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PERIODIC COPRIME MATRIX FRACTION DECOMPOSITIONS*

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Abstract. A study is presented of right (left) coprime decompositions of a collection of N-periodic rational matrices, with some ordered structure. From a block-ordered right coprime decomposition of a rational matrix of the given periodic collection, the corresponding block-ordered right coprime decompositions of the remaining matrices of the collection are constructed. In addition, those decompositions are N-periodic.

Key words. Periodic rational matrices, coprime decompositions, Smith canonical form

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1. Introduction. It is well-known that in control theory there are different approaches for studying linear multivariable systems. One of the most important advantages of the fraction matrix approach is that it permits the use of polynomial matrices which form the decomposition of the transfer matrix. Those polynomial matrices have all information about the system. In the study of rational matrices the right and left coprime polynomial fraction decompositions play an important role in control theory, because they are directly related to the concept of controllability and observability; see [4] and [6]. Further, those kind of decompositions apply to a lot of problems of multivariable systems as the minimal realization problem, the regulator problem with internal stability, the output feedback compensator problem, etc. Different studies of fraction decompositions with applications in control invariant systems are [3], [5], [7]. and [8]

A discrete-time linear periodic system in the z-domain can be defined by a periodic collection of rational matrices of the following form (see [2]),

$$\{H_s(z), s\in\mathbb{Z}\},\ \ H_{s+N}(z) = H_s(z) \in \mathbb{R}^{pN\times mN}[z], \tag{1}$$

such that

$$H_{s+1}(z) = S_{pN,p}(z)H_s(z)S^{-1}_{mN,m}(z),\ \ s\in\mathbb{Z}, \tag{2}$$

where $S_{pN,p}(z)$ and $S_{mN,m}(z)$ are given in (3), and one of the matrix of the collection, say $H_0(z)$, is proper with lower block triangular polynomial part.

It seems that the knowledge of coprime decompositions of the collection (1)–(2) can help to solve problems of discrete-time linear periodic systems

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as those mentioned above in the invariant case. For instance, one can use those decompositions to obtain, with an output feedback, a stable closed-loop periodic system of minimal dimension. The main aim of this work is to obtain a right (left) coprime decomposition of a collection of periodic rational matrices and study the relationship among them. This paper is structured as follows: In Section 2 we state our definitions and give some notation to simplify the paper. Then, in Section 3 we study the block-ordered right coprime decomposition of a rational matrix. Next, in Section 4, we study the Smith canonical form of the natural decomposition of H_{s+1}, from a given decomposition of H_s. That canonical form is basic in the last section, where we shall construct a block-ordered right coprime decomposition of H_{s+1} from one of H_s.

2. Preliminaries and notation. Given two nonnegative integers α, β with $\alpha \geq \beta$, we define the polynomial matrices:

$$S_{\alpha,\beta}(z) = \begin{bmatrix} O & I_{\alpha-\beta} \\ zI_\beta & O \end{bmatrix}. \tag{3}$$

Some properties of those matrices can be found in [1]. Here, we explicitly state the following properties which are complementaries to those given in [1].

Lemma 2.1. *The matrices given in (3) satisfy the following properties:*

(i) $S_{\alpha,\beta_1}(z) \cdot S_{\alpha,\beta_2}(z) = \begin{cases} S_{\alpha,\beta_1+\beta_2}(z) & if \quad \beta_1+\beta_2 \leq \alpha \\ zS_{\alpha,\beta_1+\beta_2-\alpha}(z) & if \quad \beta_1+\beta_2 > \alpha \end{cases}$

(ii) $det\ S_{\alpha,\beta}(z) = (-1)^{\beta(\alpha-\beta)} z^\beta$

(iii) $S^{-1}_{\alpha,\beta}(z) = z^{-1} S_{\alpha,\alpha-\beta}(z)$.

In the rest of the paper we shall denote, in general, rational matrices by H_s, because most of the results will be applied in the context of rational matrices defined in (1) satisfying property (2). We recall that a pair of polynomial matrices $\big(D_s(z), N_s(z)\big)$, with $D_s(z) \in \mathbb{R}^{mN\times mN}[z]$ and $N_s(z) \in \mathbb{R}^{pN\times mN}[z]$, is a *right decomposition* of $H_s(z)$ if

$$H_s(z) = N_s(z) D_s^{-1}(z)\ . \tag{4}$$

In what follows we shall work with right *coprime* decompositions. A pair of polynomial matrices $\big(D(z), N(z)\big)$ is said to be *right coprime* if the Smith canonical form of the matrix

$$\mathcal{F}\Big(D(z), N(z)\Big) = \begin{bmatrix} D(z) \\ N(z) \end{bmatrix} \in \mathbb{R}^{(m+p)N\times mN}[z]$$

is the matrix

$$\begin{bmatrix} I_{mN} \\ O \end{bmatrix}\ . \tag{5}$$

From the right decomposition (4) of $H_s(z)$ and the relation (2) we have

$$\begin{aligned} H_{s+1}(z) &= S_{pN,p}(z)H_s(z)S^{-1}_{mN,m}(z) = S_{pN,p}(z)N_s(z)D_s^{-1}(z)S^{-1}_{mN,m}(z) \\ &= \Big(S_{pN,p}(z)N_s(z)\Big)\Big(S_{mN,m}(z)D_s(z)\Big)^{-1} , \end{aligned} \tag{6}$$

which is a right decomposition of $H_{s+1}(z)$. In general, the right decomposition (6) is not right coprime even in the case the right decomposition $\Big(D_s(z), N_s(z)\Big)$ is right coprime, as the following example shows.

