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PERMANENTS OF HESSENBERG $(0,1)$-MATRICES REVISITED*

BRENT J. DESCHAMP† AND BRYAN L. SHADER‡

Abstract. This paper considers the maximum value of the permanent over the class $\mathcal{H}(m,n)$ of $n \times n$ Hessenberg, $(0,1)$-matrices with m 1's, and shows that among those matrices that attain the maximum value there exists a matrix with a special form. This special form determines the exact value of the maximum permanent on $\mathcal{H}(m,n)$ for certain values of m and n.

Key words. Hessenberg matrices, Permanents.

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1. Introduction.

Let $A = [a_{ij}]$ be an $n \times n$ $(0,1)$-matrix. The *permanent* of A, denoted $\operatorname{per} A$, is defined as:

$$\operatorname{per} A = \sum_{\pi \in S_n} a_{1\pi(1)} a_{2\pi(2)} \dots a_{n\pi(n)}, \tag{1.1}$$

where S_n denotes the set of all permutations of $\{1, 2, \dots, n\}$. A *transversal* of an $n \times n$ $(0,1)$-matrix is a collection of n 1's of A with no two in the same row or column. The permanent of A can be equivalently defined as the number of transversals of A. See [4] for classic results concerning the permanent.

An $n \times n$ matrix is *lower Hessenberg* if $a_{ij} = 0$ for $j > i + 1$. Henceforth, we abbreviate lower Hessenberg to Hessenberg. The *full* Hessenberg matrix of order n, H_n, is the $n \times n$ lower Hessenberg $(0,1)$-matrix with 1 in position (i,j) for all i and j with $j \leq i + 1$. Define $\mathcal{H}(m,n)$ to be the class of $n \times n$ Hessenberg $(0,1)$-matrices with m 1's, and let $P(m,n)$ denote the maximum value of the permanent on $\mathcal{H}(m,n)$. A matrix in $\mathcal{H}(m,n)$ with permanent $P(m,n)$ is called a *maximizer* of $\mathcal{H}(m,n)$.

In [5], the value of $P(m,n)$ is determined for m and n satisfying $m \geq 3(n^2 + 3n - 2)/8$ or $n \leq m \leq 8n/3$, and for other m and n it is shown that $\mathcal{H}(m,n)$ contains a maximizer with a special form. This paper significantly refines the special form found in [5], and determines $P(m,n)$ for many new values of m and n.

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†Department of Mathematics & Statistics, California State Polytechnic University, Pomona, Pomona, California 91768, USA (bjdeschamp@csupomona.edu).

‡Department of Mathematics, University of Wyoming, Laramie, Wyoming 82071, USA (bshader@uwyo.edu).

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First, some notation and terminology are stated consistent with [5] and [1], which considers the maximum permanent over the class of arbitrary $(0,1)$-matrices with a specified number of 0's.

Let M by an $m \times n$ matrix. For integers i and j with $i \leq j$ we let $\langle i,j \rangle$ denote the set $\{i, i+1, \ldots, j\}$. For subsets α of $\langle 1,m \rangle$ and β of $\langle 1,n \rangle$, $M[\alpha,\beta]$ denotes the submatrix of M whose rows, respectively columns, are indexed by the elements of α, respectively, of β. The complementary submatrix is denoted by $M(\alpha,\beta)$. When $m = n$ we abbreviate $M[\alpha,\alpha]$ to $M[\alpha]$, and $M(\alpha,\alpha)$ to $M(\alpha)$.

The square matrix A is *partly decomposable* if there exist permutation matrices P and Q such that PAQ has the form

$$\begin{bmatrix} B & O \\ C & D \end{bmatrix},$$

where B and D are square, nonvacuous, matrices. Equivalently, A is partly decomposable if and only if it contains a zero submatrix whose dimensions sum to n. If A is not partly decomposable, then A is *fully indecomposable.*

If the square $(0,1)$-matrix A has a transversal, then there exist permutation matrices P and Q, and a positive integer b such that PAQ has the form:

$$\begin{bmatrix} A_1 & O & O & \cdots & O \\ A_{21} & A_2 & O & \cdots & O \\ \vdots & & \ddots & & \vdots \\ A_{b-1,1} & A_{b-1,2} & & A_{b-1} & O \\ A_{b1} & A_{b2} & \cdots & A_{b,b-1} & A_b \end{bmatrix}, \tag{1.2}$$

where the matrices $A_1, \ldots, A_b$ are fully indecomposable. The $n_i \times n_i$ matrices A_i are the *fully indecomposable components* of A and are unique up to permutation of rows and columns. Note that $\operatorname{per} A = \prod_{i=1}^{b} \operatorname{per} A_i$.

Let $\operatorname{nnz} A$ denote the number of nonzero entries of A. Note that for an $n \times n$ Hessenberg matrix $\operatorname{nnz} A \leq (n^2 + 3n - 2)/2$.

A Hessenberg $(0,1)$-matrix A is *staircased* if whenever $i \geq j$ and $a_{ij} = 0$, then $a_{kj} = 0$ for $k = i+1, \ldots, n$ and $a_{il} = 0$ for $l = 1, \ldots, j-1$. Note that if A is staircased and $a_{ij} = 0$, then $a_{kl} = 0$ for all $i \leq k \leq n$ and $1 \leq l \leq j$.

Let J_n, I_n, and O_n denote the $n \times n$ matrix of all 1's, the identity matrix, and the zero matrix, respectively. Let E_{ij} denote the matrix of size appropriate to the context with a 1 in the (i,j)-position and 0's elsewhere.

2. Previous Results. This section states some of the results from [5] that concern the structure of permanent maximizers in $\mathcal{H}(m,n)$. The first result shows

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that each fully indecomposable component of a Hessenberg matrix is permutationally equivalent to a Hessenberg matrix, and the second describes some structural properties of fully indecomposable and staircased Hessenberg matrices and also gives formulas for the permanent in terms of minors of Hessenberg $(0,1)$-matrices.

LEMMA 2.1. *If $A = [a_{ij}]$ is an $n \times n$ Hessenberg $(0,1)$-matrix with* $\operatorname{per} A > 0$*, then each fully indecomposable component of A is permutationally equivalent to a Hessenberg matrix.*

LEMMA 2.2. *The following hold for an $n \times n$ Hessenberg $(0,1)$-matrix $A = [a_{ij}]$:*

(a) If A is fully indecomposable, then $a_{11} = 1$, $a_{nn} = 1$ and $a_{i,i+1} = 1$ for $i = 1, 2, \ldots, n-1$.
(b) If A is fully indecomposable and staircased, then $a_{i+1,i} = 1$ for $i = 1, 2, \ldots, n-1$, and $a_{ii} = 1$ for $i = 1, 2, \ldots, n$.
(c) If each $a_{i,i+1} = 1$ for $i = 1, 2, \ldots, n-1$ and k and l are integers such that $1 \le l \le k \le n$, then

$$\operatorname{per} A(k,l) = \operatorname{per} A[\langle 1, l-1\rangle]\operatorname{per} A[\langle k+1, n\rangle],$$

where a vacuous permanent with $l = 1$ or $k = n$ is set to equal 1.

By Lemma 2.2, for $j \le i$,

$$\operatorname{per} H_n(i,j) = \begin{cases} 1 & \text{if } i = n \text{ and } j = 1 \\ 2^{j-2} & \text{if } i = n \text{ and } j \ge 2 \\ 2^{n-i-1} & \text{if } n-1 \ge i \ge 1 \text{ and } j = 1 \\ 2^{n-i+j-3} & \text{if } n-1 \ge i \ge 1 \text{ and } j \ge 2 \\ 2^{n-2} & \text{if } n-1 \ge i \ge 1 \text{ and } j = i+1 \end{cases} \tag{2.1}$$

Also, note

$$\operatorname{per} H_n = 2^{n-1}. \tag{2.2}$$

For a Hessenberg $(0,1)$-matrix A, an *interchangeable column pair* of A is a pair of entries (k,l) and $(k-1,l)$ with $k > l$ such that $a_{kl} = 1$ and $a_{k-1,l} = 0$. An *interchangeable row pair* of A is a pair of entries (k,l) and $(k,l+1)$ with $k > l$ such that $a_{kl} = 1$ and $a_{k,l+1} = 0$. Note that a Hessenberg $(0,1)$-matrix is staircased if and only if it has no interchangeable row pairs and no interchangeable column pairs.

An exchange rule for $\mathcal{H}(m,n)$ is an operation that takes a matrix $A \in \mathcal{H}(m,n)$ with certain properties and rearranges the entries of A to obtain a matrix $B \in \mathcal{H}(m,n)$ with $\operatorname{per} B \ge \operatorname{per} A$. The following is an exchange rule, which we call the *Bubbling Exchange Rule*, developed and used in [5].

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Lemma 2.3 (The Bubbling Exchange Rule). *Let A be an $n \times n$ fully indecomposable, Hessenberg $(0,1)$-matrix that has an interchangeable column pair, and let (k,l) and $(k-1,l)$ be the interchangeable column pair of A with l largest. Let B be the matrix obtained from A by interchanging the 1 in position (k,l) with the 0 in position $(k-1,l)$.*[1] *Then* $\operatorname{per} B \geq \operatorname{per} A$.

An analogous result holds for interchangeable row pairs. If A has an interchangeable row pair, and if (k,l), $(k,l+1)$ is an interchangeable row pair of A with k smallest, the matrix B obtained from A by switching the 1 in position (k,l) with the 0 in position $(k,l+1)$ satisfies $\operatorname{per} B \geq \operatorname{per} A$. By the Bubbling Exchange Rule we mean either that in Lemma 2.3 or its analog for interchangeable row pairs.

Repeated application of Lemma 2.3 yields the following theorem.

Theorem 2.4. *Let m and n be positive integers with $n \leq m \leq \frac{n^2+3n-2}{2}$. Then there exists a matrix $A \in \mathcal{H}(m,n)$ with permanent $P(m,n)$ such that A has the form* (1.2)*, where each A_i is a fully indecomposable, staircased Hessenberg matrix.*

In developing additional exchange rules, bounds will be needed on the permanent of submatrices related to the permanent of the original matrix.

Lemma 2.5. *If $A = [a_{ij}]$ is a fully indecomposable, staircased Hessenberg $(0,1)$-matrix of order $n \geq 2$, then*

$$\frac{1}{2}\operatorname{per} A \leq \operatorname{per} A(1) \leq \frac{2}{3}\operatorname{per} A.$$

Moreover, $\frac{1}{2}\operatorname{per} A = \operatorname{per} A(1)$ if and only if the first two columns of A are equal, and $\frac{2}{3}\operatorname{per} A = \operatorname{per} A(1)$ if and only if $a_{31} = 0$ and columns two and three of A disagree only in their first entries.

