

MATRICES OF POSITIVE POLYNOMIALS*

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Abstract. Associated to a square matrix all of whose entries are real Laurent polynomials in several variables with no negative coefficients is an ordered “dimension” module introduced by Tuncel, with additional structure, which acts as an invariant for topological Markov chains, and is also an invariant for actions of tori on AF C^* -algebras. In describing this invariant, we are led naturally to eventually positivity questions, which in turn lead to descriptions of the Poisson boundaries (of random walks affiliated with these processes). There is an interplay between the algebraic, dynamical, and probabilistic aspects, for example, if the (suitably defined) endomorphism ring of the dimension module is noetherian, then the boundary is more easily described, the asymptotic behaviour of powers of the matrix is tractible, and the order-theoretic aspects of the dimension module are less difficult to deal with than in general. We also show that under relatively modest conditions, the largest eigenvalue function is a complete invariant for finite equivalence (early results of Marcus and Tuncel showed that it is not a complete invariant in general, but is so if the large eigenvalue is a polynomial).

Key words. positive polynomial, trace, point evaluation, Newton polyhedron, convex polytope, primitive matrix, random walk, order ideal, (topological) shift equivalence, finite equivalence, Choquet simplex, real analytic function.

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Introduction.

Let M be a matrix whose entries are polynomials in d real variables with no negative coefficients. Tuncel [Tu] initiated a study of “dimension modules” (certain ordered vector spaces) associated to such matrices, in connection with classification problems for (topological) Markov chains. (See [Tu, Tu2] for details.) In this article, we develop structure and classification theory in somewhat different directions.

Let A be either $\mathbf{Z}[x_i^{\pm 1}]$ or $\mathbf{R}[x_i^{\pm 1}]$ (Laurent polynomial rings in d variables), and let A^+ denote the corresponding set of Laurent polynomials with no negative coefficients. Then A is a partially ordered ring with positive cone A^+ , and the space of columns of size n , denoted A^n , is an ordered A -module. If M is in $M_n A^+$ (i.e., M is an $n \times n$ matrix with entries from A^+), then M induces an order preserving A -module map, $M : A^n \rightarrow A^n$. Iterating this map, we obtain the dimension module associated to M as the direct limit,

$$G_M := \lim A^n \xrightarrow{M} A^n \xrightarrow{M} A^n \xrightarrow{M} \dots$$

This is an ordered A -module, consisting of equivalence classes, $[c, k]$ where $(c, k) \in A^n \times \mathbf{N}$, and $[c, k] = [c', k']$ if there exists $N \geq k, k'$ such that $M^{N-k}c = M^{N-k'}c'$; $[c, k]$ is in the positive cone G_M^+ if there exists N such that $M^N c \in (A^n)^+$. The shift map sending $[c, k] \mapsto [c, k + 1]$ is an order automorphism, with inverse $\hat{M} : G_M \rightarrow G_M$ defined via $\hat{M}([c, k]) = [Mc, k]$. The structural problems referred to earlier arise from attempts to say something about the positive cone of G_M . Explicitly, one of the crucial problems is

- (\mathfrak{S}) Decide in terms of M (in $M_n A^+$) and c (in A^n) whether there exists an integer N so that the column $M^N c$ has all of its entries in A^+ .

(Of course, $[c, k]$ belongs to G_M^+ if and only if such an N exists.) When $n = 1$, M is just a polynomial P in A^+ , and f is a Laurent polynomial with integer (or real) coefficients; in this case, (\mathfrak{S}) asks to decide

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if the Laurent polynomial $P^m f$ has no negative coefficients for some integer m . This was solved in [H3, Theorems A and B]. Special cases had been solved by Meissner [Me]. In this situation, the solution leads to connections with a result of Ney and Spitzer [NS], determining the Martin boundary of the random walk on \mathbf{Z}^d associated with P . There were also connections to the weighted moment map of algebraic geometry [Od, p. 94], and to an affine $\text{AGL}(d, \mathbf{Z})$ -invariant for compact lattice polytopes [H5], as well as a real affine ($\text{AGL}(d, \mathbf{R})$) invariant for general compact convex polytopes.

The problem (\mathfrak{S}) arises in other contexts than topological Markov chains. If $A = \mathbf{Z}[x_i^{\pm 1}]$, the limiting dimension module is a direct limit of partially ordered abelian groups. Let R be an AF C^* -algebra [E1] whose ordered Grothendieck group is the limit obtained by evaluating all the variables appearing everywhere in (1) at 1—in other words, each $x_i \mapsto 1$:

$$(2) \quad G_M(1) = \lim \mathbf{Z}^n \xrightarrow{M(1)} \mathbf{Z}^n \xrightarrow{M(1)} \mathbf{Z}^n \xrightarrow{M(1)} \dots$$

(Here $M(1)$ an abbreviation of the real nonnegative matrix $M(1, 1, \dots, 1)$.) In other words, $K_0(R) = G_M(1)$, as ordered abelian groups. Then there is an action α of the d -torus $T = \mathbf{T}^d$ on R so that the equivariant Grothendieck group of R , $K_0^T(R)$ is G_M when viewed as a partially ordered module over the representation ring of T , which of course is A . Alternatively, $G_M = K_0(R \times_\alpha T)$. This is a special case of more general results in [HR], which deals with classification results for locally finite actions of tori and other compact Lie groups on AF C^* -algebras.

There is a probabilistic interpretation that is closely related. Begin with the matrix $M = (M_{ij})$ with entries M_{ij} in A^+ . Form the set of sites, $S = \mathbf{Z}^d \times \{1, 2, \dots, n\}$. It is helpful to think of S as n disjoint copies of the lattice \mathbf{Z}^d piled on top of each other, with the second coordinate indexing the level. A particle at point $(z, k) \in S$ (lattice point z , level k) is constrained to jump to level j in the next unit of time; its motion in this level is governed by the entry M_{jk} . Write $M_{jk} = \sum \lambda_w x^w$ where we use multinomial notation $x^w = x_1^{w(1)} \cdot \dots \cdot x_d^{w(d)}$, λ_w are all nonnegative real numbers, and we have suppressed the dependence on jk for expository convenience. Then on level j , the particle moves from z according to the rule that the probability of displacement by w is $\lambda_w/M_{jk}(1)$. Then (\mathfrak{S}) amounts to asking which initial finitely supported signed distributions on S become nonnegative after a finite number of iterations of this process.

In the case that $n = 1$, this is related to the Poisson boundary, although knowledge of the latter is insufficient to solve (\mathfrak{S}) . In this case, let μ be the measure on \mathbf{Z}^d obtained from P : $\mu(\{w\}) = \lambda_w/P(1)$, where $P = \sum \lambda_w x^w$. Let ν be a finitely supported signed measure, and set $\nu_0 = |\nu|$ (any positive measure with the same support as ν will do just as well). The specialization of (\mathfrak{S}) asks if there exists m so that the m -fold convolution of μ , convolved (once) with ν be positive. The answer is affirmative if and only if for every extremal $[0, \infty]$ -valued harmonic function h (on \mathbf{Z}^d) finite with respect to ν_0 , $\sum_{w \in \text{supp } \nu} \nu(w)h(w) > 0$. If ν_0 is mutually absolutely continuous with respect to μ or some convolution power $\mu^{(k)} = \mu * \dots * \mu$ for some k , then the set of these extremal harmonic functions is compact (and in this situation is contractible). However, for general ν_0 , the set of such functions need not be either compact or connected (if $d = 1$, it is compact, but need not be connected).

For $n > 1$, there is a choice of initial (row of) measures so that a boundary, analogous to the Poisson boundary, occurs (it is the positive portion of the real maximal ideal space of a partially ordered ring naturally associated to M and is at least compact, although not generally connected). Other choices lead to very complicated situations that must be disentangled in order to attack (\mathfrak{S}) .

One can also put the random walk interpretation (actually these admit the same formalism of the “matrix-valued random walks” occurring in [CW]—although the problems considered are almost entirely unrelated) in the context of subshifts of finite type. By symbol splitting, we may assume that all of the entries of M are either monomials or zero (although the matrix size may change). Form $S = \mathbf{Z}^d \times \{1, 2, \dots, n\}$ as we did before, and set

$$X = S^{\mathbf{N}}$$

$$X_M = \{x = (w(x, i), k(x, i)) \in S^{\mathbf{N}} \mid M_{k(x, i+1), k(x, i)} = x^w, \quad w = w(x, i+1) - w(x, i)\}.$$

That is, if $x(i) = (w(x, i), k(x, i))$ where $w(x, i) \in \mathbf{Z}^d$ and $k(x, i) \in \{1, 2, \dots, n\}$, then the sequence $x = x(i)$ belongs to X_M provided that for each i in \mathbf{N} , the transition from $w(x, i)$ to $w(x, i+1)$ is permitted—that is, $w = w(x, i+1) - w(x, i)$ is the exponent of the monomial appearing in the $k(x, i+1), k(x, i)$ entry of M . One can think of this as a shift space on an enormous state space (S), but with relatively few transitions permitted. In general, X_M is not compact, but we may find a lot of compact cylinder sets, by selecting a finite set of elements “ z ” in S and times i in \mathbf{N} , $U = \{(z_j, i(j))\}$ (j running over a finite index set), and set

$$K_U = \{x \in X_M \mid \text{there exists } j \text{ such that } x(i(j)) = z_j\}.$$

The path x belongs to K_U if it passes through at least one of the z_j at time $i(j)$. This corresponds to the selection of an initial measure.

We normally assume M is primitive, i.e., some power of M has no zero entries; however, at times, we have to face up to the reducible case, which is somewhat more complicated.

The approach used when $n = 1$ is elaborated in order to apply to the general case. Associated to the ordered vector space G_M is the “bounded subring” (defined in section 1), denoted $E_b(G_M)$. This is a partially ordered ring of endomorphisms of G_M that are bounded in the appropriate sense. Much of the information concerning positivity of elements of G_M is reflected, although not terribly clearly, in the structure of $E_b(G_M)$. For example, the (densely defined) traces on G_M in principal permit one to decide positivity of a specific element, and these traces factor through (in the appropriate sense) $E_b(G_M)$. Recall that a trace on an ordered vector space is a nonzero positive linear functional. In this case, “densely defined” means the domain of the trace is an order ideal in G_M . The (global, that is, everywhere defined) traces of G_M yield some information about positivity (for example, necessary for $[c, k]$ to be positive is that $\tau([c, k]) > 0$ for every global trace on G_M), but these are far from sufficient to determine the ordering, even if $d = 1$ (in contrast, when $n = d = 1$, the global traces *do* determine the ordering). The pure traces on $E_b(G_M)$ that are not faithful determine ideals of $E_b(G_M)$, and part of the structure theory is to describe these order ideals, which turn out to correspond (not bijectively) with faces of the convex polytope associated to M by Marcus and Tuncel [MT]. (When $n = 1$, the association is a bijection and many very nice properties hold [H5].)

In the case that $n = 1$, the ordered ring $E_b(G_M)$ (in the context of $n = 1$, called R_P [H5]) is always a finitely generated commutative ring/algebra (depending on whether we use \mathbf{Z} or \mathbf{R} in the definition of A) with no zero divisors. It is easy to see that $E_b(G_M)$ may have zero divisors and need not be commutative, but generically the characteristic polynomial is irreducible over A , and that case, it is a commutative domain. Unfortunately, if $n \geq 2$, generically it is not finitely generated; it is not even noetherian; in fact, it does not even satisfy the ascending chain condition on order ideals. This means that the positivity problem is far more complicated than expected. We give necessary conditions in order that this ascending chain condition hold.

Among classification problems is the question of deciding finite equivalence of two such matrices; that is, given square M and M' with entries in A^+ , give criteria in order that there be a nonzero rectangular matrix X with entries in A^+ such that $MX = XM'$. A necessary condition is given by equality of the corresponding beta functions; for each $r = (r_i)$ in $(\mathbf{R}^d)^{++}$ ($= \{r \in \mathbf{R}^d \mid r_i > 0, i = 1, 2, \dots, d\}$), $M(r)$ (all the entries of M evaluated at r) is a real nonnegative matrix, and our initial primitivity assumption means the large eigenvalue, denoted $\beta_M(r)$ (or simply $\beta(r)$) has multiplicity one; it follows that $r \mapsto \beta_M(r)$ is a real analytic function on $(\mathbf{R}^d)^{++}$. A necessary condition for finite equivalence is that $\beta_M = \beta_{M'}$ on $(\mathbf{R}^d)^{++}$. This was conjectured to be sufficient as well. However, Marcus and Tuncel [MT] gave two examples to show this was not the case; I showed (independently, but only in the form of a talk at the Adler conference) that the conjecture failed, for one of the pairs that Marcus and Tuncel considered, by completely different methods. However, it had been known already that if $d = 1$, the conjecture is true (Parry), and Marcus and Tuncel [MT2] also showed that if β were a polynomial, the conjecture succeeds.

Here we give conditions on M and M' that guarantee that β is a complete invariant for finite equivalence. For any (primitive) M , we can find a left eigenvector for β whose entries are polynomials in β with coefficients

from A , i.e., in $\mathbf{R}[x_i^{\pm 1}][[\beta]]$, and each entry is real analytic and strictly positive on $(\mathbf{R}^d)^{++}$. If we can arrange that each entry of the eigenvector is bounded above and below away from zero by a quotient of elements of A^+ (this is a simplification of the real definition; the actual definition is more general), then we say M satisfies (%) (a local criterion, using the facial structure of M , can also be given, 8.7). For example, if β is polynomial, or if M is a companion matrix or the transpose thereof, or if the Newton polyhedra of all the entries of some power of M are the same, then (%) holds. We say M satisfies (#) if the ratio of the second largest absolute value of eigenvalues of $M(r)$ to $\beta_M(r)$ is bounded above by $1 - \epsilon$ as r varies over $(\mathbf{R}^d)^{++}$. This holds if for every face F of the convex set associated to M_F , there is a unique irreducible block which hits the spectral radius, and it is primitive (actually, in the statement of the result, the last “primitive” can be deleted), so it is relatively easy to test for.

Imprecisely, we show that among matrices satisfying (#) and (%), the beta function is a complete invariant for finite equivalence. It is conceivable that (#) implies (%), so the latter condition may be redundant. It is also possible that (%) alone is sufficient β to be a complete invariant for finite equivalence.

In results leading up to this, we note that analytic and algebraic functions that are positive on $(\mathbf{R}^d)^{++}$ can behave rather strangely, and this strangeness is reflected in boundary behaviour. Explicitly, if $\rho : (\mathbf{R}^d)^{++} \rightarrow \mathbf{R}^{++}$ is algebraic (i.e., satisfies a polynomial equation with coefficients from A) and real analytic, and if $X : \mathbf{R}^+ \rightarrow (\mathbf{R}^d)^{++}$ is the exponential of a ray in \mathbf{R}^d with directional derivative v , then

$$\lim_{t \rightarrow \infty} \frac{\log \rho(X(t))}{\log t}$$

may depend on the starting point of the ray; if ρ is bounded above and below by integer multiples of a rational function (whose numerator and denominator are in A^+), then the limit does not depend on the starting point. This provides a notion of (local) “linkages” between the various blocks of M_F (for a fixed face F), which would be part of the data for shift equivalence. In the survey article [Tu2], Tuncel provides a global way of linking blocks.

Principal results. Theorem 4.5 describes completely the pure (global) traces on G_M and its order ideals under the mildest of conditions, as well as the faithful pure traces on $E_b(G_M)$. Roughly speaking, they factor through the left Perron eigenvector for β evaluated at points of $(\mathbf{R}^d)^{++}$.

The unfaithful pure traces on order ideals of $E_b(G_M)$ and of G_M are described in several situations, completely if the necessary condition for noetherianness holds (sections 5 and 6). Roughly speaking, they factor through quotients corresponding to taking facial matrices. In fact, the unfaithful pure traces could be completely described in this form if an innocuous-looking conjecture were true.

The consequences, and subsequent non-genericity, of the ascending chain condition on order ideals of G_M are discussed in section 7. Some of the relevant conditions can be simply expressed in terms of the weighted graph associated to M .

The finite equivalence result discussed above, Theorem 10.7, is proved in section 10.

In section 12, necessary and sufficient conditions on a size 2 matrix M are determined in order that it satisfy the condition (%) discussed previously. It turns out to be easy to express but difficult to prove: (%) holds if and only if the discriminant of the matrix divides a polynomial with no negative coefficients, Theorem 12.1. It follows that for size 2 matrices wherein the diagonal entries have no common monomials, (%) is equivalent to $M + pI$ satisfying (#) for some p in A^+ (this property is called weak (#), and admits a much less awkward formulation).

Finally, in section 13, we have a weak realization theorem, which says that under suitable circumstances, given suitable functions β , there exists a primitive matrix M for which $\beta_M = \beta^N$ for some undetermined integer N ; moreover, M may be selected to satisfy a strong condition, (**), which asserts that in some power of M , the Newton polyhedra of all the entries are identical.

This paper is a considerable elaboration of results in the preprint, “Eventual positivity and finite equivalence for matrices of polynomials”, versions of which were distributed from 1987 on.

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1 The Bounded Endomorphism Ring

In this section, we define and develop the basic properties needed of the ring (algebra) of bounded endomorphisms. Curiously, even in the case of zero variables, the notion is of some interest.

We recall some notation and definitions from the introduction. The ring of Laurent polynomials in the d variables x_1, x_2, \dots, x_d over either the integers or reals is denoted A , and the set of elements of A having no negative coefficients is a positive cone A^+ for A , making the latter into a partially ordered ring. The ring of $n \times n$ matrices with entries from A will be denoted $M_n A$; of course, $M_n A^+$ will denote the set of matrices with entries from A^+ . For an integer n , A^n will denote the set of columns of size n , viewed as an ordered A -module (with the direct sum ordering). Fix n and M in $M_n A^+$. The real matrix obtained by evaluating all the variables at 1 will be denoted $M(1)$, an abbreviation for $M(1, 1, \dots, 1)$. Define the direct limit, as ordered A -modules,

$$(1) \quad G_M = \lim A^n \xrightarrow{M} A^n \xrightarrow{M} A^n \xrightarrow{M} \dots$$

This is the set of equivalence classes,

$$\{(a, k) \mid a \in A^n, \quad k \in \mathbf{N}\} / \sim,$$

where $(a, k) \sim (a', k')$ if there exists $m > k, k'$ such that $M^{m-k} a = M^{m-k'} a'$. The A -module structure is obvious, and G_M admits an ordering making it into a directed partially ordered A -module, by means of the positive cone

$$G_M^+ = \{[a, k] \in G_M \mid \text{there exists } (a', k') \sim (a, k) \text{ with } a' \in (A^n)^+\}.$$

We observe that $[a, k] = [Ma, k + 1]$, and M induces an order isomorphism (that commutes with the action of A) by means of $\hat{M}[a, k] = [Ma, k]$; its inverse is given by a “shift”, $\hat{M}^{-1}[a, k] = [a, k + 1]$. More generally, if N is in $M_n A$ and commutes with M , then N induces an A -module endomorphism of G_M via $\hat{N}[a, k] = [Na, k]$. This will be positive (i.e., order preserving) if and only if there exists an integer m so that NM^k belongs to $M_n A^+$. Thus $\hat{}$ describes an assignment (actually a homomorphism of A -modules and of rings), from the centralizer of M in $M_n A$, $C_A(M)$, to the ring of A -module endomorphisms of G_M , $\text{End}_A G_M$.

The matrix M is an element of the centre of $C_A(M)$, and we may formally invert M ; if the determinant of M is nonzero, this amounts to adjoining M^{-1} from the matrices with entries in the field of fractions of A (the rational functions in the d variables). In general, the construction is given as limit ring,

$$C_A(M)[M^{-1}] = \lim C_A(M) \xrightarrow{\times M} C_A(M) \xrightarrow{\times M} C_A(M) \xrightarrow{\times M} \dots$$

We have been a little sloppy in notation; if the determinant of M is zero, then the ideal of $C_A(M)$, $\{N \in C_A(M) \mid M^n N = \mathbf{0}\}$ will be factored out. So, despite the notation, $C_A(M)$ will not be a subring of $C_A(M)[M^{-1}]$. Note that since M belongs to an $n \times n$ matrix ring over a field (the rational functions in the d variables), for any column a such that $M^s a$ is zero, it follows that $M^n a$ is also zero. In any case, $\hat{\cdot}$ induces a map from $C_A(M)[M^{-1}]$ to $\text{End}_A G_M$.

Define $E(G_M) = \{e \in \text{End}_A G_M \mid e\hat{M} = \hat{M}e\}$; then the range of $\hat{\cdot} : C_A(M)[M^{-1}] \rightarrow \text{End}_A G_M$ is obviously inside $E(G_M)$. In fact the range is onto, and the map is an isomorphism.

Lemma 1.1 The map

$$\hat{\cdot} : C_A(M)[M^{-1}] \rightarrow E(G_M)$$

is an isomorphism of rings and A -modules.

Proof: It is routine to verify the map is a well-defined homomorphism of rings and A -modules. We first show the map is onto. Select e in $E(G_M)$. Let E_j ($j = 1, 2, \dots, n$) denote the standard basis elements for A^n . Then we may find elements a_j in A^n together with integers $k(j)$ such that $e[E_j, 1] = [a_j, k(j)]$. We may assume that $k(j)$ are all equal, say to k , so that $e[E_j, 1] = [a_j, k]$ for all j . Define N_0 to be the $n \times n$ matrix with entries from A whose j th column is a_j . Obviously, $(\hat{M})^{-(k-1)}\hat{N}_0$ has the same effect as e on the $[E_j, 1]$'s. It is a routine verification that $\sum_j [E_j, 1]A$ is essential (as an A -module) in G_M , so that if two A -module endomorphisms agree on $\{[E_j, 1]\}$, they are equal. As the map $C_A(M) \rightarrow C_A(M)[M^{-1}]$ is not necessarily one to one, it is not immediate that N_0 commutes with M . However, it easily follows that there exists an integer s so that $N_0 M^s$ commutes with M . Set $N = N_0 M^s$, and observe that $(\hat{M})^{-(k-1+s)}\hat{N} = e$. Thus the map is onto.

It is immediate from the definitions that the map is one to one: $\hat{N}\hat{M}^{-t} = 0$ implies $[NE_j, 1] = [0, 1]$, so that $M^s NE_j = 0$ for some s (and all j), whence $M^s N = 0$, which means N is in the kernel of the map $C_A(M) \rightarrow C_A(M)[M^{-1}]$. •

Remark. 1: The portion of the preceding that does not involve the ordering applies if A is a general commutative unital ring and M is an arbitrary square matrix (see [BoH, p. 125ff] where $\mathcal{E}(\cdot)$ is defined for square matrices with entries in a commutative ring).

Now we discuss the positive cone of $E(G_M)$. The natural definition of a positive cone on a ring of endomorphisms is the following:

$$E(G_M)^+ = \{e \in E(G_M) \mid e(G_M^+) \subseteq G_M^+\}.$$

Lemma 1.2 Under the map

$$\hat{\cdot} : C_A(M)[M^{-1}] \rightarrow E(G_M),$$

an element of $C_A(M)[M^{-1}]$ is sent to a positive element $\hat{M}^{-k}\hat{N}$ of $E(G_M)$ if and only if there exists a positive integer m such that $M^m N$ belongs to $M_n A^+$.

Proof: Let e denote the endomorphism described by $\hat{M}^{-k}\hat{N}$. If $e \geq 0$, $e[E_j, 1] \geq [0, 1]$ for all j . Hence there exists k such that for all j , $e[E_j, 1] = [a_j, k]$ where a_j belongs to $(A^n)^+$. It follows from the definitions that multiplication by some power of M will render all the columns of N positive, which is what we want. The converse is trivial. •

Now we observe that every element of $E(G_M)$ is a difference of elements of its positive cone (i.e., as a partially ordered abelian group, $E(G_M)_1$ is directed).

Finally, we can define the bounded subring of $E(G_M)$. Let $\mathbf{I} : G_M \rightarrow G_M$ denote the identity element of $E(G_M)$. Set

$$E_b(G_M) = \{e \in E(G_M) \mid \text{there exists } k \in \mathbf{N} \text{ such that } -k\mathbf{I} \leq e \leq k\mathbf{I}\}$$

$$E_b(G_M)^+ = E_b(G_M) \cap E(G_M)^+$$

We observe that with the relative ordering, $E_b(G_M)$ is a partially ordered ring; it need not be commutative, although this is generic; it is an order ideal in $E(G_M)$, and moreover I is an order unit for $E_b(G_M)$. Obviously G_M is an ordered module over $E_b(G_M)$. However, that A does not act on $E_b(G_M)$, because (informally) all of the former's elements (except the constants) are unbounded in any reasonable sense.

For now, we make the standing hypothesis that the real matrix $M(1)$ is primitive. This is equivalent to saying that there exists a power of M so that every entry is not zero. One of the reasons for doing this is to guarantee that if H is any nonzero order ideal of G_M then its bounded subring is the same as $E_b(G_M)$, so that all order ideals of G_M may be viewed as ordered modules over a common partially ordered ring.

Now let H be an order ideal of G_M . Recall from the introduction that to describe the ordering on G_M we must analyze all the pure states of the order ideals (that have order units).

Specifically, suppose that G is a partially ordered unperforated abelian group and a is an element thereof. If there exists an order ideal with order unit of G , I , containing a such that for all (pure) traces τ of I , $\tau(a) > 0$, then a is in the positive cone of G . If no such order ideal exists, then a is not in the positive cone; in particular, there need not be a minimal order ideal containing a , or there could be a trace on a minimal order ideal with order unit among those containing a , at which $\tau(a) < 0$.

To analyze the traces on an order ideal I , we examine the bounded subrings $E_b(I)$; these automatically contain I as an order ideal, and their pure trace spaces are compact. Moreover, there is a natural map from the pure trace space of the latter to that of I . The pure trace spaces are described in sections 4–6.

We may form $E(H) = \{e \in E(G_M) \mid eH \subseteq H\}$, and put the relative order on it. It is an ordered subring. Unfortunately, even if H admits an order unit, $E(H)$ need not admit the identity as an order unit, so the latter is not generally equal to its bounded subring (which we shall define shortly) even when $n = d = 1$.

Example 1.3 An order ideal H in G_M which has an order unit but for which $E(H)$ does not admit the identity as an order unit.

Here $n = d = 1$, so $M = P$ a polynomial in one variable. Set $P = 1 + x + x^3$. As in [H5], G_M is the ring $\mathbf{R}[x, x^{-1}, P^{-1}]$, and the bounded subring, $E_b(G_M)$ is just $R_P = \mathbf{R}[x/P, 1/P]$. The positive cone of the latter is generated additively and multiplicatively by $\{1/P, x/P, x^3/P\}$ [H5, I.4]. Let $H = I$ be the ideal of R_P generated by $\{1/P, x/P\}$. As in [H5], this is also an order ideal of R_P (hence of G_M), and $(1+x)/P$ is an order unit for I . Since each of x/P^2 , x^2/P^2 , and x^3/P^2 belong to I , multiplication by x^2/P is a positive endomorphism of I , which obviously extends to a positive endomorphism of G_M . However, it is not bounded by a multiple of the identity endomorphism (so that the identity is not an order unit for $E(H)$). To see this, we it is sufficient to show that there does not exist an integer k such that $(x^2/P) \cdot 1 \leq k \cdot 1$, i.e., $x^2/P \leq k$ is impossible. If $x^2/P \leq k$, then according to the definition of the ordering, there would exist K so that $x^2 P^{K-1} \leq k P^K$ with respect to the usual ordering on $A = \mathbf{R}[x^{\pm 1}]$. In other words, every monomial appearing in $x^2(1+x+x^3)^{K-1}$ would have to appear in P^K . It is a simple exercise to show that x^{3K-1} appears in the former but not in the latter.

In the case that $n = 1$, failure of $E(H)$ to admit an order unit if H is an order ideal of R_P forces the latter ring to be not integrally closed in its field of fractions (see [H5, Section III]). •

If H is an order ideal of G_M , we define its bounded subring

$$E_b(H) = \{e \in E(H) \mid \text{there exists } k \in \mathbf{N} \text{ such that } -kI \leq e \leq kI\}$$

$$E_b(H)^+ = E_b(H) \cap E(H)^+.$$

We will show that $E_b(H) = E_b(G_M)$, at least when $M(1)$ is primitive. Thus $E_b(H)$ is independent of the choice of (nonzero) H . This requires a series of lemmas.

Lemma 1.4 Suppose that $M(1)$ is primitive. If e belongs to $E(G_M)^+$ and $eh = 0$ for some h in $G_M^+ \setminus \{0\}$, then $e = 0$.

Proof. We may write $h = [a, k]$ for some a in $(A^n)^+$ and some integer k , and we may also assume that $e = \hat{M}^{-s}\hat{N}$ for some matrix N commuting with M , all of whose coefficients lie in A^+ . Then $eh = [Na, s+k] = [0, s+k]$. This forces $M^t Na = 0$ for some positive integer t . Evaluate all the variables at $(1, 1, \dots, 1)$. This yields $N(1)M^t(1)a(1) = 0$. Since $M(1)$ is primitive and the entries of $a(1)$ are all nonnegative real numbers (and not all zero, unless h were zero to start with), there exists a larger value of t so that all the entries of $M^t(1)a(1)$ are strictly positive. Since all the entries of $N(1)$ are nonnegative, $N(1)a(1) \neq 0$, a contradiction. •

Corollary 1.5 Suppose that $M(1)$ is primitive and H is an order ideal in G_M . If e belongs to $E(G_M)^+$ and $eH = 0$, then $e = 0$.

We think of M as giving rise to a graph on the n points $\{1, 2, \dots, n\}$ (each of which will be called a *state*), with edges being labelled either by the exponents or the corresponding monomials. Thus if x^w appears in the Laurent polynomial $M_{i,j}$ position, there is an edge joining i to j with weight w . (We do not worry about multiplicities for now.) This will be called the (multiplicity-free) graph of M .

Lemma 1.6 If $M(1)$ is primitive, then there exist w in \mathbf{Z}^d and a positive integer k such that $x^w M^{-k}$ belongs to $E_b(G_M)$.

Proof. The conclusion amounts to $x^w \mathbf{I} \leq KM^l$ for some positive integer K , the ordering computed with respect to that on $E(G_M)$. Form the graph of M . Select a loop from 1 to 1 that contains every state in $\{1, 2, \dots, n\}$ at least once (this is of course possible, because $M(1)$ is primitive). Let k be the length of the path; so if w is the sum of the weights along the loop, x^w appears in the $(1, 1)$ entry of M^k . For any particular state i , we can obtain a loop of the same length by starting at the first occurrence of i instead of the first 1. Since the resulting loop is a cyclic permutation of the original, it has the same sum of weights. This means that each diagonal entry of M^k contains x^w ; it follows immediately that $x^w \mathbf{I} \leq KM^l$, as desired. •

Since everything remains the same for our immediate purposes if we replace M by $x^{-w}M^k$, we may even assume that $\mathbf{I} \leq KM$ (that is, $0 \in \text{Log } M_{i,i}$ for all i). We will have to be a bit more careful about this when we come consider the invariants Δ and Γ of Krieger (our assumption for now is that they are the same).

Lemma 1.7 Suppose that $M(1)$ is primitive. If H is a nonzero order ideal of G_M , then

$$A[\hat{M}^{\pm 1}] \cdot H = G_M \quad \text{and} \quad A[\hat{M}^{\pm 1}]^+ \cdot H^+ = G_M^+.$$

Proof. The order ideal contains an element of the form $[x^w E_j, k]$ for some integers j and k , and some lattice point w . By multiplying by an element of the form ax^{-w} (for a in A^+), we obtain $[aE_j, k]$. By multiplying by a power (positive or negative) of \hat{M} , we obtain anything of this form but with arbitrary k . Finally, by primitivity of $M(1)$ and the convexity of H , we can always find w' and k' (depending on j) so that for any state j , $[x^{w'} E_j, k']$ belongs to H . Hence by applying suitable elements of a and powers of \hat{M} , an arbitrary element of G_M^+ is attainable; this yields the second assertion and the first follows immediately. •

Proposition 1.8 Suppose that $M(1)$ is primitive. Let H be a nonzero order ideal of G_M , and let e be an element of $E(G_M)$ such that $eH = \mathbf{0}$. Then e is zero.

Proof. Select $[x^w E_j, k]$ in H . Apply M using the identity $[Ma, k+1] = [a, k]$. From the convexity of H together with the primitivity of $M(1)$, we may find k' , together with lattice points w_j such that for all $j = 1, 2, \dots, n$, $[x^{w_j} E_j, k']$ belongs to H . Write $e = \hat{M}^s \hat{N}$. As \hat{M} is an automorphism, we must have $\hat{N}[x^{w_j} E_j, k'] = \mathbf{0}$ for all j . Thus $x^{w_j} M^n \hat{N} E_j$ are all zero, and this obviously forces $M^n \hat{N} = \mathbf{0}$. Hence e is zero. •

Proposition 1.9 Suppose that $M(1)$ is primitive. If H is an order ideal in G_M , then $E_b(G_M) = E_b(H)$.

Proof. We first show that $E_b(H)$ is the order ideal of $E(G_M)$ generated by the identity. By definition, $E_b(H)$ is a subring of $E(G_M)$. Suppose that e belongs to $E(G_M)$ and $e(H^+) \subseteq H^+$; we wish to show that $e(G_M^+) \subseteq G_M^+$. This follows from Lemma 1.7. Thus the relative ordering on $E_b(H)$ agrees with the natural ordering.

Since the identity element of $E_b(H)$ is automatically an order unit for it, $E_b(H)$ is directed. Finally, suppose that $e' \leq e$ where e and e' are positive elements of $E(G_M)$ and e belongs to $E_b(H)$. As $e(H^+) \subseteq H^+$ and H is an order ideal, it easily follows that $e'(H^+) \subseteq H^+$, so that $e(H) \subseteq H$. Hence e' belongs to $E(H)$; but $e' \leq KI_H$ for some K (since the same inequality holds for e), so that e' belongs to $E_b(H)$.

Hence $E_b(H)$ is an order ideal of $E(G_M)$ and has the identity as an order unit. However, by definition $E_b(G_M)$ is the order ideal generated by the identity. So the two must be equal. •

Now we show that $E_b(G_M)$ is a shift invariant for M . Recall that if \mathcal{R} is a (unital) partially ordered commutative ring, and M and M' are square matrices with entries from \mathcal{R}^+ , then we say that they are (*lag l*) *shift equivalent* (over \mathcal{R}^+) if there exist rectangular matrices R and S with entries from \mathcal{R}^+ such that

$$\begin{aligned} MR &= RM' & SM &= M'S \\ RS &= M^l & SR &= (M')^l. \end{aligned}$$

If all the matrices involved have entries in \mathcal{R} but not necessarily in the positive cone, then we say M and M' are *algebraically shift equivalent* (over \mathcal{R}).

The reason G_M itself is *not* a shift invariant is that it depends on the choice of A , that is, on the number of variables, and how they are labelled. Let B be a subgroup of \mathbf{Z}^d , and suppose that all the entries of M have all of their exponents lying in B . Let A_0 denote the subring (if $A = \mathbf{Z}[x_i^{\pm 1}]$) or subalgebra (if $A = \mathbf{R}[x_i^{\pm 1}]$) generated by B . We say that M is *defined over* A_0 . We may define $G_{M,0}$ by replacing A^n with A_0^n throughout (1). Then it is immediate that G_M is naturally isomorphic (in all possible ways) with the tensor product of ordered A_0 -modules, $G_{M,0} \otimes_{A_0} A$.

Lemma 1.10 Under these conditions,

$$E_b(G_M) = E_b(G_{M,0}).$$

Proof: Select b in $G_{M,0}^+$, and form H_0 , the order ideal generated by b therein. We claim that H is also an order ideal in G_M . Write $A = F[B_1]$ (giving the name B_1 to the standard copy of \mathbf{Z}^d) and $A_0 = F[B]$, where F is either the integers or the reals. Let T be a transversal of B in B_1 ; that is, T is a subset of B_1 such that $B = \cup_{t \in T} (t + B)$ and for all $t \neq t'$ in T , $t - t'$ does not belong to B . We may assume that zero belongs to T . Obviously, $A = \oplus_{t \in T} x^t A_0$. Since the coefficients of the entries of M all lie in A_0 , we have that for all t , $M((x^t A_0)^n) \subseteq (x^t A_0)^n$. It easily follows that G_M decomposes as $\oplus_{t \in T} x^t G_{M,0}$, as ordered $A_0[\hat{M}^{\pm 1}]$ -modules. Moreover, each summand is an order ideal in G_M , so that H (being an order ideal in the summand corresponding to $t = 0$) is also an order ideal. (Order ideals in order ideals of a dimension group are themselves order ideals in the big dimension group.)

Next, we observe that if $G = \oplus G_i$, a direct sum of partially ordered (directed) abelian groups with the coordinatewise ordering on G , and $e : G \rightarrow G$ is a positive endomorphism such that $e \leq KI_G$ (as endomorphisms), then for each i , $e(G_i) \subseteq G_i$. To see this, select g in G_i^+ . Then $e(g) \leq Kg$, so that $e(g)$ belongs to the order ideal generated by g ; this obviously is contained in G_i . Since G_i is directed, it follows that $e(G_i) \subseteq G_i$. 1

Hence if e is in $E_b(G_M)$, for all t , $e(x^t G_{M,0}) \subseteq x^t G_{M,0}$. We have an order-preserving ring homomorphism, $\Delta : E_b(G_{M,0}) \rightarrow E_b(G_M)$, by permitting e to commute with each x^t . By Proposition 1.8, an element of $E_b(G_M)$ is determined by its effect on a nonzero order ideal. Hence the map is onto (since these endomorphisms must commute with elements of A). •

2. A brief look at $E_b(G_M)$ as a shift invariant

Suppose that M and M' are matrices with entries from A^+ (a common Laurent polynomial ring) and they are shift equivalent; by enlarging the ring we may assume that the entries in the implementing matrices R and S are also in A (this is not really necessary—a result in [PT] asserts that the equivalence can be implemented by matrices with entries in A^+).

Proposition 2.1 If M and M' are defined over a common Laurent polynomial ring A , and are shift equivalent (with the shift equivalence implemented over A), then the shift equivalence induces an order-isomorphism of ordered A -algebras, $E(G_M) \rightarrow E(G_{M'})$, and an isomorphism of ordered rings $E_b(G_M) \rightarrow E_b(G_{M'})$.

Proof. The equation $MR = RM'$ induces an order-preserving A -module homomorphism, $\hat{R} : G_{M'} \rightarrow G_M$ by means of $[a, k]_{M'} \mapsto [Ra, k]_M$. It is a triviality to check that $\hat{M}\hat{R} = \hat{R}\hat{M}'$. Similarly, S induces $\hat{S} : G_M \rightarrow G_{M'}$, that also intertwines \hat{M}' and \hat{M} . In general, having two such intertwining homomorphisms is not sufficient to obtain a ring homomorphism, either $E(G_M) \rightarrow E(G_{M'})$ or between their bounded subrings (examples exist in the case that $d = 0$). However, we also see from $RS = M^l$ and $SR = (M')^l$, that $\hat{R}\hat{S} = \hat{M}^l$ and $\hat{S}\hat{R} = (\hat{M}')^l$. Define $\phi_R : E(G_{M'}) \rightarrow E(G_M)$ as follows. Pick e' in $E(G_{M'})$ and denote $\phi_R(e')$ by e . For h in G_M , define

$$e(h) = (\hat{M})^{-l} \hat{R} e' (\hat{S} h).$$

The intertwining relations guarantee that $e(h) = (\hat{R}) e' (\hat{S} (\hat{M})^{-l} h)$, as well. The so-defined e is clearly linear. It commutes with the actions of A and \hat{M} (and thus with \hat{M}^{-1}), because of the following identities:

$$\begin{aligned} e(a\hat{M}^t h) &= \hat{M}^{-l} \hat{R} e' (\hat{S} \cdot a \cdot \hat{M}^t h) \\ &= \hat{M}^{-l} \hat{R} e' (a\hat{M}'^t \hat{S} h) \\ &= \hat{M}^{-l} \hat{R} \cdot a \cdot \hat{M}'^t e' (\hat{S} h) \\ &= a\hat{M}'^t e' (\hat{S} h) \end{aligned}$$

If $e' \geq \mathbf{0}$, then it follows that $\phi_R(e')$ is also positive. So $\phi_R(e')$ is at least well-defined. We check that ϕ_R is a ring homomorphism.

$$\begin{aligned} \phi_R(e'_1 \circ e'_2)(h) &= \hat{M}^{-l} \hat{R} (e'_1 \circ e'_2) (\hat{S} h) \\ &= \hat{R} (e'_1 \circ e'_2) (\hat{S} \hat{M}^{-l} h) \\ &= \phi_R(e'_1) \left[\hat{R} e'_2 (\hat{S} \hat{M}^{-l} h) \right] \\ &= \hat{R} \left[\hat{S} \hat{M}^{-l} \hat{R} e'_2 (\hat{S} \hat{M}^{-l} h) \right] \\ &= \hat{R} e'_1 \left(e'_2 (\hat{S} \hat{M}^{-l} h) \right) = \phi_R(e'_1 \circ e'_2)(h). \end{aligned}$$

Hence under the assumptions $MR \stackrel{!}{=} RM'$, $SM = M'S$, and $SR = (M')^l$, we deduce that ϕ_R is an order-preserving A -module ring homomorphism that intertwines the action of the original pair of matrices. One defines ϕ_S in a similar fashion, and a five-line computation reveals that they are mutually inverse.

It is routine to verify that the bounded subrings are mapped isomorphically to each other under these isomorphisms. •

Beginning with shift equivalent M and M' , find a Laurent polynomial ring over which they and the implementing matrices are all defined. We have just obtained a pair of natural inverse ring isomorphisms between the rings of endomorphisms (after enlarging the polynomial rings). These isomorphisms induce an isomorphism on the bounded subrings; however, the latter are independent of the choice of coefficient ring. So the bounded subring is a shift invariant. (In fact, a little more is true—if the first three equations hold, we obtain a map induced by ϕ_R from the bounded subring corresponding to M' to that corresponding to M .)

Example 2.2 Zero variables.

Suppose that $d = 0$, so that $M = M(1) = M_0$ is a matrix of constants with entries from either \mathbf{Z} or \mathbf{R} . Assume the former, and that M_0 is primitive. Then G_{M_0} is simply Krieger's dimension group, and it is simple (as a dimension group) with a unique trace. Let v in $\mathbf{R}^{1 \times n}$ be the unique (up to positive scalar multiple) strictly positive left eigenvector of M_0 , with corresponding eigenvalue ρ , which is of course the spectral radius. The unique trace $V : G_M \rightarrow \mathbf{R}$ is determined via $V[c, k] = v \cdot c / \rho^k$. Then $G_M^+ \setminus \{0\} = V^{-1}((0, \infty))$. If N is a matrix commuting with (M_0) , then $V\hat{N} = \lambda(N)V$, where $\lambda(N)$ is the eigenvalue of N whose eigenvector is v . It follows easily that $E(G_{M_0}) = E_b(G_{M_0})$. (This equality can only occur if $d = 0$, or other equally degenerate situations.)

Thus $E_b(G_{M_0}) = C_{\mathbf{Z}}(M_0)[M_0^{-1}]$. This is a finer shift invariant (over \mathbf{Z}) than $\mathbf{Z}[1/\rho]$ studied in [BMT], but not so fine as the complete invariant (when $d = 0$) which is G_{M_0} as a module over $E_b(G_{M_0})$.

For example, let a and b be positive integers. Set

$$M_0 = \begin{bmatrix} a & 8b \\ b & a \end{bmatrix} \quad M'_0 = \begin{bmatrix} a & 4b \\ 2b & a \end{bmatrix}.$$

Both matrices have $\rho = a + 2b\sqrt{2}$. The centralizer of M_0 is the centralizer of $\begin{bmatrix} 0 & 8 \\ 1 & 0 \end{bmatrix}$ (subtract a times the identity and divide by b); this is the companion matrix for the polynomial $x^2 - 8$. Hence $C_{\mathbf{Z}}(M_0) \cong \mathbf{Z}[2\sqrt{2}]$, and thus $E_b(G_{M_0}) \cong \mathbf{Z}[2\sqrt{2}][(a + 2b\sqrt{2})^{-1}]$ (the isomorphisms are given by the effect on the left eigenvector, or the function λ defined earlier). On the other hand, the centralizer of M'_0 is that of $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$. Thus $E_b(G_{M'_0}) \cong \mathbf{Z}[\sqrt{2}][(a + 2b\sqrt{2})^{-1}]$. It is straightforward to check that the two bounded endomorphism rings are isomorphic if and only if

$$(*) \quad \mathbf{Z}[2\sqrt{2}][(a + 2b\sqrt{2})^{-1}] = \mathbf{Z}[\sqrt{2}][(a + 2b\sqrt{2})^{-1}]$$

(note the equality). This will occur if and only if there exists a positive integer k so that $(a + 2b\sqrt{2})^k \sqrt{2} = c + 2d\sqrt{2}$ for some integers c and d . On taking norms, this would imply $2(a^2 - 8b^2)^k = c^2 - 8d^2$. Hence 2 divides c , so 4 divides the right side. Thus 2 divides a^2 . Conversely, if a is even, $\rho^2 = 2(a\rho - 2a^2 - 4b^2)$, so that invertibility of ρ entails invertibility of 2 and thus $(*)$ holds.

We have deduced that $(*)$ holds if and only if a is even. However, $\mathbf{Z}[\sqrt{2}]$ is a principal ideal domain, so $\mathbf{Z}[\sqrt{2}][(a + 2b\sqrt{2})^{-1}]$ is as well. Hence, with very little effort, we obtain that M_0 is shift equivalent to M'_0 if and only if a is even.

On the other hand, if

$$M_0 = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix} \quad M'_0 = \begin{bmatrix} 4 & 15 \\ 1 & 4 \end{bmatrix},$$

we have that the bounded endomorphism rings of both matrices are $\mathbf{Z}[\sqrt{15}]$ (as $4 + \sqrt{15}$ is a unit). The matrices correspond to the two ideal classes of $\mathbf{Z}[\sqrt{15}]$, and so are not shift equivalent (this is easy to see directly—as the matrices are invertible, shift equivalence is the same as conjugacy via $\text{GL}(2, \mathbf{Z})$).

If we add the identity to both matrices, the bounded endomorphism ring is then $\mathbf{Z}[\sqrt{15}][(5 + \sqrt{15})^{-1}]$. This is a principal ideal domain (as the ideal in $\mathbf{Z}[\sqrt{15}]$ that contains 5 and $\sqrt{15}$ represents the non-trivial ideal class). Hence in this case the matrices are shift equivalent. •

Example 2.3 FOG examples (two variables).

Now let $d = 2$; instead of x_1, x_2 , we write x, y . We consider a pair of matrices suggested by Jack Wagoner and Mike Boyle in connection with FOG (“finite order generation”) and other murky conjectures, most of which have been resolved:

$$M = \begin{pmatrix} x & 1 & 0 \\ 0 & y & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad M' = \begin{pmatrix} y & 1 & 0 \\ 0 & x & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

The second matrix is obtained from the first by interchanging x and y . These matrices arise from the graphs:



(The unlabelled arrows have weight $1 = x^0y^0$.) The automorphism of $A = \mathbf{Z}[x^{\pm 1}, y^{\pm 1}]$ given by interchanging x and y induces an order isomorphism $G_M \rightarrow G_{M'}$, and correspondingly between the rings of endomorphisms, and the rings of bounded endomorphisms. These do not implement module isomorphisms, because they do not (in the first two cases) commute with the action of A . Although the matrices are conjugate via an elementary matrix of $GL(3, A)$, they are not shift equivalent; in fact they are not even finitely equivalent, a fact that was discovered independently by Marcus and Tuncel [MT2] using quite general techniques and by me (using completely ad hoc methods).

By examining the left eigenvector for the large eigenvalue function of M , it is routine to check that in fact M is conjugate (i.e., using $GL(3, A)$) to the companion matrix for its characteristic polynomial. Thus $C_A(M) = A[M]$, the polynomial algebra (over A) in M . So $E(G_M) = A[M^{\pm 1}]$ —but the bounded subring, $E_b(G_M)$ is more difficult to compute.

We shall have more to say about this example in section 7. •

Now we prove a result which is related to the converse of Proposition 2.1. This is already known.

Proposition 2.4 If M and M' are defined over a common Laurent polynomial ring (algebra) A , then any order isomorphism between G_M and $G_{M'}$ as $A[\hat{M}]$ -modules is induced by a shift equivalence between M and M' , up to a power of \hat{M} .

Proof. Suppose $\phi : G_M \rightarrow G_{M'}$ is a positive A -module homomorphism satisfying $\phi\hat{M} = \hat{M}'\phi$. Then there exists an integer k together with positive columns a_j in $A^{n'}$ such that for all $j = 1, 2, \dots, n$,

$$\phi([E_j, 1]) = [a_j, k].$$

Define the $n' \times n$ matrix R_0 by setting its j th column to be a_j . All the entries of R_0 lie in A^+ , and we observe that as ϕ intertwines the shift(s) and is an A -module homomorphism, ϕ has the same effect on G_M as applying R_0 to the equivalence classes and following this with $\hat{M}^{(k-1)}$. In particular, \hat{R}_0 is well-defined and satisfies $\hat{R}_0\hat{M} = \hat{M}'\hat{R}_0$. It follows immediately that $(R_0M - M'R_0)M^n = \mathbf{0}$. If we replace R_0 by $R_1 = R_0M^n$, then we see that R_1 has all its entries in A^+ , intertwines M and M' , and satisfies $\hat{R} = \phi\hat{M}^{n-k+1}$.

Similarly, if $\psi : G_{M'} \rightarrow G_M$ is positive and intertwines \hat{M} and \hat{M}' , there exists a rectangular matrix S with entries from A^+ that intertwines M and M' and implements ψ , up to a power of \hat{M} or \hat{M}' . By

multiplying on or the other of R_1 or S by a power of M we may assume that the power of \hat{M} that indicates the perturbation from ϕ and ψ is the same; say $\hat{R}_1 = \phi \hat{M}^l$ and $\hat{S} = \hat{M}^l \psi$.

