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BLOCK REPRESENTATIONS OF THE DRAZIN INVERSE OF A BIPARTITE MATRIX*

M. CATRAL†, D.D. OLESKY‡, AND P. VAN DEN DRIESSCHE†

Abstract. Block representations of the Drazin inverse of a bipartite matrix $A = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ in terms of the Drazin inverse of the smaller order block product BC or CB are presented. Relationships between the index of A and the index of BC are determined, and examples are given to illustrate all such possible relationships.

Key words. Drazin inverse, Bipartite matrix.

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1. Introduction. Let A be an $n \times n$ real or complex matrix. The *index* of A is the smallest nonnegative integer q such that $\operatorname{rank} A^{q+1} = \operatorname{rank} A^q$. The *Drazin inverse* of A is the unique matrix A^D satisfying

$$AA^D = A^D A \tag{1.1}$$

$$A^D A A^D = A^D \tag{1.2}$$

$$A^{q+1} A^D = A^q, \tag{1.3}$$

where $q = \text{index } A$ (see, for example, [1, Chapter 4], [2, Chapter 7]). If index $A = 0$, then A is nonsingular and $A^D = A^{-1}$. If index $A = 1$, then $A^D = A^{\#}$, the *group inverse* of A. See [1], [2], [8] and references therein for applications of the Drazin inverse.

The problem of finding explicit representations for the Drazin inverse of a general 2×2 block matrix in terms of its blocks was posed by Campbell and Meyer in [2]. Since then, special cases of this problem have been studied. Some recent papers containing representations for the Drazin inverse of such 2×2 block matrices are [3], [4], [6], [7], [8], [9], [11] and [12]; however the general problem remains open.

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†Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia, Canada V8W 3R4 (mcatral@uvic.ca, pvdd@math.uvic.ca).

‡Department of Computer Science, University of Victoria, Victoria, British Columbia, Canada V8W 3P6 (dolesky@cs.uvic.ca).

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In this article, we consider $n \times n$ block matrices of the form

$$A = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}, \tag{1.4}$$

where B is $p \times (n-p)$, C is $(n-p) \times p$ and the zero blocks are square. Since the digraph associated with a matrix of the form (1.4) is bipartite, we call such a matrix a *bipartite matrix*. In section 2, we give block representations for the Drazin inverse of a bipartite matrix. These block representations are given in terms of the Drazin inverse of either BC or CB, both of which are matrices of smaller order than A. These formulas for A^D when A has the form (1.4) cannot, to our knowledge, be obtained from known formulas for the Drazin inverse of 2×2 block matrices. In section 3, we describe relations between the index of the matrix BC and the index of A, and in section 4 we give examples to illustrate these results.

2. Block representations for A^D.

The following result gives the Drazin inverse of a bipartite matrix in terms of the Drazin inverse of a product of its submatrices.

THEOREM 2.1. *Let A be as in* (1.4). *Then*

$$A^D = \begin{bmatrix} 0 & (BC)^D B \\ C(BC)^D & 0 \end{bmatrix}. \tag{2.1}$$

Furthermore, if index $BC = s$, then index $A \le 2s+1$.

Proof. Denote the matrix on the right hand side of (2.1) by X. Then

$$AX = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \begin{bmatrix} 0 & (BC)^D B \\ C(BC)^D & 0 \end{bmatrix} = \begin{bmatrix} BC(BC)^D & 0 \\ 0 & C(BC)^D B \end{bmatrix},$$

$$XA = \begin{bmatrix} 0 & (BC)^D B \\ C(BC)^D & 0 \end{bmatrix} \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} (BC)^D BC & 0 \\ 0 & C(BC)^D B \end{bmatrix}$$

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and

$$\begin{aligned} XAX &= \begin{bmatrix} (BC)^D BC & 0 \\ 0 & C(BC)^D B \end{bmatrix} \begin{bmatrix} 0 & (BC)^D B \\ C(BC)^D & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & (BC)^D BC(BC)^D B \\ C(BC)^D BC(BC)^D & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & (BC)^D B \\ C(BC)^D & 0 \end{bmatrix} \text{ by (1.2).} \end{aligned}$$