Example 2.2. Let us consider the matrices

$$H_0(z) = \begin{bmatrix} \frac{1}{z} & 0 \\ 1 & \frac{1}{z-1} \end{bmatrix} \quad \text{and} \quad H_1(z) = \begin{bmatrix} \frac{1}{z-1} & \frac{1}{z} \\ 0 & \frac{1}{z} \end{bmatrix} .$$

The periodic set $\{H_0(z), H_1(z), H_{s+2}(z) = H_s(z), s \in \mathbb{Z}\}$ is a periodic collection of rational matrices with period $N = 2$ and, according to the notation given in (1), $p = 1$ and $m = 1$. The pair of matrices

$$\Big(D_0(z), N_0(z)\Big) = \left(\begin{bmatrix} z^2 - z & z \\ z^2 - z & z - 1 \end{bmatrix}, \begin{bmatrix} z - 1 & 1 \\ z^2 & z + 1 \end{bmatrix}\right)$$

is a right coprime decomposition of $H_0(z)$. Then, compute the matrices

$$S_{2,1}(z)D_0(z) = \begin{bmatrix} 0 & 1 \\ z & 0 \end{bmatrix} \begin{bmatrix} z^2 - z & z \\ z^2 - z & z - 1 \end{bmatrix} = \begin{bmatrix} z^2 - z & z - 1 \\ z^3 - z^2 & z^2 \end{bmatrix},$$

$$S_{2,1}(z)N_0(z) = \begin{bmatrix} 0 & 1 \\ z & 0 \end{bmatrix} \begin{bmatrix} z - 1 & 1 \\ z^2 & z + 1 \end{bmatrix} = \begin{bmatrix} z^2 & z + 1 \\ z^2 - z & z \end{bmatrix}.$$

The pair

$$\Big(S_{2,1}(z)D_0(z), S_{2,1}(z)N_0(z)\Big) \tag{7}$$

is a right decomposition of $H_1(z)$. Since the Smith canonical form of the matrix $\mathcal{F}\Big(S_{2,1}(z)D_0(z), S_{2,1}(z)N_0(z)\Big)$ is

$$\begin{bmatrix} 1 & 0 \\ 0 & z \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

the pair (7) is not right coprime.

For simplicity, we will omit the variable z in rational and polynomial matrices, however, when $z=0$, we will always make it explicit.

Let us consider the following row-block partition of the matrices D_s and N_s,

$$D_s = \begin{bmatrix} \widetilde{D}_1 \\ \widetilde{D}_2 \\ \vdots \\ \widetilde{D}_N \end{bmatrix}, \; N_s = \begin{bmatrix} \widetilde{N}_1 \\ \widetilde{N}_2 \\ \vdots \\ \widetilde{N}_N \end{bmatrix},$$

with $\widetilde{D}_i \in \mathbb{R}^{m\times mN}[z]$ and $\widetilde{N}_i \in \mathbb{R}^{p\times mN}[z]$, for $i=1,2,\ldots,N$. We construct the block-polynomial matrices

$$\mathcal{F}_k(D_s,N_s) = \begin{bmatrix} \widetilde{D}_k \\ \vdots \\ \widetilde{D}_N \\ \widetilde{N}_k \\ \vdots \\ \widetilde{N}_N \end{bmatrix} \in \mathbb{R}^{(m+p)(N-k+1)\times mN}[z],$$

for $k=1,2,\ldots,N$. Note that $\mathcal{F}$ defined in the first section is just $\mathcal{F}_1$. Denote

$$r_{sk} = \operatorname{rank} \mathcal{F}_k\Big(D_s(0), N_s(0)\Big), \; k=1,2,\ldots,N. \tag{8}$$

Definition 2.3. *Let (D_s,N_s) denote a right decomposition of a rational matrix. We shall call the sequence $(r_{s1},r_{s2},\ldots,r_{sN})$ the characteristic rank of the pair (D_s,N_s).*

Clearly,

$$mN \geq r_{s1} \geq r_{s2} \geq \cdots \geq r_{sN} \geq 0.$$

When definition 2.3 applies to a coprime decomposition then the first inequality of the above expression becomes equality, i.e., $mN = r_{s1}$.

Definition 2.4. *A right decomposition of H_s, (D_s,N_s), with characteristic rank $(r_{s1},r_{s2},\ldots,r_{sN})$ is said to be* block-ordered *if for each matrix $\mathcal{F}_k(D_s,N_s)$, $k=1,2,\ldots,N$, there exists a partition in two blocks,*

$$\mathcal{F}_k(D_s,N_s) = [\,G_k \quad \overline{G}_k\,], \; \overline{G}_k \in \mathbb{R}^{(m+p)(N-k+1)\times r_{sk}}[z] \;,$$

such that $G_k(0) = O$.

The next result gives a practical characterization of block-ordered decompositions. This characterization will be useful throughout the paper.

