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LEAST SQUARES (P,Q)-ORTHOGONAL SYMMETRIC SOLUTIONS OF THE MATRIX EQUATION AND ITS OPTIMAL APPROXIMATION*

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Abstract. In this paper, the relationship between the (P,Q)-orthogonal symmetric and symmetric matrices is derived. By applying the generalized singular value decomposition, the general expression of the least square (P,Q)-orthogonal symmetric solutions for the matrix equation $A^TXB = C$ is provided. Based on the projection theorem in inner space, and by using the canonical correlation decomposition, an analytical expression of the optimal approximate solution in the least squares (P,Q)-orthogonal symmetric solution set of the matrix equation $A^TXB = C$ to a given matrix is obtained. An explicit expression of the least square (P,Q)-orthogonal symmetric solution for the matrix equation $A^TXB = C$ with the minimum-norm is also derived. An algorithm for finding the optimal approximation solution is described and some numerical results are given.

Key words. Matrix equation, Least squares solution, (P,Q)-orthogonal symmetric matrix, Optimal approximate solution.

AMS subject classifications. 65F15, 65F20.

1. Introduction. Let $R^{m\times n}$ denote the set of all $m \times n$ real matrices, and $SR^{n\times n}$, $OR^{n\times n}$ denote the set of $n\times n$ real symmetric matrices and $n\times n$ orthogonal matrices, respectively. I_n denotes $n \times n$ unit matrix. The notations A^T, $\| A \|$ stand for the transpose and the Frobenius norm of A, respectively. For $A = (a_{ij}) \in R^{m\times n}$, $B = (b_{ij}) \in R^{m\times n}$, $A * B = (a_{ij}b_{ij}) \in R^{m\times n}$ represents the Hadamard product of matrices A and B. Let $SOR^{n\times n} = \{P \in R^{n\times n} | P^T = P, P^2 = I\}$ denote the set of $n \times n$ generalized reflection matrices.

DEFINITION 1.1. Given $P,Q \in SOR^{n\times n}$, we say that $X \in R^{n\times n}$ is (P,Q)-*orthogonal symmetric*, if

$$(PXQ)^T = PXQ.$$

We denote by $SR^{n\times n}(P,Q)$ the set of all (P,Q)-*orthogonal symmetric matrices.*

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In this paper, we consider the following problems.

Problem I. Given $P, Q \in SOR^{n\times n}$, $A \in R^{n\times m}$, $B \in R^{n\times l}$, and $C \in R^{m\times l}$, find $X \in SR^{n\times n}(P, Q)$ such that

$$\| A^T XB - C \|^2 = \min . \tag{1.1}$$

Problem II. Given $P, Q \in SOR^{n\times n}$, $A \in R^{n\times m}$, $B \in R^{n\times l}$, and $C \in R^{m\times l}$, find $Y \in SR^{n\times n}$ such that

$$\| (PA)^T Y(QB) - C \|^2 = \min . \tag{1.2}$$

Problem III. Let S_E be the solution set of Problem I. Given $X^* \in R^{n\times n}$, find $\hat{X} \in S_E$ such that

$$\| \hat{X} - X^* \|^2 = \min_{X\in S_E} \| X - X^* \|^2 .$$

Problem IV. Let S_E be the solution set of Problem I, find $\tilde{X} \in S_E$ such that

$$\| \tilde{X} \|^2 = \min .$$

An inverse problem [2, 3, 6, 7] arising in structural modification of the dynamic behavior of a structure calls for the solution of certain linear matrix equations. The matrix equation

$$A^T XB = C$$

with X being orthogonal-symmetric has been studied by Peng [15] which gives the necessary and sufficient conditions for the existence and the general solution expression. In [16], the necessary and sufficient conditions for the solvability of the matrix equation

$$A^H XB = C$$

over the sets of reflexive and anti-reflexive matrices are given, and the general expressions for the reflexive and anti-reflexive solutions are obtained. Don [9], Magnus [12], and Chu [5] have discussed the matrix equation

$$BXA^T = T$$

where the solution matrices are known to have a given structure (e.g., symmetric, triangular, diagonal), either directly from the matrix equation or indirectly from the

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equivalent vector equation. But they did not consider the least squares solutions of the equation.

For the least squares problem, the (M,N)-symmetric Procrustes problem of the matrix equation $AX = B$ has been treated in [13]. The least squares orthogonal-symmetric solutions of the matrix equation $A^T XB = C$, and the least squares symmetric, skew-symmetric solutions of the equation $BXA^T = T$ have been considered, respectively, in [14] and [8]. Recently, Qiu, Zhang and Lu in [17] have proposed an iterative method for the least squares problem of the matrix equation $BXA^T = F$.

Problem III, that is, the optimal approximation problem of a matrix with the given matrix restriction, is proposed in the processes of test or recovery of linear systems with incomplete data or revising data. The optimal estimation $\hat{X}$ is a matrix that not only satisfies the given restriction but also best approximates the given matrix.

In this paper, we will discuss the least square (P,Q)-orthogonal symmetric solutions and its optimal approximation for the matrix equation $A^T XB = C$. By using the generalized singular value decomposition (GSVD), the projection theorem and the canonical correlation decomposition (CCD), we obtain the general expressions of the solutions for Problem I, II, III and IV.

The paper is organized as follows. In section 2, we will give the general expressions of the solutions for Problem I and II. In section 3, we will discuss Problem III and IV. In section 4, we will give an algorithm to compute the solution of Problem III and numerical examples.

2. The solutions of Problem I and II. In this section, we derive the general expressions for the solutions of Problem I and II.