Thus, X satisfies $AX = XA$ by (1.1) and $XAX = X$. Let index $BC = s$. Then

$$\begin{aligned} A^{2s+2}X &= \begin{bmatrix} (BC)^{s+1} & 0 \\ 0 & (CB)^{s+1} \end{bmatrix} \begin{bmatrix} 0 & (BC)^D B \\ C(BC)^D & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & (BC)^{s+1}(BC)^D B \\ (CB)^{s+1}C(BC)^D & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & (BC)^s B \\ C(BC)^{s+1}(BC)^D & 0 \end{bmatrix} \text{ by (1.3) and associativity} \\ &= \begin{bmatrix} 0 & (BC)^s B \\ C(BC)^s & 0 \end{bmatrix} \text{ by (1.3)} \\ &= A^{2s+1}. \end{aligned}$$

By [2, Theorem 7.2.3], index $A \le 2s + 1$ and $X = A^D$. ∎

We now give three lemmas, the results of which are used to write A^D in terms of $(CB)^D$, rather than $(BC)^D$ as in (2.1). Lemma 2.2 is an easy exercise using the definition of the Drazin inverse, and Lemma 2.3 is proved in [2, p. 149] for square matrices but that proof holds for the more general case stated below.

Lemma 2.2. *If U is an $n \times n$ matrix, then $(U^2)^D = (U^D)^2$.*

Lemma 2.3. *If V is $m \times n$ and W is $n \times m$, then $(VW)^D = V[(WV)^2]^D W$.*

Lemma 2.4. *If B is $p \times (n-p)$ and C is $(n-p) \times p$, then $(BC)^D B = B(CB)^D$.*

Proof. By Lemmas 2.2 and 2.3, $(BC)^D = B[(CB)^2]^D C = B[(CB)^D]^2 C$. Using (1.1) and (1.2), this gives $(BC)^D B = B[(CB)^D]^2 CB = B(CB)^D(CB)^D CB = B(CB)^D CB(CB)^D = B(CB)^D$. ∎

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Note that Lemma 2.4 implies that $(CB)^D C = C(BC)^D$ and thus Theorem 2.1 gives the following four representations for the Drazin inverse A^D of a bipartite matrix.

COROLLARY 2.5. *Let A be as in* (1.4). *Then*

$$\begin{aligned} A^D &= \begin{bmatrix} 0 & (BC)^D B \\ C(BC)^D & 0 \end{bmatrix} = \begin{bmatrix} 0 & B(CB)^D \\ C(BC)^D & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & (BC)^D B \\ (CB)^D C & 0 \end{bmatrix} = \begin{bmatrix} 0 & B(CB)^D \\ (CB)^D C & 0 \end{bmatrix}. \end{aligned}$$

We end this section with some special cases for A^D in Corollary 2.5. If A is nonsingular, then B and C are necessarily square and nonsingular, and the formulas in Corollary 2.5 reduce to

$$A^{-1} = \begin{bmatrix} 0 & C^{-1} \\ B^{-1} & 0 \end{bmatrix}.$$

If BC is nilpotent, then both $(BC)^D$ and A^D are zero matrices. If $C = B^*$ is $(n-p)\times p$ with $\operatorname{rank} B < p$, then BB^* is singular and Hermitian; thus index $BB^* = 1$. As A is also Hermitian, index $A = 1$. In this case, $A^D = A^{\#} = A^{\dagger}$, the Moore-Penrose inverse of A, with

$$A^D = \begin{bmatrix} 0 & (BB^*)^{\dagger} B \\ B^*(BB^*)^{\dagger} & 0 \end{bmatrix} = \begin{bmatrix} 0 & B^{*\dagger} \\ B^{\dagger} & 0 \end{bmatrix} = A^{\dagger}.$$

3. Index of A from the index of BC. Results in this section that we state for index A in terms of BC and index BC could alternatively be stated in terms of CB and index CB.

Let A be as in (1.4). Then for $j = 0, 1, \ldots$

$$A^{2j} = \begin{bmatrix} (BC)^j & 0 \\ 0 & (CB)^j \end{bmatrix}$$

and

$$A^{2j+1} = \begin{bmatrix} 0 & (BC)^j B \\ C(BC)^j & 0 \end{bmatrix} = \begin{bmatrix} 0 & B(CB)^j \\ (CB)^j C & 0 \end{bmatrix}.$$

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Thus,

$$\operatorname{rank} A^{2j} = \operatorname{rank}(BC)^j + \operatorname{rank}(CB)^j \tag{3.1}$$

and

$$\operatorname{rank} A^{2j+1} = \operatorname{rank}(BC)^j B + \operatorname{rank} C(BC)^j. \tag{3.2}$$

Let $s = \text{index } BC$ and suppose that $s = 0$. If $n = 2p$, then B and C are both $p \times p$ invertible matrices and index $A = 0$. In this case, $A^D = A^{-1}$. Otherwise, if $n \neq 2p$, then $\operatorname{rank} BC = \operatorname{rank} B = \operatorname{rank} C = \operatorname{rank} CB = p$, index $A = 1$ and $A^D = A^{\#}$ as given in [5, Theorem 2.2] in terms of $(BC)^{-1}$.