LEMMA 2.5. *Let* (D_s, N_s) *be a right coprime decomposition of* H_s *with characteristic rank* $(r_{s1}, r_{s2}, \ldots, r_{sN})$. *Then* (D_s, N_s) *is block-ordered if and only if there exists a partition of the matrices* D_s *and* N_s,

$$(9)\quad \begin{array}{l} D_s = \begin{bmatrix} D_{11} & D_{12} & D_{13} & \ldots & D_{1,N-1} & D_{1N} \\ zD_{21} & D_{22} & D_{23} & \ldots & D_{2,N-1} & D_{2N} \\ zD_{31} & zD_{32} & D_{33} & \ldots & D_{3,N-1} & D_{3N} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ zD_{N-1,1} & zD_{N-1,2} & zD_{N-1,3} & \ldots & D_{N-1,N-1} & D_{N-1,N} \\ zD_{N1} & zD_{N2} & zD_{N3} & \ldots & zD_{N,N-1} & D_{NN} \end{bmatrix}, \\ N_s = \begin{bmatrix} N_{11} & N_{12} & N_{13} & \ldots & N_{1,N-1} & N_{1N} \\ zN_{21} & N_{22} & N_{23} & \ldots & N_{2,N-1} & N_{2N} \\ zN_{31} & zN_{32} & N_{33} & \ldots & N_{3,N-1} & N_{3N} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ zN_{N-1,1} & zN_{N-1,2} & zN_{N-1,3} & \ldots & N_{N-1,N-1} & N_{N-1,N} \\ zN_{N1} & zN_{N2} & zN_{N3} & \ldots & zN_{N,N-1} & N_{NN} \end{bmatrix}, \end{array}$$

with $D_{ii} \in \mathbb{R}^{m\times(r_{si}-r_{s,i+1})}[z]$, $N_{ii} \in \mathbb{R}^{p\times(r_{si}-r_{s,i+1})}[z]$ *and*

$$(10)\quad \operatorname{rank}\begin{bmatrix} D_{ii}(0) \\ N_{ii}(0) \end{bmatrix} = r_{si} - r_{s,i+1},$$

for $i = 1, 2, \ldots, N$, *where* $r_{s,N+1} = 0$.

Proof. Assume that (D_s, N_s) is block-ordered. Consider a block partition of matrices D_s and N_s with the same block sizes as partitions (9), where we denote the blocks of the lower block triangular part by $\hat{D}_{ij}$ and $\hat{N}_{ij}$, $i > j$. We have to prove that each of those blocks are multiples of z and the equation (10) holds.

Since (D_s, N_s) is block-ordered the matrix

$$\mathcal{F}_N(D_s, N_s) = \left[\begin{array}{ccc|c} \hat{D}_{N1} & \ldots & \hat{D}_{N,N-1} & D_{NN} \\ \hat{N}_{N1} & \ldots & \hat{N}_{N,N-1} & N_{NN} \end{array}\right]$$

evaluated at $z = 0$ is $\mathcal{F}_N(D_s(0), N_s(0)) = [O \quad \bar{G}_N(0)]$. Then, we can write $\hat{D}_{Nj} = zD_{Nj}$ and $\hat{N}_{Nj} = zN_{Nj}$ for $j = 1, 2, \ldots, N-1$. Moreover, by (8), rank $\begin{bmatrix} D_{NN}(0) \\ N_{NN}(0) \end{bmatrix} = r_{sN}$. Next, similarly,

$$\begin{aligned}
&\mathcal{F}_{N-1}(D_s(0), N_s(0)) \\
&\quad = \left[\begin{array}{lll|ll} \hat{D}_{N-1,1}(0) & \dots & \hat{D}_{N-1,N-2}(0) & D_{N-1,N-1}(0) & D_{N-1,N}(0) \\ O & \dots & O & O & D_{NN}(0) \\ \hat{N}_{N-1,1}(0) & \dots & \hat{N}_{N-1,N-2}(0) & N_{N-1,N-1}(0) & N_{N-1,N}(0) \\ O & \dots & O & O & N_{NN}(0) \end{array}\right] \\
&\quad = [O\ \bar{G}_{N-1}(0)],
\end{aligned}$$

implying that $\hat{D}_{N-1,j} = zD_{N-1,j}$ and $\hat{N}_{N-1,j} = zN_{N-1,j}$, for $j = 1, 2, \dots, N-2$. By the expression (8) and the characteristic rank of the decomposition, we can conclude that rank $\begin{bmatrix} D_{N-1,N-1}(0) \\ N_{N-1,N-1}(0) \end{bmatrix} = r_{s,N-1} - r_{sN}$. Reasoning in this way we complete the proof.
The converse is straightforward. □

As we said in the introduction, the aim of this paper is to find a block-ordered right coprime decomposition of matrix H_{s+1}, from a block-ordered right coprime decomposition of H_s, $s \in \mathbb{Z}$.

3. Block ordered right coprime decomposition of a rational matrix. In this section, we prove the existence of a block-ordered right coprime decomposition of an arbitrary rational matrix H_s, which will be used in the rest of the paper.