Proof. We first prove the inequalities. For the case $n = 2$,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Clearly,

$$\frac{1}{2}\operatorname{per} A = 1 = \operatorname{per} A(1) \leq \frac{4}{3} = \frac{2}{3}\operatorname{per} A.$$

For $n > 2$, write A as

$$A = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ b_1 & b_2 & b_3 & \cdots & b_n \end{bmatrix},$$

[1] That is, B is obtained from A by "bubbling" the 1 in position (k,l) up to position $(k-1,l)$

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where each b_i is an $(n-1)\times 1$ vector. Since A is staircased, $0 \le b_1 \le b_2$ (entrywise). Thus $b_1 \le b_2$ and each entry of A is nonnegative, $\operatorname{per}[b_1\ b_3\ \cdots\ b_n] \le \operatorname{per}[b_2\ b_3\ \cdots\ b_n]$, and (by Laplace expansion along the first row of A) we have:

$$\begin{aligned}\operatorname{per} A &= \operatorname{per} A(1) + \operatorname{per} A(1,2)\\ &= \operatorname{per}[b_2\ b_3\ \cdots\ b_n] + \operatorname{per}[b_1\ b_3\ \cdots\ b_n]\\ &\le 2\operatorname{per}[b_2\ b_3\ \cdots\ b_n]\\ &= 2\operatorname{per} A(1).\end{aligned} \tag{2.3}$$

Thus $\frac{1}{2}\operatorname{per} A \le \operatorname{per} A(1)$.

Since A is fully indecomposable and staircased,

$$\operatorname{per} A = \sum_{i=1}^{n} \operatorname{per} A(\langle 1,i\rangle) \ge \operatorname{per} A(1) + \operatorname{per} A(\langle 1,2\rangle). \tag{2.4}$$

Note $A(1)$ has order at least 2 since $n > 2$, and applying the previously established inequality to $A(1)$ yields $\operatorname{per} A(\langle 1,2\rangle) \ge \frac{1}{2}\operatorname{per} A(1)$. Thus by (2.4), $\operatorname{per} A \ge \frac{3}{2}\operatorname{per} A(1)$.

We now analyze the cases of equality.

Clearly, if $b_1 = b_2$, then by (2.3), $\operatorname{per} A = 2\operatorname{per} A(1)$.

Next suppose $\operatorname{per} A = 2\operatorname{per} A(1)$. Then equality holds throughout (2.3). The full indecomposability of $A(1)$ implies that $b_1 = b_2$, since if $b_1 \ne b_2$, then

$$\operatorname{per}[b_1\ b_3\ \cdots\ b_n] < \operatorname{per}[b_2\ b_3\ \cdots\ b_n].$$

Now suppose $\operatorname{per} A = \frac{3}{2}\operatorname{per} A(1)$. Then by (2.4) and the fact that $\operatorname{per} A[\langle 1,i\rangle] \ne 0$, $a_{31} = a_{41} = \cdots = a_{n1} = 0$, and by Laplace expansion of A it follows that $\operatorname{per} A = \operatorname{per} A(1) + \operatorname{per} A(1,2)$. Hence $2\operatorname{per} A(1,2) = \operatorname{per} A(1)$. Therefore, by the previous argument, the second and third columns of A agree except in the first entry, as desired.

If $a_{31} = 0$ and columns two and three disagree in only their first entries, then since A is staircased

$$A = \begin{bmatrix} 1 & 1 & 0 & \dots & 0\\ b & b_2 & b_3 & \dots & b_n \end{bmatrix}$$

where b is the column vector $(1,0,\dots,0)^T$. Now

$$\begin{aligned}\operatorname{per} A &= \operatorname{per} A(1) + \operatorname{per} A(1,2)\\ &= \operatorname{per} A(1) + \tfrac{1}{2}\operatorname{per} A(1)\\ &= \tfrac{3}{2}\operatorname{per} A(1),\end{aligned}$$

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with the second equality coming from the fact that $b_2 = b_3$. ☐

COROLLARY 2.6. *If $A = [a_{ij}]$ is a fully indecomposable, staircased Hessenberg $(0,1)$-matrix of order $n \geq 3$, then for $1 \leq s \leq n-1$,*

$$\left(\frac{1}{2}\right)^s \operatorname{per} A \leq \operatorname{per} A[\langle s+1, n\rangle] \leq \left(\frac{2}{3}\right)^s \operatorname{per} A.$$

Proof. The proof is by induction on n.

Consider the case $n = 3$, so $1 \leq s \leq 2$. Since A is staircased, A has the form:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ a & 1 & 1 \end{bmatrix}.$$

Note that $3 \leq \operatorname{per} A \leq 4$. If $s = 1$, then

$$\frac{1}{2} \operatorname{per} A \leq \frac{1}{2} 4 = 2 = \operatorname{per} A[\langle 2, 3\rangle] = \frac{2}{3} 3 \leq \frac{2}{3} \operatorname{per} A.$$

If $s = 2$, then

$$\left(\frac{1}{2}\right)^2 \operatorname{per} A \leq \frac{1}{4} 4 = 1 = \operatorname{per} A[\langle 3, 3\rangle] \leq \frac{4}{3} = \frac{4}{9} 3 \leq \left(\frac{2}{3}\right)^2 \operatorname{per} A.$$

Thus the statement is true for $n = 3$.

Consider $n > 3$. The case $s = 1$ is proved in Lemma 2.5. Consider $s \geq 2$.

By Lemma 2.5, $\frac{1}{2} \operatorname{per} A \leq \operatorname{per} A(1) \leq \frac{2}{3} \operatorname{per} A$. Since A is fully indecomposable and staircased, so is A(1). Thus we can apply induction, and since $A[\langle s+1, n\rangle] = A(1)[\langle s, n-1\rangle]$ we conclude

$$\left(\frac{1}{2}\right)^{s-1} \operatorname{per} A(1) \leq \operatorname{per} A[\langle s+1, n\rangle] \leq \left(\frac{2}{3}\right)^{s-1} \operatorname{per} A(1).$$

Therefore,

$$\left(\frac{1}{2}\right)^s \operatorname{per} A \leq \operatorname{per} A[\langle s+1, n\rangle] \leq \left(\frac{2}{3}\right)^s \operatorname{per} A. \qquad ☐$$

3. Exchange Rules. This section develops new exchange rules for Hessenberg $(0,1)$-matrices. These exchange rules will be used in Section 4 to determine the structure of a matrix in $\mathcal{H}(m,n)$ that attains $P(m,n)$.

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Let $A = [a_{ij}]$ be an $n \times n$ Hessenberg $(0,1)$-matrix, and let k be a positive integer. The *k-th stripe* of A is the vector $(a_{k,1}, a_{k+1,2}, \ldots, a_{n,n-k+1})$ for $1 \leq k \leq n$. The k-th stripe is *full* (respectively *zero*) if each entry is nonzero (respectively zero). The *width* of A, denoted $\mathrm{w}(A)$, is the largest k such that the k-th stripe is nonzero.

The Hessenberg $(0,1)$-matrix A is called *striped* provided its superdiagonal is full, $\mathrm{w}(A) \geq 2$ and its k-th stripe is full for $k = 1, 2, \ldots, \mathrm{w}(A)-1$. If in addition its $\mathrm{w}(A)$-th stripe is full, then A is *banded*.

The *s-banded* Hessenberg matrix, denoted $H_{n,s}$, of order n is the $n \times n$ $(0,1)$-matrix with 1's in the positions (i,j) with $1 - s \leq j - i \leq 1$, and 0's elsewhere. Note that $H_{n,s}$ is a banded matrix and $H_n = H_{n,n}$.

A matrix is *persymmetric* if the (i,j)-entry is equal to the $(n-j+1, n-i+1)$-entry for all i and j. Or equivalently, the matrix is symmetric about the back diagonal. Note $H_{n,s}$ is persymmetric.

Example 3.1. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

The second stripe of A is $(1,1,0,1)$, while the third stripe is $(0,1,0)$. The first stripe is full, and the fourth stripe is zero.

Let

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

The matrix B is a striped, but not banded, matrix with $\mathrm{w}(B) = 3$, and D is 2-banded.

A lexicographic order may be introduced on Hessenberg matrices by associating an order to the stripes of the matrix. As above, the k-th stripe of a Hessenberg matrix A may be associated with the vector $\vec{A}_k = (a_{k,1}, a_{k+1,2}, \ldots, a_{n,n-k+1})$. In this way the entire matrix A may be represented by concatenating the stripes in the following order: $\vec{A}_n, \vec{A}_{n-1}, \ldots, \vec{A}_2, \vec{A}_1, \vec{A}_0$ where $\vec{A}_0$ represents the superdiagonal. In this way the matrix A may be represented by the vector

$$\vec{A} = (a_{n,1}, a_{n-1,1}, a_{n,2}, a_{n-2,1}, \ldots, a_{2,1}, a_{3,2}, \ldots, a_{n,n-1}, a_{1,1}, \ldots, a_{n,n}, a_{1,2}, \ldots, a_{n-1,n}),$$

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or more simply

$$\vec{A} = (\vec{A}_n, \vec{A}_{n-1}, \ldots, \vec{A}_2, \vec{A}_1, \vec{A}_0).$$

In this way we can order the $n \times n$ Hessenberg $(0, 1)$-matrices, namely, for matrices A and B, $B \geq A$ if and only if $\vec{B} \geq \vec{A}$ in lexicographic order. This ordering will be used in Lemmas 3.3 and 3.6 and applied in Section 4.

A lemma is needed before proving the first exchange rule.