We use the following rank inequality (see [10, page 13]) of Frobenius in the proof of some of the results in this section.

Lemma 3.1. (Frobenius Inequality) *If U is $m \times k$, V is $k \times n$ and W is $n \times p$, then*

$$\operatorname{rank} UV + \operatorname{rank} VW \leq \operatorname{rank} V + \operatorname{rank} UVW.$$

Theorem 3.2. *Let A be as in* (1.4) *and suppose that index $BC = s \geq 1$. Then index $A = 2s-1,\ 2s$ or $2s+1$.*

Proof. From Theorem 2.1, index $A \leq 2s+1$. By Lemma 3.1,

$$\begin{aligned} \operatorname{rank} B(CB)^{s-1} + \operatorname{rank}(CB)^{s-1}C &\leq \operatorname{rank}(CB)^{s-1} + \operatorname{rank}(BC)^s \\ &< \operatorname{rank}(CB)^{s-1} + \operatorname{rank}(BC)^{s-1}, \end{aligned}$$

since index $BC = s$. Thus, using (3.1) and (3.2), $\operatorname{rank} A^{2s-1} < \operatorname{rank} A^{2s-2}$ and index $A > 2s-2$. Therefore, index $A = 2s-1,\ 2s$ or $2s+1$. ∎

In the following three theorems, we give necessary and sufficient conditions for each of the values of index A which are identified in Theorem 3.2.

Theorem 3.3. *Let A be as in* (1.4) *and suppose that index $BC = s \geq 1$. Then index $A = 2s-1$ if and only if*
(i) $\operatorname{rank}(BC)^s = \operatorname{rank}(BC)^{s-1}B$ *and* $\operatorname{rank}(CB)^s = \operatorname{rank}(CB)^{s-1}C$*, or*
(ii) $\operatorname{rank}(BC)^s = \operatorname{rank}(CB)^{s-1}C$ *and* $\operatorname{rank}(CB)^s = \operatorname{rank}(BC)^{s-1}B$*.*

Proof. From Theorem 3.2, index $A \geq 2s-1$. Now, using (3.1) and (3.2), $\operatorname{rank} A^{2s} = \operatorname{rank} A^{2s-1}$ if and only if $\operatorname{rank}(BC)^s + \operatorname{rank}(CB)^s = \operatorname{rank}(BC)^{s-1}B + \operatorname{rank}(CB)^{s-1}C$, or equivalently, *(i)* or *(ii)* holds. Thus, index $A = 2s-1$ if and only if either of the above rank conditions hold. ∎

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Note that if index $A = 2s - 1$, then in fact $\text{rank}(BC)^s = \text{rank}(BC)^{s-1}B = \text{rank}(CB)^s = \text{rank}(CB)^{s-1}C$. The conditions of Theorem 3.3 hold for any Hermitian bipartite matrix A (as in (1.4) with $C = B^*$) since index $BB^* = 1 =$ index A.

Lemma 3.4. *If index $BC = s$, then*

$$\text{rank}(BC)^{s+1} = \text{rank}(BC)^s = \text{rank}(BC)^s B = \text{rank}\, C(BC)^s = \text{rank}(CB)^{s+1}.$$

Proof. Let $t = \text{rank}(BC)^s$. Since index $BC = s$, it follows that $t = \text{rank}(BC)^s = \text{rank}(BC)^{s+1} = \text{rank}(BC)^s B = \text{rank}\, C(BC)^s$, where the latter two equalities hold as $t = \text{rank}(BC)^{s+1} \leq \text{rank}(BC)^s B \leq \text{rank}(BC)^s = t$ and $t = \text{rank}(BC)^{s+1} \leq \text{rank}\, C(BC)^s \leq \text{rank}(BC)^s = t$. By Lemma 3.1,

$$\text{rank}\, C(BC)^s + \text{rank}(BC)^s B \leq \text{rank}(BC)^s + \text{rank}(CB)^{s+1}$$

so

$$2t \;\leq\; t + \text{rank}(CB)^{s+1} = \text{rank}\, A^{2s+2} \leq \text{rank}\, A^{2s+1} = 2t.$$

Thus, $\text{rank}(CB)^{s+1} = t$. ◻

Theorem 3.5. *Let A be as in* (1.4) *and suppose that index $BC = s \geq 1$. Then index $A = 2s$ if and only if index $CB = s$ and (i)* $\text{rank}(BC)^s < \text{rank}(BC)^{s-1}B$ *or (ii)* $\text{rank}(CB)^s < \text{rank}(CB)^{s-1}C$.

