

ADDITIVE RANK-ONE NONINCREASING MAPS ON HERMITIAN MATRICES OVER THE FIELD $GF(2^2)^*$

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Abstract. A complete classification of additive rank-one nonincreasing maps on hermitian matrices over Galois field $GF(2^2)$ is obtained. This field is special and was not covered in a previous paper. As a consequence, some known applications, like the classification of additive rank-additivity preserving maps, are extended to arbitrary fields. An application concerning the preservers of hermitian varieties is also presented.

Key words. Hermitian matrix, Rank, Additive preserver, Galois field, Weak homomorphism of a graph.

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1. Introduction. By definition, hermitian matrices X and Y are adjacent if $\text{rk}(X - Y) = 1$. Hua's fundamental theorem of the geometry of hermitian matrices (see e.g. [32, Theorem 6.4]) classifies all bijective maps Φ on hermitian matrices which preserve adjacency in both directions, i.e., $\text{rk}(X - Y) = 1$ iff $\text{rk}(\Phi(X) - \Phi(Y)) = 1$. Recently, important generalizations of Hua's theorem on hermitian matrices were obtained. In [17, 13, 14] "the assumption of both directions" was reduced to one direction only. In the case of hermitian matrices over complex or finite field even the assumption of bijectivity was dropped [18, 27]. For fairly arbitrary division ring of characteristic not two, such a result is known for 2×2 matrices [16].

These and related problems arise also in algebraic combinatorics. More precisely, in the study of association schemes and distance regular graphs. An *association scheme* is a pair $(\mathcal{X}, \mathcal{R})$ where \mathcal{X} is a finite set and $\mathcal{R} = \{R_0, R_1, \dots, R_d\}$ is a partition of $\mathcal{X} \times \mathcal{X}$ where relations R_i satisfy some additional properties (see [5, p. 43]). An automorphism of an association scheme is a permutation Φ on \mathcal{X} , for which $(x, y) \in R_i$ implies $(\Phi(x), \Phi(y)) \in R_i$ (the definition of an automorphism of an association scheme slightly varies throughout literature). In the case of *hermitian forms scheme*, \mathcal{X} is the set of hermitian matrices over the Galois field of order q^2 (q is a power of a prime), and $(X, Y) \in R_i$ iff $\text{rk}(X - Y) = i$. *Hermitian forms graph* is a distance regular graph

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obtained from hermitian forms scheme such that the set of vertices \mathcal{X} is formed by hermitian matrices, while edges are induced by the relation R_1 , i.e., X and Y are adjacent iff $\text{rk}(X - Y) = 1$. Consequently, an automorphism of the hermitian forms graph is precisely a bijection which preserves adjacency (in both directions since \mathcal{X} is finite). Recall that such automorphisms are classified in Hua's theorem. Moreover, very recently it was shown in [27] that any endomorphism of the (finite) hermitian forms graph is necessarily bijective and hence an automorphism. We refer to [1, 5] for more on association schemes and distance regular graphs and to [21, 22, 31] for more on hermitian forms schemes/graphs.

Hermitian forms graphs can also be used to construct $(b; d)$ -disjunct matrices [19]. Such matrices are used to construct an error-tolerable pooling designs which are used to identify the set of defective items in a large population of items. These designs have applications also to the screening of DNA sequence. For more on $(b; d)$ -disjunct matrices and pooling designs see e.g. [15, 8, 33, 26].

We now turn the discussion to weak adjacency preservers (in one direction), i.e., to those maps Φ for which $\text{rk}(X - Y) = 1$ implies $\text{rk}(\Phi(X) - \Phi(Y)) \leq 1$. In the language of graph theory, such maps are also called weak endomorphisms of the (hermitian forms) graph (see e.g. [20, p. 25]). Although the definitions of preserving adjacency/weak adjacency look similar, there is a fundamental difference between them. When considering hermitian matrices over a finite field (i.e., when the hermitian forms graph is finite), any adjacency preserver is bijective, provided that the involution on the field is not the identity map [27]. Hence, any such map is of the form $\Phi(X) = TX^\sigma T^* + Y$, where the matrix T is invertible, Y is hermitian, and the automorphism σ is applied entrywise. On the contrary, the structure of weak adjacency preservers seems much more complicated. Consider for example the map defined by $\Phi(A) = A$ if $\text{rk } A = 1$ and $\Phi(A) = 0$ otherwise. For another nonstandard example fix two hermitian matrices M and N of rank one and define: $\Phi(A) = 0$ if $\text{rk } A$ is even; $\Phi(A) = M$ if the number $\text{rk } A$ equals 1 modulo 4; $\Phi(A) = N$ if the number $\text{rk } A$ equals 3 modulo 4. At the present time, we are not aware of any result that would classify all such maps. However, if we add the additivity assumption on Φ then we could still hope for a nice structural results. Nevertheless, we still can not expect to reduce the problem on 'strong' adjacency preservers, since e.g. the map $\Phi(X) = TX^\sigma T^*$ preserves adjacency in the weak sense for any T , while to preserve adjacency in the "strong" sense T must be invertible.