Lemma 3.2. *Consider an $n \times n$ fully indecomposable, staircased, Hessenberg $(0, 1)$-matrix C of the form*

$$C = \left[\begin{array}{c|c} A & E_{k1} \\ \hline D & B \end{array}\right]$$

where A is of order k, with $1 \leq k \leq n-1$, and B is of order $n-k$. Then

$$\operatorname{per} C \leq 2 \operatorname{per} A \cdot \operatorname{per} B.$$

Proof. Note that $\operatorname{per} C = \operatorname{per} A \cdot \operatorname{per} B + \operatorname{per} C(k, k+1)$. Each transversal of C containing the 1 in position $(k, k+1)$ contains exactly one entry from D. Thus as C is staircased

$$\begin{aligned} \operatorname{per} C &= \operatorname{per} A \cdot \operatorname{per} B + \operatorname{per} C(k, k+1) \\ &= \operatorname{per} A \cdot \operatorname{per} B + \textstyle\sum_{j=1}^{k} \sum_{i=1}^{n-k} d_{ij} \cdot \operatorname{per} A(k, j) \cdot \operatorname{per} B(i, 1) \\ &\leq \operatorname{per} A \cdot \operatorname{per} B + \textstyle\sum_{j=1}^{k} \sum_{i=1}^{n-k} a_{kj} \cdot b_{i1} \cdot \operatorname{per} A(k, j) \cdot \operatorname{per} B(i, 1) \\ &= \operatorname{per} A \cdot \operatorname{per} B + \operatorname{per} A \cdot \operatorname{per} B \\ &= 2 \operatorname{per} A \cdot \operatorname{per} B. \end{aligned}$$ □

Lemma 3.3. *Let $A = [a_{ij}]$ be an $n \times n$ fully indecomposable, staircased Hessenberg $(0, 1)$-matrix for which there exist integers $p < n$, with $p \geq 4$ and $n \geq 5$, and $s > 2$ such that $\mathrm{w}(A) = p$, the $(p-1)$-th stripe has $a_{p-2+i,i} = 1$ for $i = 1, \ldots, s-1$ and $a_{p-2+s,s} = 0$ and the p-th stripe has $a_{p,1} = 1$ and $a_{p-1+i,i} = 0$ for $i = 2, \ldots, s-1$.*

Let B be the matrix obtained from A by replacing $a_{p,1}$ by 0, and $a_{p+s-2,s}$ by 1. Then

$$\operatorname{per} B \geq \operatorname{per} A.$$

Proof. Let C be the matrix obtained from A by replacing $a_{p,1}$ by 0. Since $\operatorname{per} A = \operatorname{per} C + \operatorname{per} C(p, 1)$ and $\operatorname{per} B = \operatorname{per} C + \operatorname{per} C(p+s-2, s)$ it suffices to show that $\operatorname{per} C(p+s-2, s) \geq \operatorname{per} C(p, 1)$.

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First consider the case that $s \leq p$. Since C is staircased and $c_{p-1,1} = 1$, $C[\langle 1, p-1\rangle] = H_{p-1}$. Thus,

$$\operatorname{per} C[\langle 1, s-1\rangle] = \operatorname{per} H_{s-1} = 2^{s-2}.$$

By Lemma 2.2,

$$\begin{aligned} \operatorname{per} C(s+p-2, s) &= \begin{cases} \operatorname{per} C[\langle 1, s-1\rangle] \cdot \operatorname{per} C[\langle s+p-1, n\rangle] & \text{if} \quad s+p-2 < n \\ \operatorname{per} C[\langle 1, s-1\rangle] & \text{if} \quad s+p-2 = n \end{cases} \\ &= \begin{cases} 2^{s-2} \operatorname{per} C[\langle s+p-1, n\rangle] & \text{if} \quad s+p-2 < n \\ 2^{s-2} & \text{if} \quad s+p-2 = n. \end{cases} \end{aligned} \tag{3.1}$$

Also by Lemma 2.2,

$$\operatorname{per} C(p, 1) = \operatorname{per} C[\langle p+1, n\rangle]. \tag{3.2}$$

If $s + p - 2 = n$, then (since $C[\langle p+1, n\rangle]$ is $(n-p) \times (n-p)$)

$$\operatorname{per} C[\langle p+1, n\rangle] \leq \operatorname{per} H_{n-p} = 2^{n-p-1} = 2^{s-3} < 2^{s-2}. \tag{3.3}$$

If $s + p - 2 < n$, then, since $C[\langle p+1, n\rangle]$ is fully indecomposable and staircased, the leftmost inequality in Corollary 2.6 implies

$$\operatorname{per} C[\langle p+1, n\rangle] \leq 2^{s+p-1-(p+1)} \operatorname{per} C[\langle s+p-1, n\rangle] = 2^{s-2} \operatorname{per} C[\langle s+p-1, n\rangle]. \tag{3.4}$$

Hence (3.1)-(3.4) imply

$$\operatorname{per} C(p, 1) \leq \operatorname{per} C(s+p-2, s),$$

as desired.

Now consider the case when $s > p$ and $s + p - 2 < n$. As before

$$\operatorname{per} C(p, 1) = \operatorname{per} C[\langle p+1, n\rangle]$$

and

$$\operatorname{per} C(s+p-2, s) = \operatorname{per} C[\langle 1, s-1\rangle] \cdot \operatorname{per} C[\langle s+p-1, n\rangle].$$

Write $C[\langle p+1, n\rangle]$ in the form:

$$\begin{aligned} C[\langle p+1, n\rangle] &= \left[\begin{array}{c|c} C[\langle p+1, s+p-2\rangle] & C[\langle p+1, s+p-2\rangle, \langle s+p-1, n\rangle] \\ \hline X & C[\langle s+p-1, n\rangle] \end{array}\right] \\ &\leq \left[\begin{array}{c|c} H_{s-2,p-1} - E_{s-2,s-p} & E_{s-2,1} \\ \hline X & C[\langle s+p-1, n\rangle] \end{array}\right], \end{aligned}$$

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where $C[\langle p+1, s+p-2\rangle] \leq H_{s-2,p-1} - E_{s-2,s-p}$ follows from the fact that C is staircased and the structure of the stripes of C. As such, by Lemma 3.2,

$$\operatorname{per} C[\langle p+1, n\rangle] \leq 2 \operatorname{per}(H_{s-2,p-1} - E_{s-2,s-p}) \cdot \operatorname{per} C[\langle s+p-1, n\rangle]. \qquad (3.5)$$

Note that

$$C[\langle 1, s-1\rangle] = H_{s-1,p-1} \geq H_{s-1,p-1} - E_{s-2,s-p},$$

and since the last two rows of $H_{s-1,p-1} - E_{s-2,s-p}$ are equal, Laplace expansion of the permanent along the last column yields

$$\operatorname{per} C[\langle 1, s-1\rangle] \geq \operatorname{per}(H_{s-1,p-1} - E_{s-2,s-p}) = 2 \operatorname{per}(H_{s-2,p-1} - E_{s-2,s-p}). \qquad (3.6)$$

Combining (3.5) and (3.6) yields

$$\begin{aligned} \operatorname{per} C(p,1) &= \operatorname{per} C[\langle p+1, n\rangle] \\ &\leq 2 \operatorname{per}(H_{s-2,p-1} - E_{s-2,s-p}) \cdot \operatorname{per} C[\langle s+p-1, n\rangle] \\ &\leq \operatorname{per} C[\langle 1, s-1\rangle] \cdot \operatorname{per} C[\langle s+p-1, n\rangle] \\ &= \operatorname{per} C(s+p-2, s). \end{aligned}$$

Finally, the case $s > p$ and $s+p-2 = n$ is handled like the previous case by defining the vacuous matrix $C[\langle s+p-1, n\rangle]$ to have permanent 1. ☐

Example 3.4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \boxed{1} & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{0} & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Let $p = 5$. Then the $(p-1)$-th stripe is $(1, 1, 1, 0, 1, 0, 0)$, the p-th stripe is $(1, 0, 0, 0, 0, 0)$, and we may take $s = 4$. Then exchanging entries a_{51} and a_{74} by defining B as A with $b_{51} = 0$ and $b_{74} = 1$, then $\operatorname{per} B \geq \operatorname{per} A$. In fact, $\operatorname{per} B = 301$ and $\operatorname{per} A = 300$.

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Note that for A above $s \le p$, so in the proof of Lemma 3.3 the following matrices appear:

$$\operatorname{per} C(7,4) = \operatorname{per} C[\langle 1,3\rangle]\cdot\operatorname{per} C[\langle 8,10\rangle] = \operatorname{per} H_3\cdot\operatorname{per}\begin{bmatrix} 1 & 1 & 0\\ 1 & 1 & 1\\ 0 & 1 & 1\end{bmatrix} = 2^2 \operatorname{per}\begin{bmatrix} 1 & 1 & 0\\ 1 & 1 & 1\\ 0 & 1 & 1\end{bmatrix},$$

$$\operatorname{per} C(5,1) = \operatorname{per} C[\langle 6,10\rangle] = \operatorname{per}\begin{bmatrix} 1 & 1 & 0 & 0 & 0\\ 1 & 1 & 1 & 0 & 0\\ 1 & 1 & 1 & 1 & 0\\ 0 & 1 & 1 & 1 & 1\\ 0 & 0 & 0 & 1 & 1\end{bmatrix} \le 2^2 \operatorname{per}\begin{bmatrix} 1 & 1 & 0\\ 1 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}.$$

Example 3.5. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ \boxed{1} & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{0} & 1 & 1 & 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1\end{bmatrix}.$$

Here $n = 13$, $p = 5$, and $s = 7$. In this case $s > p$, and the following matrices appear:

$$\operatorname{per} C(10,7) = \operatorname{per} C[\langle 1,6\rangle]\cdot\operatorname{per} C[\langle 11,13\rangle] = \operatorname{per}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 0 & 0 & 0\\ 1 & 1 & 1 & 1 & 0 & 0\\ 1 & 1 & 1 & 1 & 1 & 0\\ 0 & 1 & 1 & 1 & 1 & 1\\ 0 & 0 & 1 & 1 & 1 & 1\end{bmatrix}\cdot\operatorname{per}\begin{bmatrix} 1 & 1 & 0\\ 1 & 1 & 1\\ 0 & 1 & 1\end{bmatrix},$$

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$$\begin{aligned}
\operatorname{per} C(6,13) &= \operatorname{per}\left[\begin{array}{ccccc|ccc}
1&1&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0\\
1&1&1&1&0&0&0&0\\
1&1&1&1&1&0&0&0\\
0&0&1&1&1&1&0&0\\ \hline
0&0&0&1&1&1&1&0\\
0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&1&1
\end{array}\right] \\
&\le 2\operatorname{per}(H_{5,4}-E_{5,2})\cdot\operatorname{per}\begin{bmatrix}1&1&0\\1&1&1\\0&1&1\end{bmatrix} \qquad . \\
&\le \operatorname{per}\begin{bmatrix}
1&1&0&0&0&0\\
1&1&1&0&0&0\\
1&1&1&1&0&0\\
1&1&1&1&1&0\\
0&1&1&1&1&1\\
0&0&1&1&1&1
\end{bmatrix}\cdot\operatorname{per}\begin{bmatrix}1&1&0\\1&1&1\\0&1&1\end{bmatrix}.
\end{aligned}$$

And so $\operatorname{per} B \geq \operatorname{per} A$. In fact, $\operatorname{per} B = 1835$ and $\operatorname{per} A = 1814$.

Note that in Lemma 3.3 the exchange decreases the lexicographic order, $\vec{B} < \vec{A}$, as the exchange moves a 1 from the p-th stripe to the $(p-1)$-th stripe.