Proof. Suppose that index $A = 2s$. Let $t = \text{rank}(BC)^s$. Then

$$\text{rank}\, A^{2s} = \text{rank}\, A^{2s+1} = 2t < \text{rank}\, A^{2s-1},$$

where the second equality follows from Lemma 3.4. Since $\text{rank}\, A^{2s} = \text{rank}(BC)^s + \text{rank}(CB)^s = 2t$, it follows that $\text{rank}(CB)^s = t$. Using (3.2),

$$\begin{aligned} \text{rank}\, A^{2s-1} &= \text{rank}(BC)^{s-1}B + \text{rank}\, C(BC)^{s-1} \\ &= \text{rank}\, B(CB)^{s-1} + \text{rank}(CB)^{s-1}C \\ &\leq \text{rank}(CB)^{s-1} + \text{rank}(BC)^s \quad \text{by Lemma 3.1} \\ &= \text{rank}(CB)^{s-1} + t. \end{aligned}$$

Thus,

$$2t \;<\; \text{rank}\, A^{2s-1} \;\leq\; \text{rank}(CB)^{s-1} + t$$

and therefore $t < \text{rank}(CB)^{s-1}$. This shows that index $CB = s$ since $\text{rank}(CB)^{s+1} = t$ by Lemma 3.4. Also, if $\text{rank}(BC)^s = \text{rank}(BC)^{s-1}B$ and $\text{rank}(CB)^s =$

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$\operatorname{rank}(CB)^{s-1}C$, then by Theorem 3.3*(i)*, index $A = 2s-1$, a contradiction. Thus, $\operatorname{rank}(BC)^s < \operatorname{rank}(BC)^{s-1}B$ or $\operatorname{rank}(CB)^s < \operatorname{rank}(CB)^{s-1}C$.

On the other hand, suppose that index $CB = s$ and *(i)* or *(ii)* holds. Since *(i)* or *(ii)* holds, index $A \neq 2s-1$ from Theorem 3.3. If index $A = 2s+1$, then $\operatorname{rank} A^{2s+1} < \operatorname{rank} A^{2s}$ or equivalently $\operatorname{rank}(BC)^sB + \operatorname{rank} C(BC)^s < \operatorname{rank}(BC)^s + \operatorname{rank}(CB)^s$. Thus, Lemma 3.4 implies that $t < \operatorname{rank}(CB)^s$, contradicting index $CB = s$ and showing by Theorem 3.2 that index $A = 2s$. □

If A is as in (1.4) with index $BC =$ index $CB = s \geq 1$, then index A is $2s$ or $2s-1$, depending on whether or not one of the rank conditions *(i)* or *(ii)* in Theorem 3.5 holds. Note that if neither of these rank conditions holds, then the rank condition *(i)* of Theorem 3.3 holds. For example, if A is as in (1.4) with

$$B = C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

then index $BC =$ index $CB = 1 = s$, and index $A = 1 = 2s-1$. Note that neither of the rank conditions in Theorem 3.5 holds.

Theorem 3.6. *Let A be as in* (1.4) *and suppose that index $BC = s \geq 1$. Then index $A = 2s+1$ if and only if* $\operatorname{rank}(CB)^s > \operatorname{rank}(CB)^sC$.