Now suppose that $\phi\psi = \hat{M}^t$ for some integer t . Then $\hat{S}\hat{R}_1 = \hat{M}^{t+2l}$. As before, $(SR_1 - M^{t+2l})M^n = \mathbf{0}$. Replacing R_1 by $R = R_1 M^n$, we obtain that $SR = M^{t+2l+n}$, and R is positive and still intertwines M and M' . By incorporating another power of M' into S , we obtain that if $\phi\psi = (\hat{M}')^t$, then RS is also a power of M' . In particular, this yields a shift equivalence. •

Remark. 2. If we let A be an arbitrary commutative unital ring and M an arbitrary square matrix (see Remark 1), then the unordered versions of Proposition 2.1 and Proposition 2.4 still hold, where we now use algebraic shift equivalence.

In view of Proposition 2.1 and Proposition 2.4, what is the point of discussing $E_b(G_M)$ (especially since it generally is more difficult to compute than G_M)? For determining the pure traces on the order ideals, it is essential (the trace space of $E(G_M)$ is inadequate for this purpose). Moreover, the preceding depends on the choice of A . There is a close relationship between $E_b(G_M)$ and $E(G_M)$ if M is defined over the minimal possible A .

Krieger [Kr] has defined two invariants for shift equivalence, which amount to the following (where tr denotes trace). If $f = \sum \lambda_w x^w$ is a polynomial in d variables expressed in monomial notation, then $\text{Log } f = \{w \in \mathbf{Z}^d \mid \lambda_w \neq 0\}$.

$$\Gamma(M) = \langle \bigcup_{j=1}^{\infty} \text{Log } \text{tr} M^j \rangle$$

$$\Delta(M) = \bigcup_{j=1}^{\infty} (\text{Log } \text{tr} M^j - \text{Log } \text{tr} M^j).$$

The angle brackets $\langle \rangle$ indicate the group generated by the contents. If S and T are subsets of \mathbf{Z}^d , then $S - T = \{s - t \mid s \in S, t \in T\}$. Note that $\Delta(M)$ is already a group.

Lemma 2.5 If $M(1)$ is primitive, then

$$\Gamma(M) = \langle \{w \in \mathbf{Z}^d \mid x^w M^{-l} \in E_b(G_M) \text{ for some } l \in \mathbf{N}\} \rangle$$

$$\Delta(M) = \langle \{w - w_0 \mid x^w M^{-l}, x^{w_0} M^{-l} \in E_b(G_M) \text{ for some } l \in \mathbf{N}\} \rangle.$$

Proof: Select z in $\text{Log } \text{tr} M^l$. This means that z is the total weight of a loop of length l whose initial and terminal state¹ is j for some j . There exists a loop of length k that passes through every state (as $M(1)$ is primitive). Let its total weight be denoted w . In this latter loop, insert the original loop immediately adjacent to an occurrence of j . This creates a loop of length $k + l$ containing every state, of weight $z + w$. As the loop of weight w contains every state, w belongs to $\text{Log } (M^k)_{ii}$ for all $i = 1, 2, \dots, n$ and thus $x^w \mathbf{I} \leq M^k$ for some positive integer K (as endomorphisms of G_M). Thus $x^w \hat{M}^{-k}$ belongs to $E_b(G_M)$. Similarly, $x^{z+w} \hat{M}^{-(k+l)}$ belongs to $E_b(G_M)$. Hence $z = z + w - w$ belongs to the right side.

Conversely, if $x^w \hat{M}^{-l}$ is a member of $E_b(G_M)$, then $x^w \mathbf{I} [E_j, 1] \leq K \hat{M}^l [E_j, 1]$ for some K and all j . Hence there exists an integer N so that $M^N (K M^l - x^w \mathbf{I}) E_j$ has all of its entries in A^+ . This simply means that all the entries of the matrix $K M^{l+N} - x^w M^N$ belong to A^+ . Taking a diagonal entry, we see that if u belongs to $\text{Log } (M^N)_{ii}$ for some¹ i , then $u + w$ belongs to $\text{Log } (M^{l+N})_{ii}$. Hence $w = u + w - u$ expresses w as the difference of two elements of $\Gamma(M)$.

The argument for $\Delta(M)$ is parallel. •

¹ We use ‘vertex’ in another context.

Parry and Schmidt [PS] (viz. [MT, 1.9]) had shown that if $M(1)$ is primitive, there exists a diagonal matrix D with monomial entries so that DMD^{-1} has all of its entries in $\mathbf{Z}[\Gamma(M)]$. Since this implements a very strong form of shift equivalence, we may assume that M already has support in $\Gamma(M)$, i.e. $\text{Log } M \subset \Gamma(M)$. For many purposes (including (\mathfrak{S})), we can replace M by $x^w M$ (for any lattice point w), and so even assume that $\text{Log } M \subset \Delta(M)$. The result in [PS] can be deduced from the following, which is interesting in itself.

Proposition 2.6 Suppose that for some i , no entry in the i th row and column of $M(1)$ zero. Then

$$\text{Log } \text{tr} M \cup \text{Log } \text{tr} M^2 \cup \text{Log } \text{tr} M^3$$

generates $\Gamma(M)$ as an abelian group, and there exists a diagonal matrix D with monomial entries such that $\text{Log } DMD^{-1} \subset \Gamma(M)$.

Proof. Let W denote the subgroup of $\Gamma(M)$ generated by the three sets. For each (k, l) , select any exponent $w_{k,l}$ appearing in $\text{Log } M_{k,l}$, unless $M_{k,l}$ is zero, in which case we avoid that choice of (k, l) for now. Whenever it makes sense, $w_{k,l} + w_{l,k} \in W$, and similarly, for any three indices, $w_{j,k} + w_{k,l} + w_{l,j} \in W$. We deduce that if $w_{k,l}$ is defined, then it is congruent modulo W to $w_{i,l} - w_{i,k}$ (which is always defined). Moreover, if $w'_{k,l}$ is another choice replacing $w_{k,l}$, then their difference belongs to W (since we may use the same $w_{i,l}$ and $w_{i,k}$ for $w'_{k,l}$). Set $D = \text{diag}(1, x^{w_{1,2}}, \dots, x^{w_{1,n}})$. Then $(DMD^{-1})_{kk} = M_{kk}$, while for $k \neq l$, $(DMD^{-1})_{kl}$ is either zero or $x^{-w_{k,l}} x^{v_{k,l}} M_{kl}$ for some $v_{k,l}$ in W . Thus $\text{Log } (DMD^{-1})_{k,l} \subset W$ (where $\text{Log } 0$ is the empty set). •

Lemma 2.7 Suppose that $M(1)$ is primitive and $\text{Log } M \subset \Delta(M)$. If H and H' are nonzero order ideals of G_M , then $H \cap H'$ is not empty. (In other words, all nonzero order ideals of G_M are essential.)

Proof. We may find (nonzero positive) elements of the form $[x^w E_j, k]$ and $[x^{w'} E_{j'}, k]$ in H and H' respectively. By replacing these by $[x^w M^t E_j, k + t]$ and $[x^{w'} M^t E_{j'}, k + t]$ for suitable t , and using the convexity property of order ideals (and the fact that $M(1)$ is primitive), we may even assume that originally $j = j'$. There exists an integer l so that $x^{w_0 - w} \hat{M}^{-l}$ is an element of $E_b(G_M)$. In particular, if it is applied to any element of an order ideal, the outcome still belongs to that order ideal. Applying it to $[x^w E_j, k]$, the outcome is not zero, positive and lies in H ; on the other hand, the image is $[x^{w_0} E_j, k + l]$. Under the current assumption that $\mathbf{I} \leq KM$, this element is dominated by an integer multiple of $[x^{w'} E_j, k]$, so belongs to H' . •

Let A be one of the ordered rings $\mathbf{Z}[x_i^{\pm 1}]$ or $\mathbf{R}[x_i^{\pm 1}]$ (Laurent polynomial rings in d variables), equipped with positive cone consisting of the Laurent polynomials with no negative coefficients. For two rectangular matrices of the same dimensions with entries from A , we write $S \prec S'$ if S' has all of its entries in A^+ and $\text{Log } S_{ij} \subseteq \text{Log } S'_{ij}$ for all coordinates.

Theorem 2.8 If $M(1)$ is primitive and $\Delta(M) = \mathbf{Z}^d$, then there exists $e_0 = x^{v_0} \hat{M}^{-l}$ in the centre of $E_b(G_M)$ such that $E(G_M) = E_b(G_M)[e_0^{-1}]$. Additionally, for h in G_M , if $e_0 h \geq 0$ then $h \geq 0$.

Proof. If f is an element of $E(G_M)$, we may replace it by f multiplied by any power of \hat{M} . Hence we may assume that $f = \hat{N}$ where N is in $C_A(M)$.

First assume that $\text{tr} M \neq 0$ and 0 belongs to $\text{Log } \text{tr} M$. As $M(1)$ is irreducible, there exists k ($k = n!$ will do) together with a weight w so that there is a loop l of length k containing every state from 1 to n with total weight w . Set $W_t = \{y \in \mathbf{Z}^d \mid x^y M^{-(t+k)} \in E_b(G_M)\}$.

Claim: $W_t \supset \bigcup_{0 \leq s \leq t} (w + \text{Log } \text{tr} M^s)$.

Select v appearing in the i, i position of M^s . This corresponds to a path l' from i to i with total weight v . Insert l' into l so that the initial state i appears adjacent to an occurrence of i in l . This creates a loop l of

length $k + s$ with total weight $w + v$. Set $m = t - s$. As 0 belongs to $\text{Log tr}M$, there exists a state j' so that 0 lies in $\text{Log } M_{j'j'}$ —yielding a loop of length 1 from j' to j' . Insert the m -fold concatenation of this loop in $l \cup l'$, adjacent to an occurrence of j' . The weight is unchanged, but of course the length is increased to $k + t$. As every state appears in the resulting loop, we have that $x^{w+v} \hat{M}^{-(k+t)}$ belongs to $E_b(G_M)$. This proves the claimed statement.

Let $s(1), s(2), \dots$ be positive integers, and let $t \geq \sum s(j)$.

Claim: $w + \sum \text{Log tr}M^{s(j)} \subset W_t$.

Select $v(j)$ in $\text{Log tr}M^{s(j)}$; let $l(j)$ be a corresponding loop with initial state $(i(j))$ and total weight $v(j)$. If $m = t - \sum s(j)$, set l_0 to be the loop consisting of the constant path at j' of length m . Insert each $l(j)$ adjacent to an occurrence of $i(j)$ in l and l_0 adjacent to an occurrence of j' . The resulting loop has total weight $w + \sum \text{Log tr}M^{s(j)}$ and is of length t . This establishes the claim.

The cone in \mathbf{Z}^d with vertex w generated by all of the $\text{Log tr}M^s$ generates \mathbf{Z}^d as an abelian group, by hypothesis ($\Delta(M) = \mathbf{Z}^d$). Hence it contain an element v_0 such that all of $w + v_0 \pm e_i$ ($i = 1, 2, \dots, n$) belong to it as well (here e_i run over the standard basis of \mathbf{Z}^d). By the last claim, there exists t_0 such that $\{w + v_0 \pm e_i\} \subseteq W_{t_0}$ —in fact, $x^{w+v_0 \pm e_i} \mathbf{I} \prec M^{t_0}$.

Choose z in \mathbf{Z}^d , and write $z' = z - w$. Now decompose

$$z' = \sum_I a_i e_i + \sum_{I'} b_{i'} e_{i'}$$

where $I \cup I'$ is a partition of a subset of $\{1, 2, \dots, n\}$, a_i are positive integers and $b_{i'}$ are negative. Set $m = \sum a_i + \sum |b_{i'}|$. We note that for all $t \geq mt_0$,

$$w + \sum a_i(v_0 + e_i) + \sum (-b_{i'})(v_0 - e_{i'}) \in W_t.$$

Thus $w + mv_0 + z$ lies in W_t , whence $x^w(x^{v_0})^m x^z \mathbf{I} \prec M^{mt_0+k}$. By the preceding, $x^w(x^{v_0})^m x^z \mathbf{I} \prec M^{m(t_0+k)}$, and so $x^z \mathbf{I} \prec (x^{-v_0} M^{t_0+k})^m$. Moreover, this holds with m replaced by any integer exceeding it.

Let E_{ij} denote the matrix with a 1 in the i, j position and zeroes elsewhere. We shall show that for any z in \mathbf{Z}^d and choice of i, j ,

$$x^z E_{ij} \prec (x^{-v_0} M^{t_0+k})^m$$

(in other words, $z \in \text{Log}((x^{-v_0} M^{t_0+k})^m)_{ij}$) for all sufficiently large m .

Choose v in $\text{Log}((x^{-v_0} M^{t_0+k})^m)_{ij}$ (note that $t_0 + k > n!$, so every entry of the matrix is nonzero). This corresponds to a path of length $t_0 + k$ from i to j with total weight v . In particular, $x^v E_{ij} \prec x^{-v_0} M^{t_0+k}$. By the preceding, for all sufficiently large m , $x^{z-v} \mathbf{I} \prec (x^{-v_0} M^{t_0+k})^m$, so the result obtains on multiplication.

Now let N be an element of $C_A(M)$. Since $\text{Log } N$ is finite, for all sufficiently large m , for all i and j ,

$$\text{Log } N_{ij} \subset \text{Log}((x^{-v_0} M^{t_0+k})^m)_{ij}.$$

Hence there exists an integer K so that $-Kx^{-v_0} M^{t_0+k} \leq N \leq Kx^{-v_0} M^{t_0+k}$. In particular, $\hat{N} \cdot (x^{v_0} \hat{M}^{-(t_0+k)})^{m_1}$ belongs to $E_b(G_M)$ for all sufficiently large m_1 . Applying the argument with $z' = -w$, we deduce that for some m_0 , the element defined as $e_0 = (x^{v_0} \hat{M}^{-(t_0+k)})^{m_0}$ belongs to $E_b(G_M)$, and is actually in the centre. Thus if m_1 is chosen to be a multiple of m_0 ,

$$\hat{N} = (\hat{N} \cdot e_0) e_0^{-1}$$

where $\hat{N}e_0$ belongs to $E_b(G_M)$ and e_0 is in the centre of $E_b(G_M)$. Moreover, e_0 is not a zero divisor in $E(G_M)$, and thus is not a zero divisor in $E_b(G_M)$.

If $\text{tr}M \neq 0$ but 0 is not in $\text{Log tr}M$, choose w_0 in $\text{Log tr}M$, and replace M by $x^{-w_0}M$.

Finally, if $\text{tr}M = 0$ we observe that $\text{tr}M^s \neq 0$ for all $s \geq n!$ (and in fact $s \geq n^2$ is sufficient). If g. c. d. $(s, n!) = 1$ and $NM^s = M^sN$, then $(M^nN)M = M(M^nN)$ (by Lemma A1.1). Thus M^nN belongs to $C_A(M)$, so $\widehat{M}^nN\widehat{M}^{-n}$ is in $E(G_M)$ and implements N . In other words, $C_A(M^s)[\widehat{M}^{-s}] = C_A(M)[\widehat{M}]$. So we can replace M by M^s in the preceding. •

The methods used here suggest the following shift invariant. For M in M_nA^+ , define the subset of $\mathbf{Z}^d \times \mathbf{N}$,

$$T(M) = \{(w, t) \in \mathbf{Z}^d \times \mathbf{N} \mid \text{there exists } s \text{ such that } x^w M^s \prec M^{s+t}\}.$$

Our first assertion is that $T(M)$ is a shift invariant for M . This follows immediately from the observation (practically a tautology) that (w, t) belongs to $T(M)$ if and only if $x^w \widehat{M}^{-t}$ is an element of the bounded endomorphism ring $E_b(G_M)$. There is a natural map from $T(M)$ to $\text{WPS}(M) := \cup(\text{Log tr } M^k)/k$, given by $\tau : (w, t) \mapsto w/t$; again verification is obvious. The image is in general a proper subset, and of course its image is also a shift invariant, but there is some information lost in going from $T(M)$ to its image. This will be seen as a consequence of the result to be proved below, that if $d = 1$, $\text{cvx } \tau T(M) = \text{cvx } \text{WPS}(M)$. (The formal definition of WPS, introduced by Marcus & Tuncel [MT], will be given in section 3.)

First, we present an example of what can happen when $d = 2$. Suppose

$$M = \begin{bmatrix} x & 0 & 1 \\ 1 & y & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

our familiar FOG example (2.3). Then it is fairly easy to see that

$$T(M) = \{((a, b), t) \in \mathbf{Z}^2 \times \mathbf{N} \mid a \geq 0, b \geq 0, \text{ and } t \geq a + b + 3\}.$$

Thus although $(0, 0)$ (one of the vertices of $\text{cvx } \text{WPS}(M)$, the standard triangle in \mathbf{R}^2) belongs to $\tau T(M)$ (via the element $((0, 0), 3) - 3$ can be replaced by any larger integer, but no smaller one), neither $(1, 0)$ nor $(0, 1)$ does. We do obtain that both of these points as limits of elements of $\tau T(M)$, e.g., $\{\tau((k, 0), k + 3)\}$, but this is a general phenomenon. In this example, τ causes a loss of information in the sense that although $(0, 0)$ is in its image, we cannot tell that $((0, 0), 1)$ is not in $T(M)$. This example $E_b(G_M)$ will be investigated more deeply in the next section. •

Valerio de Angelis asked whether the following is true, in the form given in the third sentence.

Proposition 2.9 Let M be a primitive square matrix of size n with entries from A^+ , where $d = 1$. Then $\text{cvx } \tau T(M) = \text{cvx } \text{WPS}(M)$. In particular, if 0 is the left endpoint of $\text{cvx } \text{WPS}(M)$, there exist positive integers s and t such that $M^s \prec M^{s+t}$.

Proof. The conclusion of the second sentence is that $(0, t)$ belongs to $T(M)$. It is clearly enough to prove this, since we may replace x (the lone variable) by its inverse and multiply by a suitable power of x to obtain the other endpoint. The following proof yields extravagantly large choices for both s and t , but it may be that this is necessary (particularly when n is large).

We assume that 0 is the left endpoint of $\text{cvx } \text{WPS}(M)$; in particular, 0 itself belongs to $\text{WPS}(M)$. The first step is to find a state s in $\{1, 2, \dots, n\}$ and a positive integer k such that the set $S = \text{Log}(M^k)_{ss}$ satisfies:

- (i) $0 \in S$
- (ii) $\text{gcd } S = \text{gcd}(\cup_e \text{Log tr } M^e)$.
- (iii) There exists an integer $m > 2n$ in S such that for all integers $l \geq m$ lying in the right side of (ii), l can be expressed as a nonnegative integer combination of elements of S .

Note that (ii) makes sense— S cannot contain any negative integers, as $\text{WPS}(M)$ contains no negative rationals (negative exponents may appear in entries in all powers of M , but not on the diagonal). Of course (ii) implies there exists an integer m with the properties ascribed to it in (iii), but there we also insist that m belong to S (this is easy to arrange).

Since 0 is in $\text{WPS}(M)$, there exists a loop of length k_1 (for some $k \leq n$) whose weight is zero; let s be one of the states it passes through (i.e., 0 belongs to $\text{Log } M^{k_1}_{ss}$). Now replace M by M^{k_1} (which we can do throughout). Look at all the loops on the state s . Since the matrix is primitive, we may assume the greatest common divisor of their weights, c , is that of the right side of (ii). To see this, let u be a state, and suppose there is a loop $u \rightarrow u$ (of some unspecified length) with weight v such that c does not divide v . There exist paths $s \rightarrow u$ and $u \rightarrow s$ of weights e, f respectively (by the primitivity of the matrix), so there exists a loop $s \rightarrow u \rightarrow s$ on s with weight $e + f$. Then c divides $e + f$. Inserting the loop of weight v on u into this loop, we create a loop $s \rightarrow u \rightarrow u \rightarrow s$ of weight $v + e + f$. As c must divide this, it divides v , a contradiction. Replace the current power of M by a larger one, by raising it to the power $\prod l_i$ where l_i run over the lengths of loops required to implement a set of weights whose greatest common divisor is c . By replacing this power by a still higher one (so that some “ m ” belongs, which we can arrange to exceed $2n$), we obtain condition (iii). The k in this construction is enormous.

Replace the original M by $M^{n!}$, to absorb all minimal loops. Now select a path of length N (N to be determined later, but much larger than n) from state i to state j , having weight w (this can be negative); i.e., x^w appears in M^N_{ji} .

If there exists a loop $i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_t = i_1$ appearing in this path (permitting permutations) of weight zero, there exists a loop of length 1, $i_h \rightarrow i_h$ with weight zero, where i_h is one of the states appearing in the loop. Since we may repeat $i_h \rightarrow i_h$ as often as we wish in the original path, increasing the length by 1 each time but not increasing the weight, we deduce that for all $y > 0$, x^w appears in M^{N+y}_{ji} .

If no loop of weight zero appears in the path, we perform surgery on it. Suppose there exist F loops used in the path (appearing with various multiplicities); we can ensure that $F \geq (N + n - 1)/n$, as there exists one state that appears at least N/n times in the path. Since none of these loops can have weight zero, they all have nonzero weight, hence $w \geq Fc \geq c(N/n - 1)$. There exists an edge from i to s (since we have replaced M by a large power of itself) of some weight K say, and an edge in the reverse direction of L . Then $K + L \geq 0$. Let p from i to j be an edge of least weight, say e (which can be negative). Form a new path from i to j via $i \rightarrow s \rightarrow s \rightarrow \dots \rightarrow s \rightarrow i \rightarrow p$ as follows. The i to s and $s \rightarrow i$ edges have total weight $K + L$. Let $u = w - K - L - e$ (we will adjust N later so the following construction will work). Write $u = \sum_{s_i \in S} a_i s_i + am$ where a and the a_i are positive integers; we can do this if u is sufficiently large and we arrange that a is defined so that $2m > u - am \geq m$. Use the loop of length 1 on s of weight m , a times; use the loops of length 1 and weight s_i , a_i times. The upshot is a path of length $3 + a + \sum a_i$ with total weight u . Moreover, we can throw in any number of loops of length 1 and weight zero, since the path hits s .

If we guarantee that $F \geq m + K + L + e$, e.g., if $N \geq n(m + K + L + e + 1)$ and $N \geq 3 + a + \sum a_i$, then we have that x^w appears in $\text{Log}(M^{N+y})_{ji}$ for all $y > 0$. Now $\sum a_i \leq 2m$ and $a \leq N/m$, so we only require $N(1 - 1/m) \geq m + 3$, i.e., $N \geq (m - 1)(m + 3)/m$ for the latter constraint. Now m, K, L, e depend only on i and j , so we take $N \geq n \max \{m(i, j) + K(i, j) + L(i, j) + 1\}$. We have shown that for all i and j , and for all $y > 0$, $\text{Log}(M^N)_{ij} \subseteq \text{Log}(M^{N+y})_{ij}$ for all $y > 0$. Since we have made a replacement of M by $M^{n!}$, we deduce $M^{n!N} \prec M^{(n!+y)N}$, so that $(0, n!)$ belongs to $T(M)$. •

3. The many faces of M

As usual, M will denote an $n \times n$ matrix with entries in A^+ , the set of d -variable Laurent polynomials having no negative coefficients. Define a convex polytope in \mathbf{R}^d associated to M ,

$$K(M) = \frac{1}{n!} \text{cvx Log tr } M^{n!}.$$

This has vertices with coefficients in $1/n! \mathbf{Z}$, and agrees with the weight per symbol polytope of Marcus and Tuncel [MT, Section 3], which is denoted $\overline{\text{WPS}}_M$. If $n = 1$, this is just the Newton polyhedron (see [H5]).

Marcus and Tuncel [op. cit.] considered the facial structure of $K(M)$. To every face F , they associated a matrix M_F also with entries in A^+ . We will show that there are natural maps $E_b(G_M) \rightarrow E_b(G_{M_F})$. In general, there are no maps $G_M \rightarrow G_{M_F}$ nor $E(G_M) \rightarrow E(G_{M_F})$; the bounded subring is a shift invariant that detects facial matrices. We also use the facial matrices to determine the pure traces on $E_b(G_M)$ and on order ideals in G_M . In order to proceed, we have to see how the facial matrices behave on replacing M by powers of itself.

Define $\beta_M : (\mathbf{R}^d)^{++} \rightarrow \mathbf{R}$ by setting $\beta_M(r)$ to be the spectral radius of $M(r)$. If $M(1)$ is primitive, then β is a real analytic function (by Hartogs' theorem, we may assume that there is only one variable; since β is a solution to the characteristic equation of M and the multiplicity of $\beta(r)$ is one for every r , analyticity follows).

Let $A_{\mathbf{Q}}$ denote the algebra generated by the monomials $\{x^w\}$ where w is allowed to range over rational d -tuples. Obviously, $A_{\mathbf{Q}}^+$ will denote the positive cone consisting of positive linear combinations of the monomials. If S is any convex subset of \mathbf{R}^d , we denote by $\partial_e S$, the set of vertices (extreme points) of S . If $S = \text{cvx Log } P$, we sometimes abbreviate $\partial_e \text{cvx Log } P$ to $\partial_e \text{Log } P$.

Recall from [A, p. 290–295] that if p is a (real) analytic function on $(0, \infty)$ that satisfies an equation with coefficients from $\mathbf{R}[t^{\pm 1}]$, then its behaviour at 0 is of fractional power order, that is, for some integers a and b , $\lim_{t \rightarrow 0^+} p(t)/t^{a/b}$ exists and is not zero, and similarly, $\lim_{t \rightarrow \infty} p(t)/t^{a'/b'}$ is not zero.

Lemma 3.1 Let p be a nonconstant function of one real variable t that is continuous on an interval (a, ∞) and satisfies the equation

$$p^n + \sum_{i=0}^{n-1} s_i p^i = 0$$

where the s_i are rational functions. Suppose that for all sufficiently large t , $p(t)$ is nonnegative.

- (a) If for all sufficiently large t , $p(t)$ is larger than the absolute value of all the other roots of the equation $\lambda^n + \sum s_i(t)\lambda^i = 0$, then either $\lim_{t \rightarrow \infty} p(t)$ exists or $\liminf_{t \rightarrow \infty} p(t) = \infty$.
- (b) If all of the s_i are bounded on $[a, \infty)$, then $\lim_{t \rightarrow \infty} p(t)$ exists.

Proof. Suppose that $a < a_1 < a_2 < \dots$, is a sequence with $\sup a_j = \infty$ such that $p(a_j)$ converges to a limit, r . As t increases, $p(t)/t^{c/b}$ is bounded on a real neighbourhood of ∞ for some integers b and c , with $0 \leq b \leq n$. Obviously $c > 0$ is impossible; if $c < 0$, $r = 0$ and the limit exists. This leaves the situation that $c = 0$, that is, that p be bounded. This is also a consequence of (b).

Assume p is bounded; as the complement in its Stone-Ćech compactification of $[a, \infty)$ is connected, if β has more than one limit point at infinity, it has a whole interval of them. We observe that the roots at each point are uniformly bounded, and so in case (a) holds, the symmetric functions s_i are themselves uniformly bounded. As they are rational, we may apply l'Hôpital's rule; thus $\lim_{t \rightarrow \infty} s_i(t) = r_i$ exists. Hence any limit point at infinity, r , of $\{p(t)\}$ must satisfy the monic equation $r^n + \sum r_i r^i = 0$. This equation has only finitely many roots, so $\{p(t)\}$ admits at most finitely many limit points at infinity. Since the limit points form an interval, there can only be one, so that $\lim p(t)$ exists. •

Lemma 3.2 Suppose that $M(1)$ is primitive. For $n!$ dividing s , define the Laurent polynomial $P_s = \text{tr } M^s$. Then β_M^s/P_s and P_s/β_M^s are bounded as functions on $(\mathbf{R}^d)^{++}$. Moreover, for all r in $(\mathbf{R}^d)^{++}$,

$$\lim_{s \rightarrow \infty} \frac{P_s}{\beta_M^s}(r) = 1.$$

Proof. Abbreviate β_M to β . Let $\lambda^n + \sum \sigma_i \lambda^{n-i}$ be the characteristic polynomial of M^s (so $\sigma_1 = -\text{tr } M^s$, $\sigma_n = (-1)^n \det M^s$, etc.). Set $g = \beta^s/P_s$; this is a positive real analytic function on $(\mathbf{R}^d)^{++}$ such that

$$g^n + \sum \frac{\sigma_i}{P_i} g^{n-i} = 0.$$

We observe that as functions on $(\mathbf{R}^d)^{++}$, $|\sigma_i| \leq \mathcal{O}(\text{tr } M^{is})$: for any matrix N with entries in A^+ , $\text{tr } N^j \cdot \text{tr } N^k \leq \text{tr } N^{j+k}$, and now Newton's identities apply. Next, we claim that $\text{tr } M^{is} = \mathcal{O}((\text{tr } M^s)^i)$. It is obviously sufficient to show that every lattice point w appearing in $\text{Log tr } M^{is}$ is a convex combination of lattice points appearing in $\text{Log}(\text{tr } M^s)^i$. By definition, there is a loop of length is with initial state j of total weight w . All minimal loops must be of length n or less, as there are only n states. Thus w can be decomposed as a sum of weights of minimal loops, l_1, l_2, \dots, l_k with a common initial state which can be concatenated to form the original loop (we may have to perform some surgery, e.g.,

$$1 \rightarrow 2 \rightarrow 2 \rightarrow 3 \rightarrow 3 \rightarrow 2 \rightarrow 1$$

can be re-formed into $2 \rightarrow 3 \rightarrow 3 \rightarrow 2$ and $2 \rightarrow 1 \rightarrow 2$ by cyclically permuting the original). Since $n!$ divides s , the concatenation of l_i occurs a suitable number of times in the graph associated to M^s . If the weight of l_t is $w(t)$ and the length of l_t is $L(t)$, we have $\sum L(t) = is$ and $\sum w(t) = w$. The weight of l_t concatenated $is/L(t)$ times is $w(t) \cdot is/L(t)$, and so

$$\sum_t \left(\frac{is}{L(t)} w(t) \right) \cdot \frac{L(t)}{is} = \sum w(t) = w,$$

expressing w as a convex combination of lattice points appearing in $\text{Log}(\text{tr } M^s)^i$.

So there exist constants u and U such that $|\sigma_i| \leq u \text{tr } M^{is} \leq U P_s^i$ as functions on $(\mathbf{R}^d)^{++}$. Thus σ_i/P_s^i is bounded. Hence g is bounded; in other words, $\beta^s = \mathcal{O}(P_s)$. On the other hand, $\text{tr } M^k \leq n\beta^k$, so $P_s = \mathcal{O}(\beta^s)$.

The limit result is a consequence of the simple fact that for a primitive matrix N with spectral radius ρ , $\text{tr } N^s / \rho^s \rightarrow 1$. •

Lemma 3.3 If Q and Q' belong to $A_{\mathbf{Q}}^+$ and Q/Q' is bounded on $(\mathbf{R}^d)^{++}$, then $\text{Log } Q \subseteq \text{cvx Log } Q'$. If both Q/Q' and Q'/Q are bounded, then $\text{cvx Log } Q = \text{cvx Log } Q'$.

Proof. This can be deduced from results in [H5]; we recall the method of proof. By redefining the lattice, we may replace $A_{\mathbf{Q}}^+$ by A^+ , that is, all the supporting monomials are exponentials of lattice points. If the first inclusion fails, then there exists z in $\partial_e \text{cvx Log } Q \setminus \text{cvx Log } Q'$; since $\partial_e \text{cvx Log } Q \subseteq \text{Log } Q$, z belongs to $\text{Log } Q$. There exists a linear functional $\alpha : \mathbf{R}^d \rightarrow \mathbf{R}$ such that $a = \alpha(z) > b = \sup \{\alpha \cdot w \mid w \in \text{cvx Log } Q'\}$; we may assume that α is defined over the rationals, and thus over the integers. Define the path $X(t) : \mathbf{R}^+ \rightarrow (\mathbf{R}^d)^{++}$ as $\exp t\alpha$ (that is, write α as given by the row of integers $(\alpha_1, \dots, \alpha_d)$, and set $X(t) = (t^{\alpha_1}, \dots, t^{\alpha_d})$. For a monomial, $x^w(X(t)) = t^{\alpha \cdot w}$. Since all the coefficients are nonnegative, the growth of $Q(X(t))$ (as a function of t) is at least as large as that of t^a , while the growth of $Q'(X(t))$ is at most that of t^b . Hence Q/Q' is unbounded on $(\mathbf{R}^d)^{++}$, a contradiction. The second part is an immediate consequence. •

Corollary 3.4 (a) The convex polytope $K(M)$ depends only on β , in the sense that if Q is any element of $A_{\mathbf{Q}}^+$ such that both $(\beta^m/Q)^{\pm 1}$ are bounded on $(\mathbf{R}^d)^{++}$ for some m , then $\text{cvx Log } Q = mK(M)$.

(b) For all integers k , $K(M^k) = kK(M)$.

Proof. Let Q be as described. Setting $P = \text{tr } M^{n!}$, we have that both $(\beta^{kn!}/P^k)^{\pm 1}$ are bounded. By Lemma 3.3, $n! \text{cvx Log } Q = k \text{cvx Log } P = kn!K(M)$. Hence $K(M)$ is determined entirely by the choice of β , proving (a). Part (b) follows from this and Lemma 3.2. •

Now let F be a face of $K(M)$. Marcus and Tuncel [MT] have defined a matrix associated to M and F . For each loop l of weight w and length t in the graph associated to M define its *weight per symbol*, $\text{wps}_M(l) = \frac{1}{t}w \in \mathbf{Q}^d$. Define

$$\text{WPS}(M) = \{\text{wps}_M(l) \mid l \text{ is a loop in the graph of } M\}.$$

As in [MT], $\text{cvx WPS } M = K(M)$; it also follows immediately from Lemma 3.2 and Corollary 3.4.

Define a matrix M_F . Suppose that x^v appears in M_{ij} with coefficient λ , and there exists a loop l with weight w and length t such that $w/t \in F$ and the transition $i \rightarrow j$ with weight v is part of l . Then we put λx^v into the ij position of M_F . If no such loop exists, the contribution to the ij entry from x^v is zero. Summing over all v in $\text{Log } M_{ij}$, we obtain $(M_F)_{ij}$, and this describes M_F completely.

If $N \prec M$, we may define N_F in the obvious fashion (but associated to the graph of M , not of N). Associated to the graph of M^k , we define $(M^k)_{kF}$ (by Corollary 3.4(b), kF is the face of $K(M^k)$ corresponding to F), and if $N \prec M^k$, we can define N_{kF} (arising from the graph of M^k).

Lemma 3.5 (a) For all integers k , $(M_F)^k = (M^k)_{kF}$.

(b) If $N \prec M^r$ for some integer r , then for all integers k

$$N_{rF}(M^k)_{kF} = (NM^k)_{(k+r)F}.$$

Proof. (a). As $K(M^k) = kK(M)$, the right side is at least well defined. Consider the ij entry of $(M_F)^k$; this will be a sum of positive linear multiples of monomials with exponents w that can be decomposed as $w = \sum_{s=1}^k v_s$, where (possibly after a permutation), x^{v_s} appears in the $i_s i_{s+1}$ entry of M and there exists a loop l_s of length $t(s)$ and total weight $w(l_s)$ containing the transition $i_s \rightarrow i_{s+1}$ with weight v_s such that $w(l_s)/t(s)$ belongs to F . Now we construct a loop that contains the path $i = i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_{k+1} = j$ out of the loops of l_s . Form the path $i \rightarrow j$; cyclically permute the loop l_k so that its initial state is $i_{k+1} = j$, and snip off the last transition, $i_k \rightarrow i_{k+1}$. Attach the rest of the loop l_k to the current path; the terminal point is now i_k , and we see because the original path $i \rightarrow j$ has its last transition $i_k \rightarrow i_{k+1}$ with weight v_s , a copy of the whole of l_k is embedded in the current path. Now start l_{k-1} at i_k , snip off its last transition, and attach it at the current terminal point, i_k ; again loop l_{k-1} is embedded in the current path, and it terminates at i_{k-2} . This process may be continued until we return to $i_1 = i$, and the resulting loop is a union of the loops l_s (see the illustration below).

The resulting loop l has total length $\sum t(s)$ (the sum of the lengths of the loops l_s and total weight $\sum w(s)$. Now $\sum w(s)/\sum t(s)$ is a convex combination of $\{w(s)/t(s)\}$; as each of the latter belongs to F , so does the former. Finally we note that a loop of length t corresponding to M has length t/k as a loop arising from the graph of M^k , if k divides t . Our loop l need not have length divisible by k , but we can certainly iterate it k times. The resulting loop has the same weight per symbol, and can be viewed as a loop from the graph of M^k , whose first transition has weight $\sum v(s) = w$. The outcome of this is that if λx^w appears in the ij position of $(M_F)^k$, then it also appears in the ij position of $(M^k)_{kF}$.

Conversely, suppose that x^w appears in the ij position of M^k and w is the weight at a transition $i \rightarrow j$ of a loop l (with respect to M^k) of length t such that $w(l)/t \in kF$. Viewed with respect to the graph of M , it decomposes into a number of loops of length kt ; obviously, each has its weight per symbol in F . Write $w = \sum_{s=1}^k v_s$ corresponding to a decomposition (at the transition $i \rightarrow j$). Each x^{v_s} then appears in $(M_F)_{i_s, i_{s+1}}$, and since this covers all possible ways of obtaining x^w in $(M^k)_{ij}$, we deduce that whatever appears in the ij position of $(M^k)_{kF}$ also appears in the ij position of $(M_F)^k$.

(b). By (a), it suffices to show $N_{rF}M_F = (NM)_{(r+1)F}$. However, the preceding argument does not require the use of M^r , but merely the presence of certain monomials in the same positions. •

This permits us to define an ideal of $E_b(G_M)$,

$$I_F = \left\{ e = \hat{N}\hat{M}^{-k} \in E_b(G_M) \mid (M^n N)_{(k+n)F} = \mathbf{0} \right\}.$$

We must check that this is well defined, that is, if $(M^n N)_{(k+n)F} = \mathbf{0}$, then $(M^n N)_{(k+n+1)F} = \mathbf{0}$. As N and M commute, this follows from Lemma 3.5(b) above. The rest of the properties for it to be an ideal are verified routinely. This allows us to define a positive ring homomorphism π_F , given by

$$\pi_F : E_b(G_M) \rightarrow E_b(G_{M_F}) \quad \hat{N}\hat{M}^{-k} \mapsto (\widehat{N_{kF}})(\widehat{M_{kF}})^{-k}.$$

It is clear that if $N \prec M^k$, then $N_{kF} \prec (M^k)_{kF}$. The kernel is precisely I_F .

This construction begs the following questions: Is π_F onto? If $a \in \text{Image}(\pi_F) \cap E_b(G_{M_F})^+$, does a belong to $\pi_F(E_b(G_M)^+)$? Is I_F generated by its positive elements? If $n = 1$, the answers to all three are affirmative (see [H5]). The answer to the first question is no in general, and the second and third questions are equivalent. As a very simple example for the first question, set

$$M = \begin{bmatrix} 1 & 1 \\ x & 1 \end{bmatrix}.$$

Then $K(M)$ is the interval $[0, 1/2]$; let F be the face $\{0\}$, so that M_F is the identity matrix of size 2. Thus $E_b(G_{M_F}) \cong \mathbf{R} \times \mathbf{R}$. However, it is easy to check that $E_b(G_M) \cong \mathbf{R}[1/(1 + \sqrt{x})]$ as rings, and there is only one map from this to the reals that sends the generator to zero; thus the image of $E_b(G_M)$ can only be one copy of \mathbf{R} . Hence the map π_F is not onto.

Without being able to decide whether the ideals $I_F = \ker \pi_F$ (for each face F of $K(M)$) are order ideals, we can show that there is an order ideal J contained in $I_0 := \bigcap I_F$ (the intersection over all faces F) such that I_0/J is nilpotent and moreover, every pure trace that is not faithful must kill at least one of the I_F . The methods are elementary (if somewhat tedious) in view of earlier results. Recall from [H5] the definition of R_P (which is really just $E_b(G_M)$ for M the 1×1 matrix P with P in A^+), as well as \mathbf{p}_F , the kernel of $R_P \rightarrow R_{P_F}$ with F a face of $\text{cvx Log } P$.

Lemma 3.6 Let P be an element of A^+ , with $K = \text{cvx Log } P$, and assume the origin is contained in $\text{Log } P$ and the interior of K . Set $I = \bigcap_{F \in \{\text{facets of } K\}} \mathbf{p}_F$.

- (a) There exists m such that $I^m \subseteq (1/P)R_P$.
- (b) Suppose that Q belongs to A^+ , $Q \prec P^k$ and Q/P^k belongs to I . For all sufficiently large s , there exists t such that $Q^t \prec P^{tk-s}$.
- (c) For any w in \mathbf{Z}^d , there exists l such that $x^w \prec P^l$.

Proof. (a) The minimal prime ideals that contain $1/P$ are all of the form \mathbf{p}_F where F is a facet [H5]; the hypothesis that 0 is in the interior of K ensures that every such ideal contains $1/P$. Hence every element of $I/(1/P)$ is nil, and noetherianness of R_P yields the m .

(b) It suffices to prove this with $s = 1$, since we may take powers. By (a), we may write $(Q/P)^k = (1/P)r$ where r belongs to R_P^+ . As r is bounded by a multiple of 1, it follows that there exists a positive integer j such that $Q^m P^{j-km} \prec P^{j-1}$. By increasing j , we may assume it is divisible by k , and write $j = kj'$. As $Q \prec P^k$, $Q^{j'} \prec P^j$. Hence $Q^{m+j'} \prec P^{k(m+j')-1}$, as desired.

(c) Straightforward, and already in [H5]. •

Proposition 3.7 Suppose B and M belong to $\mathbf{M}_n A^+$, $B \prec M$ and $Q := \text{tr } B^{n!}$ and $P := \text{tr } M^{n!}$ satisfy the hypotheses of Lemma 3.6. Then for all positive integers t , there exists m such that $B^m \prec M^{m-t}$.

Proof. It obviously suffices to show the inequality for some positive integer t , since we may take powers and exploit $B \prec M$. By Theorem 5.3, there exists j such that $(B^{n!})^j \prec Q(B^{n!})^{j-1}$. We raise this to a very high power, l , which will be determined shortly; we obtain $(B^{n!})^{jl} \prec Q^l (B^{n!})^{(j-1)l}$.

Given w in \mathbf{Z}^d , there exists a depending on w such that $x^{-w} \prec P^a$. Hence if $Q \prec P^{r-s}$, we have $x^w Q \prec P^{r-s+a}$. Then $Q^r \prec x^w P^{r-s+a}$, and this is true for all sufficiently large r (with fixed a and s). For all sufficiently large l , it follows that $Q^l \prec x^w P^{l-s+a}$. Thus

$$(B^{n!})^{jl} \prec x^w P^{l-s+a} (B^{n!})^{(j-1)l} \prec x^w P^{l+s-a} M^{n!(j-1)l}.$$

Abbreviate $l - s + a = t$. By the first paragraph of the proof of Theorem 2.8, there exists w such that $x^w P^t M^{-n!t-k'}$ belongs to $E_b(G_M)$ for some k' , in fact, $k' = n!$ will do but is extravagantly large. In other

words, there exists b such that $x^w P^t M^b \prec M^{n!t+k'+b}$. The a will be determined by this particular choice for w , and then require (on s) that $s - a > 1$. Combining the various inequalities,

$$\begin{aligned} B^{n!jl} &\prec M^{n!(j-1)l} M^{n!(l-s+a)+k'+b} \\ &= M^{n!(l-s+a+(j-1)l)+k'} \\ &= M^{n!(jl-s+a)+k'}, \end{aligned}$$

and since $k' \leq n!$, we are done. •

Let K be a partially ordered abelian group; a *trace* of K is a nonzero positive group homomorphism, $\gamma : K \rightarrow \mathbf{R}$. If there is a distinguished element (such as an order unit) u , we say it is *normalized* (at u) if $\gamma(u) = 1$. A trace γ is *pure* (or *extremal*) if it cannot be written as a positive linear combination of other traces, except in a trivial way; alternatively, if γ is normalized at an order unit, if it cannot be expressed as a nontrivial convex linear combination of other traces. A trace (usually assumed to be pure) is *faithful* if its kernel contains no nonzero positive elements.

Proposition 3.8 Suppose M is a primitive matrix in $\mathbf{M}_n A^+$ such that the origin is interior of $K(M)$.

- (a) If NM^{-k} is an element of $E_b(G_M)$ such that $\pi_F(NM^{-k}) = 0$ for all facets F , then there exists a power of NM^{-k} that belongs to $M^{-c}E_b(G_M)$ for some positive integer c such that M^{-c} is in $E_b(G_M)$.
- (b) If τ is pure trace of $E_b(G_M)$ that is not faithful, then there exists a facet F of $K(M)$ such that $\tau(I_F) = 0$.

Proof. (a) We may obviously replace M by M^k and so assume $k = 1$. We have that $(N_F)(M^{-1})_F = 0$ for all facets F of $K(M)$; set B to be the matrix obtained from N by replacing all coefficients appearing in every polynomial entry of N by 1 (of course, B will not commute with M ; fortunately, this does not matter). There exists a c for which M^{-c} belongs to $E_b(G_M)$ by Theorem 2.8. By the preceding result, $B^m \prec M^{m-c'}$ for all sufficiently large c' for some m , depending on c' ; we may choose c' to be a multiple of c . Then $\pm N^m \prec M^{m-c'}$, whence with respect to $E_b(G_M)$, $(NM^{-1})^m$ is bounded above by a multiple of $M^{-c'}$, and it is easy to check that this forces $(NM^{-1})^m$ to belong to the ideal generated by M^{-c} (which is also an order ideal).

(b) If $\tau(I_F)$ is not zero for all facets F , then τ does not kill their product, which is contained in their intersection. By (a), every element of the intersection raised to some power belongs to $M^{-c}E_b(G_M)$, hence $\tau(M^{-c}) \neq 0$. However, it is a triviality to see that this forces τ to be faithful. •

4. Faithful traces

In this section, we shall determine all the faithful pure traces on $E_b(G_M)$, $E(G_M)$, and G_M , as well as on the order ideals of the latter. We shall also obtain partial results about the other pure traces.

Complete knowledge of the pure traces would enable us to determine the positive cones of the partially ordered groups involved. It is equivalent to describing the boundaries for the corresponding random walks, as discussed in the Introduction. Although the pure trace space for $E_b(G_M)$ is compact, and that for $E(G_M)$ is a natural copy of Euclidean space (when $M(1)$ is primitive), in general, the descriptions are not completely straightforward. There are complications arising from the fact that at some faces, M_F need not be primitive.

In this section, we first describe some faithful, pure traces of $E_b(G_M)$, $E(G_M)$, and G_M , that are naturally obtainable from the limit process,¹ and then show that all are of this form when $\Gamma = \mathbf{Z}^d$ and $M(1)$ is primitive. Subsequent sections will deal with the non-faithful traces, and complete descriptions are possible only in somewhat special cases.

Let $r = (r_i)$ be a point in $(\mathbf{R}^d)^{++}$, and let N be a matrix in $\mathbf{M}_n A$. Let $N(r)$ be the matrix of real numbers obtained by evaluating each entry at r . If M has only entries from A^+ (and our convention is that

this is always the case), obviously $M(r)$ is a nonnegative real matrix. It will be primitive if and only if $M(1)$ is (recall that $M(1)$ is short for $M(1, \dots, 1)$). We obtain a natural positive homomorphism, $G_M \rightarrow G_{M(r)}$, from the following diagram:

$$(r) \quad \begin{array}{ccccccc} A^n & \xrightarrow{M} & A^n & \xrightarrow{M} & A^n & \xrightarrow{M} & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbf{R}^n & \xrightarrow{M(r)} & \mathbf{R}^n & \xrightarrow{M(r)} & \mathbf{R}^n & \xrightarrow{M(r)} & \dots \end{array}$$

The map $A^n \rightarrow \mathbf{R}^n$ sends the column of polynomials $(a_1, \dots, a_n)^T$ to the column of real numbers, $(a_1(r), \dots, a_n(r))^T$. The map on the limit groups is clearly onto, the image of the positive cone of G_M is all of the positive cone of $G_{M(r)}$, and there is a corresponding map, $E(G_M) \rightarrow E(G_{M(r)})$ sending $\hat{N}\hat{M}^{-k}$ to $\widehat{N(r)}\widehat{M(r)}^{-k}$. We can obtain at least one trace on $E(G_M)$ and G_M as follows. Let $v(r)$ be a nonnegative, nonzero left eigenvector for $M(r)$, with eigenvalue b . Of course, if $M(1)$ is primitive, there is only one choice for b , namely $\beta(r)$, and up to scalar multiple, there is only one choice for $v(r)$ and this is strictly positive. We can define a trace, $V(r) : G_M \rightarrow \mathbf{R}$ by defining a trace on $G_{M(r)}$ and composing with the map in (r). The trace on $G_{M(r)}$ is defined via the diagram:

$$\begin{array}{ccccccc} \mathbf{R}^n & \xrightarrow{M(r)} & \mathbf{R}^n & \xrightarrow{M(r)} & \mathbf{R}^n & \xrightarrow{M(r)} & \dots \\ \searrow v(r) & & \downarrow \frac{1}{b}v(r) & & \swarrow \frac{1}{b^2}v(r) & & \\ & & \mathbf{R} & & & & \end{array}$$

Here the row $v(r)$ acts by left multiplication on the columns \mathbf{R}^n . If b has multiplicity 1 as an eigenvalue of $M(r)$, or if $v(r)$ has been chosen so that it is a common eigenvector of the centralizer of M (this can always be arranged), define $\gamma_r : E(G_M) \rightarrow \mathbf{R}$ via $\gamma_r(\hat{N}\hat{M}^{-k}) = \alpha/b^k$, where $v(r)N(r) = \alpha v(r)$. Then γ_r is clearly a trace, and is also multiplicative; despite its notation, it depends on the choice of eigenvector, $v(r)$, unless $M(1)$ primitive.