Lemma 3.6 (The Stripe Exchange Rule). *Let $A = [a_{ij}]$ be an $n \times n$ fully indecomposable, staircased Hessenberg $(0,1)$-matrix for which there exist integers p, q, s with $p \geq 4$, $p + s - 2 \leq n$, and $s > q + 1$ such that $\mathrm{w}(A) = p$, the $(p-1)$-th stripe has $a_{p-2+i,i} = 1$ for $i = q, \ldots, s-1$ and $a_{p-2+s,s} = 0$ and the p-th stripe of A has $a_{p-1+q,q} = 1$ and $a_{p-1+i,i} = 0$ for $i = q+1, \ldots, s-1$. Let B be the matrix obtained from A by replacing $a_{p-1+q,q}$ by 0 and $a_{p+s-2,s}$ by 1. Then*

$$\operatorname{per} B \geq \operatorname{per} A.$$

Proof. The proof is by induction on q. The case $q = 1$ is handled by Lemma 3.3. Assume $q \geq 2$, and proceed by induction.

Since $q \geq 2$, the first columns of A and B are identical. Let t be the largest index such that $a_{t,1} = 1$. Then

$$\operatorname{per} A = \sum_{i=1}^{t} \operatorname{per} A[\langle i+1, n\rangle]$$

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and

$$\operatorname{per} B = \sum_{i=1}^{t} \operatorname{per} B[\langle i+1, n\rangle].$$

Hence it suffices to show that for $i \le t$

$$\operatorname{per} A[\langle i+1, n\rangle] \le \operatorname{per} B[\langle i+1, n\rangle].$$

Note that if $(p-1+q, q) \in \langle i+1, n\rangle \times \langle i+1, n\rangle$, then $(p+s-2, s) \in \langle i+1, n\rangle \times \langle i+1, n\rangle$. Hence if $(p-1+q, q) \in \langle i+1, n\rangle \times \langle i+1, n\rangle$, then $A[\langle i+1, n\rangle]$ and $B[\langle i+1, n\rangle]$ satisfy the induction hypothesis, and therefore $\operatorname{per} A[\langle i+1, n\rangle] \le \operatorname{per} B[\langle i+1, n\rangle]$.

If $(p-1+q, q) \notin \langle i+1, n\rangle \times \langle i+1, n\rangle$, then $A[\langle i+1, n\rangle] \le B[\langle i+1, n\rangle]$ as $a_{p+s-2,s} = 0$ and $b_{p+s-2,s} = 1$, and hence $\operatorname{per} A[\langle i+1, n\rangle] \le \operatorname{per} B[\langle i+1, n\rangle]$. This completes the proof. □

Example 3.7. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & \boxed{0} & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Let $p = 4$, $q = 2$, and $s = 4$. The $(p-1)$-th stripe is $(0, 1, 1, 0, 0)$, and the p-th stripe is $(0, 1, 0, 0)$. So we may exchange the entries $a_{p-1+q,q} = a_{52} = 1$ and $a_{p+s-2,s} = a_{64} = 0$ to obtain B with $b_{52} = 0$ and $b_{64} = 1$. Then $\operatorname{per} B = \operatorname{per} B(1) + \operatorname{per} B(\langle 1, 2\rangle)$, $\operatorname{per} A = \operatorname{per} A(1) + \operatorname{per} B(\langle 1, 2\rangle)$, $\operatorname{per} A(1) \le \operatorname{per} B(1)$, and $\operatorname{per} B(\langle 1, 2\rangle) \ge \operatorname{per} A(\langle 1, 2\rangle)$ by Lemma 3.6.

Many of the exchange rules consider the upper-left entries of a given stripe, but the nature of the permanent allows for similar statements about the corresponding lower-right, persymmetric, entries of the same stripe. Note both Lemmas 3.3 and 3.6 have persymmetric analogs. By the Stripe Exchange Rule, we mean either Lemma 3.6 or its persymmetric analog. Both lemmas will be needed in Section 4.

Note that in Lemma 3.6 the lexicographic order has again been decreased, $\vec{B} < \vec{A}$, as the exchange moves a 1 from the p-th stripe to the $(p-1)$-th stripe. This is also the case with Lemma 2.3. Thus each of the exchange operations decreases the lexicographic order of the matrix. This decrease will be exploited in Section 4.

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The previous two lemmas dealt with exchanging entries within fully indecomposable components in order to possibly increase the permanent. The following lemma is an exchange rule for combining two fully indecomposable components into a single fully indecomposable matrix.

Lemma 3.8 (The Two-Large-Leg Exchange Rule). *Let A and B be fully indecomposable, staircased Hessenberg $(0,1)$-matrices of orders n_A and n_B, respectively, with $n_A, n_B \geq 6$. Assume that A has k 1's in its last row, B has l 1's in its first column, $k < n_A$, and $l < n_B$.*

Set

$$M = \left[\begin{array}{c|c} A + E_{n_A, n_A-k} & O \\ \hline O & B + E_{l+1,1} \end{array}\right],$$

and

$$N = \left[\begin{array}{c|c} A & E_{n_A,1} \\ \hline E_{1,n_A} & B \end{array}\right],$$

where $E_{n_A,1}$ is $n_A \times n_B$ and E_{1,n_A} is $n_B \times n_A$, and set $m = \min\{k,l\}$.

If $m \geq 5$, then $\operatorname{per} N > \operatorname{per} M$.

Proof. Note that by Lemma 2.2,

$$\operatorname{per}(A + E_{n_A,n_A-k}) = \begin{cases} \operatorname{per} A + 1 & \text{if} \quad n_A - k = 1 \\ \operatorname{per} A + \operatorname{per} A[\langle 1, n_A - k - 1\rangle] & \text{if} \quad n_A - k > 1. \end{cases}$$

First suppose $k = n_A - 1$. Then

$$A \geq \left[\begin{array}{c|c} 1 & 1 \quad\quad\quad \\ \hline 1 & \\ 0 & \\ \vdots & H_{n_A-1} \\ 0 & \end{array}\right];$$

so by Laplace expansion along the first row or column

$$\begin{aligned} \operatorname{per} A &\geq \operatorname{per} H_{n_A-1} + \operatorname{per} H_{n_A-2} \\ &= 2^{n_A-2} + 2^{n_A-3} \\ &= 3 \cdot 2^{n_A-3} \\ &\geq \left(\tfrac{3}{2}\right)^{n_A} \\ &= \left(\tfrac{3}{2}\right)^{k+1} \quad \text{for } n_A \geq 4. \end{aligned}$$

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Thus,

$$\operatorname{per}(A+E_{n_A,n_A-k}) = \operatorname{per}A + 1 \le \operatorname{per}A + \left(\frac{2}{3}\right)^{k+1}\operatorname{per}A \le \left(1+\left(\frac{2}{3}\right)^{k+1}\right)\operatorname{per}A.$$

Next suppose $n_A - k > 1$. Since A is fully indecomposable and staircased, the persymmetric analog of Corollary 2.6 implies

$$\operatorname{per}A[\langle 1, n_A-k-1\rangle] \le \left(\frac{2}{3}\right)^{k+1}\operatorname{per}A,$$

and thus

$$\operatorname{per}(A+E_{n_A,n_A-k}) \le \left(1+\left(\frac{2}{3}\right)^{k+1}\right)\operatorname{per}A.$$

Similarly,

$$\operatorname{per}(B+E_{l+1,1}) \le \left(1+\left(\frac{2}{3}\right)^{l+1}\right)\operatorname{per}B.$$

It follows that for all k and l

$$\operatorname{per}M \le \left(1+\left(\frac{2}{3}\right)^{m+1}\right)^2\operatorname{per}A\cdot\operatorname{per}B < \frac{5}{4}\operatorname{per}A\cdot\operatorname{per}B \tag{3.7}$$

since $k, l \ge m$ and $m \ge 5$.

As A and B are fully indecomposable and staircased, then by Lemma 2.5

$$\begin{aligned}\operatorname{per}N &= \operatorname{per}A\cdot\operatorname{per}B + \operatorname{per}A(n_A)\cdot\operatorname{per}B(1)\\ &\ge \operatorname{per}A\cdot\operatorname{per}B + \tfrac{1}{2}\operatorname{per}A\cdot\tfrac{1}{2}\operatorname{per}B\\ &= \tfrac{5}{4}\operatorname{per}A\cdot\operatorname{per}B.\end{aligned}\tag{3.8}$$

Therefore, by (3.7) and (3.8), $\operatorname{per}N > \operatorname{per}M$, as desired. □

Example 3.9. In this example A and B are both fully indecomposable, $m = 4$, and $\operatorname{per}M > \operatorname{per}N$. This shows the assumption of $m \ge 5$ in Lemma 3.8 is necessary.

Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 1 & 1 & 0\\ 0 & 0 & 1 & 1 & 1 & 1\\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

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and

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then

$$M = \left[\begin{array}{cccccc|cccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 1 & 1 & 1 & 1 & \boxed{0} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \boxed{0} & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

Let $n_A = n_B = 6$, $k = 4$, and $l = 4$, so $m = 4 < 5$, $M = (A + E_{62}) \oplus (B + E_{51})$ and $N = (A \oplus B) + E_{67} + E_{76}$. Direct calculation gives $\operatorname{per} N < \operatorname{per} M$. In fact, $\operatorname{per} N = 330$ and $\operatorname{per} M = 483$.

Informally, Lemma 3.8 and the use of persymmetry shows that if R is a permanent maximizer in $\mathcal{H}(m, n)$, each of whose fully indecomposable components is staircased, then at most one of its fully indecomposable components can have more than 5 nonzero entries in its first column (or last row).[2]

Lemma 3.10. *Suppose A and B are fully indecomposable, staircased Hessenberg $(0, 1)$-matrices of order n_A and n_B, respectively, such that $A \oplus B$ is a maximizer in $\mathcal{H}(\operatorname{nnz}(A \oplus B), n_A + n_B)$. Let k be the number of 1's in the last row of A and l be the number of 1's in the first column of B. Then there exists a permanent maximizer such that $|l - k| \le 1$.*

[2]The "legs" of a component are the 1's in its first column and the 1's in its last row.

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Proof. To the contrary, assume $l \geq k+2$. Let

$$C = \left[\begin{array}{ccc|c} & A & x & \\ 0\ldots0 & \underbrace{1\ldots1}_{k} & 1 & O \\ \hline & O & & B(1) \end{array}\right]$$

where $x = (0,\ldots,0,1)^T$. Then the last two rows of the upperleft matrix are equal, and by Lemma 2.5

$$\operatorname{per} C = 2 \operatorname{per} A \cdot \operatorname{per} B(1) \geq \operatorname{per} A \cdot \operatorname{per} B = P(\operatorname{nnz}(A \oplus B), n_A + n_B). \tag{3.9}$$

Note that $\operatorname{nnz} C = \operatorname{nnz} A + k + 2 + \operatorname{nnz} B(1) = \operatorname{nnz} A + k - l + 1 + \operatorname{nnz} B < \operatorname{nnz} A + \operatorname{nnz} B = \operatorname{nnz}(A \oplus B)$, since $k - l + 1 < 0$. Thus C has at least one fewer entries equal to 1 than $A \oplus B$.