Let $A \in \mathbb{R}^{p\times q}[z]$ be a polynomial matrix, with $p \geq q$. Let S be its Smith canonical form

$$S = \begin{bmatrix} I_q \\ O \end{bmatrix}.$$

It is easily seen that there exists an unimodular polynomial matrix, $\overline{V} \in \mathbb{R}^{p\times p}[z]$, such that

$$\overline{V}A = \begin{bmatrix} I_q \\ O \end{bmatrix}. \tag{11}$$

THEOREM 3.1. *Let $H_s, s \in \mathbb{Z}$ be a rational matrix. Then, there exists a block-ordered right coprime decomposition of H_s.*

Proof. Let $H_s = N_s D_s^{-1}$ be a right coprime decomposition, with the following sizes $D_s \in \mathbb{R}^{mN\times mN}[z]$ and $N_s \in \mathbb{R}^{pN\times mN}[z]$. We know that the Smith canonical form of the matrix

$$\mathcal{F}_1(D_s, N_s) = \begin{bmatrix} D_s \\ N_s \end{bmatrix}$$

is given by (5). There exists, (see (11)), an unimodular polynomial matrix, $\overline{V}$, such that

$$\overline{V}\mathcal{F}_1(D_s, N_s) = \begin{bmatrix} I_{mN} \\ O \end{bmatrix} .$$

Let $(r_{s1}, r_{s2}, \ldots, r_{sN})$ be the characteristic rank of the pair of matrices (D_s, N_s). Then, there exists a nonsingular constant matrix W_1 of adequate size such that

$$\mathcal{F}_2\Big(D_s(0), N_s(0)\Big)W_1 = [\, O \quad \overline{G}_2(0)\,]$$

where $\overline{G}_2 \in \mathbb{R}^{(m+p)(N-1)\times r_{s2}}[z]$. Then,

$$D_sW_1 = \begin{bmatrix} D_{11} & \widetilde{D}_{12} \\ zD_{21} & \widetilde{D}_{22} \\ zD_{31} & \widetilde{D}_{32} \\ \vdots & \vdots \\ zD_{N1} & \widetilde{D}_{N2} \end{bmatrix}, \; N_sW_1 = \begin{bmatrix} N_{11} & \widetilde{N}_{12} \\ zN_{21} & \widetilde{N}_{22} \\ zN_{31} & \widetilde{N}_{32} \\ \vdots & \vdots \\ zN_{N1} & \widetilde{N}_{N2} \end{bmatrix}$$

with $D_{11} \in \mathbb{R}^{m\times(mN-r_{s2})}[z]$ and $N_{11} \in \mathbb{R}^{p\times(mN-r_{s2})}[z]$.
Obviously, the characteristic rank of the pair (D_sW_1, N_sW_1) coincides with that of (D_s, N_s). Then, there exists a nonsingular constant matrix,

$$W_2 = \begin{bmatrix} I_{r_{s2}} & O \\ O & \overline{W}_2 \end{bmatrix}$$

of adequate size such that

$$\mathcal{F}_3\Big(D_s(0)W_1, N_s(0)W_1\Big)W_2 = [\, O \quad \overline{G}_3(0)\,]$$

with $\overline{G}_3 \in \mathbb{R}^{(m+p)(N-2)\times r_{s3}}[z]$. Therefore,

$$D_sW_1W_2 = \begin{bmatrix} D_{11} & D_{12} & \widetilde{D}_{13} \\ zD_{21} & D_{22} & \widetilde{D}_{23} \\ zD_{31} & zD_{32} & \widetilde{D}_{33} \\ \vdots & \vdots & \vdots \\ zD_{N1} & zD_{N2} & \widetilde{D}_{N3} \end{bmatrix}, \; N_sW_1W_2 = \begin{bmatrix} N_{11} & N_{12} & \widetilde{N}_{13} \\ zN_{21} & N_{22} & \widetilde{N}_{23} \\ zN_{31} & zN_{32} & \widetilde{N}_{33} \\ \vdots & \vdots & \vdots \\ zN_{N1} & zN_{N2} & \widetilde{N}_{N3} \end{bmatrix}$$

with $D_{22} \in \mathbb{R}^{m\times(r_{s2}-r_{s3})}[z]$ and $N_{22} \in \mathbb{R}^{p\times(r_{s2}-r_{s3})}[z]$. The process can be repeated until we have constructed a pair of matrices $(\overline{D}_s, \overline{N}_s)$,

$$\overline{D}_s = D_sW, \; \overline{N}_s = N_sW,$$

which admits a partition of the form (9) with the conditions (10), where $W = W_1 W_2 \cdots W_{N-1}$.
The matrices $\overline{D}_s$ and $\overline{N}_s$ form a right coprime decomposition for H_s, since

$$\overline{N}_s(\overline{D}_s)^{-1} = (N_s W)(D_s W)^{-1} = (N_s W)(W^{-1} D_s^{-1}) = N_s D_s^{-1},$$

and

$$\overline{V} \begin{bmatrix} \overline{D}_s \\ \overline{N}_s \end{bmatrix} W^{-1} = \overline{V} \begin{bmatrix} \overline{D}_s W^{-1} \\ \overline{N}_s W^{-1} \end{bmatrix} = \overline{V} \begin{bmatrix} D_s \\ N_s \end{bmatrix} = \begin{bmatrix} I_{mN} \\ O \end{bmatrix}.$$