Proposition 4.1 If $V : G_M \rightarrow \mathbf{R}$ is a trace satisfying $V(e(g)) = \gamma(e)V(g)$ for e in $E(G_M)$, g in G_M , and γ is a multiplicative trace of $E(G_M)$, then there exist r and $v(r)$ such that $\gamma = \gamma_r$ and $V = V(r)$.

Proof. Restricting γ to A (obviously, A is a unital subalgebra of $E(G_M)$, acting by multiples of the identity), we obtain our candidate for r , from $r_i = \gamma(x_i \mathbf{I})$; obviously, if p belongs to A , then $\gamma(p\mathbf{I}) = p(r)$. Now let E_j denote the standard basis elements for A^n . Define $v_j = V([E_j, 1])$ and $b = \gamma(\hat{M})$. Then

$$bv_j = V(\hat{M}[E_j, 1]) = V([ME_j, 1]) = V\left(\left[\sum_i M_{ji}E_j, 1\right]\right) = \sum_i M_{ij}(r)v_i = \sum_i v_i M_{ij}(r);$$

in other words, $v = (v_j)$ is a left eigenvector for $M(r)$ with eigenvalue b ; the entries are all nonnegative as V is nonnegative. Now G_M is spanned as an A -module by $[E_j, k]$, $k = 1, 2, \dots$. As $\hat{M}[E_j, k] = [E_j, k+1]$, G_M is spanned as an $A[\hat{M}]$ -module by $[E_j^1, 1]$. It follows that the trace V is of the form described in (r) (as the trace is nonzero, neither b nor all of v_j can be zero). Let N be a matrix in the centralizer of M ; then the multiplicativity of V and γ permit us to apply the same equations to \hat{N} that we did for \hat{M} ; we thus obtain that v is a common eigenvector for $C_A(M)$, and γ restricts to the corresponding eigenvalue of the matrix evaluated at r . •

Not all pure traces of $E(G_M)$ need be multiplicative—in fact, there need not be any multiplicative ones at all (take $d = 0$, $n > 1$, M the identity matrix of size n ; then $E(G_M) \cong M_n \mathbf{R}$ with the entrywise ordering, so it has no real-valued multiplicative homomorphisms). However, if $M(1)$ is primitive, we show these ‘point evaluation’ traces exhaust the pure rays of traces. The following is a basic result that we use in various forms.

Theorem 4.2 ([H5, I.1]) Let H be a partially ordered abelian group admitting an order unit, and let R be a partially ordered unital ring having 1 as an order unit, and such that H is an ordered R -module. Then for every pure normalized trace V of (H, u) , there exists a unique multiplicative (and pure) trace γ of R such that for all e in R and h in H , $V(eh) = \gamma(e)V(h)$.

Recall that an order-preserving group homomorphism $\phi : H' \rightarrow H$ between two partially ordered abelian groups is said to be an *order-embedding* if ϕ is one to one, and $\phi(h') \geq 0$ implies $h' \geq 0$.

Lemma 4.3 If the matrix M is in $M_n A^+$, then there exists an order ideal H in G_M having an order unit u , such that there is an order-embedding, $\phi : E_b(G_M) \rightarrow \oplus^n H$.

Proof. Form the limit of ordered A -modules,

$$M_n A \xrightarrow{M \times} M_n A \xrightarrow{M \times} M_n A \xrightarrow{M \times} \dots;$$

(we are multiplying on the left). Observing that left multiplication by M acts independently on the columns, we see that as ordered $A[\hat{M}]$ -modules, this decomposes as a direct sum of n copies of G_M , even as $E(G_M)$ -modules. If we replace every occurrence of $M_n A$ by $C_A(M)$, the resulting direct limit is simply $E(G_M)$, and the inclusion (obtained from $C_A(M) \subset M_n A$) is an order-embedding as $E(G_M)$ -modules. Thus we obtain an order-embedding, $\Phi : E(G_M) \rightarrow \oplus^n G_M$; let Φ_i denote the projection to the i -th copy of G_M . Set H_i to be the order ideal of G_M generated by $\Phi_i(\mathbf{1})$ (the image of the identity). Then $H = \sum H_i$ is an order ideal in G_M (a sum of order ideals in a dimension group is an order ideal), and has $\sum \Phi_i(\mathbf{1})$ as its order unit. Set $\phi = \Phi|_{E_b(G_M)}$, and observe that the range of ϕ is contained in $\oplus^n H$; the restriction of an order-embedding is obviously an order-embedding. •

Denote by H_M the order ideal H obtained in Lemma 4.3.

Lemma 4.4 Every pure trace γ of $E_b(G_M)$ is multiplicative and there exists a pure trace V of one of the modules H_i (Lemma 4.3) such that for all e in $E_b(G_M)$ and for all h in H_i , $V(eh) = \gamma(e)V(h)$.

Proof. By Theorem 4.2, every pure trace (normalized at the identity) is multiplicative. By Lemma 4.3, there is an order-embedding $E_b(G_M) \rightarrow \oplus H_i$; by construction, the identity is sent to the natural order unit. By [GH, 1.4], there exists a trace of $\oplus H_i$ extending γ . Since the preimage of a face in the trace space of a partially ordered group is a face, it follows we may assume the extended trace is pure. A pure trace on an ordered direct sum must restrict to zero on all but one summand, and the restriction there must be pure. Hence we obtain a pure trace $V : H_i \rightarrow \mathbf{R}$ for some i such that $V(\pi_i(e)) = \gamma(e)$ for all e in $E_b(G_M)$. By Theorem 4.2, there exists a pure trace of $E_b(G_M)$, γ_0 such that $V(eh) = \gamma_0(e)V(h)$. Applying this with $h = \pi_i(\mathbf{1})$, we deduce $\gamma_0(e) = \gamma(e)$, that is, $\gamma_0 = \gamma$, as desired. •

Now suppose $M(1)$ is primitive, and let e_0 be the positive element obtained in Theorem 2.8. Let γ be any multiplicative trace of $E_b(G_M)$. If $\gamma(e_0) \neq 0$, then γ extends uniquely to a multiplicative trace on $E(G_M)$. Let $H = H_i$ as obtained in Lemma 4.4, and let V be a pure trace of the latter satisfying the properties obtained there. It follows easily from Theorem 2.8 that $G_M = \cup_k e_0^{-k} H_i$. We may extend V to a trace V' on G_M (by defining $V'(e^{-k}h) = V(h)/\gamma(e)^k$) that satisfies all the properties needed for Proposition 4.1 to apply. Hence γ and V are determined by an eigenvector coming from a point evaluation.

Now let γ be an arbitrary pure trace of $E(G_M)$. Its restriction to $E_b(G_M)$ is not zero and is pure by Theorem 2.8 and [HM, I.3]. Hence it is multiplicative.

Theorem 4.5 Suppose that $M(1)$ is primitive. Let γ be either a pure trace of $E(G_M)$ or a pure faithful trace of $E_b(G_M)$. Then γ is obtained from a point evaluation at r and a left nonnegative eigenvector of $M(r)$ as in Proposition 4.1. If V is any pure trace of G_M , then V is also obtainable from a point evaluation, as in Proposition 4.1. All of these pure traces are faithful.

Proof. The first assertion is proved in the preceding paragraphs. To establish the second, restrict V to any order ideal H that has an order unit which is not killed by V (there must exist a positive element that is not in the kernel of V ; let H be the order ideal generated by it). By Theorem 4.2, there exists a pure trace of $E_b(G_M)$, γ with the usual properties. Now the restriction of V to any order ideal containing H is also not zero; it is easy to check that the corresponding pure trace of $E_b(G_M)$ must be the same. Hence we have that for every order ideal of the form $\sum_{\text{finite}} e_0^{-k} H$, the restriction of V is not zero and is affiliated with γ . Suppose $\gamma(e_0) = 0$; from $H = e^k(e_0^{-k} H)$, we deduce that for all h in H , $V(h) = \gamma^k(e)V(e_0^{-k}h) = 0$, a contradiction. Hence γ extends to a pure trace on $E(G_M) = E_b(G_M)[e_0^{-1}]$; it follows easily that V satisfies the hypotheses of Proposition 4.1 and we are done. Faithfulness is an automatic consequence of the primitivity of $M(r)$. •

With some care, one can use similar techniques to prove analogous results even when $M(1)$ is not primitive; in this case, $E(G_M)$ should be replaced by $E_b(G_M)[e_0^{-1}]$ (where $e_0 = x^{-v_0}\hat{M}^k$ for suitable choices of v_0 and k); of course, faithfulness will generally fail. We cannot avoid the non-primitive situation, because in much of what follows, the remaining pure traces of either an order ideal H or of $E_b(G_M)$ itself emanate from the nonnegative left eigenvectors (and their relatives) of the facial matrices, M_F , which need not be primitive.

5. Attack of the perfidious traces

Here we discuss pure traces that are not faithful. A preliminary step concerns extensions of traces from subrings. If S is a partially ordered ring (with 1 in the positive cone) and s is a positive element, we may form the limit ordered ring $S[s^{-1}]$ as $\lim \times s : S \rightarrow S$ with the limit ordering,

$$S[s^{-1}]^+ = \{a/s^k \mid as^l \in S^+ \text{ for some positive integer } l\}.$$

In general, $S[s^{-1}]^+ \cap S \neq S^+$, i.e., the inclusion $S \rightarrow S[s^{-1}]$ need not be an order-embedding.

Let Q be a nonzero element of A^+ . It is easy to see that there is a natural, order-preserving map $E_b(G_M) \subset E_b(G_{QM})$ and we identify the former with its image in the latter (however, the map is not generally an order-embedding, so some care may be necessary). It turns out to be much easier to compute the pure traces on order ideals of $E_b(G_{QM})$ and G_{QM} for the appropriate choices of Q .

Lemma 5.1 Suppose that Q is a nonzero element of A^+ and $\hat{Q}\hat{M}^{-r}$ belongs to $E_b(G_M)$ for some integer r .

- (a) $E_b(G_{QM}) \subseteq E_b(G_M)[(Q/M^r)^{-1}]$ and this inclusion is an order embedding (with the limit ordering on the latter).
- (b) If $\hat{Q}\hat{M}^{-r}$ is an order unit of $E_b(G_{QM})$, then

$$E_b(G_{QM}) = E_b(G_M)[(\hat{Q}\hat{M}^{-r})^{-1}]$$

and the inclusion $E_b(G_M) \subset E_b(G_{QM})$ is an order embedding.

- (c) Suppose that $\hat{Q}\hat{M}^{-r}$ is an order unit of $E_b(G_M)$. For f in A^n , if $(QM)^m f \in (A^n)^+$ for some integer m , then there exists m' so that $M^{m'} f \in (A^n)^+$.

Proof. (a). Select e in $E_b(G_{QM})^+$, write $e = \hat{N}\hat{Q}\hat{M}^{-l}$ with N in $C_A(M)^+$ (note that $C_A(M) = C_A(QM)$) and $N \prec (QM)^l$. From $\hat{Q}\hat{M}^{-r}$ belonging to $E_b(G_M)$, it follows that there exists a positive integer k such

that $QM^k \prec M^{k+r}$. Set $t = (k+r)l$, and define $N' = NM^{t-(r+1)l} = NM^{(k-1)l}$. Clearly N lies in $C_A(M)^+$ and

$$\begin{aligned} N' &\prec (QM)^l M^{t-(r-1)l} = M^{(t-r)l} Q^l \\ &= (QM^k)^l \prec (M^{k+r})^l = M^t. \end{aligned}$$

Hence $f = N' \hat{M}^{-t}$ belongs to $E_b(G_M)^+$ and the formal equation,

$$“N' M^{-t} (Q^{-1} M^r)^l = N (QM)^{-l}”$$

translates to $e = f(\hat{Q}\hat{M}^{-r})^{-l}$. Hence the desired inclusion of rings, and this inclusion is clearly order-preserving. If f belongs to $E_b(G_M)$ and $g = f(\hat{Q}\hat{M}^{-r})^{-l} \in E_b(G_M)[Q/M^r]^+$, there exists an integer m such that $(\hat{Q}\hat{M}^{-r})^m f \in E_b(G_M)^+$, so if additionally g belongs to $E_b(G_{QM})^+$, then g belongs to $E_b(G_{QM})^+$ (as multiplication by QM is an order automorphism of G_{QM}).

(c). It is sufficient to show Qf belongs to $(A^n)^+$ implies $M^m f \in (A^n)^+$. From the proof of (a), we already have that $QM^k \prec M^{k+r}$. As $\hat{Q}\hat{M}^{-r}$ is an order unit, there exists t so that $M^{t+r} \prec QM^t$. Since we may increase either k or t , we may assume that $QM^k \prec M^{k+r} \prec QM^k$. Thus for all i and j , $\text{Log}(M^{k+r})_{ij} = \text{Log} Q + \text{Log}(M^k)_{ij}$. Write $P_{ij} = (M^{k+r})_{ij}$, and set $P'_{ij} = Q(M^k)_{ij}$. Then $\text{Log} P_{ij} = \text{Log} P'_{ij}$. Clearly $P'_{ij} f \in (A^n)^+$ for all i and j . Hence [H5, II.1] yields that there exists $n(i, j)$ such that $P^{n(i, j)} f \in (A^n)^+$. Setting $m' = \max n(i, j)$, we have $P^{m'} f \in (A^n)^+$. Hence $(M^{k+r})^{m'n^2} f \in (A^n)^+$.

(b). The second part is an immediate consequence of (c). Now $M^{t+r} \prec QM^t$ implies $\hat{M}^r \hat{Q}^{-1}$ belongs to $E_b(G_{QM})$, so part (a) yields the desired equality. •

Conditions that guarantee that the conclusion of (c) hold are obtained in the next result. If S is a subset of \mathbf{Z}^d and k is a positive integer, as usual, we define $kS = \left\{ \sum_{i=1}^k s_i \mid s_i \in S \right\}$.

Proposition 5.2 Let M be an element of $M_n A^+$, and let S be a subset of \mathbf{Z}^d such that for every i and j , there exists an integer $k(i, j)$ such that $\text{Log} M_{ij} = k(i, j)S$. Let Q be an element of A^+ such that $\text{Log} Q = S$. Suppose that $f = (f_1, \dots, f_n)^T$ is a column of size n with entries from A and there exists an integer e such that the column $(QM)^e f$ consists of Laurent polynomials with no negative coefficients. Then there exists an integer e' such that the same is true of the column $M^{e'} f$.

Proof. It is sufficient to show that if Qf has no entries with negative coefficients then $M^{e'} f$ has the same property for some e' . We recall a basic result [H5, II.1], a consequence of which is the following: If Q and Q' are elements of A^+ with $k\text{Log} Q = \text{Log} Q'$, and g is an element of A for which there exists an integer s with $Q^s g$ having no negative coefficients, then there exists t such that $(Q')^t g$ has no negative coefficients. The hypothesis on Qf asserts that Qf_r , its r th coordinate, has no negative coefficients for $1 \leq r \leq n$. The ‘ Q' ’ will vary over $\text{Log} M_{ij}$. Applying this result a finite number of times, it follows that there exists an integer N such that for all i, j , and r , for all $N' \geq N$, we have $M_{ij}^{N'} f_r$ has no negative coefficients. Now every entry of $M^{n^2 N}$ will consist of a sum of products of $n^2 N$ terms from among the M_{ij} ’s. Thus each product will have at least one of the M_{ij} ’s repeated at least N times, and so each product will make any of the f_r ’s have no negative coefficients. It follows that for all i, j , and r , $(M^{n^2 N})_{ij} f_r$ will have no negative coefficients, and thus the same will be true of every entry of $M^{n^2 N} f$. •

Later (Theorem 5.6), we will show that if merely $\text{cvx} \text{Log} M_{ij}$ are all equal (that is, for all i and j), then the conclusion applies—the result is that for such a matrix, some power satisfies the hypothesis of the preceding with all $k(i, j)$ being one. However, if $\text{cvx} \text{Log} M_{ij}$ are multiples of a common d -dimensional lattice polytope, the conclusion can fail, even in one variable. (Straightforward examples exist.)

We now study G_{QM} from a different point of view. If Q belongs to A^+ , we may view it as a matrix of size 1, and thus form $E_b(G_Q)$; this is nothing else but R_Q [H5]. In this case, all the pure traces have

been determined, together with the positive cone [H3, Theorem A or B]. For a lattice polytope K and $f = \sum \lambda_w x^w$ in A , define $f_K = \sum_{w \in K} \lambda_w x^w$. For any face F of $\text{cvx Log } Q$, there is an order-preserving onto homomorphism of algebras, $\pi_F : R_Q \rightarrow R_{Q_F}$, given by $\pi_F(f/Q^k) = f_{kF}/Q_F^k$. Moreover, the image of the positive cone of R_Q is all of the positive cone of R_{Q_F} .

Now let M be as usual, that is, an element of $M_n A^+$. There is a natural embedding $R_Q \rightarrow E_b(G_{QM})$, which is a one to one real algebra homomorphism that is order-preserving, as follows. A typical element of R_Q (or R_Q^+) can be written in the form f/Q^k with $\text{Log } f \subseteq Q^k$ (and $f \in A^+$). We observe that $fM^k \prec (MQ)^k$, so for a in A^n , $[a, m]_{MQ} \mapsto [fM^k a, m + k]$ is a well-defined map, and of course, $[QfM^k a, m + k] = [fa, m]$. This defines f/Q as an endomorphism of G_{QM} , and it is obviously in $E_b(G_{QM})$. This assignment is clearly an algebra monomorphism, and equally obviously is order-preserving.

In this generality, the map $R_Q \rightarrow E_b(G_{QM})$ need not be an order embedding, even when n (the size of the matrix M) is 1. For example, if P is an element of A^+ such that $\text{cvx Log } P$ admits a facet that is not parallel to a facet of $\text{cvx Log } Q$, there exists f in A such that Pf has no negative coefficients, while $Q^m f \notin A^+$ for all m . If additionally, $\text{cvx Log } Q$ has dimension d (the number of variables in A), we may also assume that $\text{Log } f \subseteq \text{Log } Q^k$ for some k . Thus $f/Q^k \in R_Q \setminus R_Q^+$ and $f/Q^k \in R_{QP}^+$.

In particular, if H is an order ideal of G_{QM} , it is also an ordered R_Q -module. If H is instead an order ideal of G_M , we may extend (enlarge) it to an order ideal of G_{MQ} , as follows. We note that G_{QM} is naturally isomorphic to $\lim G_M \xrightarrow{\times Q} G_M$, so $E(G_M) \otimes_A A[Q^{-1}] \cong E(G_{QM})$. Since $QH \not\subseteq H$ in general, we must be more careful in dealing with order ideals. The map $\alpha_Q : G_M \rightarrow G_{QM}$ given by $\alpha_Q[a, k]_M = [Q^{k-1}a, k]_{QM}$ describes the natural inclusion. Let H_Q denote the order ideal of G_{QM} generated by $\alpha_Q(H)$. Again there is a natural inclusion $H \rightarrow H_Q$ which is order-preserving but in general is not an order embedding. Obviously every pure trace on H_Q restricts to a trace, in fact a pure trace. The restriction map (traces on H_Q to traces of H) need not be one to one (even when $n = 1$). It can be shown to be onto (i.e., that every trace on H lifts to a trace on H_Q) in many cases; if it could be shown to be onto in all cases, we could use the results on the positive cone in G_{QM} (with $Q = \text{tr } M^{n!}$) to completely determine the positive cone in G_M . Order ideals of G_{QM} that admit order units can be realized as limits of R_Q -modules, and then we use the traces of R_Q to describe all the traces of H_Q and $E_b(G_{QM})$. In other words, we use R_Q as a replacement for the reals.

To begin the process, we wish to have elements of the form M^k/Q^l in $E_b(G_{QM})$, for suitable k and l .

Theorem 5.3 Let M be an element $M_n A^+$, and let P in A^+ be such that $\text{Log } P = \text{Log } \text{tr } M^{n!}$. There exists an integer N so that

$$M^{N+n!} \prec PM^N.$$

In particular, $M^{n!}/P \in E_b(G_{PM})$.

Before proving the result, we require some notation, formalizing what we had before Proposition 2.4, except that now we work with generic matrices, i.e., $M = (x_{ij})$ in the n^2 variables x_{ij} . In other words, we begin with the situation that the weight on the transition $i \rightarrow j$ corresponding to the ij entry of M is the ij -th basis element of \mathbf{Z}^{n^2} . A path of length N will be denoted $p = (n_1, n_2, \dots, n_{N+1})$ where $1 \leq n_j \leq n$ and describes the sequence of transitions, $n_1 \rightarrow n_2 \rightarrow \dots \rightarrow n_{N+1}$. We define the monomial attached to the path, $x^{(p)} = \prod x_{n_i, n_{i+1}}$. There is an obvious equivalence relation on paths of the same length. If in p there exist integers $l' > l > k' > k$ such that $n_k = n_l = i$ and $n_{k'} = n_{l'} = j$, interchange the paths $(n_k, \dots, n_{k'})$ and $(n_l, \dots, n_{l'})$, call the resulting path q ; let the relation $p \sim q$ generate the equivalence relation. This process will be called a ‘‘regrouping’’, so if $p \equiv q$, q is obtainable from p by a sequence of regroupings. Note that if $p \equiv q$, then obviously $x^{(p)} = x^{(q)}$, but the converse does not hold since cyclic permutations are not allowed in this equivalence relation.

Fix n . We shall show there exists an integer (depending only on n) N such that for every path $p = (n_1, n_2, \dots, n_{N+1})$ of length N , there exists a sequence of regroupings of p and j in $\{1, 2, \dots, N - k\}$ such that in the regrouped path, $n_j = n_{j+n!}$.

Lemma 5.4 For any positive integer n , there exists an integer N (depending on n), such that for every path $p = (k_1, k_2, \dots, k_{N+1})$ of length N on n symbols, there exists a regrouping of p and an integer j such that $k_j = k_{j+n!}$.

Proof. Let $\{1, 2, \dots, n\}$ denote the symbols. Fix p of length N , and suppose the conclusion fails for this path. A gap of length s in the occurrences of i describes the following segment of a path, $k_t = i, \dots, k_{t+s} = i$ with $k_{t+j} \neq i$ for $1 \leq j < s$ (that is, a minimal loop with vertex i of length s). We obtain an estimate for the number of occurrences of any one symbol. We note that there can be at most $n! - 1$ gaps of length 1 in p for any particular i (if there were more, we could regroup p to obtain a sequence of $n!$ consecutive occurrences; similarly, there can not be more than $(n! - 1)/2$ gaps of length 2 for any particular i , all the way up to at most $(n! - 1)/n$ gaps of length k . All the remaining gap lengths must be at least $n + 1$. Hence we obtain an upper bound of $(N + 1)/(n + 1) + (n! - 1)(1 + 1/2 + 1/3 + \dots + 1/n)$ for the number of occurrences of i . As there are n symbols and the path length is N , we deduce

$$N + 1 \leq \frac{n(N + 1)}{n + 1} + (n! - 1)n(\ln(n + 1)).$$

Hence $N + 1 \leq (n! - 1)n(n + 1)(\ln(n + 1))$. In particular, if we choose $N + 1 > (n + 2)! \ln(n + 1)$, we are done. •

The estimate on N obtained in the preceding is extremely crude. To begin with, $n!$ may be replaced by l.c.m. $\{2, 3, \dots, n\}$. Even then there is a lot of looseness in the argument.

Proof of Theorem 5.3: First we prove this in the case of the generic matrix, with $A = \mathbf{R}[x_{ij}]$ and $M = (x_{ij})$. A monomial appearing in M_{ij} will be of the form $x^{(p)}$ for a path on n symbols $p = (m_1, m_2, m_{N+1})$, with $m_1 = i$ and $m_{N+1} = j$. By the preceding, up to a regrouping (which does not affect the monomial), there is a subpath of length $n!$, $p' = (m_t, \dots, m_{t+n!})$, with $m_t = m_{t+n!}$. Form the path of length $N - n!$, p'' , by deleting $(m_{t+1}, \dots, m_{t+n!})$. Clearly, $x^{(p'')} \cdot x^{(p')} = x^{(p)}$. Now $x^{(p')}$ appears in $\text{tr } M^{n!}$ and $x^{(p'')}$ appears in $M^{N-n!}$, so $x^{(p)}$ appears in $PM^{N-n!}$.

Now let M be an arbitrary matrix in $M_n A^+$. Say x^w appears in M^N . Then we may write $w = w_{i_1 i_2} + w_{i_2 i_3} + \dots + w_{i_N j}$, with w_{st} in \mathbf{Z}^d belonging to $\text{Log } M_{st}$. Consider the path $p = (i, i_2, \dots, i_N, j)$. There exists a subloop $i_t, \dots, i_{t+n!}$ in (up to a regrouping) and we note that $\sum_{s=0}^{n!-1} w_{i_{t+1+s}, i_{t+s+2}}$. Now we observe that the sum of the w 's remaining belongs to $\text{Log } M^{N-n!}$, so the sum of the two belongs to $(\text{Log } M^N)_{ij}$. •

Despite its plausibility, for $P = \text{tr } M^{n!}$, it is not true that the element $M^{n!}/P$ of $E_b(G_{PM})$ is an order unit thereof.

If in Theorem 5.3, $P = Q^{n!}$ for some Q in A^+ , it is not generally true that M/Q itself belongs to $E_b(G_{QM})$, even though all powers beyond the $n!$ do belong.

Example 5.5 A matrix M in $M_3 A^+$ ($d = 9$) such that the element M^6/P ($P = \text{tr } M^6$) of $E_b(G_{PM})$ is *not* an order unit thereof.

Let $M = (x_{ij})$ be the generic matrix in the 9 variables $\{x_{ij}\}$. Then among the monomials in P are x_{22}^6 . It is clearly sufficient to show that for all positive integers $s < k$ and j , $\text{Log } (P^k M^j)_{11} \not\subseteq \text{Log } (P^{k-s} M^{j+6s})$ (the desired conclusion follows on setting $s = 1$). One of the monomials appearing in $(P^k M^j)_{11}$ is $x_{22}^{6k} x_{11}^j$. In order to obtain a monomial involving only x_{11} and x_{22} (and no other variables) in $P^{k-s} M^{j+6s}$, we must have that all the x_{22} terms must come from the power of P (else an x_{12} or x_{13} "transition" term would appear, arising from M). Hence $6(k - s)$ is the largest exponent possible for x_{22} in such a monomial. As $s > 0$, the non-inclusion is verified. Obviously we could do this with fewer variables.

The FOG examples (2.3) also have this property. We shall see later that failure of M/P or any of its powers to be an order unit in the relevant bounded subring is generic. •

Now that we have a special endomorphism, the next stage would be to find all the pure traces of $E_b(G_{PM})$ and the order ideals of G_{PM} . This we shall do in the next section. For the remainder of this

section, we analyze in detail a class of matrices, those satisfying (**). We determine the pure traces, and precise criteria for eventual positivity. A matrix M in $M_n \mathbf{R}[x_i^{\pm 1}]^+$ satisfies condition (**) if for some integer K , for all i and j , $\text{cvx Log}(M^K)_{ij}$ are equal. We show that this implies a formally stronger condition.

Theorem 5.6 If the matrix M in $M_n \mathbf{R}[x_i^{\pm 1}]^+$ satisfies condition (**), then for each sufficiently large integer U , the n^2 sets of lattice points, $\text{Log}(M^U)_{ij}$ are all equal.

We first require a combinatorial “absorption” result, extending slightly [H8, Lemma 2.1]. Recall that if A and B are subsets of \mathbf{Z}^d (or any abelian group), then for positive integers p and q , $pA + qB$ denotes the set

$$\left\{ \sum_{i=1}^p a_i + \sum_{j=1}^q b_j \mid a_i \in A, b_j \in B \right\}.$$

Lemma 5.7 (Absorption) Let m and n be positive integers, and let S and T be finite subsets of \mathbf{Z}^d such that $mT \subseteq nS$ and $m \text{cvx} T = n \text{cvx} S$. Then there exists an integer D such that for all $D' \geq D$, $D'S + mT = (D' + n)S$. Moreover, if k is any positive integer, then $D'S + mkT = (D' + nk)S$. Finally, if we assume merely that $m \text{cvx} T = n \text{cvx} S$ (and not $mT \subseteq nS$), then $(D' + nk)S \subseteq D'S + mkT$.

Proof. Choose P in $\mathbf{R}[x_i^{\pm 1}]^+$ such that $\text{Log} P = S$, and let Q in $\mathbf{R}[x_i^{\pm 1}]^+$ be such that $\text{Log} Q = mT$. Then the inclusion yields that Q/P^n is an element of R_P , and as the coefficients of Q are all nonnegative, it belongs to R_P^+ . By (e.g.) [HMem; V], the equality of the convex sets guarantees that it is an order unit of R_P . From the definition of order unit, there exists an integer D such that $\text{Log} P^D Q = \text{Log} P^{D+n}$, and this obviously holds if D is replaced by D' . Since all the coefficients are positive, the desired equality of sets results (with $k = 1$). If $k > 1$, we simply note that $D'S + 2mT = (D'S + mT) + (mT) = (D' + n)S + mT = (D' + 2n)S$, etc. The final statement comes from restricting to the subset of mT consisting of the extreme points of $\text{cvx} mT$.

Proof of Theorem 5.6. In the cases that $m = n$, we see that for m fixed, there is a unique minimal subset T satisfying the hypotheses, namely the set of extreme points of $m \text{cvx} S$. This means that the integer D can be made to depend on m alone, not on the particular choice of T . In the case of interest, S or S_{ij} will be $\text{Log} M_{ij}$, and we may assume (by (**), and replacing M by a power of itself) that $d_e \text{cvx} S_{ij} := T$ is contained in the Log set of each entry of M . Hence there exists an integer D , which we rename N such that for all i and j , $\text{Log} P^N M_{ij} = \text{Log} P^{N+1}$.

Set $m = 2(N-1)(n^2-1) + N$, and choose $U \geq n^2 m$. We will show that $\text{Log}(M^{U+1})_{ij}$ is independent of the choice of i and j . For an integer K , let $p := \{p(i, j)\}$ be a partition of K by n^2 nonnegative integers, indexed by $\{(i, j)\}$, that is, $K = \sum_{i,j=1}^n p(i, j)$. Let $\mathcal{P}(K)$ denote the set of such partitions of K . If $\{S_{ij}\}$ is a collection of finite subsets of \mathbf{Z}^d , then corresponding to the partition p , we may construct the set obtained as the sum of sets,

$$S(p) := \sum_{i,j} p(i, j) S_{ij}.$$

Now let $q = (i(1), i(2), \dots, i(K+1))$ be an ordered $K+1$ -tuple of integers in $\{1, 2, \dots, n\}$. This will be used to denote a path of length K from the symbol $i(1)$ to $i(K+1)$ by means of the sequence of transitions, $(i(1), i(2)), (i(2), i(3)), \dots, (i(K), i(K+1))$. From the path q , we similarly obtain a set,

$$S(q) := \sum_{t=1}^K S_{i(t), i(t+1)}.$$

In what follows, S_{ij} will be $\text{Log} M_{ij}$, and we shall show that for our choice of $U = K$ (on the order of

n^4N or bigger),

$$(1) \quad \bigcup_{q \in \{1,2,\dots,n\}^{U+1}} S(q) = \bigcup_{p \in \mathcal{P}(U)} S(p).$$

Fix a pair of integers $i(1)$ and $i(U+3)$, and let $\mathcal{Q}(U+2)$ denote the set of paths of length $U+2$ starting at $i(1)$ and terminating at $i(U+3)$. We note immediately that the left side of (1) is automatically contained in the right side. To prove the reverse inclusion, we select a partition of U by n^2 terms, $p \equiv p(i, j)$. As $U \geq n^2m$, there exists a pair $(i(0), j(0))$ such that $p(i(0), j(0)) \geq m$.

Define the set $D = \{(i, j) \mid p(i, j) \leq 2N\} \setminus \{(i(0), j(0))\}$, and consider the set

$$\mathcal{T} := p(i(0), j(0))S_{(i(0), j(0))} + \sum_D p(i, j)S_{(i, j)}.$$

We show that \mathcal{T} is contained in a set arising from a path of length $K := p(i(0), j(0)) + \sum_D p(i, j)$ and a partition of K of size n^2 . Of course, there has been a standing assumption that S_{ij} are all nonempty. We contemplate the set

$$(*) \quad \sum_{(i, j) \in D} p(i, j) (S_{(i(0), j)} + S_{(i, j)} + S_{(j, j(0))}) + \left(p(i(0), j(0)) - 2 \sum_D p(i, j) \right) S_{(i(0), j(0))}.$$

We remark that $\sum_D p(i, j) \leq (n^2 - 1)2N$, so that the coefficient of $S_{(i(0), j(0))}$ in (*) is at least N . Hence the absorption property described in the Lemma yields that the set in (*) contains \mathcal{T} —observe that the set $S_{(i(0), j)} + S_{(j, j(0))}$ is “absorbed” (in the sense of the final statement of Theorem 5.6) by $NS_{(i(0), j(0))}$. Note that the expression in (*) is obtainable from a path of length K .

Now let $E = \{(i, j) \mid p(i, j) \geq 2N + 1\} \setminus (i(0), j(0))$. For (i', j') belonging to E , we consider,

$$\begin{aligned} & S_{(i(0), j(0))} + S_{j(0), i'} + S_{i', j'} + S_{j', i'} + (S_{i', j'} + S_{j', i'}) + \dots \\ & + (S_{i', j'} + S_{j', i'}) + \begin{cases} S_{i', j'} \\ S_{j', i'} \end{cases} + S_{\bullet, j(0)} + S_{i(0), j(0)} \end{aligned}$$

having $\lceil [p(i', j') + 1]/2 \rceil$ occurrences of $S_{i', j'}$. The former exceeds N , and we conclude from the absorption result that this set contains $2S_{i(0), j(0)} + p(i', j')S_{i', j'}$. By concatenation (using \mathcal{T} and the set E —note that D may be empty, but this poses no problem), we conclude that the set constructed from any partition is contained in one obtained from a path of the same length as the partition, at least if the path length is sufficiently large.

Now we note that the right side of (1) can be re-expressed as $U(S_{ij})$ (as follows easily from the absorption lemma, and the fact that it cannot be any larger). Next we show that

$$\bigcup_{q \in \mathcal{Q}(U+2)} S(q) = \bigcup_{p \in \mathcal{P}(U+2)} S(p).$$

Let $W = d_e \text{Log } M_{ij} = d_e S_{ij}$ (for all i and j). The left side contains $W + (\bigcup S(q)) + W$, where the union is over all paths of length U . This is of course equal to $2W + U(\cup S_{ij})$, and by the absorption lemma, this equals $(U+2)(\cup S_{ij})$, and this equals $\bigcup_{p \in \mathcal{P}(U+2)} S(p)$. We thereby obtain

$$\bigcup_{q \in \mathcal{Q}(U+2)} S(q) = \bigcup_{p \in \mathcal{P}(U+2)} S(p) = (U+2)(\cup S_{ij}).$$

We conclude (on translating back to the matrix M) that if $U \geq n^2m$, then

$$\text{Log} (M^{U+2})_{i(1),i(U+2)} = \bigcup_{q \in \{1,2,\dots,n\}^{U+3}} S_{(q)} = (L+2) \left(\bigcup_{i,j} \text{Log} M_{ij} \right).$$

Since $i(1)$ and $i(L+2)$ are arbitrary, we deduce that $\text{Log} (M^{U+2})_{ij}$ is independent of the choice of i and j . •

Corollary 5.8 Let M and X be commuting members of $M_n A^+$.

- (a) If for some k , all the entries of M^k have the same Log set as each other, then the same is true of $M^k X$.
- (b) If for some k , all the entries of M^k have the same Newton polytope, the same is true of $M^k X$.

Proof. (a) Set $S = \text{Log} (M^k)_{ij}$ (this is independent of the choice of i and j , by hypothesis). Then $\text{Log} (M^k X)_{rc} = \cup_{t=1}^n (S + \text{Log} X_{tc})$; in other words, $\text{Log} (M^k X)_{rc} = S + \text{Log} \sum_t X_{tc}$ (the latter term is the Log set of the column sum), which is clearly independent of the row r . On the other hand, $\text{Log} (X M^k)_{rc}$ is independent of the column c , and since $X M^k = M^k X$, $\text{Log} (M^k X)_{rc}$ is independent of both r and c .

(b) The obvious parallel argument works. •

Corollary 5.9 If M satisfies (**), then for all integers l , for all w in $\text{Log tr } M^l$, (w, l) belongs to $T(M)$ (i.e., $x^w M^{-l}$ belongs to $E_b(G_M)$).

Proof. Without loss of generality, we may assume that $w = \mathbf{0}$. By Theorem 5.6, there exists k such that S defined as $\text{Log} (M^k)_{ij}$ is independent of the choice of i and j . There exists a state i such that $\mathbf{0}$ appears in $\text{Log} (M^l)_{ii}$ (by hypothesis). Thus $S + \mathbf{0} \subseteq \text{Log} (M^{k+l})_{ii}$. By Corollary 5.8(a) (or the “sufficiently large” part of Theorem 5.6, with a possibly larger choice for k), $\text{Log} (M^{k+l})_{ij}$ is independent of i and j , and thus $S \subseteq \text{Log} (M^{k+l})_{ij}$ for all i and j . Hence $M^k \prec M^{k+l}$, so that M^{-l} belongs to $E_b(G_M)$, as desired. •

If M satisfies (**), it is easy to see that for every face F , M_F is primitive, in particular, there is exactly one primitive block per face. It is conceivable that this latter property would be sufficient for shift equivalence to a matrix satisfying (**), but this is not the case. Here is a simple example illustrating this phenomenon.

Example 5.10 A primitive matrix M in $M_2 A^+$ with $d = 1$ such that for all faces F of $K(M)$, M_F has exactly one primitive block, but M is not shift equivalent to a matrix satisfying (**).

Set

$$M = \begin{bmatrix} 1 & x \\ 1 & x^2 \end{bmatrix}.$$

At the two zero-dimensional faces, M_F is a direct sum of the zero matrix with a nonzero size one matrix, so there is exactly one primitive block per face. We notice that the monomials that appear $(M^k)_{22}$ include $\{x^{2k}, x^{2k-3}, x^{2k-5}, x^{2k-6}\}$ but exclude $\{x^{2k-1}, x^{2k-2}, x^{2k-4}\}$ (this follows easily by considering the graph formulation of M). Hence $(M^{k+2})_{2,2}$ does not contain x^{2k} ; this means that for all k , $M^k \not\prec M^{k+2}$. Hence $(0, 2)$ does not belong to $T(M)$ (and all the more so $(0, 1)$ since $T(M)$ is an additive semigroup). However, $\text{Log tr } M$ includes 0, so as $\text{tr } M$ and $T(M)$ are shift invariants, no matrix shift equivalent to M can satisfy (**) by Corollary 5.9. 1

(According to Proposition 2.9, $(0, l)$ belongs to $T(M)$ for sufficiently large l ; here $l \geq 4$). •

There is a more drastic example already in the literature. We require an elementary preliminary result (likely a special case of a result asserting that if M and M' satisfy (**) and are algebraically shift equivalent, then they are shift equivalent; this is currently unknown).

We say two matrices, M and M' , are *Log P -shift equivalent*, if there exists a polynomial q in A^+ such that for some m , $\text{Log } q = \text{Log } P^m$, and the matrices qM and qM' are shift equivalent. The usual choice for P here will be $\text{tr } M^{n!}$ where n is the size of M (smaller powers might also be useful).

Lemma 5.11 Suppose M is an $n \times n$ matrix with entries from A^+ and satisfies (**). If additionally, M has zero as an eigenvalue of multiplicity $n - 1$, then M is Log P -shift equivalent to a polynomial, where $P = \text{tr } M$.

Proof. The hypothesis ensures that the characteristic polynomial of M is $(\lambda - P)\lambda^{n-1}$, so that for $n' \geq n$, the matrix $M^{n'}$ is rank one, and of course $P^{n'}$ is the only nonzero eigenvalue. By Theorem 5.6, there exists a power of M , say M^N with $N \geq n$ such that all the Log sets of the entries of M^N are equal, and since the trace is the only eigenvalue, this Log set must be $\text{Log } P^N$. Now select any row of M^N , say the first, call it $v = (q_1, \dots, q_n)$; we have that $\text{Log } q_i = \text{Log } P^N$ for all i . Now define a column $w = (f_i) \text{Tr}$ with entries rational functions, as follows. First $f_1 = 1$, and in general, f_i will be the i 1-entry of M^N divided by q_i . We note that v is a left eigenvector for M^N , and it is easy to see that because the latter is rank one, we have $M^N = wv$. To clear the denominators of the entries of w , multiply through by $q := \prod q_i$. Then $\text{Log } q = n \text{Log } q_1 = nN \text{Log } P$. Moreover, both $W := qw$ and v all of their entries in A^+ , and their product is qM . Hence qM is shift equivalent to the polynomial Wv , which of course equals qP^N . •

Example 5.12 A primitive matrix M in $\mathbf{M}_2 A^+$ with $d = 1$ such that 0 is an eigenvalue and $T(M) = T([\text{tr } M])$ is maximal, but M is not even Log $\text{tr } M$ -shift equivalent to a matrix satisfying (**).

Proof. We take the matrix given in [H4, p. 10] (there labelled A),

$$M = \begin{pmatrix} x^3 + 4x + 5 & x^3 + 2x^2 + 2x + 5 \\ 11x + 33 & 11x + 33 \end{pmatrix}$$

This has the property that $M = wv$, where $w = (x^2 - x + 5, 11) \text{Tr}$ and $v = (x + 1, x + 3)$, so that 0 is an eigenvalue. Here $\beta = \text{tr } M = (x + 2)(x^2 - 2x + 38)$. The argument in op. cit. shows (among other things) that M is not shift equivalent to the polynomial β (the only polynomial to which it could be shift equivalent). In fact, all that is necessary for the same argument to work for qM (with some polynomial q with positive coefficients) is that q have a gap at the second highest coefficient (i.e., $q = x^r + ax^{r-2} + \dots$; here β has a similar gap). It then follows from Lemma 5.11 that M cannot be even β -shift equivalent to a matrix satisfying (**).

We can also calculate the invariant $T(M)$ in this example. We observe that $M^2 = \beta M$, and it easily follows that the pair (a, t) (with a an integer since $d = 1$, and t a positive integer) belongs to $T(M)$ if and only if a belongs to $\text{Log } \beta^t$ —but this simply says that $T(M)$ and $T(\beta)$ (considering β as a size one matrix) are identical. In other words, in contrast to Example 5.10, M cannot be distinguished from the shift equivalence classes of polynomials by $T(M)$. •

Now we proceed to obtain complete descriptions of the positive cone and the extremal traces corresponding to any order ideal, in the case that (**) holds.

Let P be an element of A^+ such that $\text{Log } M \subseteq \text{Log } P$ (in what follows, we may replace M by a power of itself, if necessary). Then we may regard M/P as an element of $\mathbf{M}_n R_P^+$, and the direct limit

$$G_{M/P} = \varinjlim R_P^n \xrightarrow{\times M/P} R_P^n \xrightarrow{\times M/P} \dots$$

becomes an ordered R_P -module. There is a natural bijection between the ordered isomorphism classes of order ideals of $G_{M/P}$ that have an order unit and those of G_{MP} (except under further hypotheses on M , not all order ideals of the former admit order units, even though R_P is noetherian). More generally, there is a

natural map, $G_{M/P} \rightarrow G_{PM}$, given by $[a/P^j, k]_{M/P} \mapsto [M^j a, j + k]$ for $a \in A^n$. It is easy to check that this is well-defined, one to one, and an order-embedding; moreover, it preserves the R_P action and changes the action of multiplication by M/P to multiplication by M/P . Noting that multiplication by P is an order isomorphism of G_{PM} and the union of the ranges of the image under powers of P is all of G_{PM} , we see that solving (\mathfrak{S}) for MP is equivalent to solving the eventual positivity problem for G_{PM} .

We may form the endomorphism ring of $G_{M/P}$, and its bounded endomorphism ring. The latter is immediately seen to be naturally isomorphic (in all respects) to $E_b(G_{PM})$ (on the other hand, it may happen that the endomorphism ring of $G_{M/P}$ is the bounded subring).

An order ideal of $G_{M/P}$ is called *uniform*, if it can be written (up to order-isomorphism, as an $R_P[\widehat{M/P}]$ -module) as the direct limit,

$$G_{M/P,I} := I^n \xrightarrow{M/P} I^n \xrightarrow{M/P} I^n \xrightarrow{M/P} \dots,$$

where I is an order ideal of R_P . Recall ([G]), that an element a of a partially ordered abelian group G is an *order unit* if for all g in G there exists a positive integer n such that $g \leq na$. For $G = R_P$, this is the same as $1 \leq na$, or equivalently, $L(a) > 0$ for all pure traces of R_P .

Let $r_{k,i}$ denote the sum of the i th row of $(M/P)^k$, and as usual, $\langle s \rangle$ will denote the order ideal of R_P generated by s therein.

Proposition 5.13 Assume M/P belongs to $\mathbf{M}_n R_P^+$. If $G_{M/P}$ admits an order unit, then the order ideals generated (in R_P) by the row sums of powers of M/P eventually stabilize; that is, for all sufficiently large l ,

$$(\dagger) \quad \langle r_{l,i} \rangle = \langle r_{l+1,i} \rangle \quad \text{for all } i$$

Conversely, if (\dagger) holds, every uniform order ideal of $G_{M/P}$ admits an order unit.

Proof. Since every entry of M/P belongs to R_P , it follows that for all k , $\langle r_{k+1,i} \rangle \subseteq \langle r_{k,i} \rangle$. Say the order unit of $G_{M/P}$ is of the form $W = [(u_1, \dots, u_n)^T, k]$ for u_i in R_P and k a positive integer. Since anything bigger than an order unit is an order unit, we may assume u_i are themselves order units of R_P and all equal to 1. As W is an order unit, $[(1, 1, \dots, 1)^T, k + 1] \leq KW$ for some positive integer K . Hence there exists l such that $(M/P)^l (1, \dots, 1)^T \leq K(M/P)^{l+1} (1, \dots, 1)^T$. Note that this still holds if l is increased (permitting K to increase). Taking each entry, we obtain $r_{l,i} \leq K r_{l+1,i}$. Hence $\langle r_{l,i} \rangle \subseteq \langle r_{l+1,i} \rangle$, yielding equality for all larger l .

Suppose (\dagger) holds; let I be an order ideal of R_P and form the corresponding uniform order ideal, $G_{M/P,I}$, of $G_{M/P}$. Since R_P is noetherian, every order ideal admits an order unit (this is in [H5], but follows from the fact an order ideal in a partially ordered ring having 1 as an order unit is an ideal in the ring); let u be an order unit of I . If k is any integer such that (\dagger) holds for all $l \geq k$, we shall show that $U = [(u, u, \dots, u), k]$ is an order unit of $G_{M/P,I}$. Now (\dagger) entails that $(M/P)^k (1, \dots, 1)^T \leq K(M/P)^{k+1} (1, \dots, 1)^T$ (for some K); multiplying this by u yields that $[(u, u, \dots, u), k + 1] \leq KU$. This iterates and shows that U is indeed an order unit for $G_{M/P,I}$. •

If $P\widehat{M}^{-1}$ is in $E_b(G_M)$ (equivalently, M/P is an order unit of G_M), then (\dagger) holds; it also holds if every row sum of M/P is itself an order unit. To see this, it is clearly enough to prove that the order ideal, \mathcal{I} , in I^n generated by $(M/P)I^n$ is I^n itself. Every order ideal of I^n is of the form $\oplus J_i$. If $\mathcal{I} = \oplus J_i$ with $J_t \neq I$ for some t , there exists a pure trace γ of I such that $\gamma(J_t) = 0$ (since I/J_t is a nonzero dimension group with order unit). There is a pure trace L of R_P such that $\gamma(rs) = L(r)\gamma(s)$ for r in R_P and s in I . However, J_t is the order ideal generated by $(M/P)_{tj}I$ (as j varies). Applying γ to all of these and knowing they must all vanish, we deduce $L((M/P)_{tj}) = 0$ for all j . Hence the t th row sum vanishes at L so cannot be an order unit, a contradiction.

Conversely, if not all row sums are order units, there exists a pure trace L of R_P that wipes out the t th row for some t (if a positive element is not an order unit, it is killed by some pure trace). By [H5, V.3, p. 45], there exists a pure trace, γ , of the order ideal I , such that $\gamma(rs) = L(r)\gamma(s)$. Then γ applied to the t th coordinates of $(M/P)^k(I)^n$ will be zero for all k . This does not prevent $G_{M/P,I}$ from having an order unit (because although $u(1, \dots, 1)^T$ will not be dominated by a multiple, $(M/P)u(1, \dots, 1)^T$, applying higher powers of M/P to both sides may change this). For example, set $M = \begin{bmatrix} 1 & 2 \\ 3x & 4x \end{bmatrix}$ and $P = 1 + x$. Neither row sum is an order unit (and of course this persists in all powers), but we note that the order ideal of R_P generated by the top row of $(M/P)^k$ is the ideal $(1/P)R_P$ and that generated by the bottom row is $(x/P)R_P$ for any k ; hence the conditions of Proposition 5.13 are satisfied without any row sums being order units. It may be that this example behaves very nicely because if we change the coefficients (but not the monomials), we can transform the matrix into $\begin{bmatrix} 1 & 1 \\ x & x \end{bmatrix}$ which is lag-1 related to a 1×1 matrix, which means that all possible properties are satisfied (see section 11).

Call a column U in $(\mathbf{R}[x_i^{\pm 1}]^+)^n$ homogeneous if $\text{Log } U_i$ are all equal. Suppose that for infinitely many m , $\text{Log } (M^m U)_i$ is independent of the choice of i in $\{1, 2, \dots, n\}$. This is analogous to a uniform order ideal admitting an order unit. Note that when this happens, we may take $m = kn!$ and then it seems likely that $\text{Log } (M^{kn!} U)_i + \text{Log } \text{tr } M^{n!} = \text{Log } (M^{(k+1)n!} U)_i$ for each i and sufficiently large k .