If A or $B(1)$ is not a full Hessenberg matrix, then placing another 1 in whichever component of C is not a full Hessenberg matrix will increase the permanent of C, a contradiction on the maximality of $A \oplus B$.

Thus A and $B(1)$ are full Hessenberg matrices. In this case, then $\operatorname{per} C = \operatorname{per} H_{n_A+1} \cdot \operatorname{per} H_{n_B-1} = 2^{n_A+n_B-2} = \operatorname{per}(A \oplus B)$. Again, C has at least one fewer entries equal to 1 than $A \oplus B$, and placing an additional 1 in C will not decrease the permanent. If the permanent increases, then a contradiction arises. If the permanent remains the same, then there exists a permanent maximizer with the given property. In either case the statement holds. ∎

Example 3.11. Note that if $B(1)$ is a full Hessenberg matrix, then equality in (3.9) and Lemma 2.5 show that the first and second columns of B are equal, and thus if $B(1)$ is a full Hessenberg matrix, then so is B. The proof of Lemma 3.10 shows that for $H_{n_A} \oplus H_{n_B}$ with $|n_A - n_B| > 1$, the permanent is not decreased by considering $H_{\lfloor \frac{n}{2} \rfloor} \oplus H_{\lceil \frac{n}{2} \rceil}$, where $n = n_A + n_B$. An example in $\mathcal{H}(17,6)$ is $H_2 \oplus H_4$ versus $H_3 \oplus H_3$. Here the permanents are equal, but $\operatorname{nnz}(H_2 \oplus H_4) > \operatorname{nnz}(H_3 \oplus H_3)$, and placing an additional 1 in the latter, $(H_3 \oplus H_3) + E_{4,3}$, yields the same permanent.

Another permanent maximizer, shown below, in $\mathcal{H}(17,6)$ is fully indecomposable, showing that for a given $\mathcal{H}(m,n)$ there may exist many different permanent maximizers that satisfy the various statements in this paper.

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$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Lemma 3.10 shows that if $R \in \mathcal{H}(m, n)$ with $R = P(m, n)$, and each of R's fully indecomposable components, $R_1, R_2, \ldots, R_k$, are staircased, then there exists such an R such that the number of 1's in the last row of R_i and the number of 1's in the first column of R_j cannot differ by more than 1 for $1 \leq i, j \leq k$.

We now develop exchange rules for the final stripe.

Section 4 will show that there exists a permanent maximizer whose fully indecomposable components are striped. Thus for each fully indecomposable component, A_i, the k-th stripe of A_i is full for $2 \leq k \leq \mathrm{w}(A_i) - 1$. The $\mathrm{w}(A_i)$-th stripe may not be full.

We consider the case when $s + 1 = \mathrm{w}(A_i)$ and the $(s + 1)$-th stripe contains at most two nonzero entries.

Lemma 3.12 (The Last-Stripe Exchange Rule). *Let $A = [a_{ij}]$ be fully indecomposable, striped Hessenberg matrix with $\mathrm{w}(A) = s+1$. Suppose the $(s+1)$-th stripe of A has $a_{s+i,i} = 0$ for $1 \leq i \leq t$ and $a_{s+t+1,t+1} = 1$ where $t > 0$. Let B be the matrix obtained from A by replacing $a_{s+t+1,t+1}$ by 0 and $a_{s+1,1}$ by 1. Then* $\operatorname{per} B \geq \operatorname{per} A$.

Proof. Let C be the matrix obtained from A by replacing $a_{s+t+1,t+1}$ by 0.

Then

$$\operatorname{per} A = \operatorname{per} C + \operatorname{per} C(s+t+1, t+1) = \operatorname{per} C + \operatorname{per} C[\langle 1, t\rangle] \cdot \operatorname{per} C[\langle s+t+2, n\rangle],$$

and

$$\operatorname{per} B = \operatorname{per} C + \operatorname{per} C(s+1, 1) = \operatorname{per} C + \operatorname{per} C[\langle s+2, n\rangle].$$

Note

$$C[\langle s+2, n\rangle] \geq C[\langle s+2, s+t+1\rangle] \oplus C[\langle s+t+2, n\rangle],$$

and

$$C[\langle s+2, s+t+1\rangle] = H_{t,s} = C[\langle 1, t\rangle]$$

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as C is striped of width s and $a_{s+i,i} = 0$ for $1 \le i \le t$. Thus

$$\begin{aligned}
\operatorname{per} C(s+1,1) &= \operatorname{per} C[\langle s+2,n\rangle] \\
&\ge \operatorname{per} C[\langle s+2,s+t+1\rangle] \cdot \operatorname{per} C[\langle s+t+2,n\rangle] \\
&= \operatorname{per} C[\langle 1,t\rangle] \cdot \operatorname{per} C[\langle s+t+2,n\rangle] \\
&= \operatorname{per} C(s+t+1,t+1),
\end{aligned}$$

and $\operatorname{per} B \ge \operatorname{per} A$. □

A similar result holds for $a_{s+t+1,t+1} = 1$ and $a_{s+i,i} = 0$ for $t+2 \le i \le n-s$.

Lemma 3.12 will be used to show for every m and n there exists a permanent maximizer whose fully indecomposable components are striped and each of whose last stripe has the form $(1, 0, \ldots, 0)$ if there is a single 1 on the last stripe, the form $(1, 0, \ldots, 0, 1)$ if there are two 1's on the last stripe, and for more than two 1's on the final stripe the lemma shows that there exists a permanent maximizer with $a_{s+1,1} = a_{n,n-s} = 1$.

4. Application of Exchange Rules.

Theorem 2.4 asserts that for each m and n there exists a permanent maximizer whose fully indecomposable components are staircased, Hessenberg matrices. By Lemma 2.2 the first and second stripes of each fully indecomposable component are full. Combining these results with Lemmas 3.3 and 3.6 yields a further refinement on the structure of these fully indecomposable components.

This refinement further narrows the number of matrices that need to be considered when trying to compute $P(m, n)$. In some cases this refinement will allow the quantity $P(m, n)$ to be computed exactly. Examples of this are in the following section.

Lemma 4.1. *Let m and n be integers such that there exists $A \in \mathcal{H}(m, n)$ with $\operatorname{per} A = P(m, n)$ with the first two stripes and superdiagonal full. Then there exists a striped $B \in \mathcal{H}(m, n)$ with $\operatorname{per} B = P(m, n)$.*

Proof. Among all $B \in \mathcal{H}(m, n)$ with B fully indecomposable and $\operatorname{per} B = P(m, n)$ choose B with smallest lexicographic order. Note that the existence of A shows at least one such B exists.

By Theorem 2.4 the choice of B requires B to be staircased, otherwise there exists a top-most interchangeable row pair or a right-most interchangeable column pair that can be switched. Applying Lemma 2.3 then decreases the lexicographic order of B, a contradiction. Note that the exchange will not remove a 1 already in the superdiagonal, and as such the exchange leaves B fully indecomposable with full first and second stripes. Thus B is staircased.

Then, the matrix B is a fully indecomposable, staircased, Hessenberg matrix. Set

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$p = \mathrm{w}(B)$. If the k-th stripe of B is full for $3 \le k \le p-1$, then B is striped. Otherwise the $(p-1)$-th stripe is not full. Thus there exist two entries meeting the hypotheses of Lemma 3.6 or its persymmetric analog. (Note that Lemma 3.3 is a subcase of Lemma 3.6 or its persymmetric analog.) Without loss of generality choose such a pair of entries with $s-q$ smallest. Applying Lemma 3.6 decreases the lexicographic order of B, contrary to the choice of B. Thus B is striped with $\operatorname{per} B = P(m,n)$. ☐

The following theorem extends the results of [5] by showing the existence of a permanent maximizer in $\mathcal{H}(m,n)$ for $n \le m \le \operatorname{nnz} H_n$ with a more refined structure than that found in [5]. This new structure will allow $P(m,n)$ to be directly computed for a wide range of values of m.

Theorem 4.2. *Let m and n be positive integers with $n \le m \le \operatorname{nnz} H_n$. Then there exists a matrix $A \in \mathcal{H}(m,n)$ with permanent $P(m,n)$ such that A has the form*

$$A = \begin{bmatrix} A_1 & O & O & \cdots & O \\ * & A_2 & O & \cdots & O \\ O & * & \ddots & & \vdots \\ O & O & \ddots & A_{k-1} & O \\ O & O & O & * & A_k \end{bmatrix}, \qquad (4.1)$$

where the $*$'s may contain a single 1 *and the matrices $A_1,\ldots,A_k$ satisfy one of the following three categories: (a) $k=1$ and A is fully indecomposable and striped, (b) The matrices are fully indecomposable and striped components satisfying* $\mathrm{w}(A_i) \le 5$ *for $i=1,\ldots,k$ with $|\mathrm{w}(A_i)-\mathrm{w}(A_j)| \le 1$ for $1 \le i,j \le k$, or (c) The matrices are fully indecomposable and striped components satisfying* $\mathrm{w}(A_i) = 5$ *for $i = 1,\ldots,k-1$ and* $\mathrm{w}(A_k) = 6$.

Proof. Take $A \in \mathcal{H}(m,n)$ with $\operatorname{per} A = P(m,n)$.

The proof is separated into six claims.

Claim 1: There exists a permanent maximizer whose fully indecomposable components are staircased. Theorem 2.4 shows that there exists a permanent maximizer with the form (4.1) where each fully indecomposable component is staircased. Moreover, as each component is fully indecomposable and staircased the first and second stripes of each component are full. The 1 that may exist in the submatrices labeled $*$ is discussed in [5].

Claim 2: There exists a permanent maximizer whose fully indecomposable components are striped. Applying Lemma 4.1 to each fully indecomposable component of (4.1) shows there exists a permanent maximizer whose fully indecomposable components are striped.

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Let nnzr A_i denote the number of nonzero entries in the last row of A_i and nnzc A_i denote the number of nonzero entries in the first column of A_i.

Claim 3: If the last stripe of A_i *contains at least two* 1*'s, then there exists a permanent maximizer* A_i *that satisfies nnzc* A_i = *nnzr* $A_i = \mathrm{w}(A_i)$*, and if the last stripe contains a single* 1*, then there exists a permanent maximizer that satisfies nnzc* $A_i = \mathrm{w}(A_i)$ *or nnzr* $A_i = \mathrm{w}(A_i)$*.* Apply Lemma 3.12. Note that in the case of a single 1 in the last stripe either choice may be used as needed.