THEOREM 2.1. *Problem I has a solution if and only if Problem II has a solution.*

Proof. Suppose X be one of the solutions of Problem I, then we have $(PXQ)^T = PXQ$, and

$$\| A^T XB - C \|^2 = \min . \tag{2.1}$$

Let $Y = PXQ$, then $Y^T = Y$. From (2.1), we get $\| (PA)^T YQB - C \|^2 = \min$, that is, Y is one of least squares symmetric solutions of Problem II.

On the contrary, if Y is one of the least squares symmetric solutions of Problem II, then we have $Y^T = Y$ and

$$\| (PA)^T Y(QB) - C \|^2 = \min . \tag{2.2}$$

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Let $X = PYQ$, then $(PXQ)^T = PXQ$. From (2.2), we get $\| A^T XB - C \|^2 = \min$, that is, X is one of the least squares (P, Q)-orthogonal symmetric solutions of Problem I. ▯

LEMMA 2.2. *Let $D_1 = diag(a_1, a_2, \cdots, a_n) > 0, D_2 = diag(b_1, b_2, \cdots, b_n) > 0$, and $E = (e_{ij}) \in R^{n\times n}$, then there exists a unique $S \in SR^{n\times n}$ such that*

$$\| D_1 S D_2 - E \|^2 = \min, \tag{2.3}$$

and

$$S = \phi * (D_1 E D_2 + D_2 E^T D_1), \tag{2.4}$$

where $\phi = (\phi_{ij}),\ \phi_{ij} = \frac{1}{a_i^2 b_j^2 + a_j^2 b_i^2},\ i, j = 1, 2, \cdots, n.$

Proof. For any $S = (s_{ij}) \in SR^{n\times n}$, $E = (e_{ij}) \in R^{n\times n}$, we have

$$\| D_1 S D_2 - E \|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}(a_i s_{ij} b_j - e_{ij})^2 = \sum_{i=1}^{n}(a_i s_{ii} b_i - e_{ii})^2 + \sum_{1\le i<j\le n}[(a_i^2 b_j^2 + a_j^2 b_i^2)s_{ij}^2 - 2(a_i b_j e_{ij} + a_j b_i e_{ji})s_{ij} + (e_{ij}^2 + e_{ji}^2)].$$

Hence, there exist a unique solution $S = (s_{ij}) \in SR^{n\times n}$ such that (2.3) holds and

$$s_{ij} = \frac{a_i e_{ij} b_j + a_j e_{ji} b_i}{a_i^2 b_j^2 + a_j^2 b_i^2}, \quad 1 \le i, j \le n.$$

That is (2.4). ▯

LEMMA 2.3. *Suppose that the matrices P, Q, A, B, and C are given in Problem I. Decompose the matrix pair $[PA, QB]$ by using GSVD (see[19])as*

$$PA = W\Sigma_{PA}U^T, \quad QB = W\Sigma_{QB}V^T, \tag{2.5}$$

where W is a nonsingular $n \times n$ matrix, $U \in OR^{m\times m}$, $V \in OR^{l\times l}$, and

$$\Sigma_{PA} = \begin{pmatrix} I & & \\ & D_{PA} & \\ & & 0 \\ & 0 & \end{pmatrix} \begin{matrix} t \\ s \\ r-s-t \\ n-r \end{matrix}, \quad \Sigma_{QB} = \begin{pmatrix} 0 & & \\ & D_{QB} & \\ & & I \\ & 0 & \end{pmatrix} \begin{matrix} t \\ s \\ r-s-t \\ n-r \end{matrix}.$$

Here, $r = rank([PA, QB])$, $s = rank(PA) + rank(QB) - r$, $t = rank(PA) - s$, and $D_{PA} = diag(\alpha_1, \alpha_2, \cdots, \alpha_s) > 0$, $D_{QB} = diag(\beta_1, \beta_2, \cdots, \beta_s) > 0$ with $1 > \alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_s > 0$, $0 < \beta_1 \le \beta_2 \le \cdots \le \beta_s < 0$, and $\alpha_i^2 + \beta_i^2 = 1$, $i = 1, 2, \cdots, s$.

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Let $W = (W_1, W_2, W_3, W_4)$, $W_1 \in R^{n\times t}, W_2 \in R^{n\times s}, W_3 \in R^{n\times(r-s-t)}, W_4 \in R^{n\times(n-r)}$, and let

$$W^TYW = (\tilde{X}_{ij}), \text{ i.e., } \tilde{X}_{ij} = W_i^TYW_j, \quad i,j = 1,2,3,4, \tag{2.6}$$

$$U^TCV = \begin{array}{c} t \\ s \\ m-s-t \end{array} \left(\begin{array}{ccc} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} \\ \tilde{C}_{21} & \tilde{C}_{22} & \tilde{C}_{23} \\ \tilde{C}_{31} & \tilde{C}_{32} & \tilde{C}_{33} \end{array} \right) \begin{array}{c} \\ \\ \end{array} . \tag{2.7}$$
$$\qquad\qquad\qquad l+t-r \quad s \quad r-s-t$$