Clearly, the pair $(\overline{D}_s, \overline{N}_s)$ is a block-ordered right coprime decomposition of H_s. □

4. The Smith canonical form of the matrix $\mathcal{F}(S_{mN,m} D_s, S_{pN,p} N_s)$. Let (D_s, N_s) be a block-ordered right coprime decomposition of a rational matrix $H_s, s \in \mathbb{Z}$, and let $(r_{s1}, r_{s2}, \ldots, r_{sN})$ be its characteristic rank. For simplicity, we rewrite the partition (9) in a more compact form as

$$D_s = \begin{bmatrix} \overline{D}_{11} & \overline{D}_{12} \\ z\overline{D}_{21} & \overline{D}_{22} \end{bmatrix}, \; N_s = \begin{bmatrix} \overline{N}_{11} & \overline{N}_{12} \\ z\overline{N}_{21} & \overline{N}_{22} \end{bmatrix}, \tag{12}$$

where

$$\begin{aligned} &\overline{D}_{11} = D_{11} \in \mathbb{R}^{m \times (mN - r_{s2})}[z] \;, && \overline{D}_{12} \in \mathbb{R}^{m \times r_{s2}}[z] \;, \\ &\overline{D}_{21} \in \mathbb{R}^{m(N-1) \times (mN - r_{s2})}[z] \;, && \overline{D}_{22} \in \mathbb{R}^{m(N-1) \times r_{s2}}[z] \;, \\ &\overline{N}_{11} = N_{11} \in \mathbb{R}^{p \times (mN - r_{s2})}[z] \;, && \overline{N}_{12} \in \mathbb{R}^{p \times r_{s2}}[z] \;, \\ &\overline{N}_{21} \in \mathbb{R}^{p(N-1) \times (mN - r_{s2})}[z] \;, && \overline{N}_{22} \in \mathbb{R}^{p(N-1) \times r_{s2}}[z] \;, \end{aligned}$$

We shall construct a polynomial matrix $\overline{P}$, with a precise structure, such that $\overline{P}\ \mathcal{F}(D_s, N_s) = \begin{bmatrix} I_{mN} \\ O \end{bmatrix}$. To that end, we define the matrices

$$\overline{\mathcal{F}} = \begin{bmatrix} \overline{D}_{11} & \overline{D}_{12} \\ \overline{N}_{11} & \overline{N}_{12} \\ z\overline{D}_{21} & \overline{D}_{22} \\ z\overline{N}_{21} & \overline{N}_{22} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} I_{mN - r_{s2}} & O \\ O & O \\ O & I_{r_{s2}} \\ O & O \end{bmatrix}.$$

From (10), the matrices $\begin{bmatrix} \overline{D}_{11}(0) \\ \overline{N}_{11}(0) \end{bmatrix}$ and $\begin{bmatrix} \overline{D}_{22}(0) \\ \overline{N}_{22}(0) \end{bmatrix}$ are full column rank. Then, there exists a nonsingular constant matrix V of order $(m+p)N$ such that $V\overline{\mathcal{F}}(0) = C$, that is, $V\overline{\mathcal{F}} = C + zA$.

The structure of $\overline{\mathcal{F}}(0)$ allows us to choose the matrix V so that its lower left block, of size $(m+p)(N-1) \times (m+p)$, is null; that is,

$$V = \begin{bmatrix} * & * \\ O & * \end{bmatrix}.$$

The Smith canonical form of the matrix $V\overline{\mathcal{F}}$ is still (5). Therefore, by equation (11), with an adequate row permutation, there exists an unimodular polynomial matrix X such that

$$XV\overline{\mathcal{F}} = C, \tag{13}$$

that is, $X(C + zA) = C$. The matrix X satisfying this equality has a very special structure. It admits a partition such that

$$X = \begin{bmatrix} I_{mN-r_{s2}} + zX_{11} & X_{12} & zX_{13} & X_{14} \\ zX_{21} & X_{22} & zX_{23} & X_{24} \\ zX_{31} & X_{32} & I_{r_{s2}} + zX_{33} & X_{34} \\ zX_{41} & X_{42} & zX_{43} & X_{44} \end{bmatrix}.$$

Since this matrix is unimodular, the matrix

$$X(0) = \begin{bmatrix} I_{mN-r_{s2}} & X_{12}(0) & O & X_{14}(0) \\ O & X_{22}(0) & O & X_{24}(0) \\ O & X_{32}(0) & I_{r_{s2}} & X_{34}(0) \\ O & X_{42}(0) & O & X_{44}(0) \end{bmatrix}$$

is a nonsingular constant matrix and the submatrix

$$\begin{bmatrix} X_{22}(0) & X_{24}(0) \\ X_{42}(0) & X_{44}(0) \end{bmatrix}$$

is also nonsingular. Then, we can find a nonsingular constant matrix,

$$Y = \begin{bmatrix} I_{mN-r_{s2}} & O & O & O \\ O & Y_{22} & O & Y_{24} \\ O & Y_{32} & I_{r_{s2}} & Y_{34} \\ O & Y_{42} & O & Y_{44} \end{bmatrix},$$

such that

$$YX = \begin{bmatrix} I_{mN-r_{s2}} + zX_{11} & X_{12} & zX_{13} & X_{14} \\ zX'_{21} & X'_{22} & zX'_{23} & X'_{24} \\ zX'_{31} & zX'_{32} & I_{r_{s2}} + zX'_{33} & X'_{34} \\ zX'_{41} & zX'_{42} & zX'_{43} & X'_{44} \end{bmatrix}.$$