Claim 4: There exists a permanent maximizer whose fully indecomposable, striped components satisfy $\mathrm{w}(A_i) \leq 6$ *for all* i*.* Assume to the contrary that A_i and A_j are fully indecomposable, staircased Hessenberg matrices with $\min\{\mathrm{w}(A_i), \mathrm{w}(A_j)\} \geq 6$. Then applying Lemma 3.8 produces a single fully indecomposable, staircased matrix, called A_i', with larger permanent. This is a contradiction on the maximality of A. Thus no such components exists, and Claim 4 is satisfied.

Note that if $\mathrm{w}(A_i) = 5$ and $\mathrm{w}(A_j) = 6$, then Lemma 3.8 does not apply and a single component of width 6 can exist.

Claim 5: There exists a permanent maximizer for which its fully indecomposable, striped components satisfy $|\mathrm{w}(A_i) - \mathrm{w}(A_j)| \leq 1$ *for* $1 \leq i, j \leq k$*.* To the contrary, assume two fully indecomposable, striped components with $|\mathrm{w}(A_i) - \mathrm{w}(A_j)| \geq 2$ exist. Lemma 3.10 immediately provides a contradiction. Thus the criterion of Claim 5 is met.

Claim 6: There exists a permanent maximizer with one of the following three forms: (a) The permanent maximizer is fully indecomposable and striped, (b) The fully indecomposable, striped components satisfy $\mathrm{w}(A_i) \leq 5$ *for* $i = 1, \ldots, k$ *with* $|\mathrm{w}(A_i) - \mathrm{w}(A_j)| \leq 1$ *for* $1 \leq i, j \leq k$*, or (c) The fully indecomposable, striped components satisfy* $\mathrm{w}(A_i) = 5$ *for* $i = 1, \ldots, k-1$ *and* $\mathrm{w}(A_k) = 6$*.* Case (a) is satisfied if $k = 1$ by Claim 2. Case (b) follows from Claims 4 and 5 when no component has width greater than 5. Case (c) follows from Claims 4 and 5 when there exists exactly one component with width 6. Note that the possibility of having more than one component with width 6 is ruled out by Claim 5.

Thus the theorem is proved. □

As mentioned before, this shows that in the search for a permanent maximizer and an exact value for $P(m, n)$, one need only consider a subclass of $\mathcal{H}(m, n)$ whose matrices have the form of (4.1).

Examples of Case (a) can be found in the following corollary, while an example of Case (b) can be found in Example 3.11. Lemma 6.4, Theorem 6.5, Corollary 4.3, and counting arguments for the number of nonzero entries in fully indecomposable versus

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partly decomposable matrices can be used to limit the number of possible matrices for Case (c). For $10 \leq n \leq 30$ the possibilities can be checked by hand, but no examples of Case (c) were found.

Corollary 4.3. *Let m and n be positive integers with $m \geq 7n$. If $A \in H(m,n)$ with* $\operatorname{per} A = P(m,n)$*, then A is fully indecomposable.*

Proof. Assume $\mathcal{H}(m,n)$ has a permanent maximizer A that is not fully indecomposable. As in the proof of Theorem 4.2, the exchange rules may be applied to A to obtain a permanent maximizer B satisfying either case (b) or case (c). Note the number of fully indecomposable components of B, l, is no greater than that of A.

First consider case (b) where the fully indecomposable components are striped and $\mathrm{w}(B_i) \leq 5$ for $i = 1, \ldots, l$ with $|\mathrm{w}(B_i) - \mathrm{w}(B_j)| \leq 1$ for $1 \leq i,j \leq l$. Then each row of B has at most six 1's, though the exchanges described by Lemma 2.3 may produce an extra 1 on the subdiagonal between two fully indecomposable components, and hence $m \leq 6n+1 < 7n$, a contradiction on m. Thus a permanent maximizer cannot satisfy case (b).

Now consider case (c) where the fully indecomposable components are striped and $\mathrm{w}(B_i) = 5$ for $i = 1, \ldots, l-1$ and $\mathrm{w}(B_l) = 6$. Now $m < 7n - 2 + 1 < 7n$ since each row has at most seven 1's and at least two rows (the first and last rows of B) have at most six 1's, along with the extra 1 mentioned in the previous paragraph. This contradicts the fact that $m \geq 7n$. Thus a permanent maximizer cannot satisfy case (c).

Thus a permanent maximizer with $m \geq 7n$ is fully indecomposable. ◻

5. Consequences. This section applies Theorem 4.2 and Corollary 4.3 to determine $P(m,n)$ exactly for certain values of n and m.

A transversal in a striped, Hessenberg matrix A through an entry $(i, i-s+1)$, where $s = \mathrm{w}(A)$, necessarily contains the entries $(i-(s-k), i-s+1+k)$ for $1 \leq k \leq s-1$ and $s \leq i \leq n$. As in Lemma 2.2 the number of transversals of this type are $\operatorname{per} A[\langle 1, i-s\rangle] \cdot \operatorname{per} A[\langle i+1, n\rangle]$. Note that $A[\langle 1, i-s\rangle]$ has order $i-s$ and $A[\langle i+1, n\rangle]$ has order $n-i$. If neither submatrix is to contain an entry from the s-th stripe, then $(i-s)+(n-i) < s$, or $s > \frac{n}{2}$. Thus for $s > \frac{n}{2}$ no transversal of A can contain two entries from the s-th stripe of A.

We first show the existence of a permanent maximizer of a special form in $\mathcal{H}(\operatorname{nnz} H_{n,s-1} + k, n)$ in the case $s > \frac{n}{2}$.

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LEMMA 5.1. *If $s > \frac{n}{2}$, then*

$$\operatorname{per}(H_{n,s-1} + \sum_{j=1}^{k} E_{i_j,i_j-s+1}) = \operatorname{per} H_{n,s-1} + \sum_{j=1}^{k} \operatorname{per} H_{n,s-1}(i_j, i_j - s + 1)$$

for $1 \le k \le n-s+1$ and $s \le i_j \le n$.

Proof. As described above, if $s > \frac{n}{2}$, then no transversal containing the $(i, i-s+1)$-entry contains another entry of similar form. Thus the number of transversals containing the $(i, i-s+1)$-entry is independent of the number of transversals containing the, say, $(i, i-s+1)$-entry. The statement now follows. □

LEMMA 5.2. *Let $C = H_{n,s-1}$ with $s > \frac{n}{2}$. Then for $s+1 \le i \le n-1$, $\operatorname{per} C(i, i-s+1) = 2^{n-s-2}$.*

Proof. By Lemma 2.2

$$\operatorname{per} C(i, i-s+1) = \operatorname{per} C[\langle 1, i-s\rangle] \cdot \operatorname{per} C[\langle i+1, n\rangle].$$

As $s > \frac{n}{2}$, each of the submatrices is a fully indecomposable, full Hessenberg matrix. Thus

$$\begin{aligned}\operatorname{per} C(i, i-s+1) &= \operatorname{per} C[\langle 1, i-s\rangle] \cdot \operatorname{per} C[\langle i+1, n\rangle] \\ &= \operatorname{per} H_{i-s} \cdot \operatorname{per} H_{n-i} \\ &= 2^{i-s-1} \cdot 2^{n-i-1} \\ &= 2^{n-s-2}.\end{aligned}$$ □

LEMMA 5.3. *Let $C = H_{n,s-1}$ with $s > \frac{n}{2}$. Then* $\operatorname{per} C(s,1) = \operatorname{per} C(n, n-s+1) > \operatorname{per} C(i, i-s+1)$ *where $s \le i < n$.*

Proof. The equality follows from persymmetry. By Lemma 2.2

$$\operatorname{per} C(s,1) = \operatorname{per} C[\langle s+1, n\rangle] = \operatorname{per} H_{n-s} = 2^{n-s-1}$$

since $s \ge \lfloor \frac{n-2}{2} \rfloor$, and by Lemma 5.2

$$\operatorname{per} C(i,j) = 2^{n-s-2}.$$ □

Note $\operatorname{nnz} H_{n,s} = n - 1 + s(n+1) - \frac{s(s+1)}{2}$.

By Corollary 4.3 the following theorem particularly applies when $s \ge 7$.

THEOREM 5.4. *Consider $\mathcal{H}(\operatorname{nnz} H_{n,s-1} + k, n)$ with $s > \frac{n}{2}$ and $k \le n-s+1$. If there exists a fully indecomposable permanent maximizer, then each matrix of the form $H_{n,s-1} + E_{s,1} + E_{n,n-s+1} + K$ where K has $k-2$ nonzero entries $(i, i-s+1)$ with $s < i < n$ is a permanent maximizer.*

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Proof. By the hypothesis there exists a fully indecomposable permanent maximizer. Applying Lemma 4.1 such a maximizer may be taken to be striped. It then suffices to consider the s-th stripe and the final k 1's. By Lemma 5.1 each 1 may be considered separately.

By Lemmas 5.2 and 5.3 the $(s,1)$ and $(n,n-s+1)$ yield a larger contribution to the permanent, and the remaining $k-2$ entries may be placed arbitrarily amongst the remaining entries of the s-th stripe.

Thus there exists a permanent maximizer of the given form. ◻

Section 4 and Theorem 5.4 provide enough structure for a special permanent maximizer that for certain m and n, $P(m,n)$ can be computed directly. The remainder of this section provides specific values for $P(m,n)$ for certain m and n, and thereby extends the results in [5].

Corollary 5.5.