THEOREM 2.4. *Given $A \in R^{n\times m}$, $B \in R^{n\times l}$, $C \in R^{m\times l}$, and $P, Q \in SOR^{n\times n}$. Let the GSVD of the matrix pair $[PA, QB]$ be of form (2.5). Partition W^TYW and U^TCV according to (2.6) and (2.7), respectively. Then the general solution of Problem II can be expressed as*

$$Y = W^{-T} \left(\begin{array}{cccc} \tilde{X}_{11} & \tilde{C}_{12}D_{QB}^{-1} & \tilde{C}_{13} & \tilde{X}_{14} \\ D_{QB}^{-1}\tilde{C}_{12}^T & \tilde{X}_{22} & D_{PA}^{-1}\tilde{C}_{23} & \tilde{X}_{24} \\ \tilde{C}_{13}^T & \tilde{C}_{23}^TD_{PA}^{-1} & \tilde{X}_{33} & \tilde{X}_{34} \\ \tilde{X}_{14}^T & \tilde{X}_{24}^T & \tilde{X}_{34}^T & \tilde{X}_{44} \end{array} \right) W^{-1}, \tag{2.8}$$

where

$$\tilde{X}_{22} = \tilde{\phi} * (D_{PA}\tilde{C}_{22}D_{QB} + D_{QB}\tilde{C}_{22}^TD_{PA}), \ \tilde{\phi} = (\tilde{\phi}_{ij}),$$

$$\tilde{\phi}_{ij} = \frac{1}{\alpha_i^2\beta_j^2 + \alpha_j^2\beta_i^2}, \ i,j = 1,2,\cdots,s.$$

The matrices $\tilde{X}_{11}, \tilde{X}_{33}, \tilde{X}_{44}$ are arbitrary symmetric, $\tilde{X}_{14}, \tilde{X}_{24}, \tilde{X}_{34}$ are arbitrary.

Proof. Suppose that Y is one of the least squares symmetric solutions for Problem II, then $Y^T = Y$, and so $(W^TYW)^T = W^TYW$, i.e., $\tilde{X}_{ij} = \tilde{X}_{ji}^T$, $i,j = 1,2,3,4$.

Substitute the matrices PA, QB in (2.5) into (1.2), from orthogonal invariance of the Frobenius norm together with (2.6) and (2.7), we have

$$\begin{aligned} & \| (PA)^TY(QB) - C \|^2 \\ = & \| U\Sigma_{PA}^TW^TYW\Sigma_{QB}V^T - C \|^2 \\ = & \| \Sigma_{PA}^T(W^TYW)\Sigma_{QB} - U^TCV \|^2 \\ = & \left\| \left(\begin{array}{ccc} 0 & \tilde{X}_{12}D_{QB} & \tilde{X}_{13} \\ 0 & D_{PA}\tilde{X}_{22}D_{QB} & D_{PA}\tilde{X}_{23} \\ 0 & 0 & 0 \end{array} \right) - \left(\begin{array}{ccc} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} \\ \tilde{C}_{21} & \tilde{C}_{22} & \tilde{C}_{23} \\ \tilde{C}_{31} & \tilde{C}_{32} & \tilde{C}_{33} \end{array} \right) \right\|^2 . \end{aligned}$$

So the condition $\| (PA)^TY(QB) - C \|^2 = \min$ is equivalent to the following conditions:

$$\| \tilde{X}_{12}D_{QB} - \tilde{C}_{12} \|^2 = \min, \quad \| D_{PA}\tilde{X}_{22}D_{QB} - \tilde{C}_{22} \|^2 = \min,$$
$$\| \tilde{X}_{13} - \tilde{C}_{13} \|^2 = \min, \qquad \| D_{PA}\tilde{X}_{23} - \tilde{C}_{23} \|^2 = \min\, .$$

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Then, $\tilde{X}_{12} = \tilde{C}_{12}D_{QB}^{-1}$, $\tilde{X}_{13} = \tilde{C}_{13}$, $\tilde{X}_{23} = D_{PA}^{-1}\tilde{C}_{23}$. From Lemma 2.2, $\tilde{X}_{22} = \tilde{\phi} * (D_{PA}\tilde{C}_{22}D_{QB} + D_{QB}\tilde{C}_{22}^T D_{PA})$ with $\tilde{\phi} = (\tilde{\phi}_{ij})$, $\tilde{\phi}_{ij} = \frac{1}{\alpha_i^2\beta_j^2+\alpha_j^2\beta_i^2}$, $i, j = 1, 2, \cdots, s$. Then the general solution of Problem II can be expressed as by (2.8). The proof is completed. ◻

THEOREM 2.5. *Let $A \in R^{n\times m}, B \in R^{n\times l}, C \in R^{m\times l}$, and $P, Q \in SOR^{n\times n}$. If the GSVD of $[PA, QB]$ is of form (2.5), W^TYW and U^TCV are partitioned into (2.6) and (2.7) respectively, then the general solution of Problem I can be expressed as*

$$X = PW^{-T}\begin{pmatrix} \tilde{X}_{11} & \tilde{C}_{12}D_{QB}^{-1} & \tilde{C}_{13} & \tilde{X}_{14} \\ D_{QB}^{-1}\tilde{C}_{12}^T & \tilde{X}_{22} & D_{PA}^{-1}\tilde{C}_{23} & \tilde{X}_{24} \\ \tilde{C}_{13}^T & \tilde{C}_{23}^T D_{PA}^{-1} & \tilde{X}_{33} & \tilde{X}_{34} \\ \tilde{X}_{14}^T & \tilde{X}_{24}^T & \tilde{X}_{34}^T & \tilde{X}_{44} \end{pmatrix} W^{-1}Q, \tag{2.9}$$

where $\tilde{X}_{11}$, $\tilde{X}_{22}$, $\tilde{X}_{33}$, $\tilde{X}_{44}$, $\tilde{X}_{14}$, $\tilde{X}_{24}$, $\tilde{X}_{34}$ are the same as in Theorem 2.4

Proof. From Theorem 2.1 and Theorem 2.4, it can be easily proved. ◻

3. The solutions of Problem III and IV. In this section, we derive analytical expressions of the solutions for Problem III and IV. To this end, we first transform the least squares problem (1.1) with respect to the matrix equation $A^TXB = C$ into a consistent matrix equation, by using the projection theorem.