Note that under the additivity assumption, the adjacency/weak adjacency preservers are precisely those maps, which preserve/do not increase rank-one. It is common practice in solving additive preserver problems to reduce them to rank-one preservers. Frequently, it is easier to check whether a given map does not increase rank-one rather than to check if it does indeed preserve it (see e.g. [9, proof of The-

orem 1.2] or [10, Theorem 2.1, proof of Theorem 2.2]). We refer to [12, 24] for a nice survey of the methods and historical remarks about preserver problems.

Recently, additive maps on hermitian matrices that do not increase rank–one were classified in [25]. The only essential assumption made in this article was that the characteristic of the underlying field differs from two and from three. The classification for these two characteristics was obtained by the present authors in [28]. However, the field $GF(2^2)$, the Galois field of order four, was not covered. Presently, we were able to overcome even this. Our main result, together with [25] and [28, Main Theorem], gives us a complete classification of additive rank–one nonincreasing maps. We will see that the field $GF(2^2)$ demands new techniques. Moreover, beside the maps given in [25, 28] new nonstandard maps appear (see (iii) in Theorem 2.1). We use a Bose–Chakravarti’s result [4] to classify them.

Before proceeding to Section 2, it is perhaps worthwhile to interpret the additivity assumption also in terms of endomorphisms of the hermitian forms graph. To do this, observe that the hermitian forms graph is the Cayley graph $Cay(G, \mathcal{S})$, where G is the additive group of all hermitian matrices and the set \mathcal{S} consists of all hermitian matrices of rank one (for the definition of the Cayley graph see e.g. [3, p. 106]). Hence, additive rank–one nonincreasing maps are precisely those weak endomorphisms of $Cay(G, \mathcal{S})$, which are endomorphisms of the group G that satisfy $\Phi(\mathcal{S}) \subseteq \mathcal{S} \cup \{e\}$. Here, e denotes the group identity, that is, the zero matrix in our case.

The rest of the paper is organized as follows. In Section 2 we recall the necessary definitions and state the main result. In Section 3 the proofs are given, while in Section 4 some applications of the main theorem are presented.

2. Preliminaries and statement of the main result. Throughout this paper, the field $GF(2^2)$ is shortly denoted by \mathbb{G} . Recall that the characteristic of \mathbb{G} , $\text{char } \mathbb{G}$, equals 2, and $\mathbb{G} = \{0, 1, \iota, 1 + \iota\} = \{0, 1\} \oplus \iota\{0, 1\}$, where the multiplication is given by $\iota^2 := 1 + \iota$. If not said otherwise, the symbol σ denotes a field automorphism of \mathbb{G} . Apart the identity map, id , there exists only one more automorphism $\bar{} : \mathbb{G} \rightarrow \mathbb{G}$ defined by $\bar{\iota} := 1 + \iota$. Note that the automorphism $\bar{}$ is an *involution*, i.e., $\overline{\bar{x}} = x$ for all $x \in \mathbb{G}$. Moreover, if $x \neq 0$ then $x\bar{x} = 1$, i.e., $x^{-1} = \bar{x}$.

We use \mathbb{G}^n to denote the space of all column–vectors of length n . Given a vector $\mathbf{x} \in \mathbb{G}^n$ we can, and will do so, identify it with an $n \times 1$ matrix. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard basis of \mathbb{G}^n , and let $\{E_{11}, E_{12}, \dots, E_{mn}\}$ be the standard basis in the space $\mathcal{M}_{m \times n}(\mathbb{G})$ of all rectangular $m \times n$ matrices with entries from \mathbb{G} . When matrices are quadratic, i.e, when $m = n$, we write shortly $\mathcal{M}_n(\mathbb{G})$.

Given a matrix X and an automorphism σ we let X^σ be the matrix obtained from X by applying σ entry–wise. When $\sigma = \bar{}$, we write \overline{X} . The transpose of a

matrix X is denoted by X^{tr} . A matrix $X \in \mathcal{M}_n(\mathbb{G})$ is *hermitian* if $X^* := \overline{X}^{\text{tr}} = X$. Let $\mathcal{H}_n(\mathbb{G})$ be the set of all such matrices. Likewise we denote by $\mathcal{H}_n(\mathbb{K})$ the set of all hermitian matrices with entries from a field \mathbb{K} possessing an involution $\hat{}$, which is assumed to be *nonidentical* ($\hat{} \neq \text{id}$) unless otherwise stated. Note that $\mathcal{H}_n(\mathbb{G})$ is a vector space over the subfield $\{x \in \mathbb{G} \mid \overline{x} = x\} = \{0, 1\} =: \mathbb{Z}_2$. A map $\Phi : \mathcal{H}_n(\mathbb{G}) \rightarrow \mathcal{H}_m(\mathbb{G})$ is *additive* if $\Phi(X+Y) = \Phi(X) + \Phi(Y)$ for all $X, Y \in \mathcal{H}_n(\mathbb{G})$, and is *rank-one nonincreasing* if $\text{rk } X = 1$ implies $\text{rk } \Phi(X) \leq 1$. Note that for any $X \in \mathcal{H}_n(\mathbb{G})$ of rank r there exists an invertible matrix $Q \in \mathcal{M}_n(\mathbb{G})$ such that $X = Q(\sum_{j=1}^r E_{jj})Q^*$ (see e.g. [4, Theorem 4.1]). Consequently, any additive rank-one nonincreasing map $\Phi : \mathcal{H}_n(\mathbb{G}) \rightarrow \mathcal{H}_m(\mathbb{G})$ is also rank- r nonincreasing for any r , i.e., $\text{rk } X = r$ implies $\text{rk } \Phi(X) \leq r$. It also follows that any rank-one matrix $M \in \mathcal{H}_n(\mathbb{G})$ can be written as $M = \mathbf{x}\mathbf{x}^*$ for some nonzero $\mathbf{x} \in \mathbb{G}^n$.