Clearly, it is necessary that the sum of all the coefficients of M/P be an order unit, but this is far from sufficient. Note that one consequence of the stronger condition that all row sums be order units is that no row of $L(M/P)$ (or of a power) can vanish identically for some pure trace L of R_P . Conversely, if no row of $L(M/P)$ vanishes merely for each pure trace L corresponding to extreme points of $\text{cvx } \text{Log } P$ (there are only finitely many of them), then each row sum of all the powers are order units. Simply note that by [HM, V.3(d), p. 47], a positive element of R_P (in this case the relevant row sum) is an order unit if it is strictly positive at this finite set of pure traces.

In particular, the condition (**) is much stronger than (†). The condition (†) may fail, however, for essentially trivial reasons. It is unknown at present if sufficient for (†) to hold is that *there exist* a nonzero uniform order ideal of $G_{M/P}$ that admits an order unit.

The pure traces on a uniform order ideal of $G_{M/P}$ are relatively easy to describe. Begin with a pure trace of γ of the order ideal I . Associated to γ is a (unique) pure trace L of R_P (hence L is a multiplicative positive function) such that for all s in R_P and t in I , $\gamma(st) = L(s)\gamma(t)$ (and either L restricts to a scalar multiple of γ , or $L(I) = 0$) [HM, Lemma I.2(c), p. 4]. The matrix $L(M/P)$ obtained by evaluating every entry of M/P at L is a nonnegative real square matrix (it need not be irreducible). By raising it to the $n!$ power, we may assume all irreducible blocks are primitive. The Perron-Frobenius theorem applies and it has at least one nonnegative left eigenvector. Let v be a extremal left nonnegative eigenvector (that is, not a nonnegative combination of other nonnegative left eigenvectors, except in a trivial way) whose eigenvalue is not zero. This is supported on a union of the supports of primitive blocks, and the corresponding cut down square matrix is called indecomposable (e.g., $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ has two extremal nonnegative left eigenvectors $(1, 1)$ and $(0, 1)$ —the former is supported on both blocks) and denote the eigenvalue λ . We construct a trace $\tau \equiv \tau_{\gamma,v}$ by means of the following limit diagram:

$$\begin{array}{ccccccc}
 I^n & \xrightarrow{M/P} & I^n & \xrightarrow{M/P} & I^n & \xrightarrow{M/P} & \dots, \\
 \gamma^n \downarrow & & \gamma^n \downarrow & & \gamma^n \downarrow & & \\
 \mathbf{R}^n & \xrightarrow{L(M/P)} & \mathbf{R}^n & \xrightarrow{L(M/P)} & \mathbf{R}^n & \xrightarrow{L(M/P)} & \dots, \\
 v \searrow & & \frac{v}{\lambda} \downarrow & & \frac{v}{\lambda^2} \swarrow & & \\
 & & \mathbf{R} & & & &
 \end{array}
 \tag{5.1}$$

In other words,

$$\tau[(a_1, a_2, \dots, a_n)^T, k] = \frac{v}{\lambda^{k-1}} \cdot (\gamma(a_1), \gamma(a_2), \dots, \gamma(a_n))^T.$$

Notice that if L is a point evaluation trace on R_P , then γ must be its restriction to I (up to multiplication by a scalar), and our hypotheses on M/P yield that $L(M/P) = (M/P)(r_i)$ (evaluation a strictly positive d -tuple) is a primitive matrix, hence the “ v ” is unique. Thus τ is merely point evaluation followed by hitting with a strictly positive left eigenvector; such τ will also be called point evaluations.

At the other extreme, occurring when γ is not a point evaluation (and kills I^2), there can be many extremal nonnegative left eigenvectors.

Theorem 5.14 Let $G_{M/P,I}$ be a uniform order ideal of $G_{M/P}$ that admits an order unit. Then the pure traces of it are precisely of the form $\tau \equiv \tau_{\gamma,v}$, every such is pure, and every γ will yield at least one such τ . A pure trace is a point evaluation if and only if $\gamma = L|I$ (up to scalar multiple).

Proof. The diagram above commutes, and it follows that any such τ is a trace (even if v is merely a nonnegative eigenvector, not just one obtained from an irreducible block—only the latter yield extreme traces). First, we show that each such trace is extreme. We use the purity criterion given in [GH2, Theorem 3.1]: If a and b are positive elements of a dimension group G (with order unit, although this is not essential), and ρ is a trace thereof, then ρ is pure (i.e., extreme) if and only if for all $\epsilon > 0$, there exists z in G^+ such that $z \leq a, b$ and $\tau(z) \geq \min\{\tau(a), \tau(b)\} - \epsilon$.

For $G \equiv G_{M/P,I}$ and $\rho \equiv \tau_{\gamma,v}$, it is easy to see that we may reduce to the case that $a = [(A_1, \dots, A_n)^T, k]$ and $b = [(B_1, \dots, B_n)^T, k]$, with A_i and B_j being positive elements of I . Set $A = (A_1, \dots, A_n)^T$, and $B = (B_1, \dots, B_n)^T$. As usual, λ will denote the spectral radius of the relevant primitive block of $L(M/P)$. We may also assume that the sum of the coefficients of v is 1.

If c is a square ($n \times n$) matrix or a column of size n , denote its cut-down to the first m indices, $c_{(m)}$. For convenience, we may reorder the index set $\{1, 2, \dots, n\}$ and so assume that the indecomposable block of $L(M/P)$ emanates from which v emanates is supported on indices $\{1, \dots, m\}$; the primitive block with eigenvalue λ in the support of v (there is only one) is the support of the right eigenvector of $L(M/P)_{(m)}$ with eigenvalue λ , and we may assume the support is $\{1, 2, \dots, m'\}$ for some $m' \leq m$; we denote this right eigenvector w . In particular, $v_i > 0$ if and only if $i \leq m$ and $w_i > 0$ if and only if $i \leq m'$, and note that normalized powers of $L(M/P)_{(m)}$ converge to the matrix $wv_{(m)}/v_{(m)} \cdot w$. Also for convenience, $\gamma^n(A) := (\gamma(A_1), \dots, \gamma(A_n))^T$ in \mathbf{R}^n will be denoted $\gamma(A)$.

Suppose $\tau(a) \leq \tau(b)$. If $\tau(a) = 0$, set $z = 0$ and we are done. Otherwise, since the ordered abelian groups with which we are dealing are real vector spaces, we may assume $\tau(a) = \tau(b) = 1$, so that $v_{(m)} \cdot \gamma(A)_{(m)} > 0$. Normalize w so that $v_{(m)} \cdot w = 1$. Thus $(L(M/P)_{(m)})^l/\lambda^l$ converges to wv . Hence $((L(M/P)_{(m)})^l/\lambda^l)C$ converges to $(v \cdot C)w$ for C an m -tuple of real numbers. Applying this with $C = \gamma(A)_{(m)}$ and $C = \gamma(B)_{(m)}$, we deduce the existence of an integer l such that each entry of $((L(M/P)_{(m)})^l/\lambda^l)(\gamma(B)_{(m)} - \gamma(A)_{(m)})$ has absolute value less than ϵ (recall that $\tau(a) = v \cdot \gamma(A)/\lambda^{k-1}$).

Now consider the elements $A' = (M/P)^{l+k}A/(\lambda^l)$ and $B' = (M/P)^{l+k}B/(\lambda^l)$. By the result of the previous paragraph, $\gamma(A'_j) - \gamma(B'_j) < \epsilon$ for each $j \leq m$. Since γ is a pure trace of the dimension group I , there exist z_j in I^+ such that $z_j \leq A'_j, B'_j$ (as elements of I) and $\gamma(z_j) > \min\{\tau(A'_j), \tau(B'_j)\} - \epsilon$. Define Z in $(I^n)^+$ by setting its j th entry to be z_j if $j \leq m'$ and zero if $j > m'$. Then $z = [\lambda^l Z, l + k]$ satisfies all the required properties to verify the extremal criterion.

Now we show that any limit of traces of the form $\tau_{\gamma,v}$ is of the same form except that v need not be an extremal left eigenvector of $L(M/P)_{(m)}$ (merely a strictly positive left eigenvector, not necessarily irreducible) and γ while being a trace of I need not be pure. That is, if τ is a limit of these special traces, there exists L a pure trace of R_P , a trace γ (not necessarily pure) of I such that $\gamma(sa) = L(s)\gamma(a)$, and a nonnegative left eigenvector v of $L(M/P)$ so that τ is given as in (5.1).

If τ is such a limit, say of $\{\tau_k\}$ where τ_k depends on γ_k , L_k , and v_k , we first observe that for any element u such that $\tau(u) > 0$ and s in R_P , $L_k(s) = \tau_k(su)/\tau_k(u)$. Since each L_k is a multiplicative trace of R_P , L_k converge to a multiplicative (hence pure) trace of R_P , L , such that $\tau(su) = L(s)\tau(u)$. Similarly, $\gamma_k(a)$ is a scalar multiple of each of $\tau_k([ae_i, j])$ (except those i for which the latter are zero), we find easily that γ_k (normalized) must converge to a trace on I (limits of such traces need not be pure, of course). By recalculating the coefficients of γ_k that appear in $\tau_k([ae_i, j])$, we easily find the row v that implements τ , and it is easy to check it is an eigenvector.

Let X denote the set of normalized pure traces of $G_{M/P,I}$ of the form described in (5.1), and let \overline{X} denote its closure in the set of normalized traces; we denote the latter S (S is a Choquet simplex). We have just seen that $\overline{X} \setminus X$ contains no pure traces. If there exists a pure trace ψ not in X , then by the standard separation theorem for Choquet simplices, e.g., [AE, Theorem 1.1], of pure traces from compact sets of them, there exists an affine continuous function $\alpha : S \rightarrow \mathbf{R}$ such that $\alpha(\tau) \geq 1$ for all τ in \overline{X} and $\alpha(\psi) \leq -1$. The image of $G_{M/P,I}$ in $\text{Aff}(S)$ is dense since the former is a real vector space. Hence there exists a in $G_{M/P,I}$ such that $\tau(a) \geq 1$ for all τ in \overline{X} and $\psi(a) \leq -1$. We will deduce a contradiction by showing that a is positive (in fact it is an order unit, but this follows from the conclusion).

Write $a = [(a_1, \dots, a_n)^T, k]$ with a_i in I . Set $A = (a_1, \dots, a_n)^T$ in I^n . By hypothesis, $v \cdot \gamma(A) > 0$ for all pure traces γ (and the corresponding pure eigenvectors of $L(M/P)$, where L corresponds to $\tau_{\gamma,v}$). Fix a pure trace γ of I , and its induced pure trace L of R_P . The hypotheses assert that for all pure left eigenvectors v of $L(M/P)$, $v \cdot \gamma(A) > 0$. Hence (an easy argument even in this reducible case—see Appendix 2 for a more complicated version), there exists $m \equiv m(\gamma)$ such that the real column $L(M/P)^m \gamma(A)$ is strictly positive. For L fixed, there are only finitely many pure traces γ such that $\gamma(sj) = L(s)\gamma(j)$ (just note that $I/I\ker L$ is a finite dimensional real vector space). Hence we obtain m depending only on L , not on γ , and moreover, the least entry of $L(M/P)^m \gamma(A)$ (as γ varies over the finite set) exceeds zero, say is ϵ_L . There thus exists an open subset of the (compact) space of pure traces on R_P such that L' in this set entails the least entry of $L(M/P)^{mL} \gamma(A)$ exceeds $\epsilon_L/2$. These open sets cover the space, and by compactness, we obtain a finite subcover. Thus there exists a fixed m such that every entry of $L(M/P)^m \gamma(A)$ is strictly positive. Hence $\gamma(((M/P)^m A)_i) > 0$ for all γ and all i . Thus each $((M/P)^m A)_i$ is a positive element of I , so that a is in the positive cone of $G_{M/P,I}$. Of course, this contradicts $\psi(a) < 0$, so no such ψ exists. •

The map $I \rightarrow G_{M/P,I}$ given by $b \mapsto [(b, b, \dots, b)^T, 1]$ induces an affine (convex linear) map $S(G_{M/P}) \rightarrow S(I)$ (pick order units and normalizations to be compatible; the map on pure traces is then $\tau_{\gamma,v} \mapsto \gamma$). A particular consequence of the previous result is that this map is onto and sends pure traces to pure traces. If $L(M/P)$ has only one nonnegative left eigenvector for each L , then the map is a homeomorphism on the pure traces, hence is a homeomorphism between the traces spaces. In particular, this applies if M is a large enough power of a matrix satisfying (**) and $\text{Log } P = \text{tr } M$ (since $L(M/P)$ is then primitive). Something pleasant also occurs if (**) holds: every ideal of $G_{M/P}$ is uniform and admits an order unit.

Lemma 5.15 If every entry of some power of M/P is an order unit of R_P , then every order ideal of $G_{M/P}$ is uniform.

Proof. We may replace M/P by the appropriate power. Let D be the order ideal of $G_{M/P}$. Denote the order ideal of R_P generated by

$$\{b \in R_P^+ \mid \text{there exists } B = (B_i)^T \in R_P^+ \text{ with } [B, k] \in D \text{ and } b = B_j\}$$

by $I_{k,j}$. Since each entry of M/P is an order unit, we see immediately that $I_{k+1,i} = \sum_j I_{k,j}$ for all i and k . Hence $I_{k,j} \equiv I$ is independent of k and j , so $D = G_{M/P,I}$. •

In the following, note the appearance of G_M , not simply $G_{M/P}$.

Theorem 5.16 If M satisfies (**), then the collection of order ideals of G are in a natural bijection with the order ideals of R_P where $P = \text{tr } M^{n!}$, and the pure trace space of any order ideal of

G_M is naturally homeomorphic to that of the corresponding order ideal I of R_P , and all pure traces are given by restriction from $G_{M^{n!}/P, I}$.

Proof. Follows immediately from the previous results and standard results on extensions of traces. •

6. Return of the perfidious traces

The notion of uniformity in section 5 was useful for showing that at least some of the time, the pure (non-faithful) traces on order ideals of G_{MP} can be completely described in terms of the facial structure (Theorem 5.14). In this section, we obtain a partial description of pure traces but in a slightly more general setting, exploiting the idea of the argument involved in the proof of Theorem A1.3. The method involves the order ideal in $\lim M : \mathbf{R}^n \rightarrow \mathbf{R}^n$ in which the image of f sits; its extremal traces (corresponding to nonnegative left eigenvectors of the cut-down matrix M'') are then determined. In the case of $d > 0$ variables, we use this and try to reduce to this case.

We now describe one family of traces on order ideals (essentially the same as those described in Theorem 5.14) of order ideals of G_{MP} ; then we show that every pure trace is of this form when $\lambda(\widehat{M}/P) \neq 0$ (λ being the trace). Let $\{I_i\}_{1 \leq i \leq n}$ be a set of order ideals of R_P , set $W = \bigoplus I_i E_i$. Assume that W is invariant under the action of M/P . Let H be the order ideal of G_M generated by all choices for $[z_i E_i, 1]$ with z_i varying over I_i . Since each I_i admits an order unit, so does H . (It turns out that up to order-isomorphism, all order ideals with order units are of this form.) We may also consider the slightly larger order ideal, $\tilde{H} = \lim M/P : W \rightarrow W$, obtained by “inverting” \widehat{M}/P . It frequently happens that $\tilde{H} = H$. Now let L be a pure trace of R_P such that $L(M/P)$ is not a nilpotent matrix.

If L is a faithful trace, it must be a point evaluation, given as $b \mapsto b(r)$ for some fixed $r \in (\mathbf{R}^d)^{++}$. Since we are assuming $M(1)$ (and therefore $M(r)$ is primitive), there is then only one choice for the corresponding trace on H —it must be faithful, so Theorem 4.5 applies. Hence we may assume L is not faithful. Then, as in [HM, VII.5], there exists an order ideal \mathfrak{p} such that $\ker L \cap R_P^+$ generates \mathfrak{p} as an ideal and \mathfrak{p} corresponds to a face F of $\text{cvx Log } P$, in the sense that \mathfrak{p} is generated as an ideal (not simply as an order ideal) by $\{x^w/P \mid w \in \text{Log } P \setminus F\}$. If we define $P_F = \sum_{w \in \text{Log } P \cap F} c_w x^w$ (where $P = \sum c_w x^w$), then the natural map $R_P \rightarrow R_{P_F}$ given by $g/P^k \mapsto g_{kF}/P_F^k$ (where g_{kF} is the polynomial obtained from g by removing all the terms x^w for which $w \notin kF$) has kernel the prime ideal \mathfrak{p} , and the quotient ordering on R_{P_F} agrees with its own natural ordering.

Let $I = \sum I_i$, and let $\phi : I \rightarrow I$ be a pure trace affiliated with L (that is, $\phi(ae) = L(a)\phi(e)$ for $a \in R_P$ and $e \in I$). Let $S = \{i \mid \phi|_{I_i} \neq 0\}$ and cut down the columns and M/P itself, by deleting all the coordinates not in S . Let $(M/P)_S$ be the resulting $|S| \times |S|$ matrix. Form the corresponding W_S, H_S, \tilde{H}_S , etc.

Let ρ be a positive eigenvalue of the real matrix $L((M/P)_S)$ for which there exists a nonnegative left eigenvector. Let $v = (v(i))_{i \in S}$ be an extremal nonnegative left eigenvector for ρ (that is, v is a nonnegative eigenvector and it cannot be expressed as a non-trivial sum of other other nonnegative left eigenvectors of $L((M/P)_S)$ for ρ). Define a trace γ_S on \tilde{H}_S using v, ϕ, ρ , via $[f, k] \mapsto (v \cdot \phi(f))/\rho^{k-1}$. Then we define a trace on \tilde{H} via the diagram:

Note that the trace is actually defined not just on H but on the possibly larger order ideal, \tilde{H} . It is fairly straightforward to show that any trace constructed in this manner is a pure trace, and the corresponding $\lambda(\widehat{M}/P) = \rho \neq 0$. Our task is to show the converse.

We require a few elementary results about pure traces on dimension groups.

Lemma 6.1 Let $\{A_i\}_{i \in \Omega}$ be a set of order ideals in a dimension group C , and let $A = \sum A_i$.

- (a) If γ is a pure trace of C , then for each i , either $\gamma(A_i) = 0$ or $\gamma|_{A_i}$ is a pure trace of A_i .
- (b) Suppose that $|\Omega| < \infty$ and γ is a pure trace of A that kills $\bigcap A_i$. Then γ kills at least one of the A_i .
- (c) Suppose that all of the A_i 's, and C , admit an order unit. Let γ and γ' be pure traces of C

$$\begin{array}{ccccccc}
 \tilde{H} & = & \lim & W & \xrightarrow{M/P} & W & \xrightarrow{M/P} & \dots \\
 \downarrow & & & \downarrow & & \downarrow & & \\
 \tilde{H}_S & = & \lim & W_S & \xrightarrow{(M/P)_S} & W_S & \xrightarrow{(M/P)_S} & \dots \\
 & & \searrow & & & & & \\
 & & \gamma_S & & & & &
 \end{array}$$

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- such that $\gamma|_{A_1} = \gamma'|_{A_1}$, and this is nonzero. Then $\gamma = \gamma'$.
- (d) For each index i , let α_i be a trace on A_i , such that for all i and j , $\alpha_i|_{A_i \cap A_j} = \alpha_j|_{A_i \cap A_j}$. Then there exists a unique trace α of $A = \sum A_i$ such that $\alpha|_{A_i} = \alpha_i$ for all i .

Proof. (a) This follows from the extremal criterion in [GH2, Theorem 3.1].

(b) First observe that $A_1/(A_1 \cap A_2) \oplus A_2/(A_1 \cap A_2) \cong (A_1 + A_2)/(A_1 \cap A_2)$ (since all things in sight are dimension groups, this type of result follows from the corresponding results for \mathbf{Z}^k with the usual ordering). If $|\Omega| = 2$, any trace on $A_1 + A_2$ which kills $A_1 \cap A_2$ will thus decompose into the sum of its restrictions to the two factor groups; hence any such trace which is pure must kill one of them.

If $|\Omega| > 2$, proceed by induction. Suppose $\gamma(A_1) \neq 0$. From $\gamma(A_1 \cap (A_2 \cap \dots \cap A_n)) = 0$, we deduce from the direct sum decomposition above, that $\gamma(A_2 \cap \dots \cap A_n) = 0$, and now we apply induction to $\Omega' := \Omega \setminus \{1\}$.

(c) The difference $\gamma - \gamma'$ induces a bounded linear functional on the quotient C/A_1 , which is of course a dimension group. Hence we may write the induced bounded linear functional as $\alpha v - \beta w$ where α and β are positive real numbers and v and w are normalized traces on C/A_1 . Viewing them as traces on C , we obtain $\gamma - \gamma' = \alpha v - \beta w$, so $\gamma + \beta w = \alpha v + \gamma'$. Now $\gamma + \beta w$ can be represented as a nonnegative measure over the extremal boundary (the measure is finite, not necessarily 1, since we have not bothered normalizing $\gamma + \beta w$), and since γ is extremal, the measure must include a point mass at γ (with some positive weight). For a Choquet simplex, the representing measure is unique, and if $\gamma' \neq \gamma$, it follows that the representing measure for v must contain the point mass at γ . However, this entails that $v|_{A_1} \neq 0$, a contradiction. Hence $\gamma = \gamma'$.

(d) As is well-known, A itself is an order ideal in C (as C is a dimension group); moreover, it also follows that $A^+ = \sum A_i^+$. Given finite sets $\{a_j\}$ and $\{a'_j\}$ such that for all j , a_j, a'_j in A_j and $\sum a_j = \sum a'_j$, we wish to show that $\sum \alpha_j(a_j) = \sum \alpha_j(a'_j)$. From $\sum a_j = \sum a'_j$, it follows by the Riesz decomposition property that there exist a_{jk} in C^+ such that for all j , $a_j = \sum_k a_{jk}$, and for all k , $a'_k = \sum_j a_{jk}$. Note that $a_{jk} \leq a_j$ and $a_{jk} \leq a_k$, so that a_{jk} belongs to $A_j \cap A_k$. We observe:

$$\alpha_j(a_j) = \sum_k \alpha_j(a_{jk}) = \sum_k \alpha_k(a_{jk}).$$

Thus $\sum_j \alpha_j(a_j) = \sum_{j,k} \alpha_k(a_{jk}) = \sum_k \sum_j \alpha_k(a_{jk}) = \sum_k \alpha_k(a'_k)$, as desired.

If $a = \sum a_i = \sum a'_i$ with a, a_i, a'_i not necessarily positive, we can add to them multiples of any positive element in $\sum A_i$ that dominates all of the members of the finite set $\{\pm a_i, \pm a'_i\}$, and so assume they are all positive. We deduce that $\sum \alpha_i(a_i)$ depends only on a . So the assignment $a \mapsto \sum \alpha_i(a_i)$ is well-defined, and it is easy to check that it is additive and positive, and thus a trace. Uniqueness is trivial. •

Choose a pure trace γ on our order ideal H , which we may assume to be obtained from W in the manner described earlier. Affiliated with γ are pure traces λ on $E_b(G_{PM})$ and L on R_P such that $\gamma(eh) = \lambda(e)\gamma(h)$

for all e in $E_b(G_{PM})$ and h in H , and $\lambda|R_P = L$. We also assume for now that $\lambda(\widehat{M/P}) \neq 0$. A consequence of this assumption is that γ extends to a trace on \widehat{H} , and so it makes sense to talk about γ evaluated at elements of the form $[\sum z_i E_i, k]$ for z_i in I_i ; if we were restricted to dealing with H , then the k could only be 1.

We shall prove the result first in the case that W is of a special form. For $1 \leq i, t \leq n$, define $J_{i,t}$ to be the order ideal of R_P generated by the (i, t) entries of all powers of M/P . Since R_P is noetherian and all of its order ideals are ideals, only finitely many powers are required to generate the J 's. Now for any nonzero order ideal I of R_P , let H be the order ideal (depending on a choice for t in $\{1, 2, \dots, n\}$) generated by $\{[zE_t, 1] \mid z \in I\}$. Hence, we set $W = IE_t \oplus J_{1,t}IE_1 \oplus J_{2,t}IE_2 \oplus \dots \oplus J_{n,t}IE_n$ and we may take H to be obtained from $[W, 1]$ (we are permitted to apply the shift, since we may invert $\widehat{M/P}$).

We may assume that $\gamma|[J_{i,t}IE_t, 1] \neq 0$ (for some i and t , since $\lambda(\widehat{M/P}) \neq 0$). We perform an initial cut-down. Set

$$S = \{i \mid \gamma|[J_{i,t}IE_t, 1] \neq 0\}.$$

For s in S , we say that s is an *anomaly* if $J_{s,t} \subseteq \mathfrak{p}$, i.e., $L(J_{s,t}) = 0$. (Note that we cannot conclude from $J_{s,t} \subseteq \mathfrak{p}$ that $\gamma|[J_{i,t}IE_t, 1]$ is zero, since we do not know that $[IE_t, 1]$ belongs to the module of which γ is a pure trace.) We will show that no anomalies exist later on, but for now we *assume that no anomalies exist*. (The proof that no anomalies exist uses the fact that when no anomalies exist, the traces are what they are supposed to be, hence this rather convoluted argument.) A consequence is that L does not kill any of the order ideals $J_{i,t}$.

At this point, we realize that in order to avoid periodicity, we should have replaced M/P by its n th power or any (fixed) higher power; this of course has no effect on the direct limit ordered modules, so we may do so. It follows that in all sufficiently high powers of $L((M/P)_S)$ all of the entries in column t are not zero; hence we may assume the same of $L((M/P)_S)$, and moreover this is true for the j th column for any j in S such that j and t are states in the same irreducible block (with respect to the real matrix $L((M/P)_S)$, see the preamble to the proof of Theorem A1.3).

Now \mathfrak{p} is prime as an ideal and since none of $J_{i,t}$ ($i \in S$) is contained in \mathfrak{p} , neither is their product $J_t = \prod_{i \in S} J_{i,t}$. In R_P , a product of order ideals is an order ideal,¹ and thus J_t is generated by positive elements as an ideal. There thus exists a positive element r in $J_t \setminus \mathfrak{p}$. Obviously, $L(r) > 0$, and for z in I and i in S , $[rzE_i, 1]$ is an element of H . If s is another element of $J_t^+ \setminus \mathfrak{p}$, then evaluating $\gamma([rszE_i, 1])$ in two different ways, we deduce $L(s)\gamma([rzE_i, 1]) = L(r)\gamma([szE_i, 1])$. Hence the function $\phi_i : I \rightarrow \mathbf{R}$ given by $\phi_i(z) = \gamma([rzE_i, 1])/L(r)$ is independent of the choice of r , subject to $r \in J_t^+ \setminus \mathfrak{p}$, and it is immediate that if r is positive, then ϕ_i is a trace of I . Of course, we will end up proving that up to scalar multiples, all the ϕ_i are the same, and they are pure traces of I .

Let $\rho = \lambda(\widehat{M/P})$ (ρ is used to remind the reader of spectral radius). For every i in S , we have $\gamma([rzE_i, 1]) = \frac{1}{\rho}\gamma([(M/P)_S rzE_i, 1])$; expanding this, we deduce (writing $M/P = (a_{qp})$)

$$\phi_i(z)L(r) = \gamma([rzE_i, 1]) = \frac{1}{\rho} \sum_{j \in S} \gamma([rzE_j, 1])L(a_{ji}) = L(r) \frac{1}{\rho} \sum_j \phi_j(z)L(a_{ji}).$$

In other words, the row $\Phi(z) := (\phi_j(z)L(r))$ is a nonnegative left eigenvector for ρ of $(M/P)_S$. We will exploit purity of the original trace γ to show that Φ is independent of the choice of z (subject to $z \geq 0$ and $\Phi(z) \neq \mathbf{0}$), and is a pure nonnegative eigenvector of $L((M/P)_S)$. Let Z denote the cone of nonnegative left eigenvectors of $L((M/P)_S)$ for the eigenvalue ρ . Then $\Phi : I^+ \rightarrow Z$ is additive, and obviously extends

¹ Products of order ideals in partially ordered commutative domains (that are even dimension groups admitting 1 as an order unit) need not be order ideals; however, in R_P products of order ideals really are order ideals—this follows from the positive cone being generated by elements of the form x^w/P^k .

to a positive homomorphism $I \rightarrow Z - Z$; of course $Z - Z$ is a subspace of the eigenspace for ρ . As is well-known, Z is simplicial. This means that there is a basis for the vector space $Z - Z$, $\{v_q\}$ whose \mathbf{R}^+ span is precisely Z . Pick some z in I^+ such that $\Phi(z) \neq \mathbf{0}$, and relabel the basis so that v_1 has a component in $\Phi(z)$ relative to the basis. Let $\Psi : Z \rightarrow v_1\mathbf{R}^+$ be the projection relative to this basis. We use this to construct a trace Γ on H_S (in fact, it is defined on a larger order ideal, in general) that is dominated by γ ; purity of γ entails equality.

Let I_S denote $\lim(M/P)_S : I^{\#(S)} \rightarrow I^{\#(S)}$. Obviously, this contains H_S , even \tilde{H}_S . Our candidate for a trace on I_S is given by

$$\Gamma([zE_i, k]) = \frac{\{\Psi\Phi(z)\}_i}{\rho^{k-1}} \quad \text{for } z \in I, i \in S, \text{ and } k \in \mathbf{N}.$$

The numerator is the i th entry of the vector $\Psi\Phi(z)$. To check that Γ extends to a well-defined function, it is enough to check that $\Gamma([zE_i, k]) = \Gamma([(M/P)_S zE_i, k + 1])$. We compute:

$$\begin{aligned} \Gamma([(M/P)_S zE_i, k + 1]) &= \sum_{j \in S} \Gamma([a_{ji}zE_j, k + 1]) = \frac{1}{\rho^k} \sum_j \{\Psi\Phi(a_{ji}z)\}_j \\ &= \frac{1}{\rho^k} \sum_j \{\Psi\Phi(L(a_{ji})z)\}_j \\ &= \frac{1}{\rho^k} \sum_j L(a_{ji})\{\Psi\Phi(z)\}_j \\ &= \frac{1}{\rho^k} \rho \{\Psi\Phi(z)\}_i \\ &= \frac{1}{\rho^{k-1}} \{\Psi\Phi(z)\}_i = \Gamma([zE_i, k]). \end{aligned}$$

The fourth line comes from the fact that $\Psi\Phi(z)$ is a scalar multiple of a fixed eigenvector of $(M/P)_S$. Now that Γ is known to be well-defined, it is immediate that it is also additive and positive, and so is a trace on I_S . Now we claim that $\Gamma|_{H_S} \leq \gamma$. Suppose that i belongs to S , a belongs to $J_{i,t}$, z belongs to I and az belongs to R_P^+ . Then $az \in J_{i,t}I$, and we calculate:

$$\begin{aligned} \Gamma([azE_i, k]) &= \frac{1}{\rho^{k-1}} \{\Psi\Phi(az)\}_i \leq \frac{1}{\rho^{k-1}} \{\Phi(az)\}_i = \frac{1}{\rho^{k-1}} \phi_i(az) \\ &= \frac{1}{\rho^{k-1}} \cdot \frac{\gamma([razE_i, 1])}{L(r)} \\ &= \frac{1}{\rho^{k-1}} \gamma([azE_i, 1]) \\ &= \gamma([azE_i, k]). \end{aligned}$$

So $\Gamma|_{H_S} \leq \gamma$, and by purity of the latter, we must have equality (up to a scalar multiple); note that the assumption that v_1 actually appears in the decomposition of at least one $\Phi(z)$ ensures that Γ is not zero. If instead of choosing v_1 , we selected some other eigenvector, we would obtain a corresponding equality of the restriction with γ . It easily follows that there is only one v_1 that appears, and so Φ has one-dimensional range. Thus the ϕ 's are all scalar multiples of each other. Let ϕ denote any one of them. To see that ϕ is a pure trace of I , suppose that $\phi' \leq \phi$ (we have not bothered to normalize any traces here, so we never have to deal with scalar multiples). Then ϕ' inherits the multiplicative property, $\phi'(az) = L(a)\phi'(z)$. Now define

Γ' using ϕ' , and we similarly see that its restriction is dominated by γ ; hence equality holds, and so ϕ' is a scalar multiple of ϕ .

We have deduced that (if no anomalies exist) γ (actually γ_S , but this is obtained by factoring out an order ideal in the kernel of γ) is of the desired form, and it can be defined on a natural order ideal of $\lim(M/P)_S : R_P^S \rightarrow R_P^S$.

Now we prove that no anomalies exist. We have defined $I, S, L, \mathbf{p}, \lambda$; suppose that s is an anomaly, that is, $s \in S$ and $J_{s,t} \subseteq \mathbf{p}$. For an order ideal I_0 of R_P and s_0 in S , define $H(I_0, s_0, k)$ to be the order ideal of $G_{\mathcal{M}}$ generated by $[I_0 E_{s_0}, k]$, and $\tilde{H}(I_0, s_0, k)$ to be the enlargement obtained by inverting $\widehat{M/P}$. Notice that we have been working with $H = H(I, t, 1)$. Now $H(J_{s,t} I E_s, 2)$ is an order ideal of H and from the definition of S , γ does not vanish on it. Since γ is a pure trace of H and the latter is a dimension group, $\gamma_2 := \gamma|_{H(J_{s,t} I E_s, 2)}$ is also a pure trace.

Now let i an element of S that is not an anomaly. Then there exists a positive element r in $J_{i,t} \setminus \mathbf{p}$ and necessarily $[rzE_i, 2] \in H$ for all z in I . From the computation,

$$L(r)\gamma([J_{i,s} J_{s,t} I E_i, 2]) = \gamma([r J_{i,s} J_{s,t} I E_i, 2]) = L(J_{s,t} J_{i,s})\gamma([r I E_i, 2]) = 0 \quad \text{and} \quad L(r) \neq 0,$$

we see that $\gamma([J_{i,s} J_{s,t} I E_i, 3]) = 0$; extending this via $(\widehat{M/P})^{-1}$, we see $\gamma([J_{i,s} J_{s,t} I E_i, k]) = 0$ for all k . Hence γ_2 kills all the non-anomalous coordinates. The S_2 we obtain for γ_2 is a proper subset of S , unless S consists entirely of anomalies. This last possibility cannot occur, since the trace of some power of M/P must be nonzero. If γ_2 admits anomalies, we may continue the process, and at some stage we arrive at a situation where there are no anomalies (as γ_2 is not zero, the same holds for its successors). Hence we assume that γ_2 admits no anomalies, γ does, and deduce a contradiction.

For γ_2 (which is defined on a smaller order ideal), there is a cut down to a smaller set S_2 . By the preceding, S_2 consists of anomalies for S , but by our induction assumption, admits no relative anomalies itself(!). Let T be the complement of S_2 in S , and let U be a subset of S . Suppose that r and z are positive elements of R_P such that $rzI \subseteq J_{i,t}I$ for all i in U , and for at least one such i , $\gamma([rzE_i, 2]) > 0$ (the first condition implies $[rzE_i, 2]$ belongs to H , obviously a necessary condition for the second condition to make sense; indeed all the difficulties in the arguments here emanate from the apparent restrictions on domains of the various traces). From the equation $\rho\gamma([rzE_i, 2]) = \gamma([rz(M/P)E_i, 2])$ for all i in U , we deduce (recalling that the entries of M/P are denoted a_{pq})

$$(1) \quad \rho\gamma([rzE_i, 2]) = \sum_{j \in U} L(a_{pq})\gamma([rzE_j, 2]) + \sum_{j \in U^c} \gamma([a_{ji} rzE_j, 2]) \quad \text{for all } i \in U$$

We make various choices for r and z . First, set $J_t = \prod_{i \in T} J_{i,t}$. Since \mathbf{p} is prime (as an ideal in the ordinary ring sense), $J_t \not\subseteq \mathbf{p}$; since J_t and \mathbf{p} are order ideals, there exists $r_t \in J_t^+ \setminus \mathbf{p}$; necessarily, $L(r_t) > 0$ (since $\ker L \cap R_P^+ = \mathbf{p} \cap R_P^+$). For fixed s in S_2 , set $U_s = T \cup \{s\}$. There exists $z \in J_{s,t}I \cap R_P^+$ such that $\gamma([zE_s, 2]) > 0$ (from the definition of S). Then we notice that $rI \subseteq J_{i,t}I$ for all i in T and so $rz \in J_{i,t}I$ for all i in U_s . Now we see that $\gamma([rzE_i, 2]) = L(z)\gamma([rE_i, 2]) = 0$ for i in T , and $\gamma([rzE_s, 2]) = L(r)\gamma([zE_s, 2]) > 0$. Plugging this into (1), we deduce $L(a_{si}) = 0$ for all $i \in T$. Hence $L(a_{si}) = 0$ for all i in T and s in S_2 . Thus if we write the real matrix $L(M/P)$ according to the partition $S = T \cup S_2$, it appears in block upper triangular form,

$$T \{ \begin{matrix} 1 \\ \left(\begin{array}{cc} \overbrace{L\left(\left(\frac{M}{P}\right)_T} & \overbrace{L\left(\left(\frac{M}{P}\right)_{S_2}\right)} \\ \mathbf{0} & * \end{array} \right) \end{matrix} \right.$$

Now we notice that for γ_2 , the “ L ” and the “ λ ” are just the originals. Since γ_2 admits no anomalies, we can reconstruct it as follows. Cut down to S_2 , form the order ideal $I_2 = \sum_{s \in S_2} J_{s,t} I$. Then there exists a pure trace ϕ on I_2 and a left nonnegative eigenvector v_2 of $L((M/P)_{S_2})$ of $\rho = \lambda(\widehat{M/P})$ that implements the trace. As a result of the block upper triangular form, we may enlarge v_2 to a nonnegative left eigenvector of $L((M/P)_S)$ simply by inserting to the left of v_2 as many zeroes as the cardinality of T . This permits us to define an extension of γ_2 to H_S , which wipes out the coordinates corresponding to T , and is clearly dominated by the original trace γ (as γ_2 is the restriction of γ). Purity of γ forces equality with this weird trace, but this is impossible as γ does not wipe out any of the T -coordinates. So there are no anomalies.

To recap: If γ is a pure trace with $\lambda(\widehat{M/P}) \neq 0$, and if the order ideal is generated by $[IE_i, k]$ for some integers i and k , and I an order ideal in R_P , then γ is given by the prescription above; that is, we cut down, and locate a pure trace on I , a left nonnegative eigenvector for the cut-down real matrix $L((M/P)_S)$, and use it, exactly as in the case of $d = 0$ (c.f., the proof of Theorem A1.3). Notice by the way that the proof that no anomalous coordinates existed used an induction argument and the smaller order ideal was of the same type. The constraint on the order ideal may now be dropped by Lemma 6.1.

Now we can extend our characterization of the pure traces that do not wipe out $\widehat{M/P}$ to arbitrary order ideals of G_M . We notice that an arbitrary order ideal of G_M will be a sum of the order ideals of the type considered above; hence the restriction of γ to at least one of these will be nonzero. This restriction is of the desired type—but we note that it can be extended to one of the same type (this was our original construction of candidate traces) which is also extremal. By Lemma 6.1(c), γ must itself be of this type.

This completes the proof of the following.

Theorem 6.2 Let γ be a pure trace of an order ideal H of G_M such that $\lambda(\widehat{M/P}) \neq 0$. Then γ is obtained from the following prescription.

Let S denote the complement of the set of coordinates i such that $\gamma([zE_i, k]) = 0$ whenever $[zE_i, k]$ belongs to H . Form H_S by throwing away all the coordinates not in S . There exists an order ideal I of R_P such that H_S is contained in the order ideal of G_M generated by $\sum_{i \in S} [IE_i, k]$ for some integer k . There exists a pure trace ϕ on I and a left nonnegative eigenvector v for the cut-down real matrix $L((M/P)_S)$ with eigenvalue $\rho = \lambda(\widehat{M/P})$, such that the trace given by $[\sum_S z_i E_i, l] \mapsto v \cdot (\phi(z_i)) / \rho^{l-1}$ restricts on H_S to a multiple of γ ; moreover, the extended trace is also pure.

Suppose that for all pure traces λ of $E_b(G_{PM})$, $\lambda(\widehat{M/P}) \neq 0$; then (as $\widehat{M/P}$ is a positive element of $E_b(G_{PM})$), $\widehat{M/P}$ is an order unit. Thus there exist integers k and l so that

$$(3) \quad M^k P^l \prec M^{k+1} P^{l-1}$$

This is precisely the condition to be discussed in Proposition 7.5; it is a consequence of noetherianness of $E_b(G_M)$. It is usually easy to verify. In this case, by Lemma 5.1, the ordering on G_M is completely determined by that of G_M . We have a situation in which all the pure traces on every order ideal are known, and in principal, this permits us to determine the positive cone of G_M .

Another consequence of (3) is that $(\widehat{M/P})^{-1}$ is already in $E_b(G_{PM})$. This means that if $u = \sum [u_i E_i, k]$ is an order unit for H , so is its right shift, $\sum [u_i E_i, k + 1]$. In particular, G_M admits an order unit (the column consisting of 1’s at level one). Conversely, if for all i , the order ideal generated by $[E_i, 1]$ contains $[E_i, k]$ for some k , then $\widehat{M/P}$ is invertible in the bounded subring.

A sufficient condition for G_M to admit an order unit, is that the sum of the columns of M/P consist of order units; this is not necessary, nor does it imply that $(\widehat{M/P})^{-1}$ is invertible—consider the example,

$$M = \begin{bmatrix} 1+x & x \\ 1+x & 0 \end{bmatrix} \quad P = 1+x.$$

One very special and particularly strong property is condition (**); see section 5. Then with P in A^+ such that $\text{Log } P = \text{Log } M^k$, $\widehat{M^k/P}$ is an order unit, and all the preceding applies. We give some examples.

Example 6.3 $d = 1$ and $n = 4$.

Here $d = 1$; set

$$M = \begin{bmatrix} x^2 & 1 & x & x^2 \\ 1 & 1+x^2 & x^2 & 1 \\ 1 & 2 & x^2+2 & 1 \\ 1 & 1 & 1 & x^2+2 \end{bmatrix} \quad P = 1 + x + x^2.$$

We observe that $(1+x^4)I \leq M^2$, so that $P^3I \leq M^2P^2$; thus M^2 satisfies condition (3) above (with respect to P^2) to guarantee that $\widehat{M/P}^{-1}$ is bounded, so that $\lambda(\widehat{M/P}) > 0$ for all pure traces of $E_b(G_{PM})$. (Obviously for the positivity problem, we may always replace M by any power of itself.) The pure traces of R_P are given by point evaluations, $x \mapsto r \in [0, \infty]$. If r belongs to $(0, \infty)$, then $(M/P)(r)$ (the matrix obtained by evaluating all entries of M/P at $x = r$) is primitive (in fact, all of its entries are strictly positive); if $r = 0$, again primitivity results. In these cases, there is a unique nonnegative left eigenvector (up to scalar multiple), and it is strictly positive. On the other hand, if $r = \infty$ (that is, $(M/P)(\infty) = \lim_{r \rightarrow \infty} (M/P)(r)$) yields the pure trace L , we have

$$L(M/P) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

There are three nonnegative left eigenvectors, $(1\ 0\ 0\ 0)$, $(0\ 0\ 1\ 0)$, and $(0\ 0\ 0\ 1)$. However, cutting down $L(M/P)$ to its $(2, 2)$ entry, there is another candidate for an eigenvector that induces a trace, $(0\ 1\ 0\ 0)$. Whether this yields a trace depends on the choice of (M/P) -invariant order ideal, W , of R_P^4 .

For example, if $W = R_P^4$, then no matter what our initial choice for eigenvector of $L(M/P)$, the big ideal I will be all of R_P itself, and ϕ must equal L . Thus in this case, there are exactly three pure traces of H affiliated to L , which is evaluation at ∞ .

On the other hand, let I be the ideal generated by $\{x/P, 1/P\}$ in R_P ; this is an order ideal by [H4, II.2A]. Let W be the order ideal of R_P^4 given as $I \oplus R_P \oplus I \oplus I$. It is routine to check that $(M/P)W \subseteq W$; in other words, W is invariant. Now let ϕ be L itself; we note that $L(I) = 0$ (evaluate x/P and $1/P$ at $\infty!$), so that we can cut down to the $(2, 2)$ entry. We obtain a pure trace that simply picks out the second entry of W and evaluates it at L . Note that in this case, the corresponding eigenvalue of $L(M)$ is $L((1+x^2)/P)$, which equals 1. This yields one additional pure trace affiliated to L . For elements $h = f/P^k$ in I , we remark that $\lim_{r \rightarrow \infty} h(r) = 0$, and it follows from h being rational that $\phi(h) = \lim_{r \rightarrow \infty} h(r)/r$ exists. This defines a linear functional ϕ on I (obviously not extendible to all of R_P) which is a trace and it is routine to verify that L is affiliated to it. (Also verifiable: All other pure traces of I are restrictions of the point evaluations determined by $x \mapsto r \in [0, \infty)$.) •

Example 6.4 $d = 2$ and $n = 3$.

Here $d = 2$, M satisfies (**), and I is more interesting. Set

$$M = \begin{bmatrix} 1+2x+3y & 1+2x+3y & 1+x+y \\ 1+x+y & 1+2x+y & 1+x+y \\ 1+x+y & 1+3x+y & 2x+y \end{bmatrix}.$$

We can take $P = 1 + x + y$. We note first that $P^2I \leq M^2$; then a computation reveals that for all $1 \leq i, j \leq 3$, $\text{cvx Log } (M^2)_{ij} = \text{Log } P^2$, and so M satisfies (**). Following [H4, Example 1A, p. 7], set

$X = x/P$, $Y = y/P$, and $Z = 1 - X - Y$. Then by [op. cit.], $R_P = \mathbf{R}[X, Y]$ (the pure polynomial algebra), and $R_P^+ = \langle X, Y, Z \rangle$ (the \mathbf{R}^+ -semigroup generated additively and multiplicatively by X, Y , and Z). Set $I = (X, Y)^2$ (the square of the ideal generated by X and Y), From [H4, Remark, pp. 46-47], $I/(X, Y)^3 \cong \overline{X^2Z} \oplus \overline{XYZ} \oplus \overline{Y^2Z} \cong \mathbf{Z}^3$ (where overlined objects are the images in the quotient group) with the direct sum ordering. Any pure trace of I that is not a restriction of a pure trace of R_P must annihilate $(X, Y)^3$ and so must be the projection onto one of the three summands. The pure traces of R_P are given by point evaluations $(x, y) \mapsto (r_1, r_2) \in (\mathbf{R}^2)^{++}$ and their limit points; because of the change of variables, these correspond to $(X, Y) \mapsto (r, s)$, where $0 \leq r, s$ and $r + s \leq 1$. Observe that the pure trace L of R_P given by $(r, s) = (0, 0)$ annihilates I (and this is the only pure trace of R_P to do so).

The pure traces of I are thus either point evaluations as indicated above, except that L is excluded, or the projections onto the three summands of $I/(X, Y)^3$. The projections onto $\overline{X^2Z}$ and $\overline{Y^2Z}$ are limits of restrictions of point evaluations, whereas the projection onto \overline{XYZ} is not. We observe that since M satisfies (**), $L(M/P)$ must be primitive; moreover, by Lemma 5.15, when M satisfies (**), every limit module of the form $H = \lim W \rightarrow W$ is uniform and so we can assume that $W = I^n$ for some order ideal I . So there really is not much loss of generality in taking $W = I \oplus I \oplus I$. Now

$$L(M/P) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

and its left nonnegative eigenvector is $v = (1 \ 1 \ \sqrt{3} - 1)$, with large eigenvalue $1 + \sqrt{3}$. There are precisely three choices for ϕ that are affiliated to L , as we have indicated above. If instead we had taken $W = R_P^3$, then $\phi = L$ is the only choice. •

7. A FOG example and generic nonnoetherianness

Here we discuss the bounded endomorphism ring of the FOG matrix M given earlier (Example 2.3); explicitly,

$$M = \begin{bmatrix} X & 0 & 1 \\ 1 & Y & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

We show that $E_b(G_M)$ is a commutative ring without zero divisors; it is *not* noetherian, and has a unique maximal order ideal. It has a face F of dimension one for which the the kernel of the corresponding map $E_b(G_M) \rightarrow E_b(G_{M_F})$ is contained in the kernel of a unique pure trace (one would expect it to be contained in the kernel of a continuum of pure traces), and the kernel is an order ideal without an order unit. Other somewhat weird properties are demonstrated.

To put this in context, consider what properties are satisfied by $E_b(G_M)$ when $n = 1$. In this case, $M = P$ a polynomial, and $E_b(G_M) = R_P$. Then R_P is a commutative finitely generated algebra (in particular, it is noetherian, and thus every order ideal admits an order unit), and its positive cone is generated additively and multiplicatively by elements of the form x^w/P where x^w appears in P . For every face F of the Newton polyhedron of F , the map $R_P \rightarrow R_{P_F}$ is onto and it follows that if F is one dimensional, the kernel is contained in the kernel a “line segment” of pure traces.

We show that this example is typical if d is at least 2. For instance, if $E_b(G_M)$ is noetherian, then M must satisfy the strong condition discussed earlier in connection with calculations of traces (essentially that $M^{n!}/P$ be an order unit in $E_b(G_{MP})$ where $P = \text{tr } M^{n!}$).