LEMMA 3.1. *(Projection Theorem [18]) Let S be an inner product space, K be a subspace of S. For given $x \in S$, if there exists a $y_0 \in K$ such that $\| x - y_0 \| \leq \| x - y \|$ holds for all $y \in K$, then y_0 is unique. Moreover y_0 is the unique minimization vector in K if and only if $(x - y_0) \perp K$.*

THEOREM 3.2. *Suppose that the matrices P, Q, A, B, and C are given in Problem I, and the matrix X_0 is one of the solutions of Problem I. Let*

$$C_0 = A^TX_0B. \tag{3.1}$$

Then the (P, Q)-orthogonal symmetric solution set of the consistent matrix equation

$$A^TXB = C_0 \tag{3.2}$$

is the same as the solution set of Problem I.

Proof. Let

$$L = \{Y | Y = A^TXB, \forall X \in SR^{n\times n}(P, Q), A \in R^{n\times m}, B \in R^{n\times l}\}.$$

Then L is a subspace of $R^{m\times l}$. From (3.1), it is obvious that $C_0 \in L$, and

$$\| A^TX_0B - C \| = \min_{X\in SR^{n\times n}(P,Q)} \| A^TXB - C \| = \min_{Y\in L} \| Y - C \| .$$

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Now, by Lemma 3.1, we have

$$(A^T X_0 B - C) \perp L.$$

For all $X \in SR^{n\times n}(P,Q)$, we have

$$(A^T XB - A^T X_0 B) \in L.$$

It then follows that,

$$\begin{aligned} \| A^T XB - C \|^2 &= \| A^T XB - A^T X_0 B + A^T X_0 B - C \|^2 \\ &= \| A^T XB - A^T X_0 B \|^2 + \| A^T X_0 B - C \|^2 . \end{aligned}$$

Hence, the conclusion of this theorem holds. □

From Theorem 3.2, we easily see that the optimal approximate (P,Q)-orthogonal symmetric solution $\hat{X}$ of the consistent matrix equation (3.2) to a given matrix X^* is just the solution of Problem III. Thus, how to find C_0 is the crux for solving Problem III. So we need the following theorem.

Theorem 3.3. *Suppose that the matrices P, Q, A, B, and C are given in Problem I. Let the GSVD of the matrix pair $[PA, QB]$ be of form (2.5). Then the matrix C_0 can be expressed as*

$$C_0 = U \begin{pmatrix} 0 & \tilde{C}_{12} & \tilde{C}_{13} \\ 0 & D_{PA}\tilde{X}_{22}D_{QB} & \tilde{C}_{23} \\ 0 & 0 & 0 \end{pmatrix} V^T, \tag{3.3}$$

where $\tilde{X}_{22}$ is the same as in Theorem 2.4.

Proof. From Theorem 2.5, we know that the least squares (P,Q)-orthogonal symmetric solution X_0 of Problem I can be given by (2.9). By substituting (2.9) and (2.5) into the equation $C_0 = A^T X_0 B$, after straightforward computation, we can immediately obtain (3.3). □

Evidently, (3.3) shows that the matrix C_0 given in Theorem 3.3 is dependent only on the given matrices A, B, C, P, and Q, but independent on the least squares (P,Q)-orthogonal symmetric solution X_0 of Problem I. Furthermore, we can conclude that

$$\| C_0 - C \|^2 = \min_{X \in SR^{n\times n}(P,Q)} \| A^T XB - C \|^2 .$$

From the equation above, we know that the matrix equation $A^T XB = C$ is consistent if and only if $C_0 = C$.

To derive the solutions of Problem III and IV, we need to use the CCD of the matrix pair $[PA, QB]$.

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Lemma 3.4. *Suppose that the matrices P, Q, A, B, and C are given in Problem I. Decompose the matrix pair $[PA, QB]$ by using CCD (see[10]) as*

$$PA = M(\Pi_{PA}, 0)E_{PA}, \quad QB = M(\Pi_{QB}, 0)E_{QB}, \tag{3.4}$$

where $E_{PA} \in R^{m\times m}$, $E_{QB} \in R^{l\times l}$ are nonsingular matrices, $M \in OR^{n\times n}$, and

$$\Pi_{PA} = \begin{pmatrix} I_s & 0 & 0 \\ 0 & \Lambda_j & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \Delta_j & 0 \\ 0 & 0 & I_{t'} \end{pmatrix}, \ \Pi_{QB} = \begin{pmatrix} I_h \\ 0 \end{pmatrix},$$

are block matrices, with the diagonal matrices Λ_j and Δ_j given by

$$\Lambda_j = diag(\lambda_1, \cdots, \lambda_j),\ 1 > \lambda_1 \geq \cdots \geq \lambda_j > 0,$$
$$\Delta_j = diag(\sigma_1, \cdots, \sigma_j),\ 0 < \sigma_1 \leq \cdots \leq \sigma_j < 1, \text{ and } \Lambda_j^2 + \Delta_j^2 = I_j.$$

Here, $g = rank(PA)$, $h = rank(QB)$, $s = rank(PA)+rank(QB)-rank([PA, QB])$, $j = rank(B^TPA) - s$, $t' = rank(PA) - s - j$, and $g = s + j + t'$.