To shorten writings, we define $\mathbf{x}^2 := \mathbf{x}\mathbf{x}^*$ and $\mathbf{x} \bullet \mathbf{y} := \mathbf{x}\mathbf{y}^* + \mathbf{y}\mathbf{x}^*$. Given vectors $\mathbf{x}_1, \dots, \mathbf{x}_r \in \mathbb{G}^n$ let $\langle \mathbf{x}_1, \dots, \mathbf{x}_r \rangle$ be their \mathbb{G} -linear span, and let $\langle \mathbf{x}_1, \dots, \mathbf{x}_r \rangle^{(2)}$ be the \mathbb{Z}_2 -subspace in $\mathcal{H}_n(\mathbb{G})$ generated by matrices \mathbf{x}^2 and $\mathbf{x} \bullet \mathbf{y}$; $\mathbf{x}, \mathbf{y} \in \langle \mathbf{x}_1, \dots, \mathbf{x}_r \rangle$. It is easily checked that if $\mathbf{x}_1, \dots, \mathbf{x}_r$ are \mathbb{G} -linearly independent then matrices $\mathbf{x}_1^2, \dots, \mathbf{x}_r^2$ together with $\mathbf{x}_j \bullet \mathbf{x}_k$ and $\iota \mathbf{x}_j \bullet \mathbf{x}_k$ ($1 \leq j < k \leq r$) form a \mathbb{Z}_2 -basis of $\langle \mathbf{x}_1, \dots, \mathbf{x}_r \rangle^{(2)}$. If we replace matrices $\mathbf{x}_j \bullet \mathbf{x}_k$ and $\iota \mathbf{x}_j \bullet \mathbf{x}_k$ by $(\mathbf{x}_j + \mathbf{x}_k)^2$ and $(\iota \mathbf{x}_j + \mathbf{x}_k)^2$ then we obtain an additive basis formed by rank-one matrices. We want to point out that if $\mathbf{x}_1, \mathbf{x}_2$ are linearly independent then there are precisely five rank-one matrices in $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle^{(2)}$. Namely: $\mathbf{x}_1^2, \mathbf{x}_2^2, (\mathbf{x}_1 + \mathbf{x}_2)^2, (\iota \mathbf{x}_1 + \mathbf{x}_2)^2, (\overline{\iota} \mathbf{x}_1 + \mathbf{x}_2)^2$. Note that their sum is zero. Though being obvious, it is perhaps worthwhile to mention that if $\mathbf{x}_1 = \mathbf{e}_1$ and $\mathbf{x}_2 = \mathbf{e}_2$ are standard vectors then $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle^{(2)} = \mathcal{H}_2(\mathbb{G}) \oplus 0_{n-2}$ and the five rank-one matrices are: $E_{11}, E_{22}, E_{11} + xE_{12} + \overline{x}E_{21} + E_{22}$ ($x = 1, \iota, \overline{\iota}$).

If $\sigma \in \{\text{id}, \overline{}\}$ then we say that an additive map $g : \langle \mathbf{x}_1, \dots, \mathbf{x}_r \rangle \rightarrow \mathbb{G}^m$ is *σ -semilinear* if $g(x\mathbf{x}) = \sigma(x)g(\mathbf{x})$ holds for all $x \in \mathbb{G}$ and all $\mathbf{x} \in \langle \mathbf{x}_1, \dots, \mathbf{x}_r \rangle$. Any such g induces an additive map $\mathcal{P}(g) : \langle \mathbf{x}_1, \dots, \mathbf{x}_r \rangle^{(2)} \rightarrow \mathcal{H}_m(\mathbb{G})$ having the property $\mathcal{P}(g)(\mathbf{x}^2) = g(\mathbf{x})^2$ and $\mathcal{P}(g)(\mathbf{x} \bullet \mathbf{y}) = g(\mathbf{x}) \bullet g(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \langle \mathbf{x}_1, \dots, \mathbf{x}_r \rangle$. If $\langle \mathbf{x}_1, \dots, \mathbf{x}_r \rangle = \mathbb{G}^n$ then $\mathcal{P}(g)(X) = TX^\sigma T^*$, where $m \times n$ matrix T is defined by $T\mathbf{e}_j := g(\mathbf{e}_j)$ for $j = 1, \dots, n$.