To avoid confusion between the generic x^w (to denote a typical monomial in this case in two variables), and X^k (the monomial in only the single variable X), we have used capital letters X and Y to denote the specific variables, and $x^w = X^{w(1)}Y^{w(2)}$ where $w = (w(1), w(2))$. To start the process, we verify the earlier statement that $T(M) = \{(w, k) \in (\mathbf{Z}^+)^2 \times \mathbf{N} \mid |w| \leq k - 3\}$; in other words, $x^w M^{-k}$ belongs to $E_b(G_M)$ (for $k > 0$) if and only if w is a pair of nonnegative integers such that $|w| \leq k - 3$. We use the

notation $|w|$ to denote the sum of the entries in w , i.e., the total degree of the monomial x^w . By using the graphical formulation of M , we can check exactly which monomials appear in each entry of M^k ("deg" refers to total degree):

$$\left[\begin{array}{ccc} X^k \& \text{all deg} \leq k-3 & \text{all deg} \leq k-2 & X^{k-1}, X^{k-2}, X^{k-3} \& \text{all deg} \leq k-4 \\ \text{all deg } k-1 \& \leq k-4 & Y^k \& \text{all deg} \leq k-3 & \text{all deg} \leq k-2 \\ \text{all deg} \leq k-2 & Y^{k-1}, Y^{k-2}, Y^{k-3} \& \text{all deg} \leq k-4 & \text{all deg} \leq k-3 \end{array} \right]$$

Now if $k > 0$, $x^w M^l \prec M^{k+l}$ for some positive integer l (i.e., $x^w M^{-k}$ belongs to $E_b(G_M)$) forces $|w| + l \leq k + l - 3$ (use the diagonal entries), i.e., $|w| \leq k - 3$. Conversely, if $|w| \leq k - 3$, it is easy to check that $x^w \mathbf{I} \prec M^k$. Thus $x^w M^{-k}$ belongs to $E_b(G_M)$ for $k > 0$ if and only if $w \geq (0, 0)$ and $|w| \leq k - 3$. In particular, if p is a real polynomial in the variables X and Y , then pM^{-k} belongs to $E_b(G_M)$ if $\deg p \leq k - 3$ (when $k > 0$). Thus we have the following.

Lemma 7.1 For $k > 0$, $x^w M^{-k}$ belongs to $E_b(G_M)$ if and only if $|w| \leq k - 3$ and $w \geq (0, 0)$. In particular, if p is a real polynomial in X and Y of total degree at most $k - 3$, then pM^{-k} belongs to $E_b(G_M)$.

Now we can get a fairly good description of the centralizer of M in $M_n A$.

Lemma 7.2 The characteristic polynomial of M is $(Z - X)(Z - Y)(Z - 1) - 1 = 0$, which is irreducible over $A = \mathbf{R}[X, Y]$; thus the centralizer of M is commutative and has no zero divisors. The matrix M is $\text{GL}(3, A)$ -conjugate to a companion matrix. The centralizer of M in $M_n A$ is thus $A[M]$, the polynomials in M with coefficients from A .

The first part of the first statement is an easy calculation, and the rest of the first statement follows directly. The second statement is routine to verify, and the rest follows immediately.

Thus $E(G_M)$ is naturally isomorphic to $A[M, M^{-1}]$ (a subring of $M_n \mathbf{R}(X, Y)$, the matrix ring over the rational function field), and under this identification, an element NM^j (where N is in $A[M]$ and j is an arbitrary integer) is positive if and only if there exists a positive integer l so that all of the entries of NM^l have no nonnegative coefficients. Now as an A -module, $A[M]$ is obviously free on the set $\{\mathbf{I}, M, M^2\}$. So every element of $A[M]$ can be written uniquely in the form $N = p_0 \mathbf{I} + p_1 M + p_2 M^2$, where p_i belong to A , i.e., are polynomials. Now we decide when an element NM^{-k} belongs to $E_b(G_M)$.

We have previously noted that if p is a polynomial (with real coefficients) whose total degree is $k - 3$ or less, then pM^{-k} belongs to $E_b(G_M)$. Hence the vector space,

$$J := \{pM^{-k} \mid k \geq 2, \deg p \leq k - 3\}$$

is a (unitless) subalgebra of $E_b(G_M)$. Now we show the following.

Lemma 7.3 (a) $E_b(G_M) = J + \mathbf{R}\mathbf{I}$

(b) J is an order ideal of $E_b(G_M)$, and it contains no order unit (relative to itself); in particular, J is not a finitely generated ideal of $E_b(G_M)$.

(c) $J^2 \subseteq M^{-3}J$.

To this end, suppose that NM^{-k} belongs to $E_b(G_M)$ with N in the centralizer of M . This means that for some $l \geq 0$, $NM^l \prec M^{l+k}$. We will obtain necessary and sufficient conditions for which $N \prec M^k$; these may then be applied to $NM^l \prec M^{l+k}$, and so we may assume $l = 0$. We write $N = p_0 \mathbf{I} + p_1 M + p_2 M^2$ as before; then

$$N = \begin{bmatrix} p_0 + Xp_1 + X^2p_2 & p_2 & p_1 + (X + 1)p_2 \\ p_1 + (X + Y)p_2 & p_0 + Yp_1 + Y^2p_2 & p_2 \\ p_2 & p_1 + (Y + 1)p_2 & p_0 + p_1 + p_2 \end{bmatrix}.$$

From $N \prec M^k$ and using the form of the monomials appearing in M^k displayed previously, we deduce the following:

- (1,1) $p_0 + Xp_1 + X^2p_2 = aX^k + q$ where $\deg q \leq k - 3$, and a is a real number;
- (2,2) $p_0 + Yp_1 + Y^2p_2 = bY^k + r$ where $\deg r \leq k - 3$, and b is a real number;
- (3,3) $\deg(p_0 + p_1 + p_2) \leq k - 3$;
- (1,2) $\deg p_2 \leq k - 2$;
- (1,3) $p_1 + (X+1)p_2 = cX^{k-1} + c'X^{k-2} + c''X^{k-3} + s$ where $c, c',$ and c'' are real numbers and $\deg s \leq k-4$;
- (2,1) $p_1 + (X+Y)p_2 = t + u$ where t is homogeneous of degree $k - 1$ and $\deg u \leq k - 4$;
- (3,2) $p_1 + (Y+1)p_2 = dY^{k-1} + d'Y^{k-2} + d''Y^{k-3} + v$ where $d, d',$ and d'' are real numbers and $\deg v \leq k - 4$.

Subtracting the (2,2) entry from the (1,1) entry, we obtain $(X - Y)(p_1 + (X + Y)p_2) = aX^k - bY^k + (q - r)$. Since $X - Y$ is homogeneous and $q - r$ has total degree at most $k - 3$, $X - Y$ must divide $aX^k - bY^k$, which of course forces $a = b$. Obviously, $N' := N - aM^k \prec M^k$ in any case, and all of the diagonal entries of N' are of degree at most $k - 3$. Now we show that $N'M^{-k}$ belongs to J . Relabel the p_i so that $N' = p_0I + p_1M + p_2M^2$; as $N' \prec M^k$, all six conditions still apply, and additionally $a = b = 0$. Subtract the (3,3) entry from the (1,1) entry; we deduce $\deg(X - 1)(p_1 + (X + 1)p_2) \leq k - 3$; this forces $\deg p_1 + (X + 1)p_2 \leq k - 4$; in view of the condition from the (1,3) entry, this forces $c = c' = c'' = 0$. Similarly, subtracting the (3,3) entry from the (2,2) entry, we obtain $d = d' = d'' = 0$. Subtracting the (2,2) entry from the (1,1) entry, we obtain $\deg(X - Y)(p_1 + (X + Y)p_2) \leq k - 4$, and again this forces t to be the zero polynomial.

From $\deg p_1 + (X + 1)p_2 \leq k - 4$ and $\deg p_1 + (Y + 1)p_2 \leq k - 4$, we deduce that their difference, $(X - Y)p_2$ has degree at most $k - 4$, whence $\deg p_2 \leq k - 5$. Thus $p_2M^2 \prec M^k$. Since $\deg p_2 \leq k - 5$, it follows from $\deg p_1 + (X + 1)p_2 \leq k - 4$ that $\deg p_1 \leq k - 4$. Hence $p_1M \prec M^k$. From entry (3,3), $\deg p_0 + p_1 + p_2 \leq k - 3$, which we see now forces $\deg p_0 \leq k - 3$. Thus $p_0I \leq M^k$. So $N = aM^k + p_0I + p_1M + p_2M^2$ and each of the four terms is dominated by M^k . Thus $NM^{-k} - aI \in J$. Thus $E_b(G_M) = J + \mathbf{R}I$ as desired.

Obviously, J is generated as a vector space (or even as an abelian group) by its positive elements (positive multiples of $x^w M^{-k}$, where $|w| \leq k - 3$). The convex polytope $K(M)$ is simply the standard triangle, with vertices $\{(0, 0), (1, 0), (0, 1)\}$; let F be the face consisting of the edge joining $(1, 0)$ to $(0, 1)$. It is easy to check that the map $E_b(G_M) \rightarrow E_b(G_{M_F})$ kills J ; as J is of codimension 1, the kernel of this map is exactly J , and so the map $E_b(G_M) \rightarrow E_b(G_M)/J$ is positive. As $E_b(G_M)/J \cong \mathbf{R}$, it has a unique trace, which in turn induces a trace, necessarily pure, on $E_b(G_M)$. Any positively generated ideal that is the kernel of a pure trace is an order ideal. So J is an order ideal.

Now we show that J admits no order units (with respect to itself, obviously). Every element of J is of the form pM^{-k} where $k \geq 3 + \deg p$. Suppose $U = pM^{-k}$ is an order unit of J . Since the operations of increasing an order unit (but still staying within J) and multiplying by scalars both preserve the order unit property, we may assume that $p = (1 + X + Y)^{k-3}$. Let v be an element of $(\mathbf{Z}^+)^2$ and l a positive integer such that $|v| \leq l - 3$, so that $x^v M^{-l}$ belongs to J . There thus exists a positive real number $K \equiv K(v, l)$ such that $x^v M^{-l} \leq pM^{-k}$. Set $l = k + 1$, and let x^v vary over the monomials of degree $k - 2$ or less. We can add integer multiples of the corresponding inequalities and we obtain $(1 + X + Y)^{k-2} M^{-k-1} \leq K'(1 + X + Y)^{k-3} M^{-k}$. Hence for some integer $m > 0$, $(1 + X + Y)^{k-2} M^m \prec (1 + X + Y)^{k-3} M^{m+1}$. However, the (1, 1) entry of the left side contains the monomial $Y^{k-2} X^m$ whereas the (1, 1) entry of the right side does not (its monomials of degree $m + k - 2$ are $Y^i X^{m+k-2-i}$ with $i \leq k - 3$; a narrow miss). Hence J has no order units. If J were finitely generated as an ideal in a ring that has 1 as an order unit, it would have an order unit relative to itself (proof: decompose each member of the generating set as a difference of two positive elements; the sum of all of these elements would be an order unit, since every element of the ring is bounded above and below by a multiple of the identity).

To prove (c), it is sufficient to show the product of two of the generators of the form pM^{-k} factors as

M^{-3} times an element in J ; this is trivial. •

Next, we consider the positive kernels of the pure traces. If R is an ordered ring in which 1 is an order unit, then any pure trace is multiplicative. If τ is such a pure trace, define $\text{Ker}^+ \tau$ to be the set of differences of the positive elements in the kernel of τ ; it is easy to see that $\text{Ker}^+ \tau$ is both an ideal and an order ideal, and is the largest order ideal of R inside the kernel of τ ; moreover, τ induces a faithful pure trace on $R/\text{Ker}^+ \tau$. We first wish to determine the order ideals in $E_b(G_M)$ that are of the form $\text{Ker}^+ \tau$ for some pure trace τ .

Define

$$J_X := \{pM^{-k} \mid p \in A, \deg p \leq k - 3, \partial p / \partial X \neq 0\}$$

$$J_Y := \{pM^{-k} \mid p \in A, \deg p \leq k - 3, \partial p / \partial Y \neq 0\}.$$

It is clear that J_X and J_Y are ideals of $E_b(G_M)$ contained in J . In fact, each is of the form $\text{Ker}^+ \tau$ for suitable τ . Let F be the face of $K(M)$ (the standard triangle) consisting of the line segment $Y = 0, 0 \leq X \leq 1$. Then M_F is obtained by simply replacing Y by 0, and the map $\pi_F : E_b(G_M) \rightarrow E_b(G_{M_F})$ clearly kills J_Y and is one to one on $J \setminus J_Y$. If τ is a faithful pure trace of $E_b(G_{M_F})$ (necessarily arising from a positive left eigenvector of M_F), then it easily follows that for $\tau := \tau \circ \pi_F$, $\text{Ker}^+ \tau = J_Y$. Similarly, we obtain J_X as the positive kernel of a trace factoring through the map which sends X to 0. The singleton face consisting of the origin, corresponds to sending both X and Y to 0, and we see easily that the positive kernel of this is $J_X + J_Y$. Of course, we have seen already that J is similarly obtained from the face described by $X + Y = 1$. It turns out that this describes all the positive kernels of pure traces, and this can be used to characterize all the pure traces of $E_b(G_M)$.

Let τ be a pure trace of $E_b(G_M)$. If τ is faithful, it arises from a left eigenvector as in Theorem 4.5. If not, it kills a positive element. If we go through the procedure of 3.7–3.8, (taking a suitable power of M and dividing by a monomial in x so that the Log sets of the diagonal entries generate \mathbf{Z}^2 as a semigroup), we see that τ if it kills anything, must kill $(XM^{-3})(YM^{-3})(M^{-3})$, and so it must kill one of the three parenthetical expressions. However, if it kills the third one, it kills J^2 , hence kills J , and as J is of codimension one, τ would have to be the map $E_b(G_M) \rightarrow E_b(G_M)/J$ corresponding to the diagonal edge. If it fails to kill M^{-3} , then we can invert the latter, and τ will extend to a positive multiplicative linear functional, also called τ , on $E_b(G_M)[M^3]$; the latter contains $\mathbf{R}[X, Y, M^{\pm 1}]$, and it is straightforward to see that $E_b(G_M)[M^3]$ actually equals the former.

In any event, $\tau(X)$ and $\tau(Y)$ are now defined. If they are both not zero, then τ is faithful; if say $\tau(X) = 0$, we see that τ is now defined on $E_b(G_M)[M^3] = \mathbf{R}[X, Y, M^{\pm 1}]$ factored out by sending $X \rightarrow 0$, which yields $\mathbf{R}[Y, M_F^{\pm 1}]$ where F is the face described by $X = 0$. Since M_F is itself in companion matrix form, $\mathbf{R}[Y, M_F^{\pm 1}] = E_b(G_{M_F})[M_F^3]$. Thus τ induces a multiplicative trace on $E_b(G_{M_F})$. If $\tau(Y) = 0$, we can factor this out and repeat, and end up with τ factoring through the quotient induced by the singleton face $(0, 0)$, whose kernel is exactly $J_X + J_Y$; if $\tau(Y) \neq 0$, then the trace induced by τ on $E_b(G_{M_F})$ is faithful, and thus the positive kernel of the original τ is J_X .

Lemma 7.4 (a) Every nonzero order ideal of $E_b(G_M)$ that is of the form $\text{Ker}^+ \tau$ for some pure trace τ is one of J (corresponding to F being the diagonal edge or either vertex of the diagonal edge of $K(M)$), J_X (corresponding to the vertical edge), J_Y (corresponding to the horizontal edge), and $J_X + J_Y$ (corresponding to the vertex $(0, 0)$).

(b) Every proper order ideal of $E_b(G_M)$ is contained in J .

(c) If e is a positive element of $E_b(G_M)$ not in J , then e is an order unit of $E_b(G_M)$.

Part (a) has been proved above. Towards (b), let I be a proper order ideal, then $E_b(G_M)/I$ is partially ordered ring with the identity element as order unit; a pure, hence multiplicative trace thus exists. Hence I is in the positive kernel of a pure trace of $E_b(G_M)$, so must be one of $J_X, J_Y, J_X + J_Y$, or J . However, all of them are contained in J , and we are done.

Now (c) is trivial. If e were not an order unit, being positive, it would generate a proper order ideal, hence belong to J (by (b)), a contradiction. •

Lemma 7.4(c) is of course the analogue of [HM, V.3(d)] which holds for R_P ; it asserts that there is a finite set of maximal order ideals (one for each vertex of the Newton polyhedron of P) such that if e in R_P^+ does not belong to any of them, then e is an order unit.

In view of the structure of J , it is reasonable to ask if the positive cone of either $E_b(G_M)$ or J is generated by terms of the form $x^w M^{-k}$ and the identity (i.e., can a positive element be expressed as a positive linear combination of products of such elements). Multiplying by suitable powers of M , this is equivalent to asking that if N is a matrix in $M_n A$ centralizing M all of whose entries have no negative coefficients, then there exists l such that NM^l is a positive linear combination of terms of the form $x^v M^j$ (j being nonnegative, $v \geq 0$, with no restriction on the degrees of the monomials). This would be the precise analogue of the result for R_P [HM, Section I], which asserts that the positive cone of R_P is generated multiplicatively and additively by terms of the form x^w/P .

Unfortunately, this property fails here. As a simple example, begin with $N := M^3 - I$. We see by direct calculation that this is positive, and so $e := NM^{-3}$ is a positive element of $E_b(G_M)$, and its image modulo J is 1. If e were in the positive cone generated by $x^w M^{-k}$ and the identity, we see that since all the generators but the identity are in J , it would have to be written as a positive element of J plus I . However, it is clearly in the form, a negative element of J plus I , which is a contradiction. It follows of course that eM^{-3} is an element of J which is also not so expressible.

In any event, we can write down all the pure traces of $E_b(G_M)$. The faithful ones correspond to evaluating X and Y at nonzero real numbers and are implemented by eigenvectors as in Theorem 5.14. The non-faithful ones factor through the appropriate facial map and are obtained as faithful traces on the appropriate quotients, which are obtained by either evaluating at least one of X or Y at 0, or by killing J . Note that the natural order structure on the faces is not precisely reflected in the corresponding ideals—e.g., the diagonal edge is disjoint from the origin, but the order ideal corresponding to the former (J) contains the order ideal corresponding to the latter ($J_X + J_Y$).

In this example, what seems to cause the most effects (the weirdest of which is nonnoetherianness) is the fact that XM^{-1} is not in $E_b(G_M)$, even though X appears in $\text{tr } M$. Many of these phenomena hold much more generally. For example, although for $M' = (1 + X + Y)M$, the facial ideals are ordered corresponding to the faces, $E_b(G_{M'})$ is still not noetherian, even though it now contains lots of elements, e.g., it contains IR_P where $P = \text{tr } M$.

There is a quite strict necessary condition in order that $E_b(G_M)$ be noetherian; in the one variable case, this condition is always satisfied (by Proposition 2.9—and presumably $E_b(G_M)$ is noetherian in this case), but with more variables, it fails generically.

Proposition 7.5 Let M belong to $M_n A^+$, and let P be an element of A^+ such that $\text{Log } P = \text{Log } \text{tr } M^{n!}$. If every order ideal of $E_b(G_M)$ admits an order unit, then there exists m such that $P^{m+1} M^{n!m} \prec P^m M^{n!(m+1)}$. In particular, this applies if $E_b(G_M)$ is noetherian.

Proof. First, replace M by $M^{n!}$, so that $\text{Log } P = \text{Log } \text{tr } M$, and $\text{Log } P^k = \text{Log } \text{tr } M^k$ for all k . By dividing by any monomial x^w such that $x^w M^{-1}$ belongs to $E_b(G_M)$, we may assume that 0 is in $\text{Log } P$ as well. By Theorem 5.3, we may further replace the current M by some higher power, in order to guarantee that $M^2 \prec QM$, where $\text{Log } Q = \text{Log } \text{tr } M$ and $\text{Log } Q = l \text{Log } P$ for some integer l .

Now the first claim in the proof of Theorem 2.8 asserts that there is a point w in \mathbf{Z}^d and an integer k such that for all $0 \leq s \leq t$, $(w + \text{Log } \text{tr } M^s, t + k)$ belongs to $T(M)$. That is, $x^{w+v} M^{-(k+t)}$ belongs to $E_b(G_M)$ for all v in $\text{Log } \text{tr } M^s = \text{Log } Q^s$ for all s between 0 and t . Since 0 belongs to $\text{Log } P$ and hence to $\text{Log } Q$, and thus $\text{Log } Q^s \subseteq \text{Log } Q^t$; we thus have that everything of the following form also belongs to $E_b(G_M)$,

$$x^w M^{-k} \cdot p M^{-t} \quad \text{for any } p \text{ in } A \text{ with } p \prec Q^t.$$

Define J_t to be the order ideal of $E_b(G_M)$ generated by $x^{w+v}M^{-(k+t)}$ where v ranges over $\text{Log } Q^t$. Obviously, $x^w M^{-k} Q^t M^{-t}$ is an order unit for J_t . We claim that for $t > 1$, $J_t \subseteq J_{t+1}$. This follows from $M^{t+1} \prec Q M^t$ —multiply both sides by $x^w Q^t M^{-2t-1-k}$ to obtain that $x^w M^{-k} Q^t M^{-t}$ is in the order ideal generated by $x^w M^{-k} Q^{t+1} M^{-t-1}$.

Hence $\{J_t\}$ is an increasing family of order ideals of $E_b(G_M)$; set J to be their union. Obviously J is an order ideal of $E_b(G_M)$. If it had an order unit, that order unit would have to belong to one of the J_t , and we would deduce that for some t , $J_t = J_{t+1}$. However, this means that there would exist a positive integer K such that $x^w M^{-k} Q^{t+1} M^{-t-1} \leq K x^w M^{-k} Q^t M^{-t}$ in $E_b(G_M)$. This translates to $x^w Q^{t+1} M^u \prec x^w Q^t M^{u+1}$ in $M_n A^+$ for some positive integer u , and hence $Q^{t+1} M^u \prec Q^t M^{u+1}$. We can certainly increase the powers t and u appearing there, and after we translate back to our original choice for M and P , we obtain the desired result.

In a partially ordered noetherian ring for which 1 is an order unit, every order ideal admits its own order unit (every order ideal is an ideal, therefore requires finitely many generators as a right ideal; express each of the generators as a difference of positive elements in the order ideal, and add the positive elements; the result is an order unit for the ideal). •

For example, if $M = \begin{bmatrix} x & 1 \\ 1 & y \end{bmatrix}$, then it is an easy exercise to check that for $k \geq 2$, $\text{Log } (M^k)_{1,1}$ is $\{(k, 0)\} \cup \{(a, b) \in (\mathbf{Z}^+)^2 \mid a + b \leq k - 2\}$. We note that $(x + y + 1)^{k+1} M^k$ always contains a $y^{k+1} x^k$ term in the (1, 1) entry whereas $(x + y + 1)^k M^{k+1}$ does not, this for every k . Since $\text{Log } (x + y + 1)^2 = \text{cvx tr } M^2 \cap \mathbf{Z}^2$, it follows that the criterion of Proposition 7.5 fails in this example, and hence $E_b(G_M)$ is not noetherian. Similar computations apply to M of the previous example, showing again that $E_b(G_M)$ is not noetherian.

It is an easy exercise to verify that every order ideal admitting an order unit is equivalent to the ascending chain condition on order ideals. I still don't know if the condition obtained here, or the stronger condition (that $\text{cvx } T(M)$ be closed) is sufficient to imply every order ideal has an order unit, although this seems plausible. If M is any primitive matrix in $M_n A^+$ and $P = \text{tr } \text{Log } M^{n!}$, then $M^{n!} + P I$ satisfies the necessary conditions of Lemma 7.4.

If l is a loop of length k determined by the directed weighted graph (with weights in \mathbf{Z}^d) corresponding to M , define $\text{wt}(l)$ to be the average value of the weights of the arcs that constitute it; that is, take the sum of the weights and divide by k . This is the “weight per symbol” of the loop ($[MT]$). Let $K(M)$ be the normalized convex polyhedron associated to M , as usual. For a face E of $K(M)$, define

$$S(E) := \{i \in \{1, 2, \dots, n\} \mid \text{there exists a loop } l \text{ containing } i \text{ such that } \text{wt}(l) \in E\}$$

Say a matrix M in $M_n A^+$ is *drawn* if for $P = \text{tr } M^{n!}$, $P^{m+1} M^{n!m} \prec P^m M^{n!(m+1)}$ for some integer m . So if every order ideal in $E_b(G_M)$ has an order unit, e.g., if $E_b(G_M)$ is noetherian, then M is drawn. Call M *tight* if $Q M^{-n!k}$ belongs to $E_b(G_M)$ for some k wherein $\text{Log } Q = \partial_e k K(M)$; this is equivalent to the convex hull of the range of $T(M)$ (c.f., section 2) being closed. From the absorption lemma 5.7, it follows that tight implies drawn.

First we give a necessary condition in order that M be drawn. Let K be a (compact) convex polyhedron in \mathbf{R}^d (the definitions, with minor modifications, can be made to work in infinite dimensions), and let F be a face, and suppose that F is the convex hull of two disjoint closed faces, say $F = \text{cvx } \{E, F'\}$. We say the triple $(F; E, F')$ has *parallel faces* if no matter how an arbitrary element f of F is written as a convex combination of elements of E and F' , say $f = \alpha e + (1 - \alpha) f'$ (with e in E , f' in F' , and α in $[0, 1]$), the coefficient α is unique. In case F is a simplex, or if one or the other of E or F' is a singleton (i.e., a face consisting of an extreme point), then of course this property holds, because the e and f' themselves are unique. An example of a complemented face $F = E \vee F'$ wherein even the uniqueness of the coefficient fails occurs with E and F' being two line segments in the plane (placed so that each is still an edge of the convex hull) that are *not* parallel. On the other hand, if there exists a codimension one affine subspace, V ,

of the affine span of F such that E and F are contained in translates of V , then uniqueness of the coefficient holds, even though uniqueness of the decomposition may fail (as it does for any nontrivial trapezoid, with E and F' being the parallel edges).

The principal use of the following occurs when E and F' consist of the endpoints of an edge F .

Lemma 7.6 Let M be a primitive element of $M_n A^+$ that is drawn. Let i be any state, and let F be a face of $K(M)$ and suppose that $(F; E, F')$ has parallel faces. For any state i , if there exists a loop l containing i such that $\text{wt}(l)$ lies in E , then there exists a loop l' containing i such that $\text{wt}(l') \in F \setminus E$.

Proof. If not, for some i , there exists a loop passing through i with weight in E , and every loop passing through i having weight in F has weight in E . Say w in E is realized by one such loop l ; by the facial property, it is easy to see that we may assume that l is minimal, so its length is n or less. Hence there exists an integer $t \leq n$ such that tw appears in $\text{Log}(M^t)_{ii}$; as a consequence, $w' := n!w$ appears in $\text{Log}(M^{n!})_{ii}$, and for any positive integer k , kw' appears in $\text{Log}(M^{kn!})_{ii}$.

Pick an extreme point, f' , of F not in E (necessarily, f' belongs to F'). Then $(k+1)n!f'$ appears in $\text{Log} P^{k+1}$. Thus

$$(k+1)n!f' + kn!w \in \text{Log}(P^{k+1}M^{kn!})_{ii}.$$

Write this as

$$z := (2k+1)n! \left(\frac{k+1}{2k+1} f' + \frac{k}{2k+1} w \right) \in (2k+1)n!F.$$

By hypothesis, there exists k so that z (which obviously depends on k) belongs to $\text{Log}(P^k M^{(k+1)n!})_{ii}$. By the facial property, z belongs to the F -component of this, i.e., z belongs to

$$\text{Log}(P_{n!F}^k) \left(M_{(k+1)n!F}^{n!(k+1)} \right)_{ii}.$$

Thus we may write $z = (kn!)f_1 + (k+1)n!e_1$ where f_1 and e_1 belong to F and there is a loop passing through i (of length $(k+1)n!$) of weight e_1 . By hypothesis, e_1 belongs to E .

When we decompose $z/(2k+1)n!$ according to the first decomposition, we see that the coefficient of the E term (w) is $k/(2k+1)$. However, when we decompose f_1 into its F' and E components, we see that the coefficient of the E component in the second decomposition of $z/(2k+1)n!$ is at least $(k+1)/(2k+1)$, arriving at two decompositions with distinct coefficients, contradicting the parallel faces hypothesis. •

Here is a sufficient condition for M to be tight, which as we saw above, is stronger than being drawn. This is usually very efficient, as it only involves minimal loops (a distinct advantage over the previous necessary condition).

Lemma 7.7 Let M be a primitive element of $M_n A^+$. Sufficient for M to be tight is the following:
 For all states i , for all minimal loops l passing through i , for all v in $\partial_e K(M)$, there exist minimal loops l' and l'' such that l' passes through i , $\text{wt}(l) = \text{wt}(l')$, $\text{wt}(l'') = \text{wt}(l)$, and l' has a state in common with l'' .

Proof. This is practically tautological. It is sufficient to show that $n!v + \text{Log}(M^{n!})_{ij} \subseteq \text{Log}(M^{2n!})_{ij}$ for all extreme points v . Decompose an arbitrary path of length $n!$ from i to j as the union of a bunch of minimal loops plus a short path with no repetitions of states from i to j passing through into minimal loops. Then perform the obvious surgery on the loops to “multiply” by $x^{n!v}$ ($n!$ is excessively large; $\text{lcm}\{2, 3, 4, \dots, n\}$ would do just as well) and see the new path of length $2n!$. •

Here are a few easy illustrative examples. They become almost transparent when written as graphs.

Example 7.8 Let

$$A = \begin{bmatrix} x & 1 \\ 1 & y \end{bmatrix} \quad B = \begin{bmatrix} 0 & x+y & 0 \\ y & 0 & x \\ x & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & x+y & 0 \\ y & 0 & x \\ x & 0 & 0 \end{bmatrix}$$

Then A and B fail to be drawn, but C is tight, hence drawn. All are primitive.

Proof. Primitivity is obvious for A , and it follows for B and C from the fact that there are minimal loops of lengths 2 and 3 in matrices of size 3.

For A , there are three minimal loops with average weights the three vertices, respectively, of the standard triangle in \mathbf{Z}^2 . The only loop (minimal or not) containing state 1 (the top entry) with weight in the face, F , spanned by $\{(0, 1), (1, 0)\}$ is the self loop with weight $(1, 0)$, hence the conditions of Lemma 7.6 are violated (here $E = \{(1, 0)\}$ and $F' = \{(0, 1)\}$).

Concerning B , there are three minimal loops of lengths 3, 2, and 1 with average weights respectively $(1, 0)$, $(0, 1)$, and $(0, 0)$ respectively. With face $F = \text{cvx} \{(0, 0), (0, 1)\}$, the only loop containing state 3 and having weight in F is the self-loop, and again the conditions of Lemma 7.6 are violated.

The matrix C is obtained from B by moving the self-loop from state 3 to state 1 (moving it to state 2 would also yield a tight matrix). Now we note that the three minimal loops have a state in common, and it follows that the criterion of Lemma 7.7 is satisfied. •

Along these lines, there is also a necessary condition for a matrix to be tight. (Recall that an *edge* of a convex polyhedron is a one dimensional face.)

Lemma 7.9 If M is a primitive tight matrix in $M_n A^+$, then for each minimal loop l of weight v in $\partial_e K(M)$, for all states i which l passes through, and for all w in $\partial_e K(M) \setminus \{v\}$ such that $\text{cvx} \{v, w\}$ is an edge of $K(M)$, there exist minimal loops l' and l'' such that $\text{wt}(l') = v$, l' passes through i , $\text{wt}(l'') = w$, and there is a common state through which l' and l'' pass.

The proof of this is more or less straightforward.

A more drastic condition than tightness is that $PM^{-n!}$ belong to $E_b(G_M)$ (where $\text{Log } P = \text{Log tr } M^{n!}$ as usual). By Lemma 5.1, this forces $PM^{-n!}$ to be an order unit of $E_b(G_M)$. Much more is true.

If R is an ordered ring, and r is a positive central element (which may be a zero divisor), then we may form what we shall abusively call $R[r^{-1}]$, obtained as the limit of repeated multiplication by r , with the resulting direct limit ordering. In case r is a zero divisor, the ring is obtained by factoring out from R all elements that multiply some power of r to zero (this is a two-sided ideal), and inverting the image of r (possible because r and its image are central). If r has the property that $rr' \geq 0$ implies $r' \geq 0$ in R , then of course $R \rightarrow R[r^{-1}]$ is a relative order embedding (in the sense that if s in R has positive nonzero image in $R[r^{-1}]$, then s already is in R^+). We usually assume that any element s such that $r^m s = 0$ is not positive, which is what happens below.

In the present situation, $R = E_b(G_M)$; let Q be a polynomial such that QM^{-k} belongs to $E_b(G_M)$. In general, there is no guarantee that $rr' \geq 0$ implies $r' \geq 0$ for r' in $E_b(G_M)$. Related to this is the natural map $E_b(G_M) \rightarrow E_b(G_{MP})$; (a) of interest is when is it a relative order embedding, or at least (b) every pure trace on the former lifts to the latter. We conjecture that (b) is always true; if this were the case, we would obtain more information on traces on $E_b(G_M)$.

Proposition 7.10 Let M be a primitive matrix in $M_n A^+$, and suppose that for some (hence any) P in A^+ such that $\text{Log } P = \text{Log tr } M^{n!}$, $PM^{-n!}$ belongs to $E_b(G_M)$. The following hold:

- (a) If u is a column with entries from A^n such that $P^f u$ belongs to $(A^n)^+$ for some integer f , then there exists k such that $M^k u$ belongs to $(A^n)^+$.
- (b) As ordered rings, $E_b(G_{MP}) = E_b(G_M)[(P/M^{-n!})^{-1}]$, and the map $E_b(G_M) \rightarrow E_b(G_{MP})$ is an order embedding inducing a bijection on order ideals.

(c) All traces of $E_b(G_M)$ extend uniquely to traces of $E_b(G_{MP})$.

Proof. (a) By Theorem 5.3, for all sufficiently large k , $M^{kn!} \prec PM^{(k-1)n!}$. From $PM^{-n!}$ in $E_b(G_M)$, we obtain that for all sufficiently large l , $PM^{(l-1)n!} \prec M^{n!l}$. Thus with $m \geq \max\{k, l\}$, we have $PM^{(m-1)n!} \prec M^{n!m} \prec PM^{(m-1)n!}$. Hence for all i and j , we have that

$$\text{Log} (M^{n!m})_{ij} = \text{Log } P + (M^{n!(m-1)})_{ij}.$$

Set $N = M^{n!m}$, and write $u = (u_s)^T$; from $P^f u_s$ in A^+ , it obviously follows that for all i and j , $(P(M^{n!(m-1)})_{ij})^f u_s$ also belongs to A^+ . By [H2, Theorem 2.2], there exists a positive integer $h_0 \equiv h_0(i, j, s)$ such that $(N_{ij})^{h_0} u_s$ belongs to A^+ . Set $h = \max_{i,j,s} \{h_0(i, j, s)\}$.

Now consider $N^{n^2 h}$. Every entry is a sum of products of hn^2 entries of N , hence every such product must contain a product of at least h identical terms, i.e., $(N_{ij})^h$ for some ij . Each such product thus renders any u_s positive (all coefficients nonnegative), so that every entry of $N^{n^2 h}$ will do the same. Thus $N^{n^2 h} u$ will belong to $(A^n)^+$, so $k = n^2 m h n!$ will do.

(b) Suppose $e = N(MP)^{-k}$ is in $E_b(G_{MP})$; then there exists l such that $N(MP)^l \prec (MP)^{l+k}$; we may increase l as much as we like. As in the beginning of the proof of (a), $(MP)^{l+k} \prec M^{l+k+n!(l+k)}$ (for sufficiently large choice of l). Hence $NP^l M^{-(k+n!(l+k))} := e'$ belongs to $E_b(G_M)$. Then $e' \cdot (M^{n!}/P)^{l+k} = e$ expresses e as an element of $E_b(G_M)[(P/M^{-n!})^{-1}]$. Now part (a) shows that the two possible positive cones on the latter are equal (since we just apply it to all columns of the relevant matrices). It is an easy exercise to check that the map is an embedding, and it then follows from (a) that it is an order embedding. It is also easy to see that two distinct order ideals in $E_b(G_M)$ will remain so in the larger ring.

(c) If $A \rightarrow B$ is an order preserving map of unperforated partially ordered abelian groups with A admitting an order unit which is also an order unit for B such that if any element of A that becomes an order unit for B is then an order unit for A , then traces extend [H8, Lemma 1.4]. Uniqueness follows from uniqueness for the pure traces, which follows from the fact that the extension of the traces is uniquely determined by the its value on the inverted element. •

The really interesting property is (c); that is because (c) is possibly true for *all* primitive matrices M . If this were the case, then we could analyze the pure traces on order ideals of G_{MP} and then get information about the traces of corresponding order ideals of G_M (pure traces may extend in more than one way in this more general setting, as in the standard FOG example; however, we would obtain the trace space as a quotient of the known trace space modulo the obvious equivalence relation). Moreover, traces on order ideals of G_{MP} and $E_b(G_{MP})$ are easier to analyze, because we can use R_P . We notice that $M^{n!}/P$ is a bounded endomorphism of $(R_P)^n$, and the limit is order isomorphic to G_{MP} . The matrix M is drawn if and only if the automorphism of G_{MP} induced by $M^{n!}/P$ is an order unit in $E_b(G_{MP})$ (which entails that $E_b(G_{MP})$ is not generally obtainable from $E_b(G_M)$ by inverting $P/M^{n!}$). The traces of R_P are completely known, and in almost all cases, the traces of order ideals factor through the latter.

A plausible argument (that comes very close to succeeding) that would prove that traces do extend (from $E_b(G_M)$ to $E_b(G_{MP})$) is the following. Pick a pure trace of $E_b(G_M)$; if it is faithful, it is given by a point evaluation and an eigenvector, and it is clear how to extend it. Otherwise, it must kill one of the facial ideals I_F (Proposition 3.8), hence if we pick the smallest F (largest I_F) for which this holds, we obtain a trace on $E_b(G_M)/I_F$. The natural map $E_b(G_M)/I_F \rightarrow E_b(G_{M_F})$ is clearly one to one and an order-preserving. If we knew that traces extend here, we would be done. It is quite likely that this map is even an order embedding, in which case of course, trace extension is immediate. If the range is one dimensional (as occurs in the FOG example, for any face F), then trace extension is automatic.

The problem is that we do not know what the quotient ordering really looks like; explicitly, if N is in the centralizer of M , $N \prec M^{-k}$, and N_{kF} has all of its entries in A^+ , does it follow that there is another matrix $N' \prec M^l$ commuting with M such that all entries of N' have no negative coefficients and $N' M^{-l}$

has the same image as NM^{-k} ; in other words, do positive elements of $E_b(G_{M_F})$ that come from $E_b(G_M)$ lift to positive elements? (To complete the argument, if the trace does extend to $E_b(G_{M_F})$, then modulo some primitive blocks of M_F which it kills, it is a faithful trace and would then extend to an even larger ring than $E_b(G_{M_F P_{n!F}})$ which is more than enough to extend it to $E_b(G_{MP})$.)

It also would be desirable to answer the following questions, for M a primitive matrix in $M_n A^+$:

- 1 If M is drawn, then is it tight?
- 2 If M is tight, i.e., for some k , $QM^{-n!k}$ (where $\text{Log } Q = \partial_e k K(M)$ and Q has no negative coefficients) is in $E_b(G_M)$, then is it an order unit?

Whenever the trace extension question (for $E_b(G_M) \rightarrow E_b(G_{MP})$) has an affirmative answer, then so does (2). Moreover, regardless of the answer to (2), the element Q satisfies Proposition 7.10(a) (virtually the same argument, since the crucial tool [H2, Theorem 2.2] was proved in just the right generality to obtain this). This in turn leads to another question, which unfortunately has a negative answer.

One of the most interesting results in the study of R_P is the fact that in R_P , if u is an order unit and r is an arbitrary element, then $ur \geq 0$ implies $r \geq 0$ (an alternative formulation of [op. cit.]). For an ordered ring R , let us call this the *order unit cancellation property*. This also holds for the ordered rings arising from certain characters of compact Lie groups, although it is not explicitly stated in this fashion in [H8]. The comment of the previous paragraph says that in $E_b(G_M)$ (and somewhat more generally), order unit cancellation holds for elements of the form $QM^{-n!k}$ (at least when they are order units). It would be useful to know that order unit cancellation holds for all order units in $E_b(G_M)$ (this would give a quick proof of trace extension in the case that M is tight). Unfortunately, it fails, even for size two matrices in one variable!

The example is ludicrously simple; it is just a translation of [H9, Example 5.3]. Set

$$M = (x+1) \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad N = (x+1) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} x+2 & -x-1 \\ -x-1 & x+2 \end{bmatrix}$$

Referring to [op.cit.], M is the matrix representing χ , and C is the matrix representing ψ . We notice that C and N commute with M , and each of $u := NM^{-1}$ and $c := CM^{-1}$ are elements of $E_b(G_M)$. The argument there shows that for no k do all the entries of $M^k C$ have no negative coefficients—in other words, c is not in $E_b(G_M)^+$. However, NC has all of its entries in $\mathbf{R}[x^{\pm 1}]^+$, so that $uc \geq 0$. Finally, we see that $M \prec N \prec M$, so that u is an order unit of $E_b(G_M)$, and thus order unit cancellation fails here.

As a general comment on order unit cancellation, in any unperforated ordered ring with 1 as order unit, if $ur \geq 0$ (u an order unit) and there exists a positive integer k such that one of $kur \pm r$ is positive, then r is positive. (See [H2, Chapter 2].) So order unit cancellation is very close to holding in general, and for a generic order unit, it should hold.

In this example, $E_b(G_M)$ was commutative and had zero divisors. It is possible that order unit cancellation holds if $E_b(G_M)$ has no zero divisors (which is the generic case in more than one variable— $E_b(G_M)$ will have no zero divisors if the characteristic polynomial of M is irreducible away from zero).

The result that $E_b(G_M)$ noetherian implies M is drawn was the main reason for introducing the property of being drawn. Does a converse hold? Perhaps the converse should be put in the form, if M is drawn, then every ascending chain of order ideals in $E_b(G_M)$ terminates, equivalently, every order ideal admits an order unit. There are some (restrictive) cases in which a converse of some sort is true.

Suppose that the conclusion of Proposition 7.10(b) were true. If we knew that every matrix with entries in R_P (that commutes with $M^{n!}/P$) is bounded by a multiple of the identity (essentially this means that if N is a matrix in $M_n R_P$, possibly commuting with $M^{n!}/P$, then $\pm N(M^{n!}/P)^k \leq K(M^{n!}/P)^{k+1}$, entrywise in R_P for some positive integers k and K), then $E_b(G_{MP})$ is just the centralizer of $M^{n!}/P$ in $M_n R_P$ with the inverse adjoined. Since R_P is noetherian, the centralizer of a matrix with entries from it is a finitely generated R_P -module (follows easily from the fact that if V is a finite type module over a commutative noetherian ring, then the endomorphism ring of V is of finite type as a module; to see that just

apply $\text{Hom}[-, V]$ to a presentation of V , and observe that the functor is one-sided exact). Hence in this case, $E_b(G_{MP})$ is noetherian; moreover, the bijection between the order ideals shows that every order ideal in $E_b(G_M)$ has an order unit.

This argument requires that every matrix (centralizing $M^{n!}/P$ inside $M_n R_P$) induce a bounded endomorphism. Certainly if $(**)$ holds, this is true, so that $E_b(G_{MP})$ is noetherian. (This is hardly surprising—it is easy to check directly if that $(**)$ holds, the order ideals of $E_b(G_M)$ are in bijection with those of R_P , hence satisfy the ascending chain condition, hence every order ideal admits an order unit).

Traces, faces, and extensions. A natural and useful result would be a determination of all the pure, i.e., multiplicative, traces on $E_b(G_M)$ (say for M primitive). We already know that the faithful pure traces are given by Perron eigenvectors obtained from point evaluation—yielding a parameterization of the faithful pure traces by $(\mathbf{R}^d)^{++}$. In the case that $n = 1$, it was shown in [H5, V.1] that these are dense, and we can obtain the limit points by following certain families of paths (exponentials of rays) obtained from the dual of the Newton polytope. For $n > 1$, density fails. However, by Proposition 3.8, a non-faithful pure trace must kill at least one of the ideals, $\ker \pi_F$ for some face, thereby inducing a positive multiplicative map on the factor algebra, $E_b(G_M)/I_F$.

Q1 Is I_F an order ideal of $E_b(G_M)$?

We do not know whether I_F is an order ideal, so that although we can impose a pre-ordering on $E_b(G_M)/I_F$ (with positive cone the image of the positive cone of $E_b(G_M)$), we cannot deduce many properties of this factor algebra. For example, in some cases, it is possible to prove directly that $E_b(G_M)$ is a dimension group (for instance, if M is in companion matrix form with no repeated factors in its characteristic polynomial—it should be possible to prove $E_b(G_M)$ is a dimension group even without the companion matrix hypothesis), and if I_F were an order ideal, then $E_b(G_M)/I_F$ would be a dimension group. The result (Proposition 3.8(a)) that $\cap I_F$ is nil modulo an ideal of the form $M^{-c}E_b(G_M)$ (and this ideal is an order ideal) is a partial and useful result in this direction.

Q2 Does every pure trace on $E_b(G_M)$ that kills I_F extend to a pure trace on $E_b(G_{M_F})$ (via $\pi_F : E_b(G_M) \rightarrow E_b(G_{M_F})$)?

Q2' Is the map $E_b(G_M)/I_F \rightarrow E_b(G_{M_F})$ induced by π_F is an order-embedding?

An affirmative answer to question Q2' would imply an affirmative answer to Q2. A serious problem is whether we can use $\pi_F : E_b(G_M) \rightarrow E_b(G_{M_F})$ more fully. We would like to conclude that every pure trace on $E_b(G_M)/I_F$ (meaning, a pure trace on $E_b(G_M)$ that kills I_F) extends to $E_b(G_{M_F})$. The reason is the following.

The matrix $M_F = \oplus M_{F,\alpha}$ is a direct sum of irreducible matrices. The bounded endomorphism ring is just the direct product (with the product ordering) of the bounded endomorphism rings $E_b(G_{M_{F,\alpha}})$, and modulo irreducible matrices which are not primitive (which can be dealt with separately), we are back in the primitive case, except this time in effectively fewer variables (the number of variables measured by the dimension of the affine span of F , which is always smaller than the original number). Every pure trace on each $E_b(G_{M_{F,\alpha}})$ of course induces a pure trace on $E_b(G_M)$, although of course different choices of primitive component, $M_{F,\alpha}$, need not yield different pure traces (e.g., if for two α 's, the corresponding $M_{F,\alpha}$ have the same β function, then the faithful pure traces on $E_b(G_{M_{F,\alpha}})$ for one of the α 's will restrict to the same pure trace on $E_b(G_M)$ as their counterparts from the other choice for α). However, this is a rather minor point. Since the effective number of variables drops in going down to a face, there is an obvious opportunity for an induction argument (in some cases, there is also a drop in the size of the primitive/irreducible matrix, which can be exploited).

If we consider non-faithful pure traces¹ of one of the $E_b(G_{M_{F,\alpha}})$ (and assume for simplicity that $M_{F,\alpha}$ is primitive and not just irreducible), then of course we can apply the same process. However, all we know about the polytope $K(M_{F,\alpha})$ is that it is a non-empty subset of F —it is easy to construct examples wherein it is an arbitrary convex polyhedral subset of F —the only constraint being on the union over α , namely that $\text{cvx} \cup_{\alpha} K(M_{F,\alpha}) = F$ (every possibility can be realized with this constraint).

If for fixed α , $K(M_{F,\alpha})$ has a proper face G , then of course we can iterate the process and consider the facial matrix of $M_{F,\alpha}$ corresponding to G (I am not going to complicate the notation further), and so on. Note that G might be in the interior of F , so that it bears no relation to any faces of F or $K(M)$ (except that it has strictly smaller affine dimension than that of F).

As an amusing observation, if M satisfies the separation condition of section 10, (#), (which asserts that the ratio of the second largest absolute value of the eigenvalues of $M(r)$ to the Perron eigenvalue is bounded above away from 1, as r runs over $(\mathbf{R}^d)^{++}$), then for at least one α , $K(M_{F,\alpha}) = F$. If M satisfies the fairly strong condition that for each extreme point w of $K(M)$, some power of $x^w M^{-n!}$ belongs to $E_b(G_M)$ (this condition implies the formally weaker condition $P^{k-1} M^{kn!} \prec P^k M^{(k-1)n!}$, which is implied by noetherianness of $E_b(G_M)$), then for every α , $K(M_{F,\alpha}) = F$, which forces the “ G ” above to be a face of F ; moreover, this fairly strong condition is inherited by subsequent iterates of this process, so that every iterate only involves faces of K , not just faces of lower dimensional convex subsets. The extremely strong condition (***) forces there to be only one block per face, and the block is full size.

If we knew that question Q2 had an affirmative answer, then every pure trace on $E_b(G_M)$ would either be faithful, or factor through one of the $E_b(G_{M_{F,\alpha}})$ for some face F ; moreover, a finer description would be obtained by applying the same process (reduction to a relative face), inductively to the relevant irreducible component, until we reach a situation where the induced trace is faithful, whereupon it corresponds to a point evaluation and a corresponding Perron eigenvector. The next step would be to decide given two pure traces constructed by this iterated (relative) face business restrict to equal traces on $E_b(G_M)$; this looks relatively easy.

Q3 For $P = \text{Log tr } M^{n!}$, do traces extend from $E_b(G_M) \rightarrow E_b(G_{MP})$? Is this true for arbitrary P in A^{++} ?

This holds for $n = 1$ (not explicitly stated, but deducible from results in [H5]), and is also true whenever Q2 has an affirmative answer, as a study of the faces and the relative faces arising from the iterated process described above reveals. If we could show that the first part were true, then we could probably obtain that the second part were true, and we would get some results with respect to Q2. The idea is that the $E_b(G_{MP})$ is an algebra over R_P , and every pure trace on $E_b(G_{MP})$ induces a pure trace of R_P and the latter are completely understood. These would play the role of point evaluations, except now we are working over a compact space (in fact a compactification of $(\mathbf{R}^d)^{++}$), and not just $(\mathbf{R}^d)^{++}$, and things seem easier.