Let $M = (M_1, M_2, M_3, M_4, M_5, M_6)$, $M_1 \in R^{n\times s}$, $M_2 \in R^{n\times j}$, $M_3 \in R^{n\times(h-s-j)}$, $M_4 \in R^{n\times(n-h-j-t')}$, $M_5 \in R^{n\times j}$, $M_6 \in R^{n\times t'}$. Partition M^TYM and $E_{PA}^{-T}C_0E_{QB}^{-1}$ into the following forms:

$$M^TYM = (X_{ij}),\ X_{ij} = M_i^TYM_j,\ i,j = 1, 2, \cdots, 6, \tag{3.5}$$

$$E_{PA}^{-T}C_0E_{QB}^{-1} = \begin{matrix} s \\ j \\ t' \\ m-g \end{matrix} \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{pmatrix}. \tag{3.6}$$
$$\qquad\qquad\qquad\qquad s \qquad j \qquad h-s-j \qquad l-h$$

Theorem 3.5. *Suppose that the matrices P,Q,A,B, and C are given in Problem I, then the equation $A^TXB = C$ has a solution $X \in SR^{n\times n}(P, Q)$ if and only if the equation $(PA)^TYQB = C$ has a solution $Y \in SR^{n\times n}$, and $X = PYQ$.*

Proof. Suppose X be one of the (P, Q)-orthogonal symmetric solutions of the equation $A^TXB = C$, and let $Y = PXQ$. Then, we have $Y^T = Y$ and $(PA)^TYQB = C$, that is, Y is one of the symmetric solutions of the equation $(PA)^TYQB = C$.

Conversely, if the equation $(PA)^TYQB = C$ has a solution $Y \in SR^{n\times n}$, then let $X = PYQ$, we have $(PXQ)^T = PXQ$ and $A^TXB = C$, that is, X is one of the (P, Q)-orthogonal symmetric solutions of the equation $A^TXB = C$, and $X = PYQ$. The proof is completed. ∎

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THEOREM 3.6. *Suppose that matrices P, Q, A, B are given in Problem I, and C_0 is given by (3.3). Let the CCD of the matrix pair $[PA, QB]$ be of form (3.4). Partition the matrices M^TYM and $E_{PA}^{-T}C_0E_{QB}^{-1}$ according to (3.5) and (3.6), respectively. Then the general (P,Q)-orthogonal symmetric solution of the equation $A^TXB = C_0$ can be expressed as*

$$X = PM \begin{pmatrix} C_{11} & C_{12} & C_{13} & X_{14} & X_{15} & C_{31}^T \\ C_{12}^T & X_{22} & X_{23} & X_{24} & X_{25} & C_{32}^T \\ C_{13}^T & X_{23}^T & X_{33} & X_{34} & X_{35} & C_{33}^T \\ X_{14}^T & X_{24}^T & X_{34}^T & X_{44} & X_{45} & X_{46} \\ X_{15}^T & X_{25}^T & X_{35}^T & X_{45}^T & X_{55} & X_{56} \\ C_{31} & C_{32} & C_{33} & X_{46}^T & X_{56}^T & X_{66} \end{pmatrix} M^TQ, \tag{3.7}$$

where

$$X_{15} = (C_{21}^T - C_{12}\Lambda_j)\Delta_j^{-1},\ X_{25} = (C_{22}^T - X_{22}^T\Lambda_j)\Delta_j^{-1},\ X_{35} = (C_{23}^T - X_{23}^T\Lambda_j)\Delta_j^{-1},$$

the matrices X_{ii}, $i = 2,3,4,5,6$ are symmetric matrices with suitable dimensions, and other unknown X_{ij} are arbitrary.

Proof. From Theorem 3.5, we first consider the symmetric solutions of the equation $(PA)^TYQB = C_0$. Since $Y^T = Y$, we have M^TYM is symmetric, i.e, $X_{ij} = X_{ji}^T, i,j = 1,2,\cdots,6$. By inserting the matrices PA and QB in (3.4) into the equation $(PA)^TYQB = C_0$, we get

$$E_{PA}^T(\Pi_{PA}, 0)^T M^TYM(\Pi_{QB}, 0)E_{QB} = C_0$$

Since E_{PA}, E_{QB} are nonsingular, then

$$(\Pi_{PA}, 0)^T M^TYM(\Pi_{QB}, 0) = E_{PA}^{-T}C_0E_{QB}^{-1}$$

According to (3.5) and (3.6), we have

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} & 0 \\ \Lambda_jX_{21} + \Delta_jX_{51} & \Lambda_jX_{22} + \Delta_jX_{52} & \Lambda_jX_{23} + \Delta_jX_{53} & 0 \\ X_{61} & X_{62} & X_{63} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{pmatrix}.$$

From the above equation, we get

$$C_{11}^T = C_{11},\ C_{i4} = 0,\ C_{4j} = 0,\ (i,j = 1,2,3,4).$$

and

$$X_{11} = C_{11},\ X_{12} = C_{12},\ X_{13} = C_{13},\ X_{61} = C_{31}, X_{62} = C_{32},\ X_{63} = C_{33},$$
$$\Lambda_jX_{21} + \Delta_jX_{51} = C_{21},\ \Lambda_jX_{22} + \Delta_jX_{52} = C_{22},\ \Lambda_jX_{23} + \Delta_jX_{53} = C_{23}.$$

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After straightforward computation, and from Theorem 3.5, the (P, Q)-orthogonal symmetric solution for the equation $A^TXB = C_0$ can be expressed as (3.7). □

The following lemmas are important for deriving an analytical formula of the solution for Problem III.

Lemma 3.7. *([15]) Suppose that the matrices $E \in R^{n\times m}$, $K \in R^{m\times n}$, $F \in R^{n\times m}$, $H \in R^{m\times n}$, and $D = diag(a_1, a_2, \cdots, a_n) > 0$.*

(1) There exists a unique $G \in R^{n\times m}$ such that

$$g(G) = \| G - E \|^2 + \| G^T - K \|^2 = \min,$$

and

$$G = \frac{1}{2}(E + K^T).$$

(2) There exists a unique $G \in R^{n\times m}$ such that

$$g(G) = \| G - E \|^2 + \| G^T - K \|^2 + \| DG - F \|^2 + \| G^TD - H \|^2 = \min,$$

and

$$G = \frac{1}{2}\varphi_1 * (E + K^T + DF + DH^T),$$

with $\varphi_1 = (\varphi_{ij}),\ \varphi_{ij} = 1/(1 + a_i^2),\ (i, j = 1, 2, \cdots, n)$.