We are now ready to state the main theorem of this paper.

THEOREM 2.1. *If $n, m \geq 2$ are integers then a map $\Phi : \mathcal{H}_n(\mathbb{G}) \rightarrow \mathcal{H}_m(\mathbb{G})$ is additive rank-one nonincreasing if and only if it takes one of the following forms:*

- (i) $\Phi(X) = TX^\sigma T^*$,
- (ii) $\Phi(X) = \mathfrak{s}(H \circ X) \cdot M$,
- (iii) $\Phi(X) = P\left(\left([QQQ^*]_{11} + [QQQ^*]_{22}\right)E_{11} + \left([QQQ^*]_{12} + [QQQ^*]_{21}\right)E_{22}\right)P^*$.

Here, $T \in \mathcal{M}_{m \times n}(\mathbb{G})$, $\sigma \in \{\text{id}, \overline{}\}$, $M \in \mathcal{H}_m(\mathbb{G})$ is of rank one, $H \in \mathcal{H}_n(\mathbb{G})$, \circ denotes

the Schur (i.e., entrywise) product, $\mathfrak{s}(Y)$ is the sum of all entries of the matrix Y , $P \in \mathcal{M}_m(\mathbb{G})$ and $Q \in \mathcal{M}_n(\mathbb{G})$ are invertible, while $[QXQ^*]_{jk}$ is the (j, k) -entry of the matrix QXQ^* .

We call the maps of the form (i) *standard*, while those of the forms (ii)–(iii) *nonstandard*.

REMARK 2.2. Note that in (ii), the map $f(X) := \mathfrak{s}(H \circ X)$ is just an additive functional $f : \mathcal{H}_n(\mathbb{G}) \rightarrow \mathbb{Z}_2$ (see Lemma 3.9).

REMARK 2.3. It follows immediately from Theorem 2.1 that, in the case $m = n$, the semigroup of additive rank-one nonincreasing maps is generated by standard maps, maps of the form $\Phi(X) = f(X)E_{11}$, where f is an additive functional, and by the map $\Phi(X) = ([X]_{11} + [X]_{22})E_{11} + ([X]_{12} + [X]_{21})E_{22}$.

REMARK 2.4. Let \mathbb{K} be a field and $\widehat{}$ an involution on it (we do not assume that $\widehat{} \neq \text{id}$). If an additive rank-one nonincreasing map $\Phi : \mathcal{H}_n(\mathbb{G}) \rightarrow \mathcal{H}_m(\mathbb{K})$ has in its image a matrix of rank at least 3 then it can be shown that it is of the standard form, i.e., $\Phi(X) = \xi T X \sigma T^*$, where $\xi = \widehat{\xi}$ is a scalar and $\sigma : \mathbb{G} \rightarrow \mathbb{K}$ is a field homomorphism which intertwines the two involutions, i.e., $\widehat{\sigma(x)} = \sigma(\widehat{x})$ for all $x \in \mathbb{G}$. In particular $\widehat{} \neq \text{id}$, since $\widehat{\sigma(\iota)} = \sigma(\widehat{\iota}) \neq \sigma(\iota)$. Moreover, the additivity of Φ forces that $\text{char } \mathbb{K} = 2$. To keep the paper concise we will not prove this here.

EXAMPLE 2.5. It already follows from Theorem 2.1 that the conclusion in Remark 2.4 does not hold if $\max(\text{rk } \Phi(X)) \leq 2$. However, if $\mathbb{K} \neq \mathbb{G}$ then there exist different nonstandard additive rank-one nonincreasing maps $\Phi : \mathcal{H}_n(\mathbb{G}) \rightarrow \mathcal{H}_m(\mathbb{K})$. Consider the field $\mathbb{K} := GF(4^2) = \{0, 1, a^2 + a, a^2 + a + 1\} \oplus a \cdot \{0, 1, a^2 + a, a^2 + a + 1\}$ where the multiplication and the involution is given by the rule $a^4 = a + 1 = \widehat{a}$. The additive map $\tau : \mathbb{G} \rightarrow GF(4^2)$ defined by $\tau(1) := 1$ and $\tau(\iota) := a^2 + a$ is a field homomorphism, though it does not intertwine the two involutions. In fact, $\widehat{\tau(\iota)} \neq \tau(\widehat{\iota})$, since $\widehat{\tau(\iota)} = \widehat{a^2 + a} = (\widehat{a})^2 + \widehat{a} = (a + 1)^2 + (a + 1) = a^2 + 3a + 2 = a^2 + a$, while $\tau(\widehat{\iota}) = \tau(\iota + 1) = \tau(\iota) + \tau(1) = a^2 + a + 1$. Note that the map $x \mapsto x^2$ is additive in a field of characteristic two. Hence, the map $\Phi : \mathcal{H}_2(\mathbb{G}) \rightarrow \mathcal{H}_2(\mathbb{K})$ defined by

$$\Phi \left(\begin{bmatrix} \alpha_1 & x \\ \bar{x} & \alpha_2 \end{bmatrix} \right) := \begin{bmatrix} \tau(\alpha_1 + x^2 + \iota x) & \tau(x) + a\tau(x^2) \\ \widehat{\tau(x)} + \widehat{a\tau(x^2)} & \tau(\alpha_2) \end{bmatrix}$$

is additive as well. We claim that it does not increase rank-one. Recall that E_{11} , E_{22} , $E_{11} + xE_{12} + \bar{x}E_{21} + E_{22}$ ($x = 1, \iota, \bar{\iota}$) are the only rank-one matrices in $\mathcal{H}_2(\mathbb{G})$.