Moreover, it is easy to see that

$$YC = C. \tag{14}$$

Let us define the matrix $P = YXV$. Equations (13) and (14) imply that $P\overline{\mathcal{F}} = C$. By construction, the matrix P has the structure,

$$P = \begin{bmatrix} * & * \\ z \cdot * & * \end{bmatrix},$$

where the lower left block has size $(m+p)(N-1)\times(m+p)$. We can construct a partition of the matrix P, in blocks with adequate sizes, such that

$$P\overline{\mathcal{F}} = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ zP_{31} & zP_{32} & P_{33} & P_{34} \\ zP_{41} & zP_{42} & P_{43} & P_{44} \end{bmatrix} \begin{bmatrix} \overline{D}_{11} & \overline{D}_{12} \\ \overline{N}_{11} & \overline{N}_{12} \\ z\overline{D}_{21} & \overline{D}_{22} \\ z\overline{N}_{21} & \overline{N}_{22} \end{bmatrix} = C.$$

Hence, the matrix

$$\overline{P} = \begin{bmatrix} P_{11} & P_{13} & P_{12} & P_{14} \\ zP_{31} & P_{33} & zP_{32} & P_{34} \\ P_{21} & P_{23} & P_{22} & P_{24} \\ zP_{41} & P_{43} & zP_{42} & P_{44} \end{bmatrix}$$

satisfies

$$\overline{P}\mathcal{F}(D_s, N_s) = \begin{bmatrix} I_{mN} \\ O \end{bmatrix}.$$

This whole discussion can be summarized in the following resulr.

PROPOSITION 4.1. *Let (D_s, N_s) be a block-ordered right coprime decomposition of H_s with characteristic rank $(r_{s1}, r_{s2}, \ldots, r_{sN})$. There exists an unimodular polynomial matrix V, such that*

$$V\mathcal{F}(D_s, N_s) = \begin{bmatrix} I_{mN} \\ O \end{bmatrix}$$

with the following structure

$$V = \begin{bmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ zV_{21} & V_{22} & zV_{23} & V_{24} \\ V_{31} & V_{32} & V_{33} & V_{34} \\ zV_{41} & V_{42} & zV_{43} & V_{44} \end{bmatrix},$$

where $V_{21} \in \mathbb{R}^{r_{s2}\times m}[z]$, $V_{23} \in \mathbb{R}^{r_{s2}\times p}[z]$, $V_{41} \in \mathbb{R}^{[(m+p)(N-1)-r_{s2}]\times m}[z]$, $V_{43} \in \mathbb{R}^{[(m+p)(N-1)-r_{s2}]\times p}[z]$.

Next, define the matrix

$$V' = \begin{bmatrix} S_{mN,r_{s2}} & O \\ O & S_{pN,(p+m)(N-1)-r_{s2}} \end{bmatrix}^{-1} V \begin{bmatrix} S_{mN,m(N-1)} & O \\ O & S_{pN,p(N-1)} \end{bmatrix}. \tag{15}$$

It is easy to check that the matrix V' is unimodular.

We know, by Example 2.2, that the coprimeness of the decomposition (D_s, N_s) does not remain, in general, in the decomposition at time $s+1$. Now we can prove a special canonical form of the decomposition at time $s+1$.

THEOREM 4.2. *Let (D_s, N_s) be a block-ordered right coprime decomposition of H_s. Then, the Smith canonical form of the matrix $\mathcal{F}(S_{mN,m}D_s, S_{pN,p}N_s)$ is*

$$\begin{bmatrix} I_{r_{s2}} & O \\ O & zI_{mN-r_{s2}} \\ O & O \end{bmatrix}.$$

Proof. Let V' be the matrix defined in (15). Then,

$$V'\mathcal{F}(S_{mN,m}D_s, S_{pN,p}N_s) = V' \begin{bmatrix} S_{mN,m} & O \\ O & S_{pN,p} \end{bmatrix} \begin{bmatrix} D_s \\ N_s \end{bmatrix}$$
$$= z^{-1} \begin{bmatrix} S_{mN,mN-r_{s2}} & O \\ O & S_{pN,m(1-N)+p+r_{s2}} \end{bmatrix} V z \begin{bmatrix} D_s \\ N_s \end{bmatrix} = \begin{bmatrix} S_{mN,mN-r_{s2}} \\ O \end{bmatrix} \tag{16}$$

Multiplying this equation on the left by (the permutation matrix) $S_{mN,r_{s2}}(1)$, we have

$$V'\mathcal{F}(S_{mN,m}D_s, S_{pN,p}N_s)S_{mN,r_{s2}}(1) = \begin{bmatrix} I_{r_{s2}} & O \\ O & zI_{mN-r_{s2}} \\ O & O \\ O & O \end{bmatrix}. \quad \square$$