Lemma 3.8. *([15]) Suppose that $D = diag(a_1, a_2, \cdots, a_n) > 0$, and $E, F, H \in R^{n\times n}$, then there exists a unique $G \in SR^{n\times n}$ such that*

$$g(G) = \| G - E \|^2 + \| DG - F \|^2 + \| G^TD - H \|^2 = \min,$$

and

$$G = \frac{1}{2}\varphi_2 * (E + E^T + D(F + H^T) + (F^T + H)D),$$

with $\varphi_2 = (\varphi_{ij}),\ \varphi_{ij} = 1/(1 + a_i^2 + a_j^2),\ (i, j = 1, 2, \cdots, n)$.

Theorem 3.9. *Given $X^* \in R^{n\times n}$, and the matrices P, Q, A, B, C are the same as in Problem I. Partition the matrix M^TPX^*QM into the following form*

$$M^TPX^*QM = (\bar{X}_{ij})_{6\times 6} \qquad (3.8)$$

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compatibly with the row partitioning of Π_{PA}*, where the matrix* M *is given in (3.4). Then there exists a unique solution* $\hat{X}$ *for Problem III and* $\hat{X}$ *can be expressed as*

$$\hat{X} = PM \begin{pmatrix} C_{11} & C_{12} & C_{13} & \hat{X}_{14} & \hat{X}_{15} & C_{31}^T \\ C_{12}^T & \hat{X}_{22} & \hat{X}_{23} & \hat{X}_{24} & \hat{X}_{25} & C_{32}^T \\ C_{13}^T & \hat{X}_{23}^T & \hat{X}_{33} & \hat{X}_{34} & \hat{X}_{35} & C_{33}^T \\ \hat{X}_{14}^T & \hat{X}_{24}^T & \hat{X}_{34}^T & \hat{X}_{44} & \hat{X}_{45} & \hat{X}_{46} \\ \hat{X}_{15}^T & \hat{X}_{25}^T & \hat{X}_{35}^T & \hat{X}_{45}^T & \hat{X}_{55} & \hat{X}_{56} \\ C_{31} & C_{32} & C_{33} & \hat{X}_{46}^T & \hat{X}_{56}^T & \hat{X}_{66} \end{pmatrix} M^T Q, \tag{3.9}$$

where

$$\begin{aligned} \hat{X}_{22} &= \frac{1}{2}\hat{\phi} * [\Delta_j^2(\bar{X}_{22} + \bar{X}_{22}^T)\Delta_j^2 + 2(\Lambda_j C_{22}\Delta_j^2 + \Delta_j^2 C_{22}^T \Lambda_j) \\ &\quad -\Delta_j\Lambda_j(\bar{X}_{52} + \bar{X}_{25}^T)\Delta_j^2 - \Delta_j^2(\bar{X}_{52}^T + \bar{X}_{25})\Lambda_j\Delta_j], \\ \hat{X}_{23} &= \frac{1}{2}\Delta_j^2(\bar{X}_{23} + \bar{X}_{32}^T) + \Lambda_j C_{23} - \frac{1}{2}\Lambda_j\Delta_j(\bar{X}_{35}^T + \bar{X}_{53}), \\ \hat{X}_{15} &= (C_{21}^T - C_{12}\Lambda_j)\Delta_j^{-1},\ \hat{X}_{25} = (C_{22}^T - \hat{X}_{22}^T\Lambda_j)\Delta_j^{-1},\ \hat{X}_{35} = (C_{23}^T - \hat{X}_{23}^T\Lambda_j)\Delta_j^{-1}, \end{aligned}$$

with $\hat{\phi} = (\hat{\phi}_{ij}) \in R^{s\times s}$, $\hat{\phi}_{ij} = 1/(\sigma_i^2\sigma_j^2 + \lambda_i^2\sigma_j^2 + \lambda_j^2\sigma_i^2)$, $(i,j = 1,2,\cdots,s)$*, and other unknown* $\hat{X}_{ij} = \frac{1}{2}(\bar{X}_{ij} + \bar{X}_{ji}^T)$.

Proof. It is easy to verify that the solution set S_E is nonempty and is a closed convex set. Therefore, there exists a unique solution for Problem III [17]. From Theorems 3.2 and 3.3, we know that the solution set S_E of Problem I is the same as the (P,Q)-orthogonal symmetric solution set of the consistent equation (3.2). From Theorem 3.6, we know that the (P,Q)-orthogonal symmetric solution of the consistent equation (3.2) can be expressed as (3.7).