The determinants of their Φ -images

$$(2.1) \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a^2 + a & a + 1 \\ a & 1 \end{bmatrix}, \\ \begin{bmatrix} 1 & a^3 \\ a^3 + a^2 + a + 1 & 1 \end{bmatrix}, \begin{bmatrix} a^2 + a & a^3 + a + 1 \\ a^3 + a^2 + 1 & 1 \end{bmatrix}$$

all vanish, since $a^4 = a + 1$. Hence, all of the above matrices are of rank one indeed. Note that Φ is neither of the form (ii), since its image contains a matrix of rank two, nor of the form (i), since in this case $\Phi(E_{ii}) = E_{ii}$ ($i = 1, 2$) would imply $T = t_1 E_{11} + t_2 E_{22}$, and consequently the matrix $\Phi(E_{12} + E_{21})$ would have zero entry at position (1,1), which is a contradiction. The map Φ is also completely different from the map (iii), because its image contains more than two linearly independent matrices (see e.g. the matrices in (2.1)).

EXAMPLE 2.6. In the case $\max(\text{rk } \Phi(X)) \leq 2$ we have new nonstandard additive rank-one nonincreasing maps $\Phi : \mathcal{H}_n(\mathbb{G}) \rightarrow \mathcal{H}_m(\mathbb{K})$ even when $(\mathbb{K}, \wedge) = (\mathbb{G}, \text{id})$, i.e., when the codomain $\mathcal{H}_m(\mathbb{K})$ is the set $\mathcal{S}_m(\mathbb{G})$ of all symmetric $m \times m$ matrices over the field \mathbb{G} . The map $\Phi : \mathcal{H}_2(\mathbb{G}) \rightarrow \mathcal{S}_2(\mathbb{G})$ defined by

$$\Phi \left(\begin{bmatrix} \alpha_1 & x \\ \bar{x} & \alpha_2 \end{bmatrix} \right) := \begin{bmatrix} \alpha_1 + x & x + \bar{x} \\ x + \bar{x} & \alpha_2 + \bar{x} \end{bmatrix}$$

is additive and rank-one nonincreasing. It is not of the forms (i), (ii), (iii) since the same arguments as in Example 2.5 apply.

3. Proofs. Throughout this section, $n, m \geq 2$ are integers and $\Phi : \mathcal{H}_n(\mathbb{G}) \rightarrow \mathcal{H}_m(\mathbb{G})$ is an additive rank-one nonincreasing map.

3.1. Standard maps. We start with lemmas needed to classify maps, which turn out to be standard. The proof of the first of them can be easily deduced from [28, Lemma 3.1] and will not be given here.

LEMMA 3.1. *Let r be an integer and suppose that $\text{rk}(\sum_{j=1}^r \mathbf{y}_j^2) = r$. If a hermitian rank-one matrix M satisfies $M \notin \langle \mathbf{y}_1, \dots, \mathbf{y}_r \rangle^{(2)}$ then $\text{rk}(\sum_{j=1}^r \mathbf{y}_j^2 + M) = r + 1$.*

LEMMA 3.2. *Let r be an integer and suppose that $\text{rk } \Phi(\sum_{j=1}^r \mathbf{x}_j^2) = r$. Then, the image $\text{Im } \Phi|_{\langle \mathbf{x}_1, \dots, \mathbf{x}_r \rangle^{(2)}}$ is contained in $\langle \mathbf{y}_1, \dots, \mathbf{y}_r \rangle^{(2)}$, where $\Phi(\mathbf{x}_j^2) =: \mathbf{y}_j^2$. Moreover, for arbitrary $1 \leq j < k \leq r$, the set $\{\Phi((\mathbf{x}_j + \mathbf{x}_k)^2), \Phi((i\mathbf{x}_j + \mathbf{x}_k)^2), \Phi((\bar{i}\mathbf{x}_j + \mathbf{x}_k)^2)\}$ equals $\{(\mathbf{y}_j + \mathbf{y}_k)^2, (i\mathbf{y}_j + \mathbf{y}_k)^2, (\bar{i}\mathbf{y}_j + \mathbf{y}_k)^2\}$ or $\{0, \mathbf{y}_j^2, \mathbf{y}_k^2\}$.*

Proof. As Φ is rank nonincreasing, $\text{Im } \Phi|_{\langle \mathbf{x}_1, \dots, \mathbf{x}_r \rangle^{(2)}}$ contains only matrices of rank $\leq r$. Since the space $\langle \mathbf{x}_1, \dots, \mathbf{x}_r \rangle^{(2)}$ is additively spanned by rank-one matrices, the first conclusion, i.e., $\text{Im } \Phi|_{\langle \mathbf{x}_1, \dots, \mathbf{x}_r \rangle^{(2)}} \subseteq \langle \mathbf{y}_1, \dots, \mathbf{y}_r \rangle^{(2)}$ follows by Lemma 3.1.

