatrix}$$

with $\alpha = ||B_0|| < 1$ for a suitable norm. Here r is the dimension of $N(I - B)$ and we assume $r > 0$. Otherwise nothing has to be proved. Partitioning $A = (A_1, A_2)$, where A_1 contains the first r columns of A, we get $AB^n = (A_1, A_2 B_0^n)$, and we see from (ii) that $A_1 = 0$. But then clearly

$$||AB^n|| = ||(0, A_2 B_0^n)|| \le C\alpha^n$$

for a suitable C. □

Proof of Theorem 4.2. Obviously we need only to show the implication $(i) \Longrightarrow (ii)$.
Let $|| \; ||$ denote a vector norm satisfying $||P_i x|| \leq ||x||, i = 1, 2, x \in \mathbb{C}^k$ (See Lemma 2.1, (ii)) and define for $n \geq 0$

$$\begin{aligned} a_n(x) &= ||(P_1 - I)(P_2 P_1)^n x|| \\ b_n(x) &= ||(P_2 - I)P_1(P_2 P_1)^n x|| \\ c_n(x) &= ||(P_2 - I)(P_1 P_2)^n x|| \\ d_n(x) &= ||(P_1 - I)P_2(P_1 P_2)^n x|| \end{aligned}$$

By (i) the sequence

$$x_0 = x, x_{2i+1} = P_1 x_{2i}, x_{2i+2} = P_2 x_{2i+1}, i = 0, \ldots$$

is convergent, which gives that $a_n(x) = ||x_{2n+1} - x_{2n}|| \to 0$ and $b_n(x) = ||x_{2n+2} - x_{2n+1}|| \to 0$. The analogous result holds for c_n and d_n. Similarly we prove that the matrices $P_1 P_2$ and $P_2 P_1$ are convergent. Hence by the previous Lemma $r_n(x) \leq C\alpha^n$ for suitable $C > 0, \alpha \in (0,1)$ and $r = a, b, c, d$. This shows that the following expression

$$||x||_* = ||x|| + \max\left(\sum_{n=0}^{\infty}(a_n(x) + b_n(x)),\ \sum_{n=0}^{\infty}(c_n(x) + d_n(x))\right)$$

is finite, and it is easy to see that $||x||_* = 0$ if and only if $x = 0$. Hence it is a norm in $\mathbb{C}^k$. (This is essentially the same construction as in (7), but in this special case we can give a closed expression for the norm). By some simple manipulations we get

$$||P_1 x||_* \leq ||x||_* - a_0(x) = ||x||_* - ||P_1 x - x||$$

and the same result for P_2. As there is a $\gamma > 0$ satisfying $||x|| \geq \gamma ||x||_*$ we see that $\{P_1, P_2\} \subset \mathcal{N}_\gamma(|| \; ||_*)$. $\square$

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