With the first choice for P , the condition that some power of $PM^{-n!}$ belong to $E_b(G_{MP})$ implies the necessary condition for noetherianness of $E_b(G_M)$, namely $P^{k+1} M^{kn!} \prec P^k M^{(k+1)n!}$. In this case, the traces on $E_b(G_{MP})$ are completely determined in section 5 (since in this case, M/P is an order unit of $E_b(G_{MP})$).

The usual (and effectively the only) way to show that pure traces extend, say from $\pi : (A, u) \rightarrow (B, \pi(u))$ where u and $\pi(u)$ are order units and π is order preserving, is to show that $\pi^{-1}(B^{++})$ is contained in the positive cone of A (which is in turn equivalent to $\pi^{-1}(B^{++}) = A^{++}$). Thus to answer Q3, it would be sufficient to show that if NM^{-k} belongs to $E_b(G_M)$ and $PM^k \prec PN \prec PM^k$, then there exists l such that NM^l is nonnegative. For Q2, the problem is to show the same type of result with $E_b(G_M)/I_F \rightarrow E_b(G_{M_F})$. Attempts to attack this and Q3 via local methods (i.e., by means of the faces) boil down to the following: Is it true that if $\pi_F(e) \geq 0$ (or is an order unit) with respect to the ordering on $E_b(G_{M_F})$ (or one of its factor rings coming from an irreducible block of M_F), then there exists e' in $E_b(G_M)^+$ with $\pi_F(e') = \pi_F(e)$. In other words, do positive elements (or even just order units) lift to positive elements? This is closely connected with Q1.

A trivial case in which this holds occurs when the range of $\pi_{F,\alpha}$ is one dimensional at each α —of course, the range simply consists of the scalars, and positive scalars obviously lift to positive elements. (This occurs in the FOG example.) Anyway, when this lifting property holds for all faces, then both Q2 and Q3 have an affirmative answer.

8. Linking M & M s:

As usual, let M be a primitive $n \times n$ matrix with entries in A^+ , Laurent polynomials in d variables. Let

$v = (v(1), \dots, v(d))$ be a nonzero element of \mathbf{R}^d , and let $B = (B_1, B_2, \dots, B_d)$ be an element of $(\mathbf{R}^d)^{++}$. Define the path in $(\mathbf{R}^d)^{++}$, $X := X_{B,v} : \mathbf{R}^+ \rightarrow (\mathbf{R}^d)^{++}$ via $X(t) = (B_1 t^{v(1)}, B_2 t^{v(2)}, \dots, B_d t^{v(d)})$; thus $\log X$ is a ray in \mathbf{R}^d passing through $\log B$ with directional derivative v .

Now consider the entries of $M(X_{B,v}(t))$, particularly as $t \rightarrow \infty$. Let K be $K(M)$; then the vector v exposes a face F ; more precisely, let $s = \sup\{v \cdot k \mid k \in K\}$ (this will be abbreviated $\max v \cdot K$); then F defined as $\{k \in K \mid v \cdot k = s\}$ is a nonempty face of K . Each $M_{i,j}(X_{B,v}(t))$ is a formal polynomial in the variable t ("formal" means a sum $\sum r_i t^{g(i)}$ where r_i are real and $g(i)$ real) with no negative coefficients. For any face F , we may find a vector v exposing F relative to K that has integer coefficients (since all the vertices of K are lattice points, and we can find rational solutions to systems of linear equations with rational coefficients if we can find a real solution; clear the denominators). So let us assume v actually has integer entries. This means that each $M_{i,j}(X_{B,v}(t))$ is a Laurent polynomial with positive coefficients; we can of course work with rational entries as well, which would correspond to polynomials fractional exponents (and in this case, we would have to replace the -1 appearing in the big Oh term by $-1/n!$).

Hence we may write $M_{i,j}(X_{B,v}(t)) = g_{i,j}(B)t^{d(i,j)} + \mathbf{O}(t^{d(i,j)-1})$ for large t , where $g_{i,j}$ is a Laurent polynomial in d variables with positive coefficients (so $g_{i,j}(B)$ is an abbreviation for $g_{i,j}(B_1, B_2, \dots, B_d)$); however, we will hold B constant for a while), $d(i, j)$ is the degree (in t) of $M_{i,j}(X_{B,v}(t))$, and of course $d(i, j) = \max\{v \cdot w \mid w \in \text{Log } M_{i,j}\}$.

Now we can recover M_F as an asymptotic limit, as follows. Recall that M_F is a direct sum of irreducible matrices and a zero matrix. The *blocks* are the irreducible components or the singleton sets whose state does not belong to any irreducible block. We say two states i and j belong to the same block if either $i = j$ or the (i, j) entry of M_F lies in some irreducible block.

Lemma 8.1 For all choices of B in $(\mathbf{R}^d)^{++}$ and rational v in $\mathbf{R}^d \setminus \{0\}$, if $(M_F)_{i,j} \neq 0$, then

$$M_{i,j}(X_{B,v}(t)) = (M_F)_{i,j} (B)t^{d(i,j)} + \mathbf{O}(t^{d(i,j)-1/n!})$$

Remark. The expression $M_F(B) \equiv M_F(B_1, \dots, B_d)$ of course makes sense, since M_F was defined as a matrix with entries in A^+ . The $n!$ term could be replaced by the least common multiple of $\{2, 3, \dots, n\}$, but there is no obvious advantage. We could even allow the v to have real entries, and then there is some $\delta > 0$ that replaces the $1/n!$ term, independently of i and j .

Proof. Fix i, j and expand $M_{i,j} = \sum_{\text{Log } M_{i,j}} \lambda_z x^z$ where $\lambda_z > 0$. For each z in the Log set, either $\lambda_z x^z$ appears in $(M_F)_{i,j}$ or zero does, and the former occurs if and only if there is a cycle $i \rightarrow j \rightarrow \dots \rightarrow i$ whose first weight is z and whose average weight, a , is in F , i.e., satisfies $v \cdot a = s := \max v \cdot K = v \cdot F$.

Suppose i and j belong to the irreducible block α (just a name; $M_{F,\alpha}$ is the corresponding irreducible matrix). If $(M_F)_{i,j}$ is not zero, then there exists a cycle $i \rightarrow j \rightarrow \dots \rightarrow i$ whose first weight is y (i.e., $\lambda_y x^y$ appears in $(M_F)_{i,j}$) and average weight a lying in F . We may replace the first arc ($i \rightarrow j$) by one with weight z , and we obtain an average weight for the new cycle, $a + (z - y)/l$ where l is the length of the cycle. Since $v \cdot a = \max v \cdot K$, it follows immediately that $v \cdot z \leq v \cdot y$ for and z in $\text{Log } M_{i,j}$. Moreover, if equality holds, then $\lambda_z x^z$ does appear in $(M_F)_{i,j}$.

On the other hand, suppose $\lambda_z x^z$ and $\lambda_y x^y$ appear in $(M_F)_{i,j}$. Interchanging the roles of z and y , we see that $v \cdot y \leq v \cdot z$, so equality holds.

Thus, if $(M_F)_{i,j} \neq 0$, $\lambda_z x^z$ appears in $(M_F)_{i,j}$ if and only if $v \cdot z = \max v \cdot \text{Log } M_{i,j}$. The latter is precisely $d(i, j)$. Everything now follows. •

A minor problem that arises is that it can certainly happen that $(M_F)_{i,j} = 0$ for some entry of an irreducible block, in which case the previous result tells us nothing. This will be remedied below.

Let $u : (\mathbf{R}^d)^{++} \rightarrow \mathbf{R}$ be an algebraic function over $\mathbf{R}[x_i^{\pm 1}]$ (that is, u satisfies a polynomial, $u^n + \sum_{i=0}^{n-1} p_i u^i = 0$ where all the p_i are rational functions with real coefficients). We say that an algebraic function u exhibits *fractional polynomial behaviour* if (i) u is strictly positive as a function on $(\mathbf{R}^d)^{++}$,

and (ii) there exists an element J in $A_{\mathbb{Q}}^+$ such that u/J is bounded above and below (away from zero) on $(\mathbf{R}^d)^{++}$. If additionally J can be chosen in $\mathbf{R}[x_i^{\pm 1}]^+$, then u is said to exhibit *polynomial behaviour*.

For example, by Lemma 3.2, if β is the large eigenvalue function of the $n \times n$ primitive matrix M in $M_n \mathbf{R}[x_i^{\pm 1}]^+$, then $\beta^{n!}$ exhibits polynomial behaviour (take $J = \text{tr } M^{n!}$) and of course β itself exhibits fractional polynomial behaviour (take $J = \sum x^w$ where w varies over exponents of the form $v/n!$ where v runs over $\partial_e \text{Log tr } M^{n!}$). For many problems dealing with finite equivalence, we need to show that M has a left or right eigenvector (for the eigenvalue function β) all of whose entries exhibit fractional algebraic behaviour. This does not always occur (the FOG examples), but it will under some hypotheses, given later.

A more general property for the entries of the eigenvector is *fractional rational asymptotic behaviour*; this means there exist J and J' in $A_{\mathbb{Q}}^+$ such that uJ'/J is bounded above and below away from zero on $(\mathbf{R}^d)^{++}$.

Now let $C := (c_1, \dots, c_n)$ be a nonnegative left eigenvector for M at the eigenfunction β ; we may choose the c_i to be both algebraic and real analytic on $(\mathbf{R}^d)^{++}$, e.g., by using the method of adjoints. In general, as we shall see, we cannot choose c_i to have fractional polynomial behaviour (we obtain conditions on M , which it turns out are generic if $d > 1$, that proscribe fractional polynomial behaviour). Nonetheless, $C(X_{B,v}(t))$ does exhibit this growth (when the entries are viewed as analytic functions in t) (as we have chosen v to be rational—drop the “algebraic” if v is not necessarily rational), because it is the left eigenvector of the matrix $M(X_{B,v})$ (viewed as a matrix whose entries are polynomials in fractional powers of t), and an analytic algebraic function in one variable is of the form $Lt^{a/b} + \mathbf{O}t^{(a-1)/b}$ for some integers a and b and nonzero constant L —[A, pp. 290–295]. The point sets of the paths $X_{B,v}$ and $X_{B',v}$ are equal for sufficiently large t if and only if $\log B - \log B' \in v\mathbf{R}$, i.e., the paths may be taken to be parameterized by elements of $\mathbf{R}^d/v\mathbf{R}$.

In particular, we may write $c_j(X_{B,v}(t)) = h_j(B)t^{m(j)} + \mathbf{O}(t^{m(j)-1/n!})$ where h_j is a function (it is not true that $B \mapsto h_j(B)$ is real analytic in this context), $h_j(B) > 0$, and $m(j)$ is a rational whose denominator divides $n!$ and probably divides the least common multiple of $\{2, 3, 4, \dots, n\}$. Obviously, $C(X_{B,v})$ is a left nonnegative eigenvector for $M(X_{B,v})$ with eigenvalue $\beta(X_{B,v})$ (all are functions of t).

Now we may write $\beta(X_{B,v}(t)) = f(B)t^s + \mathbf{O}t^{s-\delta}$ from [A, pp. 290–295] and Lemma 3.1, and the fact that $\beta(X_{B,v})$ is the large eigenvalue of the primitive matrix (viewed with entries from t) $M(X_{B,v})$. (Again, while $f(B) > 0$, we are not claiming that $B \mapsto f(B)$ is real analytic.) Finally, let $d(i, j)$ denote $\deg M(X_{B,v})_{ij}$.

Call the (i, j) entry of $M(X_{B,v})$ *normal* if $d(i, j) = s + m(j) - m(i)$. We sometimes refer to (i, j) being normal by abuse of notation.

Lemma 8.2

- (a) If i and j are in the same block and $(M_F)_{ij} \neq 0$, then the (i, j) entry is normal.
- (b) If the (i, j) entry is normal, then $(M_F)_{ij} \neq 0$.

Proof. (a) For any i and j , $d(i, j) \leq s + m(j) - m(i)$. Suppose that $(M_F)_{ij} \neq 0$. There exists a loop $i = j_0 \rightarrow j = j_1 \rightarrow \dots \rightarrow j_l = i$ with average weight a which belongs to F (i.e., $v \cdot a = s$). Thus $d(i, j) + \sum_{k=0}^{l-1} d(j_k, j_{k+1}) = ls$. The left side is dominated by a telescoping sum, $ls + \sum (m(j_{k+1}) - m(j_k)) = ls$, with equality if and only if each $d(j_k, j_{k+1}) = s + m(j_{k+1}) - m(j_k)$; hence each of the consecutive pairs appearing in the path are normal, in particular, (i, j) is.

(b) From irreducibility and the definition of the blocks of M_F , there exists a path $j = j_0 \rightarrow j_1 \dots \rightarrow j_l = i$ with each entry, $(M_F)_{j_k, j_{k+1}}$ not zero. By (a), $d(j_k, j_{k+1}) = s + m(j_{k+1}) - m(j_k)$. It follows easily that if a is the average weight of the loop obtained by concatenating the path with $i \rightarrow j$, then $v \cdot a = s$, so that a belongs to F . Hence $(M_F)_{ij} \neq 0$. •

A useful tool is conjugation with a suitable diagonal matrix whose entries are monomials. This forces the path of matrices, $M(X_{B,v}(t))_{t \rightarrow \infty}$, converge, but to a matrix generally different than $M_F(B)$; it is the local version of the trick to be used in section 10, and is a form of the H-transform used in Markov chains.

Let $\Delta = \text{diag}(t^{m(1)}, \dots, t^{m(j)})$; then $M' := \Delta M(X_{B,v}) \Delta^{-1}$ has $C' := C(X_{B,v}) \Delta^{-1}$ as its left eigenvector, and of course M' still has as its entries Laurent polynomials in fractional powers of t . However, C' consists of analytic and algebraic functions that are bounded above and below (in t). This forces all the column sums of M' to be bounded above and below away from zero by a multiple of $\beta(X_{B,v})$, in particular, all the entries are bounded above by a multiple of this, so that their growth is always at most $O t^s$. The path of matrices $M'(t)/t^s$ need not converge to $M_F(B)$ as $t \rightarrow \infty$ (since it can happen that some of the entries not in a nonnull block can grow as t^s —examples in several variables of this phenomenon are unfortunately ubiquitous, as we shall see, and the phenomenon represents an obstruction of an unpleasant sort); convergence does occur in the entries for which M_F is not zero, by Lemma 8.1.

In particular, the limit point of the path of matrices is a nonnegative real matrix which is entrywise at least as large as $M_F(B)$ and therefore, the spectral radius of the latter is dominated by the limit matrix, call it M_∞ . However, it is easy to see that M_∞ has $f(B)$ (the asymptotic value of β) as an eigenvalue and the corresponding left eigenvector is strictly positive, by construction. It follows immediately that $f(B)$ is the spectral radius of M_∞ (proof: if not, by the Perron-Frobenius theorem applied to right eigenvectors, there exists $r > f(B)$ and a nonzero nonnegative *right* eigenvector w for eigenvalue r ; since $r \neq f(B)$, the inner product of the strictly positive left eigenvector for $f(B)$ with the nonnegative w must be zero, which is obviously impossible).

When we view $M(X_{B,v}(t))$ as a matrix with entries which are Laurent polynomials (with rational exponents) in the single variable t (or $t^{1/k}$ for some integer k) and no negative coefficients, conjugation with the diagonal matrix Δ is a (very) strong shift equivalence, and it is usually very easy to check that specific properties are preserved by this transformation.

Let α and α' correspond to two distinct blocks (one of which might be null). We say α is *linked to* α' , denoted $\alpha \subset \alpha'$ relative to B, F , etc., if there exists i in α and j in α' such that

$$\lim_{t \rightarrow \infty} \frac{c_j(X_{B,v}(t)) M_{j,i}(X_{B,v}(t))}{c_i(X_{B,v}(t)) \beta(X_{B,v}(t))} \neq 0;$$

in other words, $m(j) + d(j, i) = m(i) + s$ (the left side is always less than or equal to the right, as follows from the eigenvector equation). If i and j belong to different blocks, α and α' respectively, then normality of the pair (j, i) implies $\alpha' \subset \alpha$. (It perhaps is unfortunate that I called this property linking, which suggests a symmetric relation—the relation is practically anti-symmetric.)

A linkage means that there is a contribution to the eigenvector product that vanishes when we go to M_F , but does not vanish in the (asymptotic) limit, as $t \rightarrow \infty$. (So the asymptotic version of the eigenvector will not be an eigenvector for M_F , nor will the asymptotic limit of the eigenvalue necessarily be an eigenvalue for M_F when linkages exist.)

Consider the example

$$M = \begin{bmatrix} 1+x & 1 \\ y & x+y \end{bmatrix}.$$

Here $K(M)$ is the standard triangle, the left eigenvector is $(1 \ 1)$ and $\beta = 1 + x + y$. The face F spanned by $(1, 0)$ and $(0, 1)$ is exposed by the vector $(1, 1)$ (a coincidence that it is the left eigenvector too), and M_F is simply $\text{diag}(x, x+y)$. If we take the path $X(t) = (B_1 t, B_2 t)$ (any $B_i > 0$), we see that the first block is linked to the second, but not conversely. The limiting value of β is of course $B_1 + B_2$, but the asymptotic spectral radius of the top block is B_1 , strictly less than this—this is what permits the linking to take place. On the other hand, the second block has “full” value, and is not linked to anything.

Certain types of shift equivalence preserve linking with an obvious proof; specifically, the one outlined above, conjugating with a diagonal matrix whose entries are monomials—this type of conjugation obviously preserves $d(j, i) + m(j) - m(i)$ and s , so links are preserved and no new ones introduced.

Another simple example is FOG (section 2),

$$M = \begin{bmatrix} x & 0 & 1 \\ 1 & y & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Here $K(M)$ is the standard triangle, and we set F to be the diagonal edge of the triangle. Then the exposing vector v is $(1, 1)$, and so the relevant paths are $X(t) := X_{B,v}(t) = (B_1t, B_2t)$. As $\beta \geq \max\{1, x, y\}$ on $(\mathbf{R}^2)^{++}$, it follows that $\beta \geq k(1+x+y)$ for some $k > 0$, along a path of this type, $\beta(X(t)) \geq k(B_1 + B_2)t$, but $M_F = \text{diag}(x, y, 0)$, so its spectral radius at B is $\max\{B_1, B_2\}$; for some choices of B is the limiting value obtained as the spectral radius. Note that there are three blocks in M_F (labelled 1, 2, 3), and the third is the null block.

To see which pairs of blocks are linked, we have to do some actual matrix computations. The first (easy) observation is that the row $(\beta - 1, (\beta - 1)(\beta - x), 1)$ is a left eigenvector of M for the eigenvalue β . The space of paths of interest are parameterized by $X_b(t) = (t, bt)$ (where $b = B_2/B_1$). We have to determine the rate of growth of the entries of the eigenvector along these paths. The third entry is constant, so there is no problem there. Next, we observe that $\beta - 1 \geq k(x + y)$ for some $k > 0$ (on $(\mathbf{R}^2)^{++}$), and on $(\mathbf{R}^2)^{++}$, $\beta \geq \beta(0, 0) = r$, where r is the real root of $r^3 - r^2 - 1 = 0$. As $r > 1$, it follows that $\beta - 1 \geq r - 1$, and so $\beta - 1$ is bounded below by a multiple of $1 + x + y$; since β is bounded above by a multiple of the same thing, we have that $(\beta - 1)/(1 + x + y)$ is bounded above and below away from zero on $(\mathbf{R}^d)^{++}$. Hence along X_b , $(\beta - 1) \circ (X_b)(t) = c(b)t + \mathbf{O}t^{1-\delta}$, for some value $c(b) > 0$.

Now something odd happens when we try to do the same process with $\beta - x$. The characteristic polynomial of M translates to $(\beta - x)(\beta - y)(\beta - 1) = 1$. Let $\lambda \equiv \lambda(t)$ be $(\beta - x)(X_b)$. Then we have $\lambda(\lambda - (b - 1)t)(\lambda - (1 - t)) = 1$. If we write $\lambda(t) = kt^r + \mathbf{O}t^{<r}$ (k and possibly r depending on b , with $k > 0$) which of course we may, since λ is a nonnegative real analytic algebraic function of t , then we see there are three cases to consider.

If $b > 1$, the middle term $(\lambda - (b - 1)t)$ is negative and the other two positive if $r < 1$, while if $r > 1$, the growth of the first and last terms is at least $t^{2r} > t$, while the middle term is at least t , so the product cannot be bounded (1). Hence $r = 1$, and this last argument also shows that $\lambda - (b - 1)t$ is of order $1/t^2$.

If $b \leq 1$, the middle term and the first term are positive, and so the third term must be positive; hence $r \leq t$. The first term is of order t^r , and the third is of order t . If $b = 1$, the second term is of order t^r , which forces $2r + 1 = 0$, i.e., $r = -1/2$. If $b < 1$, the second term is of order t , which forces $r + 2 = 0$, i.e., $r = -2$.

The fact that r changes as b varies indicates that $\beta - x$ does not have fractional polynomial behaviour, as is easy to check. In any event, the eigenvector has growth rates (t, t^2, t^0) if $b > 1$, $(t, t^{1/2}, t^0)$ if $b = 1$, and (t, t^{-1}, t^0) if $b < 1$.

Now we can check which blocks are linked. Notice that λ grows as t . Regardless of the value of b , block 3 is linked to block 1. If $b > 1$, then block 1 is linked to block 2. If $b < 1$, then (weirdly) block 2 is linked to block 3, the null block. There are no other links. •

Lemma 8.3 Every null block is linked to some other block.

Proof. If $\{i\}$ is a null block of M_F , the equation, $\sum_j c_j M_{ji} = \beta c_i$ entails $\max_j \{m(j) + d(j, i)\} = s + m(i)$. As $d(i, i) < s$ (else $\{i\}$ would not be a null block), there exists j unequal to i such that $m(j) + d(j, i) = s + m(i)$; thus $\{i\}$ is linked to the block containing j .

Lemma 8.4 If $\alpha_1 \subset \alpha_2 \subset \dots \subset \alpha_k$ is a chain of linkages, then the blocks $\alpha_1, \alpha_2, \dots, \alpha_k$ are distinct.

Proof. Otherwise, we may suppose that $\alpha_1 = \alpha_k$, $k > 1$, and no smaller value of k is possible. We have pairs $(i_1, j_2), \dots, (i_{k-1}, j_1)$ where i_l and j_l belong to α_l , and each (j_{l+1}, i_l) is normal. If α_l is a singleton

(this means that the block size is one, e.g., if α_l is a null block), then of course $j_l = i_l$. Otherwise, there exists a path $j_l = j_{l,1} \rightarrow j_{l,2} \rightarrow \dots \rightarrow j_{l,L(l)} = i_l$ where each ordered pair $(j_{l,m}, j_{l,m+1})$ is normal and all states appearing in the path belong to α_l . Thus for each l , either $j_l = i_l$ or there is a path of this type from j_l to i_l . Hence we obtain a loop $i_1 \rightarrow j_2 \rightarrow \dots \rightarrow j_1 \rightarrow \dots \rightarrow i_1$ (the last step may simply be $\rightarrow j_1 = i_1$ if α_1 is a singleton), where each transition is a normal pair. As we have seen in the previous argument, this implies that the average degree of the loop is s , so that all the states appearing in the loop belong to the same block, a contradiction. •

Proposition 8.5 The transitive relation on blocks generated by linkages is antisymmetric. The maximal elements are not null, and $\rho(M_{F,\alpha}) = f(B)$ if and only if α is a maximal element with respect to this relation.

Proof. The previous result yields antisymmetry, and the penultimate result yields that null blocks cannot be maximal.

From the relation $\sum c_j M_{ji} = \beta c_i$, we have that for large t ,

$$f(B)t^{s+m(i)} \left(1 + o\left(t^{s+m(i)}\right)\right) = \sum' c_j(X_{B,v})M_{ji}(X_{B,v})$$

where the prime denotes the sum only over those j such that (j, i) is normal. If i belongs to a maximal block, then (j, i) being normal entails that j also belongs to the same block, α , and thus $(M_F)_{ji} \neq 0$. It follows from dividing by $t^{s+m(i)}$ that $\sum_{j \in \alpha} h_j(B)(M_F)_{ji} = f(B)h_i(B)$. Hence $H_\alpha := (h_j(B))$ is a strictly positive left eigenvector for $f(B)$ of $(M_{F,\alpha})$. A nonnegative matrix with a strictly positive left eigenvector must have that eigenvalue as its spectral radius (by applying the Perron Frobenius theorem to the right eigenvectors). Hence $f(B)$ is the spectral radius of $(M_{F,\alpha})$.

Conversely, if i belongs to an irreducible block, we apply the following, Lemma 8.6, to the matrix obtained by replacing all nonnormal entries by zero (and the others by their asymptotic limits); it is useful to perform the conjugation with the Δ matrix as described earlier in this section. •

Lemma 8.6 Let A be a square matrix with nonnegative real entries which is irreducible. Let Z be a nonnegative real matrix which partitions as

$$Z = \begin{bmatrix} A' & R \\ S & T \end{bmatrix}.$$

If $A \leq A'$ and Z admits a strictly positive left eigenvector with eigenvalue equaling that of the spectral radius of A , then S consists entirely of zeroes.

Proof. Let r be the spectral radius. By conjugating Z with the diagonal matrix arising from the eigenvector (and simultaneously conjugating A with the corresponding block of the diagonal matrix), we may assume the left eigenvector for Z is just $(1 \ 1 \ \dots \ 1 \ 1)$. Hence all the column sums of A' are bounded above by r , and at least one of them is strictly less than r if S is not the zero matrix. Obviously, the same applies to the columns sums of A . An irreducible matrix with some columns less than r and none exceeding it must have spectral radius strictly less than r (for primitive matrices, this follows from the strict monotonicity of the spectral radius function; for irreducible matrices, apply the primitive result to $(I + A)/2$, which is primitive). Thus S zero, so that $A = A'$. •

In the FOG example, we saw that linkages between the various blocks (arising from a fixed face) depended on the choice of the strictly positive parameter, B . This phenomenon is a consequence of the fact that the rates of growth—i.e., the exponent of t —of the entries of a strictly positive eigenvector along the path $X_{B,v}$ depend (for v fixed) on the choice of B . One situation in which this phenomenon cannot occur is when the matrix M satisfies the growth condition on the entries of a left eigenvector of section 10. Say M

satisfies (%) if it admits a left eigenvector whose entries satisfy fractional rational algebraic behaviour. In fact, a formally weaker condition is sufficient (it turns out of course that the “formally weaker” condition is equivalent to (%)).

Proposition 8.7 Let M be a primitive matrix in $M_n A^+$, and suppose that M admits a left positive eigenvector, $U = (U_i)$ for the analytic eigenfunction $\beta : (\mathbf{R}^d)^{++} \rightarrow \mathbf{R}^+$, with the property that for each i , there exist elements of $A_{\mathbf{Q}}^+$, p_i and q_i such that $U_i q_i / p_i : (\mathbf{R}^d)^{++} \rightarrow \mathbf{R}^+$ is bounded above and below away from zero. Then for every face F (and corresponding choice of exposing vector v), the linkages between the blocks of M_F are independent of the choice of parameter B .

Proof. We observe that for an element of $A_{\mathbf{Q}}^+$, p , $\lim_{t \rightarrow \infty} \log p(X_{B,v}(t)) / \log t$ is exactly $\max v \cdot K(p)$, where $K(p)$ denotes the convex hull of $\text{Log } p$, i.e., the Newton polyhedron of p . It follows that for each i ,

$$\lim_{t \rightarrow \infty} \log U_i(X_{B,v}(t)) / \log t = (\max v \cdot K(p_i)) - (\max v \cdot K(q_i)).$$

This is obviously independent of the choice of B . Since the partition of the states into blocks depends only on F (and not the path, $X_{B,v}$), and the presence or absence of linkages between blocks depends only on the rates of growth of the entries in M along the paths (i.e., $d(i, j)$), which of course is $\lim_{t \rightarrow \infty} \log M_{ij}(X_{B,v}(t)) / \log t$ and the entries of the eigenvector, and all of these are independent of the choice of B , it follows that linkages themselves are so independent. •

In particular, this yields a fairly simple criterion for (%) to fail; namely, if there is a block which is maximal for one value of the parameter B and not maximal for another (it being understood that the exposing vector v is fixed as usual). This provides yet another proof that the FOG example fails to satisfy (%), using the diagonal face of the standard triangle. In more than one variable, failure of (%) is generic; on the other hand, in one variable, (%) holds.

The condition on the entries given in the preceding, that there is a choice of eigenvector so that U_i has fractional polynomial growth (without assuming analyticity or algebraicity) is formally weaker than (%). However, if it holds, then consider the new eigenvector (U_i / U_1) ; the first entry is the constant 1 as a function on $(\mathbf{R}^d)^{++}$; if $U' := (U'_i)$ is an algebraic and analytic left strictly positive eigenvector for β (one exists!), then by uniqueness of the Perron eigenvector for each point of $(\mathbf{R}^d)^{++}$, we must have $U / U_1 = U' / U'_1$; hence the former is analytic and algebraic as well. Obviously, if the entries of U satisfy fractional algebraic growth, then so do the entries of U / U_1 . Thus the apparently weaker formulation is equivalent to (%).

The condition (%) is close to implying the other condition needed in section 10, namely (#) (which asserts that the large eigenvalue is uniformly bigger than the second largest eigenvalue over all points of $(\mathbf{R}^d)^{++}$). Condition (%) asserts that for every face, there is a block whose spectral radius is the spectral radius of the facial matrix, independently of the choice of B , although some other blocks may hit the spectral radius, and if they do so at one point, they do so uniformly as well; on the other hand (#) is equivalent to there being a unique (irreducible) block of maximal spectral radius, and it must be aperiodic (that is, primitive).

Obviously, (%) is an invariant of finite equivalence. It would be interesting to find a matrix M which satisfies (%), but $M \text{Tr}$ does not. (Such an example would have to be at least 3×3 , because of a simple argument connecting the eigenvector of a size 2 matrix with that of its transpose.)

Now to deal with intertwiners. Suppose that $MY = YM'$, where all the relevant matrices have entries from A^+ . We will show that Y induces nontrivial intertwining between at least one pair of blocks in M_F and M'_F respectively (with maximal spectral radius for some value of the parameter), and as much as possible, the intertwining preserves linkages. Of course, $\beta = \beta'$ entails that $K(M) = K(M')$, so that we may use the same vector v to expose the same face F of these polytopes. Fix the parameter B , and the corresponding path $X_{B,v}$.

We now deal with the paths of matrices, $Y(X_{B,v})$, $M(X_{B,v})$, and $M'(X_{B,v})$; we abbreviate the first one Z (or $Z(t)$ when convenient), and we abbreviate $X_{B,v}$ to X if no confusion results. A left eigenvector

(which we take to be strictly positive, algebraic, and analytic) C of M yields an eigenvector $C' := CY$ of M' . If we set $\Delta = \text{diag}(\dots, h_i(B)t^{m(i)}, \dots)$, then on setting $N = \Delta M(X)\Delta^{-1}$, we have that $C(X)\Delta^{-1} = (1, 1, \dots, 1)(1 + \mathbf{o}(1))$ (for large values of t) is a left eigenvector for N , and similarly we may conjugate $M'(X)$ with a diagonal matrix with monomial entries to obtain N' , still intertwining with N whose eigenvector has the same asymptotic behaviour in each coordinate. The intertwiner induced by Z will be called Z' , so that $NZ = Z'N'$.

Now $C(X)\Delta^{-1}Z'$ is of the form $t^{a/b}(1 + \mathbf{o}(1))(1, \dots, 1)$ for some integers a and b , as Z' has entries in $A_{\mathbb{Q}}^+$. Dividing Z' by the monomial $t^{a/b}$, we may assume in what follows that $a = 1 = b$.

With our current eigenvectors, the (i, j) entry of N (or of N') is normal if and only if $d(i, j) = s$, since now $m(i) = m(j) = 0$. The column sums of Z' are all $1 + \mathbf{o}(1)$; hence if $d(i, j) < s$, the contribution of N_{ij} in the product NZ' is $\mathbf{o}(t^s)$. Thus $(NZ')_{kl} = \sum'_m N_{km}Z'_{ml} + \mathbf{o}(t^s)$, where \sum' is the sum, as m varies, of the normal entries N_{km} . Similarly, $(Z'N')_{kl} = \sum'_m Z'_{km}N'_{ml} + \mathbf{o}(t^s)$.

Divide each of N and N' by t^s , so that $\beta(X_{B,v}) = f(B) + \mathbf{o}(1)$. Now the equation $NZ' = Z'N'$ consists of matrices with bounded entries as $t \rightarrow \infty$, and the limits exist. The limiting matrices satisfy $N_{\infty}Z'_{\infty} = Z'_{\infty}N'_{\infty}$, the columns sums of Z'_{∞} are all 1, and $\rho(N_{\infty}) = \rho(N'_{\infty}) = f(B) \neq 0$. We also observe that N_{∞} agrees with M_F in the entries appearing in the blocks of the latter, and moreover, there are no new cycles appearing in the former; hence the characteristic polynomials of N_{∞} and M_F are equal.

Lemma 8.8 Let T and T' be square nonnegative real matrices, and let S be a nonnegative real matrix such that $TS = ST'$, and suppose that T and T' admit strictly positive left eigenvectors V and V' respectively such that $VS = V'$. Then there exist irreducible blocks, T_{α} and T'_{α} of T and T' respectively such that $\rho(T_{\alpha}) = \rho(T)$ and the restriction of S, S_{α} , to the corresponding subspace is nonzero and intertwines T_{α} and T'_{α} .

Proof. If U is an T -invariant cone (with T acting from the right), then $U' := US$ is an invariant cone for T' . All the blocks above and below a block with full spectral radius must be zero (this applies to both T and T' , and follows from the fact that the left eigenvector is strictly positive). Hence we obtain an obvious invariant cone for T , that spanned by $\{e_i \mid i \in \alpha, \rho(T_{\alpha}) = \rho\}$; call this cone U_{ρ} . We similarly define U'_{ρ} , its counterpart for T' .

It is a triviality to check that $U_{\rho}S \subseteq U'_{\rho}$. Let T_{ρ} and S_{ρ} denote the restrictions of T and S to the vector space spanned by U_{ρ} , and T'_{ρ} will denote the restriction of T' to U'_{ρ} ; then each of the three matrices is nonnegative and they satisfy $T_{\rho}S_{\rho} = S_{\rho}T'_{\rho}$. Of course T_{ρ} and T'_{ρ} are just matrix direct sums of the blocks of T and T' respectively that have full spectral radius. Now $T_{\rho}S_{\rho}$ is not zero. To see this, let V_{ρ} be the corresponding truncation of the eigenvector to the states appearing in the blocks of spectral radius ρ . Since all blocks in T above and below a block of maximal spectral radius must be zero, it follows immediately that V_{ρ} is an eigenvector for T_{ρ} with eigenvalue ρ , and it is obviously strictly positive. It follows from the original intertwining relation that $V_{\rho}S_{\rho} = V'_{\rho}$, whence S_{ρ} is not zero.

We have thus reduced the original problem to the case wherein both matrices are direct sums of irreducible matrices all with the same spectral radius, and it is easy to verify that in this case the conclusion holds. •

We note that conjugation with the diagonal matrices in the earlier constructions amounts to strong shift equivalence (over the Laurent polynomial ring in fractional powers of t), since the entries are monomial.

A few additional tidbits follow from the properties of intertwiners. If for some value of the parameter B (with v held fixed), an intertwiner induces a nontrivial intertwining involving the blocks $\alpha_1, \dots, \alpha_k$ and $\alpha'_1, \dots, \alpha'_k$ respectively, then it induces a non-trivial intertwiner involving the same pairs of blocks for *all* values of $B \gg 0$. This follows from the continuity in B of the asymptotic values of polynomials—that is, if p is a Laurent polynomial with positive coefficients, and $p(X_{B,v}(t)) = P(B)t^s + \mathbf{o}(t^s)$, then $B \mapsto P(B)$ is continuous, in fact, it is real analytic.

In general, if $q : (\mathbf{R}^d)^{++} \rightarrow \mathbf{R}$ is algebraic, real analytic, and bounded above and below, and we define

a function Q via $B \mapsto Q(B)$ where $q(X_{B,v}(t)) = Q(B) + o(1)$ ($Q(B)$ exists and is strictly greater than zero, as $q(X(t))$ is bounded real analytic and algebraic), then it is not true that Q is real analytic. To see this, simply take the function β of the FOG example, and set $q = \beta/(1+x+y)$.

We have seen that q is bounded above and below, real analytic, and algebraic. Parameterize the paths by $X_b(t) = (t, bt)$ as we had done previously. It is easy to check that $Q(b) = \max\{1, b\}$.

Sometimes we can obtain negative information about the solution to (\mathfrak{S}) by using the behaviour along paths. For example, consider the FOG matrix (section 2) and a specific column,

$$M = \begin{bmatrix} x & 0 & 1 \\ 1 & y & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} y-x \\ 1 \\ 0 \end{bmatrix} \quad V = (\beta-1 \quad (\beta-1)(\beta-x) \quad 1)$$

We show that for no N with entries in A^+ centralizing M is it true that NC belongs to $(A^3)^+$. The row V of course is the Perron eigenvector for the eigenvalue β .

We first note that $V \cdot C = (\beta-1)(\beta-x)$, which equals $1/(\beta-x)$ (this uses the fact that $(\beta-x)(\beta-y)(\beta-1) = 1$). As before in this section, let $X_b(t) = (t, bt)$. For a positive real analytic and algebraic function f and parameter b , define

$$\psi_f(b) = \lim_{t \rightarrow \infty} \frac{\log f(X_b(t))}{\log t}$$

(if the limit exists). If $f = \beta - x$, we have already seen that $\psi_f(b)$ is 2 if $b > 1$ and -1 if $b < 1$. Hence for $g = 1/(\beta-x)$, $\psi_g(2) - \psi_g(1/2) = -3$. Now if N centralizes M , then $V \cdot NC = \beta_N V \cdot C = \beta_N/(\beta-x)$; as we are also assuming N has entries in A^+ , β_N has fractional polynomial growth, so that $\psi_{\beta_N}(b)$ is constant in b . Thus

$$\begin{aligned} \psi_{\beta_N g}(2) - \psi_{\beta_N g}(1/2) &= \psi_g(2) - \psi_g(1/2) \\ &= -3. \end{aligned}$$

In addition, we have that NC has entries in A^+ . Let $U = (p_1, p_2, p_3)^T$ be an arbitrary column with entries in A^+ . Then $V \cdot U = (\beta-1)p_1 + (\beta-1)(\beta-x)p_2 + p_3 := h$. Since p_i are positive as functions on $(\mathbf{R}^2)^{++}$, we have that $\psi_h(b) = \max\{\psi_{\beta-1}(b) + \psi_{p_1}(b), \psi_{\beta-1}(b) + \psi_{\beta-x}(b) + \psi_{p_2}(b), \psi_{p_3}(b)\}$. We had already seen that $\psi_{\beta-1}(b)$ is the constant 1, and $\psi_p(b) = \max(1, 1) \cdot \text{Log } p$ for p in A^+ . It easily follows that $\psi_h(2) - \psi_h(1/2) \geq 0$ (the point is, that the only term within the set of three that is not constant in b is $\psi_{\beta-x}(b)$; however, this is non-decreasing in b). With $U = NC$, we have shown that this function applied to h is -3 , a contradiction. So no such N exists. (Of course, $1/(\beta-x)$ is strictly positive on $(\mathbf{R}^2)^{++}$, so there was no obvious reason why N could not exist.) •

Companion matrices. These are relatively simple to deal with, and except degenerately satisfy all the usual properties. Suppose M is a companion matrix with entries from A^+ (that is, M is in the form of companion matrix, not merely that M is algebraically shift equivalent to one). So

$$M = \begin{bmatrix} 0 & 0 & 0 & \dots & p_n \\ 1 & 0 & 0 & \dots & p_{n-1} \\ 0 & 1 & 0 & \dots & p_{n-2} \\ \vdots & \ddots & & \dots & \vdots \\ 0 & 0 & \dots & 1 & p_1 \end{bmatrix}$$

where p_i belong to A^+ . Let β denote the largest eigenvalue function, as usual. If $p_n = 0$, we may remove the first row and column, and obtain a smaller size companion matrix strongly shift equivalent to the original; so there is no harm in assuming $p_n \neq 0$. Necessary and sufficient for M to be primitive is

that $\gcd\{i \mid p_i \neq 0\} = 1$. The left eigenvector function for β is $(1, \beta, \beta^2, \dots, \beta^{n-1})$, and in particular, M satisfies (%). Note that $K(M)$ is the convex hull of $\cup_i \frac{1}{i} \text{Log } p_i$ (with the convention that $\text{Log } 0$ is the empty set), as is easy to check by calculating the graph of M . The right eigenvector is almost as easy to calculate. We first notice that the characteristic polynomial of M is of course $\chi(\lambda) := \lambda^n - \sum_{i=1}^n p_{n-i} \lambda^{i-1}$; define $\chi_j(\lambda) = (\lambda^n - \sum_{i=1}^j p_{n-i} \lambda^{i-1}) / \lambda^{n-j}$ (that is, truncate the characteristic polynomial, and divide by the appropriate power of λ so that the degree is j). In particular, $\chi_1(\lambda) = \beta - p_1$ and $\chi_2(\lambda) = \beta(\beta - p_1) - p_2$ (the χ_j can be defined recursively as well, via $\chi_{j+1} = \lambda \chi_j - p_{j+1}$, as is well known). Then the right eigenvector function for β is $(1, \chi_1(\beta), \chi_2(\beta), \dots, \chi_{n-1}(\beta)) \text{Tr}$.

To verify that $M \text{Tr}$ satisfies (%), we observe that $\chi_j(\beta) = \beta^{j-n} (\sum_{i=1}^{n-j} p_{n-i} \beta^{i-1})$. Since β is bounded above and below by a polynomial (possibly with fractional exponents), so is each $\chi_j(\beta)$. (It is conceivable that in general, M satisfying (%) implies $M \text{Tr}$ does as well.)

The graph of a companion matrix consists of loops with a common vertex (we assume that not all the p_i are zero). It follows that there is exactly one nonnull block at every face; the block will be primitive (i.e., aperiodic, since nonnull blocks are automatically irreducible) provided the gcd of the surviving periods of cycles is one. Sufficient for this is that the trace, p_1 , behave asymptotically like β .

Now suppose that w is in $\text{Log } p_j$. Then we claim that $x^w M^{-j}$ belongs to the bounded subring (in other words, (w, j) belongs to $T(M)$). It suffices to show that $x^w M^{n-j} \prec M^n$ —but this is clear from the graph of M —any path of length $n - j$ beginning at a state not in a cycle corresponding to p_j must simply run along the arrows of weight 1 until it hits a state on the cycle, where upon the latter can be folded in to create a path of length n .

An even stronger property is satisfied by M . Let $P = \text{tr } M^{n!}$; then $P M^{n!} \prec M^{2n!}$ (viz., section 7). The minimal loops all pass through the same distinguished state (in this case, state n) and are obtained from the various p_i , and P is obtained from all loops of length $n!$. (This can also be obtained by modifying the criterion of Lemma 7.7 for this stronger property). It is at least plausible that a primitive companion matrix with the block at each face being aperiodic is shift equivalent to a matrix satisfying (**).

9. Local and global characterization of (#):

In this section, we express one of the many properties we have discussed, (#), in terms of the behaviour of the facial matrices M_F . Earlier, the property (#) was defined in terms of the existence uniform gap in the spectrum; explicitly, if $\beta_2(r)$ denotes the second largest absolute eigenvalue of the real matrix $M(r)$ for a strictly positive d -tuple r , and then (#) is the property that

$$(1) \quad \inf_{r \in (\mathbf{R}^d)^{++}} 1 - \frac{\beta_2(r)}{\beta(r)} > 0$$

(or simply that β_2/β is uniformly bounded away from 1 over $(\mathbf{R}^d)^{++}$). In this formulation, it is expressed globally. However, it admits a relatively simple local formulation: (#) holds if and only if for every face F of $K(M)$, M_F admits just one irreducible block whose spectral radius is that of M_F and moreover, this block is primitive.

This suggests a weakening of (#), defined in local terms only, namely *weak* (#) holds if for every face F of $K(M)$, M_F contains only one block of maximal spectral radius. (The block need not be primitive.)

Weak (#) also admits a global characterization, wherein instead of taking $\beta_2(r)$ as the second largest absolute value of the eigenvalues of $M(r)$, take it to be the eigenvalue closest to $\beta(r)$ (that is, to minimize $|\alpha - \beta(r)|$, where α varies over the eigenvalues of $M(r)$).

We begin by examining the connections between M_F and M . Set $N = M^{n!}$, and $Q = \text{tr } M^{n!}$. We have that N/Q lies in $E_b(G_{M^Q})$ by Theorem 5.3, although it need not be true that all of the entries of N/Q lie in R_Q . However, it is easy to check that all of the symmetric functions of the matrix N/Q do lie in R_Q . We have that $K(N) = n!K(M)$, and let F be a face of $K(M)$, so that $F' := n!F$ is a face of $K(N)$. For a square matrix A (with entries in some commutative ring) denote by f_A the characteristic polynomial of

$A, \det(zI - A)$, using the variable z . As the coefficients of $f_{N/Q}$ are the symmetric functions of N/Q , obviously $f_{N/Q}$ belongs to the polynomial ring $R_Q[z]$.

Proposition 9.1 Let M be a primitive matrix with entries in A_Q^+ . Then M satisfies (#) if and only if for all faces F of $K(M)$, M_F has (uniformly) one block of maximal spectral radius, and it is primitive.

Proof. Let $\pi_F : R_Q \rightarrow R_{Q_{n!F}}$ be the usual quotient map, and use the same symbol to denote the induced map $R_Q[z] \rightarrow R_{Q_{n!F}}[z]$ on the polynomial rings. We claim that $\pi_F(f_{N/Q}) = f_{N_{n!F}/Q_{n!F}}$. There is no assumption that all the entries of $N_{n!F}/Q_{n!F}$ lie in $R_{Q_{n!F}}$, merely that the symmetric functions of the matrix do; this and the claim are immediate consequences of the definitions. If $'$ denotes differentiation with respect to the variable z , then it follows obviously that $(\pi_F(f_{N/Q}))' = f'_{N_{n!F}/Q_{n!F}}$.

Now suppose that (#) fails. Then there is a sequence of points in $(\mathbf{R}^d)^{++}$, $\{r_i\}$ such that $(\beta(r_i) - \beta_2(r_i))/\beta(r_i) \rightarrow 0$; thus $(\beta^{n!}(r_i) - \beta_2^{n!}(r_i))/\beta^{n!}(r_i) \rightarrow 0$. By taking a subsequence of $\ln r_i$ (as in [H5, p. 125–126]), we may find a path of the form $X(t) = \exp(tv + w)$ (where v and w are in \mathbf{R}^d) such that $\lim_{t \rightarrow \infty} (\beta^{n!}(X(t)) - \beta_2^{n!}(X(t)))/\beta^{n!}(X(t)) \rightarrow 0$. Let F be the face of $K(M)$ exposed by v (so that the maximum, $v \cdot K(M)$, is attained precisely at F); then $n!F$ is exposed by v in $K(N)$.

Now we wish to show that $H(t) := (f'_{N/Q})(\beta^{n!}(X(t))/Q(X(t))) \rightarrow 0$ as $t \rightarrow \infty$. If g is a monic polynomial of degree n with roots $\{b, b_2, \dots, b_n\}$ with $|b| \geq |b_i|$, then $g'(b) = \prod_{i \geq 2} (b - b_i)$, and of course $|g'(b)| \leq |b|^{n-2} \inf_{i \geq 2} |b - b_i|$. Since $\beta^{n!}/Q$ is bounded on $(\mathbf{R}^d)^{++}$, we infer that $|H(t)| < K_1 |\beta^{n!}(X(t)) - \beta_2^{n!}(X(t))|/Q(X(t))$ (the K_1 is independent of t). Since $(\beta^{n!}/Q)^{\pm 1}$ are both bounded on $(\mathbf{R}^d)^{++}$, we deduce $|H(t)| \leq K |\beta^{n!}(X(t)) - \beta_2^{n!}(X(t))|/\beta^{n!}(X(t))$, and this goes to zero.

Now consider each of the coefficients (in z) of $f'_{N/Q}$ along $X(t)$; they are elements of R_Q , there is a pure trace α such that for all c in R_Q , $\lim c(X(t)) = \alpha(c)$ and α induces a faithful pure trace on $R_{Q_{n!F}}$. The pure faithful traces on the latter are given by point evaluations, so we can find a point s in the corresponding positive orthant such that α is obtained by evaluating the elements of $R_{Q_{n!F}}$ at this point. Automatically, $N_{n!F}(s)$ is defined, and since $Q_{n!F}(s)$ is not zero (since its coefficients are automatically nonnegative and not all of them are zero), we infer that $f'_{N_{n!F}/Q_{n!F}}(\beta_{n!F}(s)) = 0$ (we have removed the denominators). This entails that the real matrix $N_{n!F}(s)$ has $\beta_{n!F}(s)$ as a multiple eigenvalue (since we know already that $\beta_{n!F}$ is an eigenvalue of $N_{n!F}$). In particular, $N_{n!F}$ cannot have just a single irreducible block with maximal spectral radius, hence the same is true of M_F .

Conversely, suppose that M_F has two blocks (at some real point of the underlying positive orthant) with maximal spectral radius, and that (#) holds for M . Then the same is true for $N_{n!F}$ and $N = M^{n!}$ respectively. Recalling our g from several paragraphs above, we see that $|g'(b)| = \prod_{i \geq 2} |b - b_i| \geq (\inf \{|b - b_i|\})^{n-1}$. From (#), we have that for $b = \beta^{n!}(r)/Q(r)$, $|f'_{N/Q}(b)|$ is uniformly bounded away from zero on $(\mathbf{R}^d)^{++}$. The same thing then applies to the derivative of the characteristic polynomials of the matrices corresponding to the quotients, i.e., $N_{n!F}/Q_{n!F}$, so the latter cannot have a multiple root for the large eigenvalue at any positive real evaluation. •

The first and second matrices in Example 7.8 do not satisfy (#), but the third one does.