Consequently, since $\text{rk } \Phi(\mathbf{x}_j^2 + \mathbf{x}_k^2) = 2$, we have also $\text{Im } \Phi|_{\langle \mathbf{x}_j, \mathbf{x}_k \rangle^{(2)}} \subseteq \langle \mathbf{y}_j, \mathbf{y}_k \rangle^{(2)}$ for arbitrary $1 \leq j < k \leq r$. In particular, $\{\Phi((\mathbf{x}_j + \mathbf{x}_k)^2), \Phi((i\mathbf{x}_j + \mathbf{x}_k)^2), \Phi((\bar{i}\mathbf{x}_j + \mathbf{x}_k)^2)\} \subseteq \{0, \mathbf{y}_j^2, \mathbf{y}_k^2, (\mathbf{y}_j + \mathbf{y}_k)^2, (i\mathbf{y}_j + \mathbf{y}_k)^2, (\bar{i}\mathbf{y}_j + \mathbf{y}_k)^2\}$. The rest follows from the equation

$$(3.1) \quad \begin{aligned} 0 &= \Phi(0) = \Phi(\mathbf{x}_j^2 + \mathbf{x}_k^2 + (\mathbf{x}_j + \mathbf{x}_k)^2 + (i\mathbf{x}_j + \mathbf{x}_k)^2 + (\bar{i}\mathbf{x}_j + \mathbf{x}_k)^2) \\ &= \mathbf{y}_j^2 + \mathbf{y}_k^2 + \Phi((\mathbf{x}_j + \mathbf{x}_k)^2) + \Phi((i\mathbf{x}_j + \mathbf{x}_k)^2) + \Phi((\bar{i}\mathbf{x}_j + \mathbf{x}_k)^2). \quad \square \end{aligned}$$

The next lemma can be proved via [27, Theorem 3.1], however we give a self-contained proof which is not much longer.

LEMMA 3.3. *Let $r \in \{2, 3\}$ and $\mathbf{x}_1, \dots, \mathbf{x}_r \in \mathbb{G}^n$. If $\Phi|_{\langle \mathbf{x}_1, \dots, \mathbf{x}_r \rangle^{(2)}}$ preserves rank-one and $\text{rk } \Phi(\sum_{j=1}^r \mathbf{x}_j^2) = r$ then $\Phi|_{\langle \mathbf{x}_1, \dots, \mathbf{x}_r \rangle^{(2)}} = \mathcal{P}(g)$ for some σ -semilinear map $g : \langle \mathbf{x}_1, \dots, \mathbf{x}_r \rangle \rightarrow \mathbb{G}^m$.*

Proof. By the assumption, $\Phi|_{\langle \mathbf{x}_1, \dots, \mathbf{x}_r \rangle^{(2)}}$ preserves rank-one, i.e., it can not annihilate rank-one matrices. Hence, if we denote $\Phi(\mathbf{x}_j^2) =: \mathbf{y}_j^2$ then Lemma 3.2 implies

$$\{\Phi((\mathbf{x}_j + \mathbf{x}_k)^2), \Phi((i\mathbf{x}_j + \mathbf{x}_k)^2), \Phi((\bar{i}\mathbf{x}_j + \mathbf{x}_k)^2)\} = \{(\mathbf{y}_j + \mathbf{y}_k)^2, (i\mathbf{y}_j + \mathbf{y}_k)^2, (\bar{i}\mathbf{y}_j + \mathbf{y}_k)^2\}$$

for arbitrary $1 \leq j < k \leq r$. In other words, $\Phi((x\mathbf{x}_j + \mathbf{x}_k)^2) = (\nu_{jk}(x)\mathbf{y}_j + \mathbf{y}_k)^2$ for some function $\nu_{jk} : \mathbb{G} \rightarrow \mathbb{G}$ which satisfies $\nu_{jk}(0) = 0$ and $\nu_{jk}(x) \neq 0$ whenever $x \neq 0$. Moreover, from (3.1) we deduce that $\nu_{jk}(1) + \nu_{jk}(i) + \nu_{jk}(\bar{i}) = 0$. Therefore, if $\sigma_{jk} : \mathbb{G} \rightarrow \mathbb{G}$ is defined by $\sigma_{jk}(x) := \frac{\nu_{jk}(x)}{\nu_{jk}(1)}$ then $1 + \sigma_{jk}(i) + \sigma_{jk}(\bar{i}) = 0$, i.e., $\{\sigma_{jk}(i), \sigma_{jk}(\bar{i})\} = \{i, \bar{i}\}$. Consequently, $\sigma_{jk} \in \{\text{id}, \bar{\cdot}\}$.