From the orthogonal invariance of the Frobenius norm together with (3.8) and (3.7), we have

$$\begin{aligned} &\| X - X^* \|^2 \\ &= \| M^T PXQM - M^T PX^*QM \|^2 \\ &= \left\| \begin{pmatrix} C_{11}-\bar{X}_{11} & C_{12}-\bar{X}_{12} & C_{13}-\bar{X}_{13} & X_{14}-\bar{X}_{14} & X_{15}-\bar{X}_{15} & C_{31}^T-\bar{X}_{16} \\ C_{12}^T-\bar{X}_{21} & X_{22}-\bar{X}_{22} & X_{23}-\bar{X}_{23} & X_{24}-\bar{X}_{24} & X_{25}-\bar{X}_{25} & C_{32}^T-\bar{X}_{26} \\ C_{13}^T-\bar{X}_{31} & X_{23}^T-\bar{X}_{32} & X_{33}-\bar{X}_{33} & X_{34}-\bar{X}_{34} & X_{35}-\bar{X}_{35} & C_{33}^T-\bar{X}_{36} \\ X_{14}^T-\bar{X}_{41} & X_{24}^T-\bar{X}_{42} & X_{34}^T-\bar{X}_{43} & X_{44}-\bar{X}_{44} & X_{45}-\bar{X}_{45} & X_{46}-\bar{X}_{46} \\ X_{15}^T-\bar{X}_{51} & X_{25}^T-\bar{X}_{52} & X_{35}^T-\bar{X}_{53} & X_{45}^T-\bar{X}_{54} & X_{55}-\bar{X}_{55} & X_{56}-\bar{X}_{56} \\ C_{31}-\bar{X}_{61} & C_{32}-\bar{X}_{62} & C_{33}-\bar{X}_{63} & X_{46}^T-\bar{X}_{64} & X_{56}^T-\bar{X}_{65} & X_{66}-\bar{X}_{66} \end{pmatrix} \right\|^2 \end{aligned}$$

Hence,

$$\| X - X^* \|^2 = \min, \quad \forall X \in S_E$$

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if and only if

$$\begin{aligned} &\| X_{ii} - \bar{X}_{ii} \|^2 = \min, \quad i = 3, 4, 5, 6. \\ &\| X_{i4} - \bar{X}_{i4} \|^2 + \| X_{i4}^T - \bar{X}_{4i} \|^2 = \min, \quad i = 1, 2, 3. \\ &\| X_{4j} - \bar{X}_{4j} \|^2 + \| X_{4j}^T - \bar{X}_{j4} \|^2 = \min, \quad j = 5, 6. \\ &\| X_{56} - \bar{X}_{56} \|^2 + \| X_{56}^T - \bar{X}_{65} \|^2 = \min, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} &\| X_{22} - \bar{X}_{22} \|^2 + \| (C_{22}^T - X_{22}^T \Lambda_j) \Delta_j^{-1} - \bar{X}_{25} \|^2 \\ &\qquad + \| \Delta_j^{-1} (C_{22} - \Lambda_j X_{22}) - \bar{X}_{52} \|^2 = \min, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &\| X_{23} - \bar{X}_{23} \|^2 + \| \Delta_j^{-1} (C_{23} - \Lambda_j X_{23}) - \bar{X}_{53} \|^2 \\ &+ \| X_{23}^T - \bar{X}_{32} \|^2 + \| (C_{23}^T - X_{23}^T \Lambda_j) \Delta_j^{-1} - \bar{X}_{35} \|^2 = \min. \end{aligned} \tag{3.12}$$

By making use of Lemma 3.7 (1) and Lemma 2.1, we know that the solution of (3.10) is of the form

$$\hat{X}_{ij} = \frac{1}{2} (\bar{X}_{ij} + \bar{X}_{ji}^T).$$

By Lemma 3.8, we know that the solution of (3.11) is

$$\begin{aligned} \hat{X}_{22} = \frac{1}{2} \hat{\phi} * [&\Delta_j^2 (\bar{X}_{22} + \bar{X}_{22}^T) \Delta_j^2 + 2(\Lambda_j C_{22} \Delta_j^2 + \Delta_j^2 C_{22}^T \Lambda_j) \\ &- \Delta_j \Lambda_j (\bar{X}_{52} + \bar{X}_{25}^T) \Delta_j^2 - \Delta_j^2 (\bar{X}_{52}^T + \bar{X}_{25}) \Lambda_j \Delta_j], \end{aligned}$$

with $\hat{\phi} = (\hat{\phi}_{ij}) \in R^{s \times s}$, $\hat{\phi}_{ij} = 1/(\sigma_i^2 \sigma_j^2 + \lambda_i^2 \sigma_j^2 + \lambda_j^2 \sigma_i^2)$, $(i, j = 1, 2, \cdots, s)$. From Lemma 3.7 (2) and (3.12), we get

$$\hat{X}_{23} = \frac{1}{2} \Delta_j^2 (\bar{X}_{23} + \bar{X}_{32}^T) + \Lambda_j C_{23} - \frac{1}{2} \Lambda_j \Delta_j (\bar{X}_{35}^T + \bar{X}_{53}).$$

From Theorem 3.6, we immediately get $\hat{X}_{25} = (C_{22}^T - \hat{X}_{22}^T \Lambda_j) \Delta_j^{-1}$, $\hat{X}_{35} = (C_{23}^T - \hat{X}_{23}^T \Lambda_j) \Delta_j^{-1}$. Then, the proof is completed. □

In Theorem 3.9, if $X^* = 0$, then we will derive an analytical expression of the solution for Problem IV.

THEOREM 3.10. *Let matrices P, Q, A, B, C be given in Problem I. Then there exists a unique solution $\tilde{X}$ for Problem IV and $\tilde{X}$ can be expressed as*

$$\tilde{X} = PM \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & \hat{X}_{15} & C_{31}^T \\ C_{12}^T & \hat{X}_{22} & \Lambda_j C_{23} & 0 & \hat{X}_{25} & C_{32}^T \\ C_{13}^T & C_{23}^T \Lambda_j & 0 & 0 & C_{23}^T \Delta_j & C_{33}^T \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hat{X}_{15}^T & \hat{X}_{25}^T & \Delta_j C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \end{pmatrix} M^T Q, \tag{3.13}$$

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where $\hat{X}_{15}$, $\hat{X}_{25}$, $\hat{X}_{22}$ *are the same as in Theorem 3.9.*

Proof. The proof of this theorem is similar to that of Theorem 3.9, and it only needs to let $X^* = 0$. From (3.9), we can easily get (3.13). ▯

4. Numerical examples. Based on Theorem 3.9, we formulate the following algorithm to find the solution $\hat{X}$ of problem III.