10. Finite equivalence For matrices satisfying (%) & (#)

Here we show that under reasonable conditions, specifically, (%) & (#) together, the PT question has an affirmative answer.

The $n \times n$ matrix M with entries from $\mathbf{R}[x_i^{\pm 1}]^+$ is said to satisfy (*) if there exists an integer m such that for all vertices v of $K = \text{cvx Log}(M^m)_{ij}$, the real matrix obtained by taking the coefficient of x^v in M_{ij} is not nilpotent. This is equivalent to the following. Set $P = \sum_{w \in K \cap \mathbf{Z}^d} x^w$; if $\beta : (\mathbf{R}^d)^{++} \rightarrow \mathbf{R}^+$ denotes the large eigenvalue of M as an analytic function, then there exist positive real numbers, e and d so that as a function on $(\mathbf{R}^d)^{++}$, $e < \beta/P < d$.

If Y is a convex set, $\partial_e Y$ denotes the set of extreme points of Y . The following is probably well known.

Lemma 10.1 Let L, V , and V' be compact convex polyhedra in \mathbf{R}^d . If $L + V \subseteq L + V'$, then $V \subseteq V'$.

Proof. Set $W = \text{cvx } V \cup V'$. Then $\partial_e W \subseteq \partial_e V \cup \partial_e V'$. We will show that $L + W = L + V'$. Select x in $\partial_e(L + W)$; this entails $x = t + w$ for t, w vertices in L, W respectively [H5, VIII.5, p. 68]. If w belongs to $\partial_e V'$, then x belongs to $L + V'$; otherwise, w belongs to $\partial_e V$, so x belongs to $L + V \subseteq L + V'$. Hence $L + W \subseteq L + V'$, and the other inequality is trivial. Now the Cancellation Lemma [H5, p. 71] yields $W = V'$, so that $V \subseteq W = V'$. •

If u exhibits fractional polynomial behaviour, then we define $\text{cvx Log } u$ to be $\text{cvx Log } J$; this is well defined (by Lemma 3.3 applied to the quotients). There is no obvious way to define $\text{Log } u$.

Lemma 10.2 Let u_j ($j = 1, 2, \dots$) and u exhibit fractional polynomial behaviour, and let m_j and P belong to $A_{\mathbf{Q}}^+$. If the inequality $\sum u_j m_j \leq Pu$ holds on $(\mathbf{R}^d)^{++}$, then for all j ,

$$\text{cvx Log } u_j m_j \subseteq \text{cvx Log } Pu.$$

Proof. If not, there exists w in $\partial_e \text{cvx } \{\cup \text{cvx Log } u_j m_j\}$ that does not belong to $\text{cvx Log } Pu$. We may select Q in $A_{\mathbf{Q}}^+$ with the property that $\text{cvx Log } Q = \text{cvx } \{(\cup \text{cvx Log } u_j m_j) \cup \text{cvx Log } Pu\}$. We see that the original inequality transforms into an inequality with respect to R_Q ,

$$\sum \frac{u_j m_j}{Q} \leq \frac{Pu}{Q}.$$

Evaluating at the pure state of R_Q corresponding to the vertex w of $\text{cvx Log } Q$, we deduce

$$\sum \gamma_w \left(\frac{u_j m_j}{Q} \right) \leq 0.$$

However, we know that γ_w is a limit of point evaluation states, so that, as $u_j m_j$ are each strictly positive on $(\mathbf{R}^d)^{++}$, we deduce for each j , $\gamma_w((u_j m_j)/Q)$ must be nonnegative, hence by the equation above, must be zero. However, there exists j such that w belongs to $\text{cvx Log } u_j m_j$, and necessarily to $\partial_e \text{cvx Log } u_j m_j$ (for the same j), and so w must belong to $\text{cvx Log } u_j m_j$. Hence for this j , $\gamma_w((u_j m_j)/Q)$ must be nonzero, a contradiction. •

Proposition 10.3 Let a belong to $A_{\mathbf{Q}}^+$ and y exhibit fractional polynomial behaviour. Then $\text{cvx Log } ay = \text{cvx Log } a + \text{cvx Log } y$.

Proof. Pick P in $A_{\mathbf{Q}}^+$ such that y/P is bounded above and below on $(\mathbf{R}^d)^{++}$. Let z'' be any vertex of $\text{cvx Log } a + \text{cvx Log } y$. By [H5, VIII.5, p. 68], there will exist z in $\partial_e \text{cvx Log } a$ and z' in $\partial_e \text{cvx Log } y$ such that $z'' = z' + z$. There exists u in \mathbf{R}^d that simultaneously exposes z, z' , and z'' with respect to $\text{cvx Log } a$, $\text{cvx Log } y$, and $\text{cvx Log } a + \text{cvx Log } y$, respectively (see the proof of the Cancellation Lemma, [H5, p. 71]). Now define $X(t) = t^u = (t^{u(1)}, \dots, t^{u(d)})$ and consider $\lim_{t \rightarrow \infty} (av/aP)(X(t)) = \lim_{t \rightarrow \infty} (v/P)(X(t)) > 0$. Thus $\text{cvx Log } a + \text{cvx Log } y \subseteq \text{cvx Log } ay$, and the converse is trivial. •

Suppose M is a matrix in $M_n \mathbf{R}[x_i^{\pm 1}]^+$ for which $M(1)$ is primitive. There exists a left eigenvector for β whose entries are real analytic (that is, with no branch points) and algebraic (or even integral) over $\mathbf{R}[x_i^{\pm 1}]^+$, v . By the usual Perron-Frobenius theorem, each entry of v is strictly positive (or simultaneously strictly negative) as a function on $(\mathbf{R}^d)^{++}$. Assume for now that (*) holds, i.e., we have chosen $v = (v_i)$ such that there exist P_i in $A_{\mathbf{Q}}^+$ such that v_i/P_i are bounded above and below, and pick P_0 such that β/P_0 is also (we already know that such P_0 exist).

From $vM = \beta v$, we obtain the equations,

$$\sum_j M_{ij} v_j = \beta v_i \leq CP_0 P_i$$

(as functions on $(\mathbf{R}^d)^{++}$, and C is some positive integer) for all i . Fixing i , and applying Proposition 10.3, we deduce

$$(\dagger) \quad \text{cvx Log } M_{ij}v_j \subseteq \text{cvx Log } P_0P_i = \text{cvx Log } P_0 + \text{cvx Log } P_i.$$

Now set $\Delta = \text{diag}(P_i)$, and replace M by $\overline{M} := (\prod P_i)\Delta^{-1}M\Delta$. (This is the global version of the local trick discussed in section 8.) Its right eigenvector corresponding to the large eigenvalue β is $\Delta^{-1}v = (v_i/P_i)^\top$. Multiply this by $\prod P_i$ to obtain another eigenvector, $\overline{v} := (\overline{v}_i)^\top = (v_i \prod_{k \neq i} P_k)^\top$. The outcome of this is

$$\text{cvx Log } \overline{v}_i = \sum_{k=1}^n \text{cvx Log } P_k.$$

Hence $\text{cvx Log } \overline{v}_i$ are all equal to each other (for $i = 1, 2, \dots, n$); we denote this compact polyhedron L . Together with (\dagger) applied to M , we obtain

$$\text{cvx Log } \overline{M}_{ij}\overline{v}_j \subseteq (\text{cvx Log } P_0) + L.$$

However, by Proposition 10.3,

$$\text{cvx Log } \overline{M}_{ij} + L \subseteq (\text{cvx Log } P_0) + L.$$

By Lemma 10.1,

$$\text{cvx Log } \overline{M}_{ij} \subseteq \text{cvx Log } P_0.$$

This means that the matrix \overline{M} satisfies condition (*).

We have deduced the following:

Proposition 10.4 Let M be an element of $\mathbf{M}_n\mathbf{R}[x_i^{\pm 1}]^+$ possessing a left eigenvector whose entries exhibit fractional polynomial behaviour. Then there exist P_i in $A_{\mathbf{Q}}^+$ and $\Delta = \text{diag}(P_i)$ such that the matrix $\overline{M} := (\prod P_i)\Delta^{-1}M\Delta$ belongs to $\mathbf{M}_nA_{\mathbf{Q}}^+$ and satisfies the condition (*). Moreover, there exists a right eigenvector $\overline{v} = (\overline{v}_i)^\top$ for M such that for all w in $\partial_e \text{Log } PP_0$, $\gamma_w(\overline{v}_i/P) \neq 0$, and $\gamma_w(\overline{M}/P_0)$ has no zero rows, where P is the product of the P_i .

Proof. The only thing yet to be proved is that $\gamma_w(\overline{M}/P_0)$ has no zero rows. However, we observe that $(\overline{M}/P_0)\overline{v} = (\beta/P_0)\overline{v}$; dividing by P and applying γ_w , we deduce from $\gamma_w(\overline{v}/P)$ being a strictly positive real vector and from $\gamma_w(\beta/P_0)$ being greater than zero that the real column $\gamma_w(\overline{M}/P_0)\gamma_w(\overline{v}/P)$ is strictly positive; this precludes $\gamma_w(\overline{M}/P_0)$ having a row of zeroes. •

A column (or row) $v = (v_j)^\top$ of real meromorphic functions algebraic (over $\mathbf{R}[x_i^{\pm 1}]^+$) on $(\mathbf{R}^d)^{++}$ is said to be *very positive* if there exist P_j in $A_{\mathbf{Q}}^+$ and P in $A_{\mathbf{Q}}^+$, such that v_j/P_j are bounded as functions on $(\mathbf{R}^d)^{++}$, and for all L in $\partial_e S(R_P)$, $L(v_j/P_j) > 0$. In particular, this forces each v_j to exhibit fractional polynomial behaviour (as the set of pure states of R_P is compact and contains the point evaluations). From the definitions, the presence of a left eigenvector that is very positive is equivalent to (%).

One case in which it is very easy to prove that a matrix has a very positive eigenvector is if $d = 1$, that is, if the matrix only involves one variable. To see this, we simply recall from [A, p. 290], that if p is a (real) analytic function (say on $(0, \infty)$) that satisfies an equation with coefficients from $\mathbf{R}[x_i^{\pm 1}]$, then its behaviour at 0 is of fractional power order, i.e., $\lim_{t \rightarrow 0} p(t)/t^{a/b}$ exists and is not zero, and similarly, $\lim_{t \rightarrow \infty} p(t)/t^{a'/b'}$ exists, is not zero, and in both cases, the denominators b and b' divide the degree of the polynomial satisfied by p . Then if $s = (s_i)$ is the right eigenvector, we simply define P_i to be the element of $A_{\mathbf{Q}}^+$, $\sum x^j$, j varying over the set $\frac{1}{n}\mathbf{Z} \cap [\frac{a}{b}, \frac{a'}{b'}]$.

Let $\beta_2(r)$ denote the second largest absolute value of eigenvalues of $M(r)$, for r in $(\mathbf{R}^d)^{++}$; that is, $\beta_2(r) = \max \{|z| \mid z \text{ is a complex eigenvalue of } M(r), z \neq \beta(r)\}$. Recall that M satisfies (#) if there exists a positive real number δ such that $1 - \frac{\beta_2}{\beta} \geq \delta$ on $(\mathbf{R}^d)^{++}$.

Suppose that M admits a left eigenvector $v = (v_i)^T$ for β all of whose entries exhibit fractional polynomial growth. Then we may form $\overline{M} := (\prod P_i) \Delta M \Delta^{-1}$ with $\Delta = \text{diag}(P_i)$ for P_i in $A_{\mathbf{Q}}^+$ with v_i/P_i bounded above and below. Then \overline{M} has entries in the ring $A_{\mathbf{Q}}^+$, and moreover its left eigenvector for the large eigenvalue $\beta \prod P_i$ is $v \Delta^{-1}$. Each entry of the latter is strictly positive, and bounded above and below away from zero—this is precisely the definition of a very positive (left) eigenvector, as above. Note that if M satisfies (#) so does \overline{M} (since the ratio β_2/β is unchanged). We also notice that for every point evaluation pure state, $L, L(\overline{M}/(\prod P_i))$ has a gap from the spectral radius in its spectrum that is bounded below away from zero; it follows that if L is any pure trace of R_P that can be defined coordinatewise, the large eigenvalue of $L(\overline{M}/(\prod P_i))$ has multiplicity one and the corresponding block is primitive.

Proposition 10.5 Condition (#) implies that for all $\epsilon > 0$, for all sufficiently large integers N ,

$$1 + \epsilon \geq \frac{\text{tr } M^N(r)}{\beta^N(r)} \geq 1 - \epsilon \quad \text{for all } r \text{ in } (\mathbf{R}^d)^{++}.$$

Proof. As $\text{tr } M^N(r)$ is the sum of the N th powers of the eigenvalues of $M(r)$, we deduce

$$\beta^N(r) + (n-1)\beta_2^N(r) \geq \text{tr } M^N(r) \geq \beta^N(r) - (n-1)\beta_2^N(r).$$

For all sufficiently large N , $(1-\delta)^N < \epsilon/(n-1)$. Dividing by $\beta^N(r)$, the desired inequality comes out. •

This forces $\text{tr } M^N/\beta^N$ to converge *uniformly* on $(\mathbf{R}^d)^{++}$ to the constant function 1. Note that $\text{tr } M^N/\beta^N$ converges pointwise to 1 merely if $M(1)$ is primitive (which we assume throughout anyway). Neither inequality in Proposition 10.5 need hold if we drop condition (#).

Starting with the matrix M and a left eigenvector v whose entries exhibit fractional polynomial behaviour, apply the diagonalization trick. We obtain a new matrix, which for the purposes of the next result, we also call M , with various properties. Specifically, M now satisfies (*) (since we allow polynomials with fractional exponents), and its new left eigenvector satisfies (v_i/P) is bounded above and below away from zero. Now consider the matrix $\text{Adj}((\beta/P)\mathbf{I} - M/P)$. We observe that M/P has entries in R_P , so that the entries of the adjacency matrix are integral over R_P . Let σ_i be the i -th column of $\text{Adj}((\beta/P)\mathbf{I} - M/P)$, and define σ to be the sum $\sum_i (-1)^{e(i)} \sigma_i$, where $e(i) = 0$ if $\sigma_i(1, 1, \dots, 1)$ (σ_i evaluated at 1) exceeds zero, $e(i) = 1$ if $\sigma_i(1, 1, \dots, 1)$ (σ_i evaluated at 1) is less than zero, and any σ_i whose value at 1 is zero is discarded. Notice that the entries of σ are bounded on $(\mathbf{R}^d)^{++}$, so we may apply the states of R_P to them (see the lemma about limits of bounded algebraic functions).

As in the Appendix, if B is a nonnegative real matrix with only one eigenvalue of absolute value equaling its spectral radius, then that eigenvalue belongs to a unique block of B ; this will be called the *principal block*.

Lemma 10.6 Let M be an element of $M_n A_{\mathbf{Q}}^+$ with the following properties:

- (a) M satisfies (#);
- (b) there exists P in $A_{\mathbf{Q}}^+$ such that $\text{cvx Log } M_{ij} \subseteq \text{cvx Log } P$;
- (c) there exists a left nonnegative eigenvector for β , $v = (v_i)^T$ consisting of algebraic and real analytic functions, such that v_i/P is bounded on $(\mathbf{R}^d)^{++}$;
- (d) for all vertices w of $\text{cvx Log } P$, $L_w(\sigma)$ (which equals $\lim_{t \rightarrow \infty} \sigma(\exp u \ln t)$ for u any vector in \mathbf{R}^d that exposes $\{w\}$ as a face of $\text{cvx Log } P$) is strictly positive.

Then for all pure states L of R_P , $L(\sigma)$ is strictly positive.

Proof. We have that $L(\text{Adj}((\beta/P)\mathbf{I} - M/P)) = \text{Adj}(L(\beta/P)\mathbf{I} - L(M/P))$ for any pure trace L of R_P . Let F denote the face of $K = \text{cvx Log } P$ that contains $\Lambda_P(L)$, and let $P_F = \sum_{w \in F} \lambda_w x^w$ (where $P = \sum \lambda_w x^w$; that is, disregard the terms in P whose exponents do not lie on F). Then following [H5, Section VII], we may define R_{P_F} , and there is a natural map $\pi_F : R_P \rightarrow R_{P_F}$, which simply replaces k th powers of P in the denominator by the same powers of P_F , and changes the numerators by discarding the exponents not in kF (as occurs in going from P to P_F). The map π_F extends in a natural way to functions algebraic over R_P , via Lemma 3.1. Let $(\mathbf{R}^{d'})^{++}$ parameterize the pure faithful traces of R_{P_F} (that is, there is a change of variables so that elements of R_{P_F} are rational functions in d' variables, etc.). Now, L induces a pure state on R_{P_F}, \bar{L} , which corresponds to a point evaluation (since $\Lambda_P(L)$ is in the relative interior of F [H5, IV]) on this ring of rational functions (effectively in fewer variables than R_P).

Let \mathbf{M} denote the $n \times n$ matrix (with entries from R_{P_F}) $\pi_F(M/P)$, and let w be a vertex of F . We have that $L(\mathbf{M})$ has a strictly positive eigenvector, namely the image of that obtained from the adjacency matrix as described in the paragraph preceding this lemma. As L_w is a limit of point evaluation states on R_{P_F} , it follows that for at least at one point, r' (of $\mathbf{R}^{d'}$), $L_{r'}(\pi_F(\sigma))$ is a strictly positive row. If we now multiply \mathbf{M} by a suitable power of P_F so as to remove the denominators, we obtain a matrix, call it \mathbf{N} , with entries in $\mathbf{R}[x_i^{\pm 1}]^+$, with the property that for at least one point, r' of $\mathbf{R}^{d'}$, $\mathbf{N}(r')$ has a strictly positive eigenvector. Inasmuch as the entries of \mathbf{N} are Laurent polynomials, the 0 – 1 “skeleton” matrix of $\mathbf{N}(r)$ obtained by replacing every nonzero entry of $\mathbf{N}(r)$ by 1, is independent of the choice of r . A real matrix with nonnegative entries and having a largest eigenvalue of multiplicity just one has a strictly positive left eigenvector (see the Appendix) if and only if there exists an index i belonging to the irreducible block with the largest eigenvalue such that for all sufficiently large powers of the matrix, the i th row is strictly positive (if B is such a real matrix with large eigenvalue ρ , consider the asymptotic behaviour of $\{(B/\rho)^m\}$ as $m \rightarrow \infty$). Although the skeleton matrix of \mathbf{N} is independent of the choice of r , the equivalence class of indices corresponding to the largest eigenvalue could change, as r varies.

We show in fact that if for all r , $\mathbf{N}(r)$ has its largest eigenvalue of multiplicity 1, then even the equivalence classes are independent of the choice of r . Suppose for some index i , there exist r^1 and r^2 so that i belongs to the block with the largest eigenvalue for $\mathbf{N}(r^1)$, but not for $\mathbf{N}(r^2)$. Let $Y : [0, 1] \rightarrow (\mathbf{R}^{d'})^{++}$ be any (continuous) path with $Y(0) = \mathbf{N}(r^1)$ and $Y(1) = \mathbf{N}(r^2)$. We note that $\mathbf{N}(Y(t))$ has its largest eigenvalue of multiplicity 1 for any value of t . Define

$$\begin{aligned} \mathbf{O} &= \{t \in [0, 1] \mid i \text{ belongs to the principal block } \mathbf{N}(Y(t))\} \\ \mathbf{O}' &= \{t \in [0, 1] \mid i \text{ belongs to a block of } \mathbf{N}(Y(t)) \text{ other than the principal one}\}. \end{aligned}$$

Clearly, both \mathbf{O} and \mathbf{O}' are open, but because $\mathbf{N}(Y(t))$ has its largest eigenvalue of multiplicity one for all choices of t , we have that $\mathbf{O} \cup \mathbf{O}' = [0, 1]$. This forces one of \mathbf{O} or \mathbf{O}' to be empty, a contradiction.

Thus it makes sense to talk of an index i that belongs to the irreducible block of \mathbf{N} (i.e., of $\mathbf{N}(r)$ for all r) with largest eigenvalue. As $\mathbf{N}(r)$ has a strictly positive eigenvector, there exists an integer m_0 such that for all $m \geq m_0$, the i th column of $\mathbf{N}^m(r)$ is strictly positive. Since the skeleton of \mathbf{N} is independent of the choice of r , it follows that for all such m , and for all r , the i th column of $\mathbf{N}^m(r)$ is also strictly positive. As i belongs to the block with largest eigenvalue, $\mathbf{N}(r)$ has a strictly positive eigenvector, and moreover, this is unique (up to positive scalar multiple). Now $L_r(\pi_F(\sigma))$ is a nonzero eigenvector for the large eigenvalue of \mathbf{M} and thus of \mathbf{N} , so it is a positive scalar multiple of the unique one, hence it is strictly positive. However, the original L factors through L_r for some r in $\mathbf{R}^{d'}$, that is, $L(\sigma) = L_r(\pi_F(\sigma))$, and we are done. •

Theorem 10.7 Let M and M' be square matrices with entries from $\mathbf{R}[x_i^{\pm 1}]^+$ satisfying the following properties:

- (a) Both M and M' satisfy (#);
- (b) $\beta_M = \beta_{M'}$;
- (c) both M^T and M' satisfy (%).

Then there exists a nonzero rectangular matrix X with entries from $\mathbf{R}[x_i^{\pm 1}]^+$, such that $XM = M'X$.

Proof. Since the process in going from M to \overline{M} described earlier is a very special finite equivalence together with multiplication by a polynomial with no negative coefficients, it is sufficient to work in the context of \overline{M} .

The matrix \overline{M} satisfies (#), and also (*) by Proposition 10.4; for any vertex w of $\text{cvx Log } P$, $L_w(M/P_0)$ is a positive matrix with no zero rows, having $L_w(b/P_0)$ as its large eigenvalue, and this has multiplicity one. If we take the canonical right eigenvector (for M) to initiate the process of getting from M to \overline{M} , (i.e., using the right eigenvector instead of the left to construct Δ), the corresponding left eigenvector (on the left, σ , has the property that for all w in $\partial_e \text{cvx Log } P$, the row $L_w(\sigma)$ is strictly positive (the hypothesis of Lemma 10.6 applies to M^T ; this strict positivity is the whole point of using Δ) and for any L in $\partial_e S(R_P, 1)$, $L(\sigma)$ is nonnegative and nonzero. By Lemma 10.1, this (right) eigenvector, obtained from $\text{Adj}(\beta I - \overline{M}/P)$, is very positive. Now from the definition of adjoint, each entry of $\text{Adj}(\beta I - \overline{M}/P)$ can be written uniquely as a sum of the form $\sum_{i=0}^{e-1} n_i \beta_0^i$, where e is the degree of the characteristic polynomial of $\beta_0 = \beta/P$ over the function field, $\mathbf{R}(x_i)$ (the field of fractions of $\mathbf{R}[x_i^{\pm 1}]$), and n_i all belong to R_P .

Next, we do the same process with M' , and so obtain a right eigenvector τ for \overline{M}'/P (note that the P can be chosen to be the same for M as for M' , since the corresponding beta functions are equal). Following the prescription in [PT], we form the matrix $X_1 = \tau\sigma$; if M is $n \times n$ and M' is $m \times m$, then X_1 is $m \times n$. Obviously X_1 intertwines \overline{M} and \overline{M}' . From the remark in the previous paragraph, we may write (uniquely) $X_1 = \sum_{i=0}^{e-1} N_i(\beta/P)^i$, where the N_i are $m \times n$ matrices with entries in the ring R_P (which is contained in $\mathbf{R}(x_i)$). Now β_0 is integral of degree e over R_P and satisfies no algebraic equation of smaller degree, and it easily follows that $\{\beta_0^i\}_{i=0}^{e-1}$ is a basis for the ring $R_P[\beta_0]$ over R_P . Thus each N_i intertwines the matrices. We obtain our preliminary candidate intertwining matrix, $N = \sum_i N_i(\overline{M}/P)^i$. Certainly N is a matrix with entries in R_P that intertwines the relevant matrices. We shall show that there exists k such that each entry of $N(\overline{M}^k)$ is positive in R_P^+ .

Let L be a pure trace of R_P . We have guaranteed that $L(M/P)$ has a strictly positive left eigenvector for the large eigenvalue $L(\beta_0)$, and no other eigenvalues have this as their absolute value. By Lemma A1.2, there exists a unique nonnegative right eigenvector, necessarily for the same eigenvalue, call it w_L .

Consider the matrix product

$$\begin{aligned} L(N)w_L &= \sum L(N_i)L\left(\frac{\overline{M}}{P}\right)^i w_L \\ &= \sum L(N_i)L\left(\frac{\beta}{P}\right)^i w_L \\ &= L(\tau\sigma)w_L \\ &= L(\tau)(L(\sigma) \cdot w_L). \end{aligned}$$

Now $L(\sigma) \cdot w_L$ is the inner product of a strictly positive vector with a nonzero nonnegative one, so is a positive real number. As $L(\tau)$ is strictly positive, this means that for each row, u of N , $L(u)w_L$ is strictly positive. As w_L is the unique nonnegative right eigenvector of $L(\overline{M}/P)$, there exists $K \equiv K(L)$ such that $L(u)L(\overline{M}/P)^K$ is strictly positive. Hence $L\left(N(\overline{M}/P)^{K'}\right)$ is strictly positive for some sufficiently large (and hence all larger) K' . Hence there exists a neighbourhood, W , of L in the pure trace space of R_P such that for all L_0 in W , $L_0\left(N(\overline{M}/P)^{K'}\right)$ is also strictly positive. The pure trace space being compact, we obtain a finite open covering and a fixed integer k such that for all L in the pure trace space, $L\left(N(\overline{M}/P)^k\right)$ is strictly positive. By e.g., [HM, I.1 & I.2], every entry of $N(\overline{M}/P)^k$ is a positive element of R_P . Hence there

exists k' such that $X^1 := N(\overline{M}/P)^k P^{k'}$ is a matrix with entries in $\mathbf{R}[x_i^{\pm 1}]^+$, and this clearly intertwines \overline{M} and \overline{M}' . Now we can unravel the Δ and obtain the X that intertwines the original M and M' . •

Corollary 10.8 Let M and M' be square matrices with entries from $\mathbf{R}[x_i^{\pm 1}]^+$. If M satisfies:

- (a) (#);
- (c) (%),

then there exists nonnegative rectangular Y with entries in A^+ such that $YM = M^T Y$ and M^T satisfies (c). If M' satisfies (a) and (c) and $\beta_M = \beta_{M'}$, then there exist rectangular X and Z with entries in A^+ such that $XM = M'X$ and $MZ = ZM'$.

Proof. Setting $M'' = M^T$ in Theorem 10.7, we see that (c) therein is satisfied; hence there exists nonnegative rectangular X such that $YM = M'Y$. If w_M is a right nonnegative eigenvector for M , then Xw_M is a right eigenvector for M'' —hence M'' satisfies (c) of the hypotheses above. Thus M admits a good left nonnegative eigenvector.

If M' satisfies (a) and (c) here, then the result of previous paragraph implies that its transpose does as well. Hence the transposes of M and M' are intertwined by an “ X ”, and the existence of the Z follows by applying the transpose to the resulting equation. •

Still open: Is it true that if M merely satisfies Corollary 10.8(c), does M^T ? (That is, drop (#); a counter-example would exhibit a matrix not finitely equivalent to its transpose.)

The proof can be modified (as indicated below) to work under weaker hypotheses.

- M' has a left nonnegative eigenvector for β all of whose entries exhibit fractional polynomial behaviour, and all of whose entries lie in $\mathbf{R}[x_i^{\pm 1}][\beta]$ (the latter is important—it is not enough that the entries be integral, since the integral closure of $\mathbf{R}[x_i^{\pm 1}]$ in $\mathbf{R}(x_i)(\beta)$ can be strictly bigger than $\mathbf{R}[x_i^{\pm 1}][\beta]$); this permits us to obtain a very positive right eigenvector τ for \overline{M}' , except that this time, we use a different Δ to define the latter; here, use Δ' whose diagonal entries are determined by entries of the right eigenvector. Correspondingly, we obtain P' , which now need not be the same as P .
- M admits a right nonnegative eigenvector $v = (v_i)$ for β with the following properties:
 - (i) there exist P_i in $A_{\mathbf{Q}}$ such that v_i/P_i is bounded above—by itself, this forces the matrix $\overline{M} := P \text{diag}(P_i) M \text{diag}(P_i)^{-1}$ to satisfy (*);
 - (ii) for all pure traces L of R_P , $L(\overline{M}/P)$ admits nonnegative right eigenvector(s) only for its large eigenvalue (which must be $L(\beta/P)$), and not for any smaller ones, and moreover, $L(\sigma)$ must be in the interior of the positive left eigenspace for the eigenvalue $L(\beta/P)$;
 - (iii) v_i lie in $\mathbf{R}[x_i^{\pm 1}][\beta]$.

We show that that these conditions are sufficient. We note that $L(\beta/P) \neq 0$ implicit in (ii) entails that $\text{cvx Log } P = \text{cvx Log } P'$. To make sure that \overline{M} and \overline{M}' have the same large eigenvalue ($\beta P P'$), we multiply the first by P' and the second by P , and so the new “ P ” is $P P'$ and it is easy to check that the τ and σ we obtain as in the proof still have entries in $R_P[\beta_0]$. The argument will now go through, on noting that although there may be several choices for w_L , each one will have nonzero inner product with $L(\sigma)$, by the interior hypotheses.

In the original proof, the reason for making a construction based on the adjoint matrix was to guarantee (iii) and its analogue for M' ; and also, we could deduce some properties of τ and σ directly from their construction, that a more arbitrary choice may not satisfy. If (#) holds, then of course (ii) is automatic. If (#) does not hold, in fact a direct construction of τ or σ could not use the adjoint matrix, since the latter will vanish at some L . (However, it might be reasonable to assume only that M' satisfies (#).)

If we assume for example, that β is rational, then we may take an eigenvector with polynomial entries, and the problem then boils down to showing that the canonical choice (we note that the kernel of the map $\beta \mathbf{I} - M : A^n \rightarrow A^n$ is free of rank one over A , so the generator is defined up to multiplication by a monomial) can be made eventually positive by multiplication by some polynomial; this amounts to showing

(in this context), that $L(v_i/P_i) \neq 0$ for every L in R_P . After factoring out the ideal $\ker \pi_F$ (see Lemma 3.6), this boils down to checking what happens at a point evaluation of the reduced polynomial obtained by restricting to a face. Of course, the usual problem rears its head—the restricted matrix need not be primitive, so we cannot conclude the entry must be nonzero.

In case the extreme points of $K(M)$ are not lattice points, we may add a polynomial with fractional exponents (i.e., Q could come from $A_{\mathbb{Q}}^+$) instead. Since the finite equivalence problem is insensitive to adding scalars, the main theorem of this section applies just as well with weak (#) replacing (#).

It might be possible to delete the (#) hypothesis altogether from results in this section. It seems plausible that if M and M' are two primitive matrices with entries from A^+ , with the same beta function, and M^T and M' satisfy (%), then there should exist a polynomial q in $A[Z]$ (Z being another, dummy, variable) such that both $q(M)$ and $q(M')$ are primitive matrices and satisfy (#) (the polynomial q could be replaced by a rational function). Obviously, both $q(M)^T$ and $q(M')$ satisfy (%) since their eigenvectors may be taken as the eigenvectors of M^T and M' respectively, so they could be intertwined by a positive rectangular matrix, by the main theorem of this section. If we could also arrange that there is a rational function p in $A[Z]$ such that $p(q(M)) = M$ and $p(q(M')) = M'$ (or some suitable weakening of this; this is very often the case), then we would deduce that M^T and M' are finitely equivalent. (In fact, we could dispense with the polynomial q , and just replace $q(M)$ and its counterpart by a matrix commuting with M whose commutant is the same as that of M and another one for M' so that the two matrices satisfy (#) and have the same beta functions as each other.)

For instance, the matrix

$$M = \begin{bmatrix} x+y & 0 & 1 \\ 1 & x & 0 \\ 0 & 1 & y \end{bmatrix}$$

satisfies (%) (an easy exercise), but fails to satisfy (#) because at the singleton faces corresponding to x and y , the matrix is a direct sum of two equal scalars and the zero matrix. The form of M (and the facts that it is $\text{GL}(3, \mathbf{Z})$ -conjugate to its companion matrix, and its characteristic polynomial is irreducible over $A := \mathbf{R}[x, y]$) suggest nothing commuting with it can satisfy (#). (Since the characteristic polynomial is irreducible, any nonscalar matrix that commutes with M is a rational function thereof, and since M is conjugate to a companion matrix, anything that commutes with it is of the form $p_0I + p_1M + p_2M^2$ where p_i belong to M —this can be seen directly by exploiting any row of M —it has a zero and a one; moreover, the commutant of any nonscalar that commutes with it is the same as that of M .) Very unexpectedly, there is a matrix commuting with it satisfying (#):

$$N = \begin{bmatrix} xy & 1 & y \\ x & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The matrices M and N satisfy $M = N^2 - xyN$ and $N = M^2 - (x+y)M + xyI$, and N is shift equivalent to its companion matrix (the characteristic polynomial is $\lambda^3 - xy\lambda^2 - (x+y)\lambda - 1$). So the proposal of the previous paragraph is at least plausible.

11. Structural similarity

Let M and M' be two matrices in $M_n A_{\mathbb{Q}}$. We say they *have the same shape* (or *are shape similar*), if for each i and j ,

$$\text{cvx} \text{Log } M_{ij} = \text{cvx} \text{Log } M'_{ij}.$$

If for some k , M^k and $(M')^k$ have the same shape, then we say that they *eventually have the same shape*. This notion will be used to extend our so far rather feeble results on finite equivalence. The two matrices M and M' will be *structurally similar* if for all i and j , $\text{Log } M_{ij} = \text{Log } M'_{ij}$.

We first note that (#) is not an invariant of shape similarity or structural similarity; take the constant 2×2 matrices I and $\text{diag}(2, 1)$ (nonconstant primitive examples can easily be built up from this). It is also not true that structural similarity of primitive matrices forces the corresponding left nonnegative eigenvectors to be bounded with respect to each other (after making some normalization). For example, it is an easy exercise to see that the respective nonnegative left eigenvectors of

$$\begin{bmatrix} 1+x & 1+x \\ x & 2+x \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2+x & 1+x \\ x & 1+x \end{bmatrix}$$

are not comparable—see what happens as $x \rightarrow 0$. A minor variation on this leads to an example showing that (%) is also not an invariant of structural similarity.

The matrices

$$M = \begin{bmatrix} 1+x & 1 \\ y & x+y \end{bmatrix} \quad \text{and} \quad M' = \begin{bmatrix} 1+2x & 1 \\ y & x+y \end{bmatrix}$$

are clearly structurally similar. The matrix M admits the constant vector $(1 \ 1)$ as its left eigenvector for the large eigenvalue function, so M obviously satisfies (%). It is not difficult to check that M' does not satisfy (%) on either side; see section 12 for details.

On the other hand, $T(M)$ is an invariant of structural similarity, as is immediate from the definitions. Also $T(M)$ is an invariant of shape similarity, and the necessary condition (Proposition 7.5) in order that all order ideals of $E_b(G_M)$ possess an order unit is an invariant of structural similarity. More generally, shape similarity also implies a bijection on the lattice of order ideals.

If M and M' eventually have the same shape, then $K(M) = K(M')$; to see this, notice that $\text{cvx Log tr } M^{kn!} = \text{cvx Log tr } (M')^{kn!}$, and these are respectively equal to $kn!K(M)$ and $kn!K(M')$.

The condition (***) is just eventual structural similarity to a (necessarily rank one) matrix all of whose entries are the same. Of some interest are those matrices that are shape similar to a matrix whose β is a polynomial. These have rather pleasant properties, as we shall see.

Proposition 11.1 Let M and M' be primitive eventually shape similar matrices in $M_n A_{\mathbb{Q}}^+$, both satisfying (#). If M satisfies (%), then so do M' and M^T .

Proof. Since the eigenvectors are unchanged on replacing the matrices by their powers, we may assume the matrices are already shape similar; we may incorporate a factor of $n!$ into the power, so assume that $\text{tr } M/\beta$ and $\text{tr } M'/\beta'$ are bounded away from zero and infinity on $(\mathbf{R}^d)^{++}$. Let $v = (v_i)^T$ be the hypothesized left eigenvector for β , and apply the now standard conjugation with $\Delta = \text{diag}(P_i)$ where P_i belong to $A_{\mathbb{Q}}^+$, and v_i/P_i are bounded away from zero and infinity on $(\mathbf{R}^d)^{++}$. Set $P_0 = \prod P_i$, and form the new matrices $\overline{M} := P_0 \Delta M \Delta^{-1}$ and $\overline{M}' := P_0 \Delta M' \Delta^{-1}$, both in $M_n A_{\mathbb{Q}}^+$.

We note that \overline{M} and \overline{M}' are still shape similar (that is, to each other), and obviously both satisfy (#). Since \overline{M} satisfies (*) (with respect to some Q in $A_{\mathbb{Q}}$; see section 10), so does \overline{M}' . We notice that $w := (v_i/P_i)$ is a left nonnegative eigenvector for \overline{M} , and it is bounded, positive, and bounded away from zero; as a consequence, for each pure state L of R_P , the row $L(w) \gg \mathbf{0}$, and $L(w)$ is thus a strictly positive left eigenvector for $L(\overline{M}/Q)$, for the large eigenvalue, $L(\beta P_0/Q)$. As a consequence of (#), the multiplicity of $L(\beta P_0/Q)$ as an eigenvalue of $L(\overline{M}/Q)$ is one; so the real matrix $L(\overline{M}/Q)$ is nonnegative, has its spectral radius of multiplicity one and no other eigenvalues of the same multiplicity, and admits a strictly positive left eigenvector for that eigenvalue. Moreover, since \overline{M} and \overline{M}' are structurally similar, $L(\overline{M}/Q)$ and $L(\overline{M}'/Q)$ have nonzero entries in exactly the same position. On replacing $N'_L := L(\overline{M}'/Q)$ by some power of itself and applying Lemma A1.2, we deduce that N_L has a strictly positive (real) left eigenvector.

Now suppose that M' has no nonnegative left eigenvector for the large eigenvalue function which is both algebraic and has fractional polynomial growth; then neither would \overline{M}' , and thus neither would \overline{M}'/Q .

There exists a left eigenvector $U := (U_i)^T$ for the large eigenvalue function of \overline{M}'/Q with algebraic entries. Form $U' := U_i / \sum U_j$, so that the sum of the entries of U' is the constant with value 1; in particular, the entries of U' are bounded algebraic functions which are strictly positive on $(\mathbf{R}^d)^{++}$.

Pick any pure trace L on R_P ; then $L(U')$ (a real vector) makes sense as follows. If L is a point evaluation at r in $(\mathbf{R}^d)^{++}$, $L(U') = (U'_i(r))^T$. Otherwise, L is obtained as a limit along the exponential of a ray in \mathbf{R}^d , and it follows from Lemma 3.1, that the limit of U'_i exists along this path. This gives the definition of $L(U')$. Clearly, $L(U')$ is a nonzero (entries add to 1) nonnegative left eigenvector of the real matrix N_L for the spectral radius. Since the multiplicity of the spectral radius is 1, $L(U')$ is strictly positive. Since this is true for any L , and the set of such L is a compactification of $(\mathbf{R}^d)^{++}$, it follows that each entry of U' is bounded above and below away from 0 on $(\mathbf{R}^d)^{++}$. Of course, this means each entry of U' satisfies fractional polynomial growth (with the polynomial 1), and when we translate back to M' by undoing the process, it is immediate that the corresponding left eigenvector of the latter satisfies fractional polynomial growth.

That M^T has a left eigenvector with the desired properties is a consequence of Corollary 10.8. •

A much easier result occurs when M is structurally similar to a rank one matrix. To avoid pathologies such as that occurring in Example 5.12, we call a matrix M of *positive rank one* if there exist a row and column v and w respectively, with entries from A^+ such that $M = vw$. Obviously, any matrix satisfying (**) is shape equivalent to a positive rank one matrix (which in turn is strongly shift equivalent to a size one matrix), but the pathological example is a rank one matrix that is not even shift equivalent to a positive rank one matrix.

Proposition 11.2 Let M and M' be primitive matrices with entries from A^+ , and that M' is of positive rank one. Suppose M is shape similar to M' .

- (a) There exists Q in A^+ such that some power of MQ is structurally similar to $M'Q$.
- (b) There exists Q' in A^+ such that some power of MQ' is shift equivalent to a matrix satisfying (**).
- (c) M satisfies (#) and both M and $M \text{ Tr}$ satisfy (%).

Proof. (a) (We may replace the matrices by their powers in order to accommodate the definition of “rank one”.) The hypotheses guarantee that there exist P_i and Q_j in A^+ such that $\text{cvx Log } M_{ij} = \text{cvx Log } P_i + \text{cvx Log } Q_j$. By the Absorption lemma (5.7), there exists $R^{(ij)}$ in A^+ such that $\text{Log } R^{(ij)} M_{ij} = \text{Log } P_i Q_j R^{(ij)}$. Set $Q = \prod R^{(ij)}$.

(b) After replacing the matrices by suitable powers of themselves, and applying (a), we have that there exist P_i and Q_j in A^+ such that $\text{Log } P_i Q_j = \text{Log } M_{ij}$. Let $W = (P_i)^T$ and $V = (Q_j)$. Define $\Delta = \text{diag } (P_i)$, and set $P_0 \prod P_i$. Then $P_0 \Delta^{-1}$ has entries in A^+ and obviously $P_0 \Delta^{-1} M \Delta$ is strongly shift equivalent to $M P_0$. The (ij) entry of the former is $M_{ij} P_j \prod_{k \neq i} P_k$. Hence its Log set is $\text{Log } P_j + \text{Log } Q_j + \sum \text{Log } P_k$ —which depends only on j .

Now consider the elementary row operations $\boxed{i} \mapsto \boxed{i} - \epsilon \boxed{k}$ (which replace row i by itself minus a small real multiple of row k) applied to $P_0 M$. The ij entry before such a transformation has Log set $\text{Log } P_j Q_j + \sum \text{Log } P_k$ and we are subtracting from it a small multiple of a polynomial with the same Log set. Hence for sufficiently small real ϵ , we can be sure that the outcome has exactly the same Log set and no negative coefficients. The inverse operation adds a small multiple of column i to column k —this increases the Log set in entry jk from $\text{Log } P_j Q_j + \text{Log } P_0$ to $(\text{Log } P_i Q_i \cup \text{Log } P_j Q_j) + \text{Log } P_0$. To avoid confusion, we do $n^2 - n$ row operations (one for each pair (i, k) , choosing various small ϵ s along the way; none of the row operations changes the Log sets or the signs of the coefficients). Applying the corresponding inverse operations, we see that every entry of the final matrix has as its Log set $\text{Log } P_0 \cup_i \text{Log } P_i Q_i$, which clearly satisfies (**). This sequence of row and column operations constitutes a strong shift equivalence.

(c) Both (#) and polynomial behaviour of the nonnegative left/right eigenvectors are preserved by shift equivalences, and we already know (**) implies both conditions, so (b) yields the result. •

If a matrix were shape similar to a rank one matrix, then it is plausible that some power of it is structurally similar to a rank one matrix. This fails for the following example,

$$M = \begin{bmatrix} x^2 & x^2 \\ 1 + x + x^3 & 1 + x^3 \end{bmatrix}.$$

It is still possible that such a matrix would be shift equivalent to a matrix satisfying (**).

Call a matrix in $M_n A^+$ *full*, if there exists k such that $x^w M^{-kn!}$ belongs to $E_b(G_M)$ for all w in $\text{Log tr } M^{kn!}$. For example, all matrices satisfying (**) are full (with $k = 1$), as is any matrix shift equivalent to a full one. Another way to say this is that some power of $PM^{-n!}$ is in $E_b(G_M)$, where $P = \text{tr } M^{n!}$.

Proposition 11.3 Let M be a primitive matrix in $M_n A^+$. If M is shape similar to a matrix of positive rank one, then it is full.

Proof. This follows from the fact that $T(M)$ is an invariant of shape similarity, as is $\text{Log } M^k$ for any k . •

12. Two by two matrices

In this section, we deduce a simple necessary and sufficient condition on 2×2 matrices with entries in A^+ in order that they satisfy (%): the discriminant must divide a polynomial with no negative coefficients. (This condition is easy to decide in this context, because the discriminant has a special form.)

Let M be a square matrix with entries from A^+ . We say M is “stripped” if $\cap_i \text{Log } M_{ii}$ is empty. In other words, if x^w appears in a diagonal entry, then there is another diagonal entry where it does not appear. Obviously, if M is size 2, then it is stripped if and only if $\text{Log } M_{11} \cap \text{Log } M_{22} = \emptyset$.

If M is an arbitrary element of $M_n A^+$, we may reduce it to a stripped form by subtracting off the following multiple of the identity matrix:

$$\sum_{w \in \bigcap_{i \leq n} \text{Log } M_{ii}} \left(\inf_{i \leq n} (M_{ii}, x^w) \right) x^w,$$

where (P, x^w) denotes the coefficient of x^w appearing in P . Obviously, the stripped form still belongs to $M_n A^+$, it has the same eigenvectors as the original form, and some other properties are preserved by this process. In particular, stripping does not affect property (%).

It turns out that for size 2 stripped matrices, (%) is equivalent to the weak version of (#) discussed briefly in section 10. Adding a multiple of the identity matrix can ruin weak (#); it can also improve weak (#) to (#); stripping will preserve weak (#), but may reduce (#) to weak (#); some criteria for (%) are most easily dealt with for stripped matrices.

The discriminant, $\mathcal{D}(M)$, of a (square) matrix M will be defined as the discriminant of its characteristic polynomial (it may be useful at times to define it to be the discriminant of the minimal polynomial of β when M belongs to $M_n A^+$; we shall not do so). In particular, $\mathcal{D}(M)$ is an element of A . For a size 2 matrix $M = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$, the discriminant is $\mathcal{D}(M) = (S - P)^2 + 4QR$, which is all we need to know about discriminants for this section. It is easy to check that the polynomial $\Delta^+ := (S + P)^2 + 4QR$ in A^+ satisfies $(\beta^2 / \Delta^+)^{\pm 1}$ is bounded on $(\mathbf{R}^d)^{++}$, so that $2K(M) = \text{cvx Log } \Delta^+$.

Now assume that M is primitive and in $M_n A^+$. Let F be a non-trivial face of M . Then M_F is either irreducible or is diagonal. If M_F reducible (i.e., diagonal), then it is of the form $\text{diag}(P_F, S_F)$ where P_F has the obvious meaning, i.e., $P_F = \sum_{w \in F} (P, x^w) x^w$, etc. Suppose that condition (%) holds. We know from section 8 that either $P_F = S_F$ or $P_F - S_F$ does not vanish on $(\mathbf{R}^d)^{++}$, this for every face F for which M_F is reducible. If M_F is reducible, then obviously it is reducible for every subface of F , and we quickly deduce (using e.g., results of [H5]), that if $P_F \neq S_F$, then either $P_F - S_F \geq \epsilon P_F$ or $S_F - P_F \geq \epsilon S_F$ as functions on $(\mathbf{R}^d)^{++}$ for some epsilon. If M were a stripped matrix, then of course $P_F \neq S_F$ (P and S

have no monomials in common, and at least one of P_F and S_F is not zero). Notice that if M_F is irreducible, (%) tells us nothing new about its entries.

If F is a (closed) face of a polyhedron, $\text{Rel int}(F)$ is the empty set if F is a singleton, and the relative interior of F otherwise; in the latter case, it is an open set in $\mathbf{R}^{\dim F}$.

Theorem 12.1 Let $M = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ in $M_n A^+$ be primitive and stripped, with discriminant $\mathcal{D} := \mathcal{D}(M) = (S - P)^2 + 4QR$, and $\Delta^+ = (S + P)^2 + 4QR$. The following are equivalent.

- (i) There exists a nonzero polynomial T in A such that $T\mathcal{D}$ has no negative coefficients.
- (ii) There exists an integer m such that $(\Delta^+)^m \mathcal{D}$ not only has no negative coefficients, but satisfies $\text{Log}(\Delta^+)^m \mathcal{D} = \text{Log}(\Delta^+)^{m+1}$.
- (iii) The condition (%) is satisfied by M .
- (iv) For all faces F of $K(M)$ such that M_F is reducible, $P_F - S_F$ has no zeroes in $(\mathbf{R}^d)^{++}$.

Remarks. 1. The property of \mathcal{D} in part (i) is called “Handelman” in [MT]. Since \mathcal{D} is a square plus a nonzero polynomial with no negative coefficients, it automatically is positive on $(\mathbf{R}^d)^{++}$, so the criterion for (i) to hold [H2] reduces to \mathcal{D}_G being strictly positive for every proper face G of $\text{Log } \mathcal{D}$. This is usually easy to check (see remark 3).

2. Property (ii) and an observation about the Log sets of Δ^+ and \mathcal{D} imply that the element \mathcal{D}/Δ^+ of R_{Δ^+} is an order unit, and indeed this is how it is proved (via results in [H2]).

3. We already knew that (%) is not a shape invariant (section 11). Here is a very simple family of examples, using the theorem.

Let $\alpha \geq 0$ be a variable constant. Define the family of matrices in $M_n \mathbf{R}[x, y]^+$,

$$M_\alpha = \begin{bmatrix} \alpha x & 1 \\ y & 1 + x^2 \end{bmatrix}.$$

When $\alpha = 2$, this becomes the matrix appearing in [MT2], as part of a counter-example to the finite equivalence conjecture. The discriminant of M_α is $(x - \alpha/2)^2 + 1 - \alpha^2/4 + 4y$. The convex hull of its Log set is $\text{cvx} \{(0, 0), (2, 0), (0, 1/2)\}$, and since the polynomial has no positive zeroes, it is sufficient (to check whether (i) holds) to test the three boundary polynomials (one for each edge of the triangle). We find immediately that the discriminant satisfies (i) if and only if $\alpha < 2$. So M_α satisfies (%) precisely when $\alpha < 2$.