If $r = 2$ then $\text{rk } \Phi(\sum_{j=1}^2 \mathbf{x}_j^2) = 2$, so \mathbf{y}_1 and \mathbf{y}_2 as well as \mathbf{x}_1 and \mathbf{x}_2 are linearly independent. Define first $g(\mathbf{x}_1) := \nu_{12}(1)\mathbf{y}_1$ and $g(\mathbf{x}_2) := \mathbf{y}_2$, and then extend the map g , on whole $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle$, σ_{12} -semilinearly. Since $\nu_{12}(1)\nu_{12}(1)$ is nonzero, it equals 1, so we deduce that $\Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2 \rangle^{(2)}} = \mathcal{P}(g)$.

If $r = 3$ we proceed similarly. Vectors $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ as well as $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent. If an invertible matrix $P \in \mathcal{M}_m(\mathbb{G})$ is such that $P\mathbf{y}_j = \mathbf{e}_j$ for all j and if $x, y \neq 0$ then any 2×2 minor of the rank-one matrix

$$\begin{aligned} P\Phi((x\mathbf{x}_1 + \mathbf{x}_2 + \bar{y}\mathbf{x}_3)^2)P^* &= P(\Phi(\mathbf{x}_1^2) + \Phi(\mathbf{x}_2^2) + \Phi(\mathbf{x}_3^2) + \\ &\quad + \Phi((x\mathbf{x}_1 + \mathbf{x}_2)^2) + \Phi((xy\mathbf{x}_1 + \mathbf{x}_3)^2) + \Phi((y\mathbf{x}_2 + \mathbf{x}_3)^2))P^* \end{aligned}$$

vanishes. From the minor at positions (1, 2), (1, 3), (2, 2), and (2, 3) we deduce that

$$(3.2) \quad \nu_{12}(1)\nu_{23}(1)\sigma_{12}(x)\sigma_{23}(y) = \nu_{13}(1)\sigma_{13}(xy).$$

Recall that $\sigma_{jk}(1) = 1$. Hence, by evaluating (3.2) at $x = 1 = y$ we obtain $\nu_{12}(1)\nu_{23}(1) = \nu_{13}(1)$. Moreover, $\nu_{jk}(1)$ is nonzero, so equation (3.2) implies that

$\sigma_{12}(x)\sigma_{23}(y) = \sigma_{13}(xy)$. By choosing $x = 1$ or $y = 1$ we deduce that $\sigma_{12} = \sigma_{13} = \sigma_{23}$. Therefore, if σ_{12} -semilinear map g is defined on the basis by $g(\mathbf{x}_1) := \nu_{12}(1)\mathbf{y}_1$, $g(\mathbf{x}_2) := \mathbf{y}_2$, and $g(\mathbf{x}_3) := \overline{\nu_{23}(1)}\mathbf{y}_3$ then $\Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle^{(2)}} = \mathcal{P}(g)$. \square

LEMMA 3.4. *Suppose that $\text{rk } \Phi(\mathbf{x}_1^2 + \mathbf{x}_2^2) = 2$ and $\mathcal{P}(g) = \Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2 \rangle^{(2)}} = \mathcal{P}(g')$ for some σ -semilinear map $g : \langle \mathbf{x}_1, \mathbf{x}_2 \rangle \rightarrow \mathbb{G}^m$ and σ' -semilinear map $g' : \langle \mathbf{x}_1, \mathbf{x}_2 \rangle \rightarrow \mathbb{G}^m$. Then $g' = t \cdot g$ for some nonzero $t \in \mathbb{G}$.*

Proof. For $j = 1, 2$ the equation $g(\mathbf{x}_j)^2 = \Phi(\mathbf{x}_j^2) = g'(\mathbf{x}_j)^2$ implies that $g'(\mathbf{x}_j) = t_j g(\mathbf{x}_j)$ for some nonzero $t_j \in \mathbb{G}$. Consequently,

$$\begin{aligned} (\sigma(x)g(\mathbf{x}_1)) \bullet g(\mathbf{x}_2) &= \Phi((x\mathbf{x}_1) \bullet \mathbf{x}_2) = (\sigma'(x)g'(\mathbf{x}_1)) \bullet g'(\mathbf{x}_2) \\ &= (\sigma'(x)t_1 g(\mathbf{x}_1)) \bullet (t_2 g(\mathbf{x}_2)) \quad (x \in \mathbb{G}). \end{aligned}$$

Since $g(\mathbf{x}_1)$ and $g(\mathbf{x}_2)$ are linearly independent, we deduce that $\sigma(x) = \sigma'(x)t_1 \bar{t}_2$. Hence, $\sigma = \sigma'$ and $t_1 \bar{t}_2 = 1 = t_1 \bar{t}_1$, i.e., $t_1 = t_2 =: t$. This ends the proof. \square

LEMMA 3.5. *Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{G}^n$ be linearly independent. If $\Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2 \rangle^{(2)}} = \mathcal{P}(g)$ for some σ -semilinear map $g : \langle \mathbf{x}_1, \mathbf{x}_2 \rangle \rightarrow \mathbb{G}^m$, and $\text{rk } \Phi(\mathbf{x}_1^2 + \mathbf{x}_2^2) = 2$, then g can be σ -semilinearly extended on $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$ such that $\Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle^{(2)}} = \mathcal{P}(g)$.*

Proof. We separate two cases.