(a) If $m \geq 7n$, then

$$P(\operatorname{nnz} H_{n,s}, n) = \operatorname{per} H_{n,s}.$$

(b) If $m \geq 7n$, then

$$P(\operatorname{nnz} H_{n,s}+1, n) = \operatorname{per} H_{n,s} + \operatorname{per} H_{n-s,s}.$$

(c) If $m \geq 7n$, then

$$P(\operatorname{nnz} H_{n,s}+2, n) = \operatorname{per} H_{n,s} + 2\operatorname{per} H_{n-s,s} + \operatorname{per} H_{n-2s,s}.$$

(d) If $m \geq 7n$, $s > \frac{n}{2}$, and $k < n-s+2$, then

$$P(\operatorname{nnz} H_{n,s}+k, n) = \operatorname{per} H_{n,s} + (k-2)\,2^{n-s-2} + 2^{n-s}.$$

Proof. Items (a), (b) and (c) follow from Corollary 4.3 and Lemma 3.12. For (b),

$$\begin{aligned} P(\operatorname{nnz} H_{n,s}+1, n) &= \operatorname{per} H_{n,s} + \operatorname{per} H_{n,s}(s+1,1) \\ &= \operatorname{per} H_{n,s} + \operatorname{per} H_{n-s,s}. \end{aligned}$$

For (c),

$$\begin{aligned} P(\operatorname{nnz} H_{n,s}+2, n) &= \operatorname{per} H_{n,s} + \operatorname{per} H_{n,s}(s+1,1) \\ &\quad + \operatorname{per} H_{n,s}(n-s,n) + \operatorname{per} H_{n,s}[\langle s+1, n-s-1\rangle] \\ &= \operatorname{per} H_{n,s} + 2\operatorname{per} H_{n-s,s} + \operatorname{per} H_{n-2s,s}. \end{aligned}$$

Item (d) follows from Corollary 4.3, Lemma 3.12 and Theorem 5.4, yielding

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$$\begin{aligned} P(\operatorname{nnz} H_{n,s}+k,n) &= \operatorname{per} H_{n,s} + \textstyle\sum_{i=1}^{k-1} \operatorname{per} H_{n,s}(s+i,i) + \operatorname{per} H_{n,s}(n-s,n) \\ &= \operatorname{per} H_{n,s} + 2^{n-s-1} + (k-2)\,2^{n-s-2} + 2^{n-s-1} \\ &= \operatorname{per} H_{n,s} + (k-2)\,2^{n-s-2} + 2^{n-s}. \end{aligned} \qquad \square$$

6. Further Results. Section 4 showed for each m and n there is a permanent maximizer of $\mathcal{H}(m,n)$ each of whose fully indecomposable components is striped. This section provides bounds on $\operatorname{per} H_{n,s}$ for large and small s. The second bound relies on a connection with the s-Generalized Fibonacci Numbers. These bounds show how $P(m,n)$ grows as a function of m. We first consider the case where s is large relative to n, and a lower bound on $\operatorname{per} H_{n,s}$ is developed.

A *directed graph*, or digraph, D consists of a set V of *vertices*, a set E of *edges*, and a mapping associating to each edge $e \in E$ an ordered pair (x,y) of vertices called the *endpoints* of e. The vertex x in the ordered pair (x,y) is called the *tail* of e while y is called the *head* of e.

A *cycle* in a digraph D is a collection of edges

$$(x_0,x_1),(x_1,x_2),\ldots,(x_{n-1},x_n),(x_n,x_0),$$

where the x_i's are distinct. A *loop* is a cycle of the form (x_0,x_0). Two cycles,

$$x = \{(x_0,x_1),(x_1,x_2),\ldots,(x_{n-1},x_n),(x_n,x_0)\}$$

and

$$y = \{(y_0,y_1),(y_1,y_2),\ldots,(y_{m-1},y_m),(y_m,y_0)\},$$

are *disjoint* if $x_i \neq y_j$ for $0 \leq i \leq n$ and $0 \leq j \leq m$. A *disjoint cycle union*, or DCU, of a digraph D is a collection of mutually disjoint cycles such that every vertex of D is contained in exactly one cycle.

The *weight* of an edge e is described by the function $\operatorname{wt} : E \to \mathbb{N}$. The weight of a cycle, x, is defined as

$$\operatorname{wt}(x) = \prod_{e \in x} \operatorname{wt}(e).$$

The weight of a DCU, τ, of D is defined as

$$\operatorname{wt}(\tau) = (-1)^{|V|-k} \prod_{x \in \tau} \operatorname{wt}(x)$$

where k is the number of cycles in τ and $|V|$ is the number of vertices in each cycle.

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If $V = \{1, 2, \ldots, n\}$ we may associate with D an $n \times n$ $(0,1)$-matrix $C = [c_{ij}]$. If (i,j) is an edge of D, then $c_{ij} = \mathrm{wt}(e)$, and if (i,j) is not an edge of D, then $c_{ij} = 0$.

Lemma 6.1.

$$\det C = \sum_{\tau \in DCU} wt(\tau). \tag{6.1}$$

where DCU is the set of disjoint cycle unions of the digraph of C.

Proof.

$$\det C = \sum_{\tau \in S_n} \mathrm{sgn}(\tau) a_{1\tau(1)} a_{2\tau(2)} \ldots a_{n\tau(n)},$$

but only those τ corresponding to a DCU of the digraph of C will have nonzero contribution, and in those cases $\mathrm{wt}\,(\tau) = \mathrm{sgn}(\tau) a_{1\tau(1)} a_{2\tau(2)} \ldots a_{n\tau(n)}$. Therefore,

$$\det C = \sum_{\tau \in DCU} \mathrm{wt}(\tau). \qquad \square$$

Before stating the first result of this section an identity concerning the computation of the permanent for Hessenberg matrices is needed (see [2]).

Lemma 6.2. *Let A be an $n \times n$ $(n \geq 2)$ Hessenberg matrix, and let B be the matrix obtained from A by replacing the $(i, i+1)$ entries by their negatives. Then $\det B = \mathrm{per}\, A$.*

Proof. Proof by induction on n. The statement holds for $n = 2$ since:

$$\mathrm{per} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} + a_{12}a_{21} = \det \begin{bmatrix} a_{11} & -a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Assume $n \geq 3$, and proceed by induction. Let A and B be such a pair of $n \times n$ matrices. By Laplace expansion of B along row 1, $\det B = a_{11} \det B(1) - (-a_{12}) \det B(1,2) = a_{11} \det B(1) + a_{12} \det B(1,2)$. By the induction hypothesis

$$\det B(1) = \mathrm{per}\, A(1) \ \text{ and } \ \det B(1,2) = \mathrm{per}\, A(1,2).$$

By the above and the Laplace expansion of $\mathrm{per}\, A$ along the first row, $\det B = a_{11}\mathrm{per}\, A(1) + a_{12}\mathrm{per}\, A(1,2) = \mathrm{per}\, A$. $\square$

Lemma 6.3.

$$\mathrm{per}\, H_{n,s} > 2^{n-2}$$

whenever $s \geq \log_2 n$.

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Proof. Assume $2^s \geq n$, and let $A = H_{n,s}$. Let B be the matrix obtained from A by negating each entry of its superdiagonal. By Lemma 6.2, $\det B = \operatorname{per} A$.

Let M be the $n \times n$ $(0,1)$-matrix with $m_{ii} = 1$ and $m_{ij} = -1$ for $j = i-1$ and zero entries elsewhere. Set $C = MB$, then C has the form:

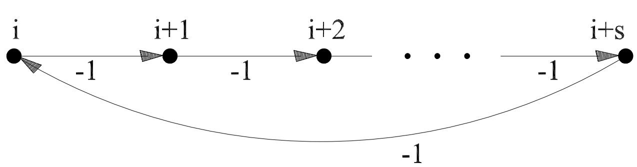
$$\begin{array}{ll} c_{11} = 1 & \\ c_{ii} = 2 & \text{for } 2 \leq i \leq n \\ c_{ij} = -1 & \text{for } j = i+1, 1 \leq i \leq n-1 \\ c_{ij} = -1 & \text{for } j = i-s, s+1 \leq i \leq n \\ c_{ij} = 0 & \text{otherwise.} \end{array}$$

meaning that the superdiagonal and the $(s+1)$-th stripe have all -1 entries.

We now compute the determinant of C via its digraph. Let D be the digraph of C, and let DCU be the set of disjoint cycle unions of D. By Lemma 6.1

$$\det C = \sum_{\tau \in DCU} \operatorname{wt}(\tau).$$

Observe that the only cycles of D are the loops (which have weight 2, except the loop at 1, which has weight 1) and the cycles of the form:



for $1 \leq i \leq n-s$, which have weight $(-1)^{s+1}(-1)^s = -1$.

Thus each DCU consists of loops and $(s+1)$-cycles. For $\tau \in DCU$, let $\kappa(\tau)$ be the number of $(s+1)$-cycles in τ, and let

$$\delta_{\tau,1} = \begin{cases} 1 & \text{if the loop of weight 1 is in } \tau \\ 0 & \text{if it is not} \end{cases}$$

Then it is easy to verify that

$$\operatorname{wt}(\tau) = (-1)^{\kappa(\tau)}\, 2^{n-\delta_{\tau,1}-\kappa(\tau)(s+1)}.$$

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Let

$$\alpha_j = \sum_{\substack{\tau \in DCU \\ \kappa(\tau)=j}} \mathrm{wt}(\tau) = (-1)^j \sum_{\substack{\tau \in DCU \\ \kappa(\tau)=j}} 2^{n-\delta_{\tau,1}-\kappa(\tau)(s+1)}.$$

Consider the map $f : DCU \backslash identity \to DCU$, defined as $f(\tau)$ is the DCU obtained from τ by replacing the $(s+1)$-cycle containing $i \in V$ with i maximal by $s+1$ loops. Note:

$$\begin{aligned}
&\kappa(f(\tau)) = \kappa(\tau) - 1, \\
&\mathrm{wt}(f(\tau)) = -2^{s+1}\mathrm{wt}(\tau) \quad \text{if } \kappa(\tau) \geq 1, \\
&|f^{-1}(\sigma)| \leq n \quad \forall \sigma \in DCU.
\end{aligned}$$

It follows that if $j \geq 2$, then

$$\begin{aligned}
|\alpha_j| &= \left| \sum_{\substack{\tau \in DCU \\ \kappa(\tau)=j}} \mathrm{wt}(\tau) \right| \\
&= \frac{1}{2^{s+1}} \left| \sum_{\substack{\tau \in DCU \\ \kappa(\tau)=j}} \mathrm{wt}(f(\tau)) \right| \\
&\leq \frac{n}{2^{s+1}} \left| \sum_{\substack{\sigma \in DCU \\ \kappa(\sigma)=j-1}} \mathrm{wt}(\sigma) \right| \\
&= \frac{n}{2^{s+1}} |\alpha_{j-1}| \\
&< |\alpha_{j-1}|
\end{aligned} \tag{6.2}$$

since $2^{s+1} \geq 2^s \geq n$.

As the α_j's alternate in sign, (6.1) and (6.2) imply

$$\begin{aligned}
\det C &> \alpha_0 + \alpha_1 \\
&= 2^{n-1} - \left(2^{n-1-(s+1)}(n-s-2) + 2^{n-(s+1)}\right) \\
&= 2^{n-2}\left[2 - \tfrac{n-s}{2^s}\right] \\
&\geq 2^{n-2}\left[2 - \tfrac{n}{2^s}\right] \\
&\geq 2^{n-2}\left[2-1\right] \quad \text{since } 2^s \geq n \\
&= 2^{n-2}
\end{aligned}.$$

Therefore, if $s \geq \log_2 n$, then $\mathrm{per}\, H_{n,s} > 2^{n-2}$. □

LEMMA 6.4. *If A is an $n \times n$ partly decomposable, Hessenberg $(0,1)$-matrix, then* $\mathrm{per}\, A \leq 2^{n-2}$.