Algorithm

Step 1. Input matrices A, B,C, P, Q and X^*;

Step 2. Make the GSVD of the matrix pair $[PA, QB]$, and partition $U^T CV$ according to (2.7);

Step 3. Compute C_0 by (3.3);

Step 4. Make the CCD of the matrix pair $[PA, QB]$, and partition $E_{PA}^{-1} C_0 E_{QB}^{-1}$ according to (3.6);

Step 5. Compute $\hat{X}$ by (3.9).

Example 4.1. *Given*

$$A = \begin{pmatrix} 0.7119 & 1.1908 & -0.1567 & -1.0565 \\ 1.2902 & -1.2025 & -1.6041 & 1.4151 \\ 0.6686 & -0.0198 & 0.2573 & -0.8051 \end{pmatrix}, \quad B = \begin{pmatrix} 0.5287 & -2.1707 & 0.6145 \\ 0.2193 & -0.0592 & 0.5077 \\ -0.9219 & -1.0106 & 1.692 \end{pmatrix},$$

$$C = \begin{pmatrix} -0.4326 & -1.1465 & 0.3273 \\ -1.6656 & 1.1909 & 0.1746 \\ 0.1253 & 1.1892 & -0.1867 \\ 0.2877 & -0.0376 & 0.7258 \end{pmatrix}, \quad X^* = \begin{pmatrix} -0.4326 & 0.2877 & 1.1892 \\ -1.6656 & -1.1465 & -0.0376 \\ 0.1253 & 1.1909 & 0.3273 \end{pmatrix},$$

and

$$P = \begin{pmatrix} -0.2105 & -0.6612 & 0.7201 \\ -0.6612 & -0.4463 & -0.6031 \\ 0.7201 & -0.6031 & -0.3432 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.3306 & 0.0408 & 0.9429 \\ 0.0408 & -0.9987 & 0.0289 \\ 0.9429 & 0.0289 & -0.3318 \end{pmatrix}.$$

By using the Algorithm, we get the matrix C_0 and the unique solution of problem III as follows:

$$C_0 = \begin{pmatrix} -0.0935 & -0.6535 & 0.5605 \\ -0.1044 & 0.0193 & 0.2883 \\ 0.1114 & 0.3393 & -0.1869 \\ 0.0095 & 0.1344 & -0.4000 \end{pmatrix}, \quad \hat{X} = \begin{pmatrix} 0.0188 & 0.1046 & 0.1768 \\ 0.0785 & -0.0959 & 0.0694 \\ 0.1123 & 0.2182 & 0.0267 \end{pmatrix}.$$

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It is easy to verify that $(PXQ)^T = PXQ$, $A^T \hat{X} B = C_0$, and $\|\hat{X} - X^*\| = 2.1273$. So the algorithm is feasible.

Example 4.2. *Let*

$$A = \begin{pmatrix} 1 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ -1.2 & -0.4 & 0.8 & -0.8 & 0 \\ -0.9 & -0.6 & 0.6 & -0.6 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 1 & -0.8 & 0 \\ 0 & 0 & 0 & 0 \\ 1.2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 3.7168 & -1.3336 & -0.2051 & 0.0036 \\ 3.9767 & -5.8704 & 3.9192 & -1.1718 \\ 3.1742 & -4.9043 & 2.9143 & -0.9511 \\ 5.9993 & -5.2680 & 3.0619 & 0.6273 \\ 2.2422 & -2.2761 & 1.3852 & 0.5735 \end{pmatrix}, \ X^* = \begin{pmatrix} 0.5 & 1 & 0.36 & 1 & 1.5 & 1.2 \\ 0.2 & 1 & 0.3 & 1.2 & 2 & 1 \\ 1 & 1.5 & 0.1 & 1.4 & 1 & 1 \\ 1.3 & 1 & 0.5 & 1.4 & 1.2 & 0.5 \\ 1 & 2.3 & 1.2 & 0.4 & 2 & 1.5 \\ 0.4 & 0.8 & 1.2 & 1.2 & 0.6 & 0 \end{pmatrix}.$$

and

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \ Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

According to the Algorithm, we get

$$C_0 = \begin{pmatrix} 3.3260 & -1.4450 & 0.1227 & 0 \\ 4.7817 & -4.9037 & 4.1157 & 0 \\ 3.5890 & -4.0614 & 3.4539 & 0 \\ 6.5907 & -4.7240 & 2.9934 & 0 \\ 2.4131 & -2.0962 & 1.3866 & 0 \end{pmatrix},$$

and

$$\hat{X} = \begin{pmatrix} 0.1221 & -1.7333 & -0.6189 & 0.3631 & -0.3802 & 0.4001 \\ -1.7333 & 1.0051 & -23.6549 & -34.8874 & 4.5926 & 1.0001 \\ -0.6189 & -23.6549 & 0.1443 & -1.2418 & 1.0103 & -1.2000 \\ -0.3631 & 34.8874 & 1.2418 & 0.6237 & 0.0003 & 0.8498 \\ -0.3802 & 4.5926 & 1.0103 & -0.0003 & 1.9557 & -0.6000 \\ -0.4001 & -1.0001 & 1.2000 & 0.8498 & 0.6000 & 0 \end{pmatrix}.$$

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It is easy to verify that X is the (P,Q)-orthogonal symmetric solution of the equation $A^TXB = C_0$, and $\|\hat{X} - X^*\| = 43.7618$.

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