5. A block-ordered right coprime decomposition of the matrix H_{s+1}. Consider the periodic collection of rational matrices $\{H_s,\ s \in \mathbb{Z}\}$ given in (1) satisfying (2). Let (D_s, N_s) be a block-ordered coprime decomposition of H_s defined in (9). Then, we define the matrices

$$D_{s+1} = S_{mN,m}D_sS^{-1}_{mN,mN-r_{s2}} \text{ and } N_{s+1} = S_{pN,p}N_sS^{-1}_{mN,mN-r_{s2}}. \tag{17}$$

Note that the block structures of D_{s+1} and N_{s+1} are parallel to those of the D_s and N_s, but now the first row block and the first column block are the second row block and second column block of (9). In general, for $1 \leq j \leq N$,

$$D_{s+j} = S_{mN,jm}D_sS^{-1}_{mN,mN-r_{s,j+1}} \text{ and } N_{s+j} = S_{pN,jp}N_sS^{-1}_{mN,mN-r_{s,j+1}}. \tag{18}$$

This last expression has been obtained using Lemma 2.1 and Lemma 5.1.

In this section we shall prove that the pair (D_{s+1}, N_{s+1}) written in (17) (more in general, (D_{s+j}, N_{s+j}), $j \geq 1$, as defined in (18)) is a block-ordered right coprime decomposition of H_{s+1}, (H_{s+j}). First, we prove the following lemma.

LEMMA 5.1. *Let $(r_{s+1,1}, r_{s+1,2}, \ldots, r_{s+1,N})$ be the characteristic rank of the right coprime pair (D_{s+1}, N_{s+1}), where the matrices D_{s+1} and N_{s+1} are given in (17). Then, $r_{s+1,k} = mN + r_{s,k+1} - r_{s2}$.*

Proof. From the definition of $r_{s+1,k}$ (see (8) and (10)), we have

$$\begin{aligned} r_{s+1,k} &= \operatorname{rank} \mathcal{F}_k\Big(D_{s+1}(0), N_{s+1}(0)\Big) \\ &= \operatorname{rank} \begin{bmatrix} D_{k+1,k+1}(0) \\ N_{k+1,k+1}(0) \end{bmatrix} + \cdots + \operatorname{rank} \begin{bmatrix} D_{NN}(0) \\ N_{NN}(0) \end{bmatrix} + \operatorname{rank} \begin{bmatrix} D_{11}(0) \\ N_{11}(0) \end{bmatrix} \\ &= r_{s,k+1} - r_{s,k+2} + r_{s,k+2} - r_{s,k+3} + \cdots + r_{sN} + r_{s1} - r_{s2} \\ &= r_{s,k+1} + mN - r_{s2} \ . \ \Box \end{aligned}$$

The result of Lemma 5.1 can be generalized to the right coprime pair (D_{s+j}, N_{s+j}), $1 \leq j \leq N$. Indeed, in this case one obtains the relations $r_{s+j,k} = mN + r_{s,k+j} - r_{s,1+j}$, with $r_{s,k+j} = r_{s,k+j-N} - mN$ if $k + j > N$.

THEOREM 5.2. *Consider the periodic collection of rational matrices defined in (1) satisfying (2), and consider the block-ordered right coprime decomposition (12) of H_s for some $s \in \mathbb{Z}$. Then the pair of matrices (D_{s+1}, N_{s+1}), defined in (17), is a block-ordered right coprime decomposition of H_{s+1}.*

Proof. First, we shall see that the pair of matrices (D_{s+1}, N_{s+1}) is a right decomposition of H_{s+1}. By equation (6), we have

$$\begin{aligned} H_{s+1} &= (S_{pN,p} N_s)(S_{mN,m} D_s)^{-1} = (N_{s+1} S_{mN,mN-r_{s2}})(D_{s+1} S_{mN,mN-r_{s2}})^{-1} \\ &= N_{s+1} D_{s+1}^{-1}, \end{aligned}$$

then, the pair of matrices (D_{s+1}, N_{s+1}) is a right decomposition of H_{s+1}. On the other hand, by Theorem 4.2, more precisely by the expression (16), we have

$$V'\mathcal{F}(D_{s+1}, N_{s+1}) = V'\mathcal{F}(S_{mN,m} D_s, S_{pN,p} N_s) S_{mN,mN-r_{s2}}^{-1} = \begin{bmatrix} I_{mN} \\ O \end{bmatrix},$$

that is, the decomposition (D_{s+1}, N_{s+1}) is right coprime (recall that the matrix V' is unimodular). Moreover,

$$\operatorname{rank} \begin{bmatrix} D_{ii}(0) \\ N_{ii}(0) \end{bmatrix} = r_{s+1,i-1} - r_{s+1,i} \ , \tag{19}$$

by Lemma 5.1. From the equalities (17) and (19), the pair of matrices (D_{s+1}, N_{s+1}) is block-ordered (see Lemma 2.5).
Hence, it is clear that (D_{s+1}, N_{s+1}) is a block-ordered right coprime decomposition of H_{s+1} . $\Box$

Notice that Theorem 5.2 applies to decompositions of H_{s+j}, $1 \leq j \leq N$, by using equation (18).

We present now some final remarks.

1. All results obtained (with the corresponding definitions) for right decompositions can be directly translated to *left* decompositions.