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Proof. As A is partly decomposable there exist permutation matrices P and Q such that PAQ has the form

$$\begin{bmatrix} B & O \\ C & D \end{bmatrix},$$

where B and D are square, nonvacuous, matrices. Also, $\operatorname{per} A = \operatorname{per} B \cdot \operatorname{per} D$. Let B have order k and therefore D has order $n-k$.

Note that by (2.2) that $\operatorname{per} B \leq 2^{k-1}$ and $\operatorname{per} D \leq 2^{n-k-1}$. Then $\operatorname{per} A \leq 2^{k-1} \cdot 2^{n-k-1} = 2^{n-2}$. □

THEOREM 6.5. *If $s \geq \log_2 n$ and $m \geq \operatorname{nnz} H_{n,s}$, then every permanent maximizer in $\mathcal{H}(m,n)$ is fully indecomposable.*

Proof. By Lemma 6.3

$$P(m,n) \geq P(\operatorname{nnz} H_{n,s}, n) \geq \operatorname{per} H_{n,s} > 2^{n-2}.$$

Thus by Lemma 6.4 every permanent maximizer in $\mathcal{H}(m,n)$ must be fully indecomposable. □

Note that Theorem 6.5 gives a lower bound on the number of nonzero entries for which the permanent maximizer is guaranteed to be fully indecomposable, but Corollary 4.3 provides a better lower bound.

EXAMPLE 6.6. Consider $\mathcal{H}(58,12)$ and a permanent maximizer $A \in \mathcal{H}(58,12)$ with $\mathrm{w}(A) = 5$ and $\operatorname{per} A = P(58,12)$. As $s \geq \log_2 12 = 3.58$, $\operatorname{per} H_{12,4} > 2^{10}$, and so $\operatorname{per} A > 2^{10}$ and A is fully indecomposable.

We now consider an upper bound on $\operatorname{per} H_{n,s}$ found by exploiting a connection with the s-Generalized Fibonacci Numbers (see [3]).

Let s and n be positive integers with $n \geq s$, $s \geq 2$, and let $H_{n,s}$ be the $n \times n$ s-banded Hessenberg $(0,1)$-matrix. Let $\{F_{n,s}\}$ be the sequence defined by

$$F_{n,s} = \begin{cases} 2^{n-1} & \text{if } n = 1,2,\ldots,s \\ F_{n-1,s} + F_{n-2,s} + \cdots + F_{n-s,s} & \text{if } n > s. \end{cases}$$

For a fixed s, $F_{1,s}, F_{2,s}, F_{3,s}, \ldots, F_{n,s}, \ldots$ is a sequence on n referred to as the *s-generalized Fibonacci numbers* (see [3]). Note that for $s = 2$ the Fibonacci Numbers, $f_n = f_{n-1} + f_{n-2}$, are realized.

PROPOSITION 6.7.

$$\operatorname{per} H_{n,s} = F_{n,s}.$$

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Proof. The proof is by strong induction on n.

First consider the case when $n \le s$. Since $n \le s$, $H_{n,s} = H_n$ and $\operatorname{per} H_n = 2^{n-1}$.

Now consider $n > s$. By Laplace expansion along the first column, it can be seen that

$$\operatorname{per} H_{n,s} = \operatorname{per} H_{n,s}[\langle 2, n\rangle] + \cdots + \operatorname{per} H_{n,s}[\langle s+1, n\rangle].$$

Note that

$$H_{n,s}[\langle j, n\rangle] = \begin{cases} H_{n-j+1,s} & \text{if } n-j+1 \ge s \\ H_{n-j+1} & \text{otherwise} \end{cases}$$

by the inductive hypothesis, for $2 \le j \le s+1$. So $\operatorname{per} H_{n,s}[\langle j, n\rangle] = F_{n-j+1,s}$.

Thus, by the Laplace expansion,

$$\operatorname{per} H_{n,s} = F_{n-1,s} + F_{n-2,s} + \ldots + F_{n-s,s} = F_{n,s}$$

by the definition of $F_{n,s}$. □

Let $p_s(x) = x^s - (x^{s-1} + x^{s-2} + \cdots + x + 1)$. Note that

$$p_s(x) = x^s - \frac{x^s - 1}{x - 1} = \frac{x^{s+1} - 2x^s + 1}{x - 1}.$$

Let $q_s(x) = x^{s+1} - 2x^s + 1$. Then the roots of $q_s(x)$ are the roots of $p_s(x)$ along with the root 1.

Note that

$$q_s\left(2 - \frac{1}{2^{s-1}}\right) = -\left(2 - \frac{1}{2^{s-1}}\right)^s \left(\frac{1}{2^{s-1}}\right) + 1 = 1 - 2\left(1 - \frac{1}{2^s}\right)^s < 0,$$

since

$$\ln 2 + s \ln\left(1 - \frac{1}{2^s}\right) = \ln 2 + s\left[\frac{-1}{2^s} + \frac{(1/2s)^2}{2} - \ldots\right] > \ln 2 - \frac{s}{2^s} \ge \ln 2 - \frac{1}{2} > 0.$$

Thus

$$1 < e^{\ln 2 + s \ln\left(1 - \frac{1}{2^s}\right)} = 2\left(1 - \frac{1}{2^s}\right)^s.$$

Also note that

$$q_s\left(2 - \frac{1}{2^s}\right) = 1 - \left(2 - \frac{1}{2^s}\right)^s \cdot \frac{1}{2^s} = 1 - \left(1 - \frac{1}{2^{s+1}}\right)^s > 0.$$

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Hence, by the Intermediate Value Theorem, $q_s(x)$ has a root in $[2 - \frac{1}{2^{s-1}}, 2 - \frac{1}{2^s}]$.

Note that $q_s'(x) = (s+1)x^s - 2sx^{s-1} = x^{s-1}((s+1)x - 2s)$ has largest root $\frac{2s}{s+1} \leq 2 - \frac{1}{2^{s-1}}$. Thus, $q_s(x)$ has at most one root in $(2 - \frac{1}{2^{s-1}}, \infty)$. Therefore, the largest real root, ω_s, of $q_s(x)$ is in $[2 - \frac{1}{2^{s-1}}, 2 - \frac{1}{2^s}]$. This proves the following:

Lemma 6.8. *The largest real root, ω_s, of $q_s(x)$ satisfies*

$$2 - \frac{1}{2^{s-1}} < \omega_s < 2 - \frac{1}{2^s}.$$

Lemma 6.9.

$$F_{n,s} \leq 2^s \omega_s^{n-s}.$$

Proof. By strong induction on n.

If $n \leq s$, then $F_{n,s} = 2^{n-1} \leq 2^s \leq 2^s \omega_s^{n-s}$, since $\omega_s \geq 1$.

If $n > s$, then

$$\begin{aligned} F_{n,s} &= F_{n-1,s} + F_{n-2,s} + \cdots + F_{n-s,s} \\ &\leq 2^s \omega_s^{n-1-s} + \cdots + 2^s \omega_s^{n-s-s} \\ &= 2^s \omega_s^{-s} [w^{n-1} + \cdots + \omega_s^{n-s}] \\ &= 2^s \omega_s^{-s} \omega_s^n \quad (\text{since } \omega_s^s = \omega_s^{s-1} + \cdots + \omega_s + 1) \\ &= 2^s \omega_s^{n-s}. \end{aligned}$$

∎

Theorem 6.10. *If $n \geq (\ln 2)(2^{s+2} + 2) + s$, then $F_{n,s} \leq 2^{n-2}$.*

Proof. By Proposition 6.7 and Lemmas 6.8 and 6.9 it suffices to show that

$$2^s \left(2 - \frac{1}{2^s}\right)^{n-s} \leq 2^{n-2},$$

or equivalently that

$$4 \leq \left(\frac{2}{2 - \frac{1}{2^s}}\right)^{n-s},$$

or equivalently that

$$4 \leq \left(\frac{1}{1 - \frac{1}{2^{s+1}}}\right)^{n-s},$$

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or equivalently that

$$\ln 4 \leq -(n-s)\ln\left(1-\frac{1}{2^{s+1}}\right).$$

Now

$$-\ln\left(1-\frac{1}{2^{s+1}}\right)=\frac{1}{2^{s+1}}-\frac{(\frac{1}{2^{s+1}})^2}{2}+\ldots,$$

so, as the series is alternating,

$$-\ln\left(1-\frac{1}{2^{s+1}}\right)\geq\frac{1}{2^{s+1}}-\frac{(\frac{1}{2^{s+1}})^2}{2}.$$

Thus

$$\begin{array}{rcll}
-(n-s)\ln\left(1-\frac{1}{2^{s+1}}\right) & \geq & (n-s)\left(\frac{1}{2^{s+1}}-\frac{1}{2^{2s+3}}\right) & \\
& \geq & \ln 2(2^{s+2}+2)\left(\frac{1}{2^{s+1}}-\frac{1}{2^{2s+3}}\right) & \text{by hypothesis} \\
& \geq & \ln 2\left(2-\frac{1}{2^{s+1}}+\frac{1}{2^{s}}-\frac{1}{2^{2s+2}}\right) & \\
& = & \ln 2\left(2+\frac{1}{2^{s+1}}-\frac{1}{2^{2s+2}}\right) & \\
& \geq & (\ln 2)2 & \\
& = & \ln 4, &
\end{array}$$

as desired. □

Considering Theorem 6.5 and Theorem 6.10 the value of s where permanent maximizers switch from being partly decomposable to fully indecomposable is approximately $\log_2 n$.

Corollary 6.11. *For A striped with* $\mathrm{w}(A)=s$, $\operatorname{per} H_{n,s-1}\leq \operatorname{per} A\leq 2^s\omega_s^{n-s}$.

Proof. First, we have $\operatorname{per} H_{n,s-1}\leq\operatorname{per} A\leq\operatorname{per} H_{n,s}$. Considering Lemma 6.9 this inequality becomes $\operatorname{per} H_{n,s-1}\leq\operatorname{per} A\leq 2^s\omega_s^{n-s}$. □

We now discuss some of the consequences of Corollary 6.11. For $s\geq\log_2 n$, $\operatorname{per} H_{n,s-1}>2^{n-2}$, but $\operatorname{per} H_n=2^{n-1}$ so

$$2^{n-2}\leq P(m,n)\leq 2^s\,\omega_s^{n-s}\leq 2^s\,\left(2-\frac{1}{2^s}\right)^{n-s},$$

and thus, for a fixed n, $P(m,n)$ as a function of m can only grow by a factor of 2 for $s\geq\log_2 n$. Also, if $n\geq(\ln 2)(2^{s+2}+2)+s$, then $F_{n,s}\leq 2^{n-2}$ so the majority of the contribution to $P(m,n)$ comes from the first few stripes.

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