Proof. Let $t = \operatorname{rank}(BC)^s$ and suppose that index $A = 2s+1$. Then as in the proof of Theorem 3.5, it follows that $t < \operatorname{rank}(CB)^s$. But by Lemma 3.4, $\operatorname{rank}(CB)^{s+1} = \operatorname{rank}(CB)^sC = \operatorname{rank} C(BC)^s = t$. Thus, $\operatorname{rank}(CB)^s > \operatorname{rank}(CB)^sC$. For the converse, $\operatorname{rank}(CB)^s > \operatorname{rank}(CB)^sC$ implies that $\operatorname{rank}(BC)^s + \operatorname{rank}(CB)^s > \operatorname{rank}(BC)^sB + \operatorname{rank} C(BC)^s$. That is, $\operatorname{rank} A^{2s} > \operatorname{rank} A^{2s+1}$ and by Theorem 3.2, index $A = 2s+1$. □

Corollary 3.7. *Let A be as in* (1.4) *and suppose that index $BC = s \geq 1$. Then index $A = 2s+1$ if and only if index $CB = s+1$.*

Proof. Suppose that index $A = 2s+1$. Theorem 3.6 gives $\operatorname{rank}(CB)^s > \operatorname{rank}(CB)^sC \geq \operatorname{rank}(CB)^{s+1}$ so index $CB \geq s+1$. The index assumptions on A and BC give $\operatorname{rank} A^{2s+2} = \operatorname{rank} A^{2s+4}$ and $\operatorname{rank}(BC)^{s+1} = \operatorname{rank}(BC)^{s+2}$, and using (3.1), these imply that $\operatorname{rank}(CB)^{s+1} = \operatorname{rank}(CB)^{s+2}$. Thus, index $CB = s+1$. For the converse, if index $CB = s+1$, then $\operatorname{rank}(CB)^s > \operatorname{rank}(CB)^{s+1}$ so $\operatorname{rank} A^{2s} > \operatorname{rank} A^{2s+2}$ by (3.1). Using (3.1), (3.2) and Lemma 3.4, $\operatorname{rank} A^{2s+2} = \operatorname{rank} A^{2s+1}$ and it follows that index $A = 2s+1$. □

4. Examples. We give examples to illustrate each of the indices in Theorem 3.2 and the associated Drazin inverses.

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Example 4.1. Let

$$A = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right].$$

Then

$$BC = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -3 \\ 0 & 0 & 1 \end{bmatrix},$$

and it is easily shown that index $BC = 2 = s$. Since $\operatorname{rank} A = 4$, $\operatorname{rank} A^2 = 3$ and $\operatorname{rank} A^3 = \operatorname{rank} A^4 = 2$, it follows that index $A = 3 = 2s - 1$. By the formula in [2, Theorem 7.7.1] for the Drazin inverse of a 2×2 block triangular matrix and noting that the leading block of BC is a 2×2 nilpotent matrix,

$$(BC)^D = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix},$$

giving

$$A^D = \left[\begin{array}{c|c} 0 & (BC)^D B \\ \hline C(BC)^D & 0 \end{array}\right] = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \\ \hline 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

It is easily verified that conditions *(i)* and *(ii)* of Theorem 3.3 are satisfied.

Example 4.2. Let

$$A = \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix},$$

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where B is a $p\times p$ singular matrix with index $B = s \geq 1$, and $C = I$, the $p\times p$ identity matrix. Then index $CB = s$ and $\text{rank}(CB)^s < \text{rank}(CB)^{s-1}C$; thus by Theorem 3.5, index $A = 2s$. In this case, by Theorem 2.1,

$$A^D = \begin{bmatrix} 0 & B^D B \\ B^D & 0 \end{bmatrix}.$$

Example 4.3. Let A be the 7×7 matrix

$$A = \left[\begin{array}{ccc|cccc} 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right],$$

for which the directed graph $D(A)$ is a path graph (see, for example, [5]). Then

$$BC = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad (BC)^2 = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & -1 \end{bmatrix},$$

so $\text{rank}\, BC = \text{rank}(BC)^2 = 2$ and index $BC = 1 = s$. Thus, $(BC)^D = (BC)^\#$ and as the directed graph $D(BC)$ is a path graph, $(BC)^\#$ is given by [5, Corollary 3.8]:

$$(BC)^\# = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

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Using Theorem 2.1,

$$A^D = \left[\begin{array}{c|c} 0 & (BC)^\# B \\ \hline C(BC)^\# & 0 \end{array}\right] = \frac{1}{2}\left[\begin{array}{ccc|cccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Note that $\operatorname{rank} CB = 3 > \operatorname{rank} CBC = 2$ and $\operatorname{index} CB = 2$; thus by Theorem 3.6 or Corollary 3.7, $\operatorname{index} A = 2s + 1 = 3$. Although $A^\#$ does not exist, $(BC)^\#$ does exist and thus results in [5] can be applied to determine A^D in this example.

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