4. Since (%) is quite a strong property, it is natural to enquire whether it implies other pleasant properties, e.g., drawn, which is a consequence of noetherianness of $E_b(G_M)$. In fact, these two properties are independent. Take $M = M_2$ of remark 3; for no positive integers $a < b$ is it true that $M^a \prec M^b$, so that M is not drawn. Since being drawn is an invariant of structural similarity, it is obvious that the same is true for any value of $\alpha > 0$ (for $\alpha = 0$, M_0 is drawn; however, M_0 is not structurally similar to the others—it is a companion matrix, so satisfies considerably stronger properties).

5. Both (i) and (iii) are unaffected by the addition of a scalar matrix to M ; hence the theorem holds if we drop (ii), (iv), and the stripped hypothesis.

Proof. (i) implies (ii). Clearly $\text{Log } \mathcal{D} \subseteq \Delta^+$. We first notice that since \mathcal{D} is a square plus a nonzero element of A^+ , \mathcal{D} is strictly positive on $(\mathbf{R}^d)^{++}$. Since M is stripped, $\text{Log } P \cap \text{Log } S$ is empty, so their respective sets of extreme points are disjoint. Suppose that w is an extreme point of $\text{cvx } \text{Log}(S + P)^2$. Then $w = 2v$ for v an extreme point of at least one of $\text{cvx } \text{Log } P$ or $\text{cvx } \text{Log } S$. Disjointness ensures that v belongs to exactly one of $\text{Log } P$ or $\text{Log } S$, and this forces the coefficient of x^w in $(S - P)^2$ to be nonzero, in fact positive. Hence $\text{cvx } \text{Log}(S + P)^2 \subseteq \text{cvx } \text{Log}(S - P)^2$. The other inclusion is trivial, so $\text{cvx } \text{Log}(S + P)^2 \subseteq \text{cvx } \text{Log}(S - P)^2$, and the coefficient of x^w in $(S - P)^2$ is positive for any extreme point w .

Next, $\text{cvx Log } \Delta^+ = \text{cvx} (\text{cvx Log } QR \cup \text{Log } (S + P)^2)$, and the previous equality together with the positivity of the coefficients at extreme points ensures that $\text{cvx Log } \Delta^+ = \text{cvx Log } \mathcal{D}$. Now (ii) follows from (i), the fact that $\mathcal{D}(1, 1, \dots) > 0$, and $\text{cvx Log } \Delta^+ = \text{cvx Log } \mathcal{D}$, by [H2, V.3A].

(ii) *implies* (iv). We know from the proof of (i) that $\text{cvx Log } \Delta^+ = \text{cvx Log } \mathcal{D}$ in any case, and it follows from (ii) and for example, [H2, V.4(b)] that \mathcal{D}/Δ^+ is an order unit of R_{Δ^+} . This means that for any face G of $\text{cvx Log } \Delta^+$, the facial polynomial \mathcal{D}_G (obtained by throwing away all the terms in monomials whose exponents do not belong to G) is strictly positive on $(\mathbf{R}^d)^{++}$. Now let F be a face of $K(M)$ such that M_F is reducible. Its two diagonal entries are P_F and S_F , and since M_F is reducible, $\text{Log } (QR)_{2F} \subseteq \text{Rel int cvx} (\text{Log } S^2 \cup \text{Log } P^2)_{2F}$. Then $G := 2F$ is a face of $\text{cvx Log } \mathcal{D}$, and $[(S - P)^2]_G$ does not vanish at any point of $(\mathbf{R}^d)^{++}$. However, this is just the square of $(S - P)_F$, so the latter does not vanish at any positive d -tuple.

(iv) *implies* (i). At those faces G of $K(M)$ for which M_G is irreducible, necessarily $\text{Log } (P - S)_G \subseteq \frac{1}{2} \text{cvx Log } (QR)_{2G}$, and obviously $(QR + (P - S)^2)_{2G}$ is strictly positive as a function. At the faces at which M is reducible, the hypotheses guarantee strict positivity of the corresponding facial function. As $\text{cvx Log } (P - S)^2 + 4QR = 2K(M)$, it follows from [H2, V.6, p. 53] that \mathcal{D} divides a polynomial with no negative coefficients.

(iv, i, ii) *imply* (iii). It is sufficient to show that $(\beta - S)^2$ is bounded above and below away from zero by a rational function whose numerator and denominator belong to A^+ . First, we notice that for any point r in $(\mathbf{R}^d)^{++}$, $M(r)$ a primitive matrix with spectral radius $\beta(r)$ and $S(r)$ is a principal block therein. Hence $\beta(r) > S(r)$ for all r in $(\mathbf{R}^d)^{++}$.

We pick the candidate f/g as follows. Call a face F of $K(M)$ of type (a) if M_F is irreducible, otherwise of type (b) if $P_F - S_F$ is positive as a function, or of type (c) if $S_F - P_F$ is positive as a function at at least one point. By hypothesis, in the latter two cases it must be positive at any point, and moreover, this persists on taking subfaces. It easily follows from [H2, V.4], that in case (b), $(P - S)_F / (P + S)_F$ is an order unit of $R_{(P+S)_F}$, and thus there exists $\epsilon > 0$ such that $(P - S)_F \geq \epsilon(P_F + S_F)$ as functions on $(\mathbf{R}^d)^{++}$. Similarly, if case (c) applies to F , $(S - P)_F \geq \epsilon(P_F + S_F)$; we may obviously arrange this to hold with the same ϵ for all faces F of types (b) and (c). We will repeatedly use the following elementary lemma.

Proposition 12.4 Let r and s be real numbers, the latter positive.

- (a) If r is constrained to be negative, the function $r + (r^2 + s^2)^{1/2}$ is bounded above and below away from zero (i.e., uniformly as r and s vary) by $s^2/(|r| + s)$. In particular, $(r + (r^2 + s^2)^{1/2})^2$ is bounded above and below away from zero by $s^4/(r^2 + s^2)$.
- (b) If r is constrained to be positive, the function $r + (s^2 + r^2)^{1/2}$ is bounded above and below away from zero by $r + s$.

Proof. (a) Set $r = -ts$ (with $t > 0$); it is easy to verify (by differentiating with respect to t) that $-t + (1 + t^2)^{1/2}$ is bounded above and below by what it's supposed to. Part (b) is even more trivial. •

For this portion of the proof only, we define the following relations. If U is a subset of $(\mathbf{R}^d)^{++}$ and $f, g : (\mathbf{R}^d)^{++} \rightarrow \mathbf{R}$ are everywhere positive functions, we write $f \lesssim g$ relative to U if f/g is bounded above as a function on U (i.e., $f|_U \leq k \cdot g|_U$ for some positive real number k). We write $f \sim g$ relative to U if both $f \lesssim g$ and $g \lesssim f$ hold relative to U . If no U is referred to in these definitions, it is assumed to be all of $(\mathbf{R}^d)^{++}$.

Let Z denote the collection of extreme points w of $K \equiv K(M)$ such that the singleton face $G := \{w\}$ is of type (b), i.e., M_G is diagonal with nonzero upper left entry. As M is stripped, and G consists of a single point, the lower right entry of M_G is zero. Now set $P' = \sum_{w \in Z} x^w$, and then define

$$H = \frac{(QR)^2 + (P')^4}{\Delta_+}.$$

Obviously $H \sim (QR + (P')^2)((QR + (P')^2)/\Delta_+)$, and since $\text{Log } P' \subseteq \text{Log } P$, it follows that $H \lesssim QR + (P')^2$. We will show that $(\beta - S)^2 \sim H$, by showing that this holds relative to each of four subsets $\{U_i\}_{i=1}^4$ whose union is $(\mathbf{R}^d)^{++}$.

If F and G are faces of types (b) and (c) respectively, then every subface is of the same type; more importantly, any path of edges (one dimensional faces) joining F to G must contain at least one edge that is of type (a), as is evident from (iv).

This implies that if v in $\mathbf{R}^d \setminus \{0\}$ is a vector exposing a type (c) face with respect to $K(M)$, then $\max v \cdot \text{Log } QR \geq 2 \max v \cdot \text{Log } P'$. Now we are almost ready to define U_1 (this is the most difficult of the four regions with which to deal).

Suppose that $\{r_n\}_{n \in \mathbf{N}}$ is a sequence of points in $(\mathbf{R}^d)^{++}$ with the property that $(P/S)(r_n) \rightarrow 0$. We show that $\liminf (QR/(P')^2)(r_n) > 0$.

To this end, set $p = (P')^2 + QR$, $a = (P')^2/p$, $b = 1 - a$, and $c = (P^2 + 4QR)/\Delta_+$. Then the ordered rings R_{p, Δ_+} and R_{Δ_+} are defined, the former contains the latter, each of a and b belong to R_{p, Δ_+}^+ , and c belongs to $R_{\Delta_+}^+$. Suppose that

- (1) $\lim_{n \rightarrow \infty} \frac{P(r_n)}{S(r_n)} = 0$
- (2) $\lim_{n \rightarrow \infty} \frac{QR(r_n)}{(P')^2(r_n)} = 0$.

(Obviously, if $\liminf (QR/(P')^2)(r_n) = 0$, we may refine the sequence $\{r_n\}$ so that (2) holds.) For each n , let $\tau_n : R_{p, \Delta_+} \rightarrow \mathbf{R}$ be the pure trace, evaluation at r_n , and let τ be a limit point of $\{\tau_n\}$ in the pure trace space of R_{p, Δ_+} . (The pure trace space of such rings is compact, e.g., [H5].) From the fact that $\text{Log } P' \subseteq \text{Log } P$, it follows from (1) and (2) that $\tau_n(c) \rightarrow 0$, so $\tau(c) = 0$. Now (2) implies that $\tau(b) = 1$, so $\tau(a) = 0$.

As a and c are nonzero and positive elements of R_{p, Δ_+} , τ is a non-faithful pure trace. By [H5, Theorem IV.1, p. 42], there exist B in $(\mathbf{R}^d)^{++}$ and v in $\mathbf{R}^d \setminus \{0\}$ such that if $X_{B,v} : \mathbf{R}^+ \rightarrow (\mathbf{R}^d)^{++}$ is the path defined by $X_{B,v}(t) = (B_1 t^{v(1)}, \dots, B_d t^{v(d)})$, then for all y in R_{p, Δ_+} ,

$$\tau(y) = \lim_{t \rightarrow \infty} y(X_{B,v}(t)).$$

(Existence of each of these limits is a simple consequence of L'Hôpital's rule.) We have already seen that for a polynomial g in A^+ , $g(X_{B,v}(t)) = f(B)t^{\max v \cdot \text{Log } g} + o(t^{\max v \cdot \text{Log } g})$, with $f(B) > 0$. From $\tau(b) = 0$, we deduce

$$(*) \quad \max v \cdot \text{Log } QR < 2 \max v \cdot \text{Log } P'$$

From $\tau(c) = 0$, it follows that

$$2 \max v \cdot \text{Log } S > \max v \cdot \text{Log } QR, \quad 2 \max v \cdot \text{Log } P.$$

As $\text{cvx Log } \Delta_+ = \text{cvx } \{\text{Log } P^2 \cup \text{Log } S^2 \cup \text{Log } QR\}$, we deduce that $\max v \cdot \text{Log } \Delta_+ = 2 \max v \cdot \text{Log } S$. As $\text{cvx Log } \Delta_+$ is $2K(M)$, we infer that the face exposed by v is of type (c). However, the earlier remark (essentially saying a line that exposes a type (c) face hits $\frac{1}{2} \text{cvx Log } QR$ at least as soon as it hits $\text{cvx Log } P'$) gives $\max v \cdot \text{Log } QR \geq 2 \max v \cdot \text{Log } P'$. This contradicts (*).

Thus there exist δ and ϵ exceeding zero such that for r in $(\mathbf{R}^d)^{++}$, $P(r) < \delta S(r)$ implies $\epsilon(P')^2(r) < QR(r)$. We may assume $\delta < 1$. Set $U_1 = \{r \in (\mathbf{R}^d)^{++} \mid P(r) \leq (\delta/2)S(r)\}$. Then U_1 is a closed subset of $(\mathbf{R}^d)^{++}$, and $(P')^2 \lesssim QR$ relative to U_1 . It is convenient to reformulate the definition of U_1 in terms of the relationship of $S - P$ to $S + P$. We simply notice that for positive numbers y and z , $y \leq (\delta/2)z$ (for $\delta/2 < 1$) is equivalent to $z - y \geq \kappa(z + y)$, where $\kappa = (2 - \delta)/(2 + \delta)$. With this choice for κ , we have

$$U_1 = \{r \in (\mathbf{R}^d)^{++} \mid (S - P)(r) \geq \kappa(S + P)(r)\}.$$

Now we compare H and $(\beta - S)^2$ on U_1 . Since $S - P$ is positive on U_1 , we deduce from Proposition 12.4 above that

$$(\beta - S)^2 \sim \frac{(QR)^2}{\mathcal{D}} \text{ relative to } U_1$$

Since $\mathcal{D} \sim \Delta_+$ (relative to $(\mathbf{R}^d)^{++}$) and $(P')^2 \lesssim QR$ relative to U_1 , it follows immediately that $(\beta - S)^2 \sim H$ relative to U_1 .

Now set $U_2 = \{r \in (\mathbf{R}^d)^{++} \mid |(P - S)(r)| \leq \epsilon_2(P + S)(r)\}$, where ϵ_2 is to be determined. Let $\theta = \inf_{r \in (\mathbf{R}^d)^{++}} \mathcal{D}(r)/\Delta_+(r)$; we know $\theta > 0$ by (ii). We have that $(P - S)^2/\Delta_+ \leq ((P - S)/(P + S))^2 \leq \epsilon_2^2$ on U_2 . So, restricted to U_2 ,

$$\frac{4QR}{\Delta_+} = \frac{\mathcal{D} - (P - S)^2}{\Delta_+} \geq \frac{\mathcal{D}}{\Delta_+} - \epsilon_2^2 \geq \theta - \epsilon_2^2.$$

Choose $\epsilon_2 < \theta^{1/2}/2$. As functions on U_2 , $QR \geq \frac{\theta - \epsilon_2^2}{4}\Delta_+$, whence $\Delta_+ \lesssim QR$ relative to U_2 . Since

$QR \lesssim \Delta_+$ (relative to $(\mathbf{R}^d)^{++}$), it follows that $QR \sim \Delta_+$ relative to U_2 . Hence relative to U_2 , $H \sim QR$.

Relative to the subregion of U_2 where $P(r) \geq S(r)$, $(\beta - S)^2 \sim (QR)^2/\mathcal{D} \sim (QR)^2/\Delta_+ \sim QR \sim H$. Relative to the subregion where $S(r) \geq P(r)$, $(\beta - S)^2 \sim (P - S)^2 + QR \sim QR \sim H$. Hence $(\beta - S)^2 \sim H$ relative to U_2 .

Now set $U_3 = \left\{r \in (\mathbf{R}^d)^{++} \mid \epsilon_2 \leq \frac{(S - P)(r)}{(S + P)(r)} \leq \kappa\right\}$. (If this turns out to be empty, ignore this part.) On U_3 , $S(r) > P(r)$, so $(\beta - S)^2 \sim (QR)^2/\mathcal{D}$ relative to U_3 . Since $\mathcal{D} \sim \Delta_+$, we obtain immediately that $(\beta - S)^2 \lesssim H$ relative to U_3 .

To show $(\beta - S)^2 \sim H$ relative to U_3 , it is thus sufficient to show that $(P')^2 \lesssim QR$ relative to U_3 (this is similar to the situation with U_1 , but considerably easier). If this condition fails, there would exist a sequence $\{r_n\}$ of elements of U_3 such that $((P')^2/QR)(r_n) \rightarrow 0$. Set $C = QR/\Delta_+$, a positive element of R_{Δ_+} . As $\text{Log } P' \subseteq \text{Log } P$ and $P \sim S$ relative to U_3 , it follows that $C(r_n) \rightarrow 0$. As before, let τ_n be the point evaluation trace at r_n (for R_{Δ_+} this time), and let τ be a limit point; then $\tau(C) = 0$, so τ is a pure non-faithful trace, and as in the argument with U_1 above, there exists a path $X_{B,v}$ such that τ is given as the limit as $t \rightarrow \infty$ of the point evaluations along the path.

Let F be the face exposed by the vector v . Set $b'' = (S - P)^2/\Delta_+$; this is an element of R_{Δ_+} and $4C + b'' = \mathcal{D}/\Delta_+$; we see that $\{b''(r_n)\}$ is bounded below away from zero, so $\tau(b'') > 0$; it is also clear that $\tau(b'') < 1$. The latter implies that F is not of type (c) (—if it were, it would force $\tau(b'') = 1$), and the former that it is not a face of type (b). Hence it must be a face of type (a), which would force $\tau(C) = 1/4$, a contradiction.

Hence $(P')^2 \lesssim QR$ relative to U_3 , and so $(\beta - S)^2 \sim H$ relative to U_3 .

Finally, define $U_4 = \left\{r \in (\mathbf{R}^d)^{++} \mid \epsilon_2 \leq \frac{(P - S)(r)}{(S + P)(r)}\right\}$. Clearly $(\beta - S)^2 \sim P^2 + QR$ relative to U_4 . It is thus sufficient to show that $(P')^2 + QR \sim P^2 + QR$ relative to U_4 (notice that the S terms in Δ_+ can be “absorbed” into their P counterparts, relative to U_4). If this fails, then there would exist a sequence $\{r_n\}$ in U_4 such that $((P')^2 + QR)/(P^2 + QR) \rightarrow 0$ along r_n . Define $f = P^2 + QR$, $c = ((P')^2 + QR)/f$ and $b' = (P^2 - S^2)/\Delta_+$ in $R_{p.\Delta_+}$. Then $c(r_n) \rightarrow 0$ and $b'(r_n)$ is bounded below away from zero. Hence if τ is a limit point of the point evaluation traces at r_n , we have that $\tau(c) = 0$ but $\tau(b') > 0$.

Once more, τ is determined by a path $X_{B,v}$, and $\tau(b') \neq 0$ entails that the face F exposed by v is not of type (c) (if it were on a type (c) face, then $\tau(b') < 0$, since the top entry has to be strictly less than the bottom entry at all traces). At a type (a) face, $\tau(c) > 0$ (arising from the QR term), so F must be type

(b). However, this forces all of its extreme points to be type (b) as well, hence they all belong to $\text{Log } P'$, which would again force $\tau(c) > 0$. This contradiction yields $(P')^2 + QR \sim P^2 + QR$ relative to U_4 , and so $(\beta - S)^2 \sim H$ relative to each of the U_i , and thus relative to their union, $(\mathbf{R}^d)^{++}$.

(iii) *implies* (iv) Follows from the remark just prior to the statement of this theorem. •

Corollary 12.3 Suppose M is a primitive size 2 matrix with entries from A^+ .

- (i) If M satisfies weak (#), then it satisfies (%).
- (ii) If M is stripped and satisfies (%), then it satisfies weak (#).

Proof. (i) Stripping preserves the weak (#) property, so we may already assume M is stripped. Then there is only one block per face that ever hits the spectral radius, as any face is of one of the types (a), (b), (c) occurring in the proof of Theorem 12.1; condition Theorem 12.1(iv) is thus satisfied.

(ii) Obvious for size 2 matrices. •

Each of the following matrices satisfies (%).

$$\begin{bmatrix} 1 & x + y \\ x + y & xy \end{bmatrix} \qquad \begin{bmatrix} 1 & x + y \\ x + y & (xy)^3 \end{bmatrix}.$$

In the first case, $K(M)$ is the standard unit square in the plane and the function H (from the proof above) can be chosen to be $((x + y)^4 + 1)/(x^2 + y^2 + x^2y^2 + 1)$; here the discriminant is $4(x + y)^2 + 1 - 2xy + x^2y^2$ which already has positive coefficients. In the second example, the discriminant is $(1 - x^3y^3)^2 + 4(x + y)^2$, $K(M)$ is the quadrilateral with vertices at $(0, 0)$, $(1, 0)$, $(0, 1)$, $(3, 3)$, and it is easy to check that at each edge, the discriminant has only positive coefficients, which is sufficient for the theorem to apply. Note that in the first case, $\beta - S$ is bounded on $(\mathbf{R}^d)^{++}$, while in the second case it is not. This shows that we cannot simply replace the complicated expression for H by something simpler, such as $(P')^2 + (QR)^2/\mathcal{D}$ (one of my earlier attempts). In either case, $P' = 1$.

In contrast, consider the one parameter family of matrices $(0 \leq \alpha)$ appearing in Remark 3,

$$M_\alpha = \begin{bmatrix} \alpha x & y \\ 1 & 1 + x^2 \end{bmatrix}.$$

Then $2K(M_\alpha)$ has vertices $\{(0, 0), (4, 0), (0, 1)\}$, and the discriminant of M_α is $(1 + x^2 - \alpha x)^2 + 4y$. Along the bottom edge, the resulting polynomial is $(1 + x^2 - \alpha x)^2$, and this will be strictly positive on $(\mathbf{R}^2)^{++}$ if and only if $\alpha < 2$, and it follows that \mathcal{D} divides a polynomial with no negative coefficients precisely when this holds. So M_α satisfies (%) if and only if $\alpha < 2$. (One could also treat α as another variable, in which case the resulting matrix fails to satisfy (%).)

It seems likely that (weak) (#) implies (%); for $n = 2$, this is Corollary 12.3; for $n = 3$, it is true provided $\text{tr } M = 0$ (in which case, (#) holds automatically); and we also know that (%) holds if either M or $M \text{ Tr}$ is in companion matrix form.

Let $\rho : (\mathbf{R}^d)^{++} \rightarrow \mathbf{R}^+$ be a nowhere vanishing real analytic algebraic function that is nonnegative. Let $B = (B_1, \dots, B_d)$ be a strictly positive d -tuple, and let $v = (v(i))$ be a nonzero real d -tuple. Define the path in $(\mathbf{R}^d)^{++}$, $X_{B,v}(t) = (B_1 t^{v(1)}, \dots, B_d t^{v(d)})$. As we have seen, $\rho(X_{B,v}(t))$ grows asymptotically like t^a for some real a (we have only seen this when v has only rational entries; we can however use the same argument with formal polynomials, although a continuity argument to go from the rationals to the reals cannot work). In other words, we can define a function, $\psi_\rho : (\mathbf{R}^d)^{++} \times (\mathbf{R}^d \setminus \{0\}) \rightarrow \mathbf{R}$ via

$$\psi_\rho(B, v) = \lim_{t \rightarrow \infty} \frac{\log \rho(X_{B,v}(t))}{\log t}$$

If ρ' is another such nonnegative function such that ρ'/ρ is bounded above and below away from zero on $(\mathbf{R}^d)^{++}$, then obviously $\psi_\rho = \psi_{\rho'}$. More generally (i.e., without the boundedness assumptions on the quotient), $\psi_{\rho/\rho'} = \psi_\rho - \psi_{\rho'}$.

If p is in $A_{\mathbf{Q}}^+$, then $\psi_p(B, v) = \max v \cdot N(p)$ (recall that $N(p) = \text{cvx Log } p$, the Newton polyhedron of p , and for a convex set K , $\max v \cdot K$ is supposed to be $\max \{v \cdot k \mid k \in K\}$; this maximum is attained at an extreme point, so $\max v \cdot N(p) = v \cdot u$ for some u in $\text{Log } p$). Thus if p and q are in $A_{\mathbf{Q}}^+$, $\psi_{p/q}(B, v) = \max v \cdot N(p) - \max v \cdot N(q)$. In particular, this is independent of the choice of B , as we had observed earlier. However, it is also continuous as a function of v (that is, $v \mapsto \phi_{p/q, B}(v)$ is continuous); in fact, it is piecewise linear, although at the moment, I do not see how to exploit this. It follows from the comments above about bounded ratios, that if ρ has fractional power asymptotic behaviour, then $\psi_{\rho, B}$, defined via $\psi_{\rho, B}(v) = \psi_\rho(B, v)$ is continuous in v and independent of B . We may also view it as a function on $S^{d-1} = (\mathbf{R}^d \setminus \{0\})/\mathbf{R}^{++}$.

For more general ρ , for almost all B , $\psi_\rho(B, v) = \max_{B'} \psi_\rho(B', v)$, i.e., the maximum value is the generic one. (This is easy to see from the polynomial satisfied by ρ ; the set of anomalous B is a finite union of varieties.)

Is it true for ρ an algebraic and analytic positive valued function on $(\mathbf{R}^d)^{++}$, that ψ_ρ being independent of B (for all v) forces ρ to have fractional rational asymptotic behaviour? This would be a useful step toward proving that (#) implies (%).

13. Eventual weak similarity to matrices satisfying (**)

In this section, we show that certain matrices can be transformed so as to satisfy (**).

Select (B, v) from the product space, $\mathbf{R}^d \times \mathbf{R}^d$. Write $B = (B(1), \dots, B(d))$ and $v = (v(1), \dots, v(d))$, and as usual define the path, $X_{B, v} : (0, \infty) \rightarrow (\mathbf{R}^d)^{++}$, via

$$X_{B, v}(t) = \left(B(1)t^{v(1)}, \dots, B(d)t^{v(d)} \right)$$

In other words, $X_{B, v}$ is just the exponential of the ray in the direction v , starting at $\log B$. Note that if v is zero, then the path is constant. If $\lambda : (\mathbf{R}^d)^{++} \rightarrow \mathbf{R}$ is real analytic and is algebraic over A , then $\lambda(X_{B, v})$ is an algebraic and real analytic function of one variable; thus if λ is additionally bounded,

$$\tau_{B, v}(\lambda) = \lim_{t \rightarrow \infty} \lambda(X_{B, v}(t))$$

exists for all choices of (B, v) . If $B - B' \in \mathbf{R} \cdot v$, then the two paths, $X_{B, v}$ and $X_{B', v}$ will differ only by an initial segment, so that the limits $\tau_{B, v}$ and $\tau_{B', v}$ will be equal for all choices of λ . This suggests that we put an equivalence relation \sim on $\mathbf{R}^d \times \mathbf{R}^d$, via

$$(B, v) \sim (B', v') \quad \text{if } v = sv' \text{ for some real } s > 0 \text{ and } B - B' \in \mathbf{R} \cdot v$$

Let \mathcal{P} denote the set of equivalence classes. Note that $\mathcal{P} \setminus \{\mathbf{R}^d \times \{0\}\}$ is a copy of the tangent space of the d -sphere (Erhard Neher pointed this out to me). Now we show that if $[B, v]$ and $[B', v']$ are two distinct equivalence classes (i.e., elements of \mathcal{P}), then there exist bounded rational (and analytic) functions that separate the linear functionals $\tau_{B, v}$ and $\tau_{B', v'}$ corresponding to $[B, v]$ and $[B', v']$, except if v and v' are positive scalar multiples of each other and not all the ratios among the entries of b are rational. In the latter exceptional case, even if we permit algebraic functions, we still could not separate them, although fractions involving x^u where u is an arbitrary point¹ of \mathbf{R}^d would suffice.

If both v and v' happen to be zero, then at least one of $\{(\sum \mu_i x_i) / (\sum x_i + 1)\}$ will have different values at B, B' , for a selection of positive integers $\{\mu_i\}$. If v is zero while v' is not, then $x_1 / (1 + x_1 + x_1^2)$ will vanish under $\tau_{B', v'}$ but not at B . So we may assume that neither b nor v' are zero. If v is not a positive scalar multiple of v' , then we may find a lattice polytope K in \mathbf{R}^d with the following property. There exist two *disjoint* faces

F and F' that are exposed by b and v' respectively (that is, $F = \{c \in K \mid v \cdot c = \max \{v \cdot k \mid k \in K\}\}$). Let $P = \sum x^w$, the w 's varying over $K \cap \mathbf{Z}^d$, and select a lattice point v in F . A routine calculation reveals that $\tau_{B,v}(x^v/P) \neq 0$ (for any choice of B , but that $\tau_{B,v'}(x^v/P) = 0$ regardless of the choice of B).

This leaves the possibility that B is a positive scalar multiple of v' . If there are ratios among the coefficients that are irrational, in general $\tau_{\alpha',b}$ and $\tau_{B,v}$ need not be distinguishable. If all the ratios are rational (permitting ∞), then we may assume b is already a lattice point, and thus we may find d affinely independent points in \mathbf{Z}^d , u_i , such that $b \cdot u_i$ is constant, as i varies. Then $x^{u_1} / \sum x^{u_i}$ will distinguish the equivalence classes.

Let $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a row or column whose entries are algebraic functions. We say Λ is *strictly positive* if each of the λ_i are positive bounded analytic functions on $(\mathbf{R}^d)^{++}$ and for all $[B, v]$ in \mathcal{P} , $\tau_{B,v}(\lambda_i)$ is bounded below away from zero. Given a matrix M with entries from A^+ , one can ask if M admits a right or left eigenvector which is strictly positive; this is equivalent to (%).

Let $\beta : (\mathbf{R}^d)^{++} \rightarrow \mathbf{R}^{++}$ be an algebraic and real analytic function, and let χ be a monic polynomial with entries that are real analytic rational functions in $\{x_1, \dots, x_d\}$ satisfied by β . For a point r in $(\mathbf{R}^d)^{++}$, let χ_r denote the polynomial whose coefficients are determined by evaluating the rational functions at r (so in particular, $\chi_r(\beta(r)) = 0$). Suppose that there exists a positive real δ , such that for all r in $(\mathbf{R}^d)^{++}$, for all other roots α of χ_r , we have that $|\alpha|/\beta(r) < 1 - \delta$. Then we say that β satisfies (#) with respect to χ . In lemma 13.1 below, we do not assume that the original matrix has entries from A^+ .

If β or one of its powers is the large eigenvalue function of some matrix M in $\mathbf{M}_n A^+$, there are certain obvious conditions that it must satisfy. For every r in $(\mathbf{R}^d)^{++}$, $\beta(r)$ must be at least large as $|\alpha|$, where the latter run over the rest of the eigenvalues of $M(r)$ —in other words, it must be at least as large as the moduli of the roots of the characteristic polynomial, evaluated at r . If we wish $M(1, 1, \dots, 1)$ to be primitive (as is usually the case), then β must be real analytic and strictly greater (at each point r) than all the other roots. Let $\chi = X^n + \sum p_i(x_1, \dots, x_d)X^i$ be the proposed characteristic polynomial; the p_i belong to A . Define the trace of χ , $\text{tr } \chi = -p_1$. Let C be the companion matrix, and let $\chi^{(N)}$ denote the characteristic polynomial of C^N (this is just a fancy way of describing something that could otherwise be obtained from Newton's identities). The condition $\beta(r) > |\alpha|$ ensures that for all r , $|\text{tr } \chi_r^{(N)} - \beta^N(r)| = o(\beta^N(r))$; but this is not sufficient to obtain another consequence of being the large eigenvalue function of M in $\mathbf{M}_n A^+$, for example that $\text{tr } \chi^{(N)}$ have no negative coefficients. So even this condition will have to be assumed.

If β satisfies (#) with respect to χ , then a stronger convergence property must hold:

$$\frac{|\text{tr } \chi_r^{(N)} - \beta^N(r)|}{\beta^N(r)} \rightarrow 0 \quad \text{uniformly in } r, \text{ as } N \rightarrow \infty.$$

Lemma 13.1 Suppose that R is a matrix with entries from A with characteristic polynomial χ , that satisfies the following properties.

- (a) R admits an eigenvalue β satisfying (#) with respect to χ .
- (b) There exist left and right eigenvectors of R for β , Λ and Θ respectively that are strictly positive.
- (c) All the entries of R^l are bounded below by integer multiples of β^l , for some integer l (i.e., for all i, j , $(R^l)_{i,j}(r)/\beta^l(r) \geq -B$ for some integer B).
- (d) For all sufficiently large integers J , there exists nonzero Q in A^+ (possibly depending on J) such that $Q \cdot \text{tr } \chi^{(J)}$ itself has no negative coefficients.

Then there exists an integer k together with an element P of A^+ such that

$$(PR)^k \in \mathbf{M}_n A^+ \quad \text{and satisfies (**).$$

Proof. As $\text{tr } \chi^{(N)}(r)$ is just the sum of the N th powers of the roots of χ_r , it follows from (#) that $|\text{tr } \chi^{(N)}(r) - \beta^N(r)| \leq (n-1)(1-\delta)^N \beta^N(r)$. Select N so that $(1-\delta)^N < 1/3(n-1)$; thus

$|\text{tr } \chi^{(N)}(r)/\beta^N(r) - 1| < 1/3$, and so $|\beta^N(r)/(\text{tr } \chi^{(N)}(r) - 1) - 1| < 1/2$. If N is increased, this inequality will be preserved.

For N sufficiently large, select Q in A^+ such that $Q \cdot \text{tr } \chi^{(N)}$ also belongs to A^+ . Set $P_N = \text{tr } \chi^{(N)} Q^N$. We may replace β^N by $\beta^N Q^N$, R by QR , $\chi = X^n + \sum p_i X^i$ by $X^n + \sum p_i Q^i X^i$, etc. With this renaming, $\|\beta^N/P_N - 1\| < 1/2$.

Define

$$P = \sum x^w \quad \text{the sum over } w \in \mathbf{Z}^d \cap \text{cvx Log } P_N^d.$$

By e.g., [HM, V.4], P_N^d/P is bounded below away from zero on $(\mathbf{R}^d)^{++}$. Clearly for all a and b , $\tau_{B,v}(\beta^N/P_N) > 0$. Thus the same is true of β^{Nd}/P .

Consider the matrix $R' = R^{Nd}/P$ in $M_n A[P^{-1}]$. We shall show that every entry of R' lies in the algebra R_P . Since $\text{cvx Log } P \cap \mathbf{Z}^d = \text{Log } P$ and $\text{cvx Log } P$ is d times a lattice polytope (this is the reason for raising P_N to the d th power), it is sufficient to show that every entry of R' is bounded as a function on $(\mathbf{R}^d)^{++}$ [HSLN, III.2 & III.4].

Reselect N so that in addition it is a multiple of l . Then it follows from $\left(\frac{\text{tr } \beta^N}{P_N}\right)^{\pm 1}$ being bounded that all of the entries of R' are bounded below. The equation $\Lambda R = \beta \Lambda$ yields $\Lambda R' = \frac{\beta^{Nd}}{P} \Lambda$. Fix j ; we deduce

$$\sum \lambda_i (R')_{ij} = \frac{\beta^{Nd}}{P} \lambda_j.$$

Evaluating at r in $(\mathbf{R}^d)^{++}$, the boundedness below hypothesis yields that the entries of R' are bounded above, and thus bounded. So, R' belongs to $M_n R_P$. Set β' to be the large eigenvalue function of R' , which we regard as a continuous functions on the pure state space of R_P .

Let γ be a pure state of R_P . Restricted to R_P , it is of the form $\tau_{B,v}$ for some (B, v) in $\mathbf{R}^d \times \mathbf{R}^d$ [HSLN, Section 3]. It is routine to verify that $\tau_{B,v}(\beta')$ is the large eigenvalue of the non-negative real matrix $\gamma(R')$, and the corresponding eigenvectors are $\tau_{B,v}(\Lambda)$ and $\tau_{B,v}(\Theta)$. Since γ is a limit of point evaluation states, (#) implies that $\tau_{B,v}(\beta')$ is larger than the modulus of all the other eigenvalues of $\gamma(R')$. By [H1, 2.1], there exists an integer D so that for all $D' \geq D$, $\gamma((R')^{D'})$ is strictly positive. (A real nonnegative matrix with a largest eigenvalue of multiplicity one whose corresponding right and left eigenvectors are strictly positive, is primitive—consider the limit of powers of the matrix normalized so that spectral radius is 1.) Hence there exists a neighbourhood U_γ of γ in the pure state space of R_P such that for all ρ in U_γ , $\rho((R')^D)$ and the same expression with D replaced by $D + 1$ are strictly positive—it follows that all sufficiently high powers are strictly positive; denote the minimum power, E_γ .

Compactness of the pure state space yields a finite open covering $\{U_i\}$ with corresponding E_i . Set $E = \max \{E_i\}$; then $\gamma((R')^{E+k})$ is strictly positive for all nonnegative integers k . If a is an element of R_P , $\gamma(a) > 0$ for all pure states γ , then not only is a positive in R_P , but it is actually an order unit. Thus all the entries of $(R')^{E+k}$ are order units. There thus exists an integer F such that all the entries of $P^F R^{E+k}$ belong to A^+ , for $k = 0, 1$. This means that all sufficiently large powers of $P^F R$ have entries from A^+ . To check that $(P^F R)^E$ satisfies (**), we simply note that this follows from the entries of $(R')^E$ being order units. •

The intricate hypotheses make the lemma quite easy to prove. However, they really are necessary. Concerning (c) for example, start with any reasonable choice for R_0 with strictly positive eigenvectors Λ and Θ and eigenvalue β etc., and find a matrix S whose entries are bounded rational functions such that $S^2 = 0$, $\Lambda S = 0$, and $S\Theta = 0$ ('reasonable' means in particular that the irreducible polynomial satisfied by β is of degree (over A) less than the size of R_0 , or else no such S can be found). Pick a lattice point w such that the growth of x^w is not dominated by that of β . Set $R = R_0 + x^w S$. Then Λ and Θ are also eigenvectors for R with eigenvalue β . If we assume (for example) that the determinant of R_0 is not zero

(almost everywhere on $(\mathbf{R}^d)^{++}$, for which it is enough that it not vanish at one point), then the argument of the proof will fail, and it is easy to construct an example where the conclusion of the Lemma fails.

Set $\Lambda = (3, 1, 2)$, $\Theta = (1, 1, 1)^T$ (no variables!), and $R_0 = \Theta\Lambda + \mathbf{I} \in M_3\mathbf{Z}^+$. Now put

$$S = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix},$$

so $S^2 = \mathbf{0}$, and ΛS and $S\Theta$ are both zero. Also $R_0S = SR_0 = S$; finally, $R = R_0 + x_1S$ admits Λ and Θ as strictly positive eigenvectors, but no power can be multiplied by a polynomial without negative coefficients to a nonnegative matrix.

As we pointed out earlier, the hypothesis concerning dividing a polynomial with no negative coefficients is necessary. Moreover, necessary and sufficient conditions are known and easy to deal with [H2, V.6]; in fact, it is sufficient that the Q of Lemma 13.1(d) belong to A and be nonzero.

To use these results, especially when R is the companion matrix of a polynomial (or a power of a companion matrix), we require some elementary algebraic results about manipulating pairs of vectors.

Lemma 13.2 Let A be a commutative ring such that all finitely generated projective modules of rank $n - 1$ are free. Let v and v' be rows of size n with entries from A , and let w and w' be columns of size n with entries from A . Suppose that $v \cdot w = v' \cdot w'$, and the latter is invertible in A . Then there exists P in $GL(n, A)$ such that $vP = v'$ and $P^{-1}w = w'$.

Proof. Obviously, we may assume $v \cdot w = 1$; it is enough to prove the result in the case that $v' = (1, 0, \dots, 0) = (w')^T$. In particular, each of v and v' is unimodular; as projectives are free, there exists P_0 in $GL(n, A)$ such that $vP_0 = (1, 0, \dots, 0)$. Relabel vP_0 and $P_0^{-1}w$ as v and w respectively, so that $w = (1, *, \dots, *)^T$. If we add any A -multiple of the first entry of the current w to any other, the inverse elementary operation subtracts a zero from the first entry of v —so v is left unchanged. We may thus assume that $w = (1, -1, *, \dots, *)^T$. Let W denote the column of size $n - 1$ obtained by removing the first entry of w .

There exists Q in $GL(n - 1, A)$ such that $Q^{-1}W = (-1, 0, \dots, 0)^T$ (size $n - 1$). Applying $P_1 = 1 \oplus Q$ to our current v and w , we may assume

$$v = (1, 0, \dots, 0) \quad \text{and} \quad w = (1, -1, 0, \dots, 0).$$

Now subtract the second entry of the current v from the first entry; the inverse operation adds the first entry of w to the second, and we are done. •

Let P be an element of A^+ , let $K = \text{cvx Log } P$, and form R_K as in [HSLN]. Then R_K embeds as a dense ring of continuous real-valued functions on the pure state space of K with the additional property that all elements of R_K that are invertible as continuous functions are already invertible in R_K . By [Sw], any invertible matrix in the (real-valued) continuous function ring can be uniformly approximated by invertible matrices with entries in R_K (and via determinants, we see that the inverses have entries in R_K as well). By [HSLN, Section 4], the pure state space of R is homeomorphic to the compact convex set K , so is contractible. Thus all projectives over the continuous function ring are free—we take the real-valued continuous function ring as our choice for A in Lemma 13.2.

Now let M be a square matrix with entries from R_P (or R_K). Suppose that M admits a largest real eigenfunction β (with values in the pure¹ state space of R_P) and there exist right and left eigenvectors v and w for β whose entries are integral over R_P (or R_K) such that $\gamma(v \cdot w) > 0$ for all pure states γ . As a continuous function on the pure state space, $v \cdot w$ is invertible. By the lemma above, there exists a strictly positive pair—that is, v' and w' , whose entries are strictly positive continuous functions on the pure state space—into which v and w are simultaneously transformed by a single invertible matrix. Since the invertible

matrix can be approximated (uniformly) by matrices with entries from R_K , the usual compactness argument yields a matrix B in $GL(R_K)$ such that both vB and $B^{-1}w$ are strictly positive. Applying the earlier lemma, we deduce the following:

Proposition 13.3 Let R be a matrix with large eigenvalue β , such that for some integer k , there exists P in A^+ satisfying:

- (i) $\text{Log } R^k \subseteq \text{Log } P$;
- (ii) β satisfies (#) with respect to χ .
- (iii) For all sufficiently large integers J , $\text{tr } \chi^{(J)}$ has no negative coefficients.

Then there exists Z in $GL(n, R_K)$ and an integer N so that for all $M \geq N$, the matrix $(P'ZRZ^{-1})^M$ has all of its entries in A^+ , and satisfies (**); here P' is an element of A^+ with $\text{Log } P' = \text{Log } \text{tr } \beta^k$ for some integer k .

This litany of conditions is actually satisfied in a number of situations. For example, if χ satisfies (iii) above and β is a root of χ satisfying (#) (with respect to χ), then we choose R to be the companion matrix for χ . To start, suppose that $(\beta/P)^{\pm 1}$ is bounded, where P belongs to A^+ . For example, relabel $\beta^{n!}$ as β , so that the trace of χ will do. We may also assume that R_P is integrally closed (by altering P somewhat, but not enough to affect the convex hull of its Log set). The companion matrix of β/P has entries in R_P , so each will belong to R_P . Then there exists a power of P so that $P^N R'$ satisfies the conclusion in the proposition above.

If instead we begin with R in $M_n A^+$, then the characteristic polynomial χ will satisfy (iii) automatically; if we assume that the large eigenvalue satisfies (#), then the preceding paragraph applies to the companion matrix of the characteristic polynomial. There is a much more difficult argument to show that a power of R can be replaced by a matrix in $M_n A^+$ for which condition (i) holds. So some power of R is conjugate to a matrix which after multiplication by a suitable polynomial, satisfies (**).

Appendix

Lemma A1.1 Let D be a commutative domain, and let M and N be $n \times n$ matrices over D . Suppose that s is an integer relatively prime to $n!$, and that $M^s N = N M^s$. If the field of fractions of D contains no non-trivial s th roots of unity, then $M^n N$ commutes with M .

Proof: We may assume D is already a field. Let K be the splitting field of the characteristic polynomial of M . The relative primeness condition guarantees that K contains no new s th roots of unity, so contains no non-trivial ones at all. We may thus assume that M is in Jordan normal form. The centralizer of M in $M_n K$ is obtained by fixing an eigenvalue λ , noting how the number of blocks of a given size occur in the generalized eigenspace attached to λ , and then taking the corresponding matrix ring over the polynomials in the block. For M^s , and $\lambda \neq 0$, the block sizes are the same: If $\lambda^s = \lambda'^s$, then the ratio λ/λ' is an s th root of unity. Since the 0 block of M^n is just the zero matrix, we see immediately that if $M^s N = N M^s$, then $(M^n N)M = M(M^n N)$. •

If M is a nonnegative real matrix with only one eigenvalue of absolute value equaling its spectral radius, then that eigenvalue belongs to a unique block of M (when M is written in block triangular form); this will be called the *principal block*. The following is probably well-known; certainly, portions of it are special cases of results in [Ga]. Compare this with [Tu2, Theorem 2, p. 293], which characterizes the existence of a positive left eigenvector.

1

Lemma A1.2 Let M be a nonnegative real matrix of size n . Suppose M has its spectral radius as an unique eigenvalue of multiplicity one, and all other eigenvalues have strictly smaller absolute value. The following are equivalent:

- (a) M has a left strictly positive eigenvector for its large eigenvalue;

- (b) in some power of M , all rows whose diagonal entry lies in the principal block of M are strictly positive;
- (c) M admits only one nonnegative right eigenvector (up to scalar multiple).
- When (b) occurs, it will occur for all powers exceeding $n^2 - n + 2$, and no other rows will be strictly positive.

Proof. Without loss of generality, we may assume the spectral radius is 1. Let v and w be the left and right nonnegative eigenvectors for 1 respectively, normalized so that the scalar product vw is 1. Then $\{M^k\}$ converges (any reasonable norm) to the matrix wv .

Hence, if v is strictly positive, then for all sufficiently large k , the rows of M^k corresponding to the positions of w that are not zero are strictly positive. Of course, these can only belong to the principal block, since the support of w must include the principal block.

Suppose (b) holds. Eventually, the rows involving the principal block are all strictly positive, which entails that no other rows are ever strictly positive (otherwise they would belong to the principal block). Moreover, it also follows that all the columns corresponding to the principal block must be zero below that block. Write $M = \begin{bmatrix} A & X \\ \mathbf{0} & C \end{bmatrix}$, where A is the principal block, and on replacing by a sufficiently high power, we may assume A and X are strictly positive, and C is merely nonnegative. Let $v = (v_A, s)$ be the partitioned left eigenvector for 1. Then v_A is the left eigenvector for A so is strictly positive; as $v_A X + sC = s$, s is strictly positive, so that so is v . Hence (b) implies (a).

Assume (a). If u is a right eigenvector for any eigenvalue other than 1, we must have $vu = 0$; as v is strictly positive, u cannot be nonnegative. So (a) implies (c).

Suppose (c) holds; write M in block upper triangular form. The right nonnegative eigenvector for the upper left block extends to an eigenvector for M by simply adjoining zeros; since M always has a nonnegative eigenvector whose eigenvalue is the spectral radius, uniqueness forces the upper left block, A , to be the principal block, and the unique nonnegative right eigenvector is simply $w = (w_A, \mathbf{0})^T$. Now a routine induction on the number blocks can proceed. •

Theorem A1.3 Let M be a square nonnegative real matrix of size n and let f be a real column of size n . Set $S = \{i \mid (M^n f)_j \neq 0 \text{ for some } j \text{ equivalent to } i\}$.

- (a) There exists an integer m such that the column $M^m f$ is strictly positive if and only if M has no zero rows and $v \cdot f > 0$ for all nonnegative left eigenvectors v of M corresponding to nonzero eigenvalues.
- (b) There exists an integer m such that the column $M^m f$ is nonnegative if and only if $v_S \cdot f_S > 0$ for all nonnegative left eigenvectors v_S of M_S that correspond to nonzero eigenvalues.

Proof: (a). Define $G_M = \lim M : \mathbf{R}^n \rightarrow \mathbf{R}^n$ equipped with the limit ordering. Being a finite dimensional vector space, it admits an order unit. By [HCM, 1.3], the pure states are all obtained from left nonnegative eigenvectors of M , in the sense that if v is one such, we define a state V on G_M via $[f, k] \mapsto (v \cdot f) / s^{k-1}$ where s is the corresponding positive eigenvalue of M . Notice that no left nonnegative eigenvector can correspond to a zero eigenvalue, as M has no rows consisting entirely of zeroes. All pure states are of this form. The stated conditions on f and M are clearly necessary. Suppose they hold. Obviously $V([f, 1]) > 0$ for all pure states V follows from the hypotheses. As G_M admits an order unit, and is obviously unperforated (even a dimension group), [HM, I.1] guarantees that $[f, 1]$ is an order unit of G_M . Set $y = (1, 1, \dots, 1)^T \in \mathbf{R}^n$. As f is an order unit, there exists a positive integer J such that $J[f, 1] - [y, 1] \geq \mathbf{0}$ in G_M . From the way the limit ordering is defined, there exists an integer N such that $M^N(Jf - y)$ has only nonnegative entries; thus every entry of $M^N f$ is at least as large as $1/J$ times that of $M^N y$, which is of course strictly positive.

(b). By (a), that there will exist an integer N (which we may take to be greater than n) such that $(M_S)^N f_S$ is strictly positive if and only if for all (extreme) left nonnegative eigenvectors v_S of M_S , $v_S \cdot f_S > 0$, and thus $M^N f \geq \mathbf{0}$.

Conversely, assume that $M^N f \geq \mathbf{0}$ for some N , and $M^N f \neq \mathbf{0}$ we may assume $N > n$. Suppose $[i] \leq [j]$, and $(M^N f)[j] \neq \mathbf{0}$. Applying a sufficiently large further power of M (in fact we only need to use the primitive matrix $M[j]$), we can even obtain that each entry of $(M^N f)[j]$ is strictly positive. It follows immediately that $(M^N f)[i]$ is not zero, and so applying more powers of M (actually $M[i]$) if necessary, we deduce that $(M^N f)[i]$ is also strictly positive. We easily conclude that for some sufficiently large N , $(M_S)^N f_S$ is strictly positive. Hence, there will be affirmative answer to the question if and only if either $M^N f = \mathbf{0}$, or for all nonnegative left eigenvectors v_S of the cut-down matrix M_S , $v_S \cdot f_S > 0$.

Suppose at the outset, we replace f by $f_0 = M^b f$, where b is $\max \{ \#([i]) \mid i \in \{1, 2, \dots, n\} \}$. Then we eliminate the indices belonging to a minimal $[i]$ if $f_0[i]$ is the zero column, and a simplification occurs. •

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