2. The definition of the matrices D_{s+1} and N_{s+1} given in (17) provides a simple algorithm for the construction of a block-ordered right coprime decomposition of the rational matrix H_{s+1} from one of H_s.

3. Repeating this algorithm, we construct a block-ordered right coprime decomposition of the rational matrices H_{s+j} defined in (18), for $1 \leq j \leq N$.

In fact, we can say more than Remark 3, that is, the decompositions of H_{s+j}, (D_{s+j}, N_{s+j}), $j \in \mathbb{Z}$, are N-*periodic* as can be seen using the expressions (18). Then, we have the next theorem.

THEOREM 5.3. *Consider the periodic collection of rational matrices defined in (1) satisfying (2). Let (D_s, N_s) be a block-ordered right coprime decomposition of the rational matrix H_s for some $s \in \mathbb{Z}$. If we construct the pairs of matrices (D_{s+j}, N_{s+j}), $j \in \mathbb{Z}^+$, using (18), the equalities*

$$D_{s+N} = D_s \text{ and } N_{s+N} = N_s$$

are satisfied.

Finally, we illustrate the above results with the following example.

EXAMPLE 5.4. The matrices

$$H_0 = \begin{bmatrix} \frac{1}{-2-z+z^2} & 0 & 0 & \frac{1}{-2-z+z^2} \\ \frac{-3-2z-z^2}{-2z-z^2+z^3} & \frac{1}{z} & -\frac{1}{z} & \frac{-5-3z}{-2z-z^2+z^3} \\ \frac{-3-6z-2z^2-z^3}{-2z-z^2+z^3} & \frac{1+2z}{z} & \frac{-1-z}{z} & \frac{-5-9z-2z^2}{-2z-z^2+z^3} \\ \frac{(-1+z)z}{-2-z+z^2} & 0 & 0 & \frac{2}{-2-z+z^2} \\ \frac{-12-7z-10z^2}{-2z-z^2+z^3} & \frac{4+z+z^2}{z} & \frac{-4-z}{z} & \frac{-20-11z-6z^2}{-2z-z^2+z^3} \\ \frac{-9-3z-8z^2-z^3}{-2z-z^2+z^3} & \frac{3+z^2}{z} & \frac{-3-z}{z} & \frac{-15-8z-6z^2}{-2z-z^2+z^3} \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 0 & \frac{2}{-2-z+z^2} & \frac{-1+z}{-2-z+z^2} & 0 \\ \frac{-4-z}{z} & \frac{-20-11z-6z^2}{-2z-z^2+z^3} & \frac{-12-7z-10z^2}{z^2(-2-z+z^2)} & \frac{4+z+z^2}{z^2} \\ \frac{-3-z}{z} & \frac{-15-8z-6z^2}{-2z-z^2+z^3} & \frac{-9-3z-8z^2-z^3}{z^2(-2-z+z^2)} & \frac{3+z^2}{z^2} \\ 0 & \frac{z}{-2-z+z^2} & \frac{1}{-2-z+z^2} & 0 \\ -1 & \frac{-5-3z}{-2-z+z^2} & \frac{-3-2z-z^2}{-2z-z^2+z^3} & \frac{1}{z} \\ -1-z & \frac{-5-9z-2z^2}{-2-z+z^2} & \frac{-3-6z-2z^2-z^3}{-2z-z^2+z^3} & \frac{1+2z}{z} \end{bmatrix},$$

form a collection of periodic rational matrices H_s with $N = 2$, since $S_{6,3}H_0S_{4,2}^{-1} = H_1$ and $S_{6,3}H_1S_{4,2}^{-1} = H_0$. We have the block-ordered right coprime decomposition $H_0 = N_0D_0^{-1}$, where

$$N_0 = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & -z & 1 & 0 \\ \hline -z & 0 & 0 & 1 \\ -2z & -4 & z & 1 \\ -z & -4 & -1+z & 0 \end{array}\right], \quad D_0 = \left[\begin{array}{c|ccc} -2-z & 0 & 0 & 1 \\ 3 & 1 & 1 & 0 \\ \hline -z & 1+z & 1 & -1 \\ z^2 & 0 & 0 & -1 \end{array}\right].$$

Further, the characteristic rank of the pair (D_0, N_0) is $(4,3)$. Then, the matrices

$$N_1 = S_{6,3}N_0S_{4,1}^{-1} = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ -4 & z & 1 & -2 \\ -4 & -1+z & 0 & -1 \\ \hline 0 & 0 & 0 & 1 \\ -z & 0 & 0 & -1 \\ -z^2 & z & 0 & 1 \end{array}\right],$$

$$D_1 = S_{4,2}D_0S_{4,1}^{-1} = \left[\begin{array}{ccc|c} 1+z & 1 & -1 & -1 \\ 0 & 0 & -1 & z \\ \hline 0 & 0 & z & -2-z \\ z & z & 0 & 3 \end{array}\right],$$

are a block-ordered right coprime decomposition of H_1, according with Theorem 5.2. By Lemma 5.1 the characteristic rank of the pair (D_1, N_1) is $(4,1)$.

In addition, it is straightforward to see that

$$S_{4,2}D_1S_{4,3}^{-1} = D_2 = D_0 \text{ and } S_{6,3}N_1S_{4,3}^{-1} = N_2 = N_0.$$

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