Case 1. Assume first that $\text{Im } \Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle^{(2)}} \neq \text{Im } \Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2 \rangle^{(2)}}$. Then there exists $\dot{\mathbf{x}}_3 \in \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$ such that $\Phi(\dot{\mathbf{x}}_3^2) \notin \text{Im } \Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2 \rangle^{(2)}}$ and $\langle \mathbf{x}_1, \mathbf{x}_2, \dot{\mathbf{x}}_3 \rangle = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$. By the assumption, $\text{Im } \Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2 \rangle^{(2)}} = \langle g(\mathbf{x}_1), g(\mathbf{x}_2) \rangle^{(2)}$, so Lemma 3.1 implies that $\text{rk } \Phi(\mathbf{x}_1^2 + \mathbf{x}_2^2 + \dot{\mathbf{x}}_3^2) = 3$.

We will show that $\Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2, \dot{\mathbf{x}}_3 \rangle^{(2)}}$ preserves rank-one, i.e., $\Phi((x_1\mathbf{x}_1 + x_2\mathbf{x}_2 + x_3\dot{\mathbf{x}}_3)^2) \neq 0$ for any nonzero tuple (x_1, x_2, x_3) of scalars. This clearly holds if $x_3 = 0$ and $(x_1, x_2) \neq (0, 0)$ or if $x_1 = 0 = x_2$, and $x_3 \neq 0$. To deal with other cases denote $\Phi(\dot{\mathbf{x}}_3^2) =: \mathbf{y}_3^2$ and choose an invertible matrix P such that $Pg(\mathbf{x}_1) = \mathbf{e}_1$, $Pg(\mathbf{x}_2) = \mathbf{e}_2$, and $P\mathbf{y}_3 = \mathbf{e}_3$. By Lemma 3.2, the set $\{\Phi((\mathbf{x}_j + \dot{\mathbf{x}}_3)^2), \Phi((i\mathbf{x}_j + \dot{\mathbf{x}}_3)^2), \Phi((\bar{i}\mathbf{x}_j + \dot{\mathbf{x}}_3)^2)\}$ equals $\{(g(\mathbf{x}_j) + \mathbf{y}_3)^2, (ig(\mathbf{x}_j) + \mathbf{y}_3)^2, (\bar{i}g(\mathbf{x}_j) + \mathbf{y}_3)^2\}$ or $\{0, g(\mathbf{x}_j)^2, \mathbf{y}_3^2\}$ for $j = 1, 2$. Assume erroneously that $\Phi((x_1\mathbf{x}_1 + x_3\dot{\mathbf{x}}_3)^2) = 0$ for some nonzero x_1 and x_3 . We can choose $x \in \{1, i, \bar{i}\}$ such that either $P\Phi((x\mathbf{x}_2 + \dot{\mathbf{x}}_3)^2)P^* = P(g(\mathbf{x}_2) + \mathbf{y}_3)^2P^* = (\mathbf{e}_2 + \mathbf{e}_3)^2$ or $P\Phi((x\mathbf{x}_2 + \dot{\mathbf{x}}_3)^2)P^* = 0$. In both cases the matrix

$$\begin{aligned} P\Phi((x_1\bar{x}_3\mathbf{x}_1 + x\mathbf{x}_2 + \dot{\mathbf{x}}_3)^2)P^* &= \\ &= P(\Phi((x_1\bar{x}_3\mathbf{x}_1) \bullet (x\mathbf{x}_2)) + \Phi(\dot{\mathbf{x}}_3^2) + \Phi((x_1\mathbf{x}_1 + x_3\dot{\mathbf{x}}_3)^2) + \Phi((x\mathbf{x}_2 + \dot{\mathbf{x}}_3)^2))P^* \\ &= (\sigma(x_1\bar{x}_3)\mathbf{e}_1) \bullet (\sigma(x)\mathbf{e}_2) + \mathbf{e}_3^2 + P\Phi((x\mathbf{x}_2 + \dot{\mathbf{x}}_3)^2)P^* \end{aligned}$$

has rank ≥ 2 , which is a contradiction. Hence, $\Phi((x_1\mathbf{x}_1 + x_3\dot{\mathbf{x}}_3)^2) \neq 0$. In the same way we show that $\Phi((x_2\mathbf{x}_2 + x_3\dot{\mathbf{x}}_3)^2) \neq 0$ whenever $x_2 \neq 0 \neq x_3$. Lastly, let all

x_1, x_2, x_3 be nonzero. Then, $\Phi((x_1\mathbf{x}_1 + x_2\mathbf{x}_2 + x_3\dot{\mathbf{x}}_3)^2) \neq 0$ since the matrix

$$P\Phi((x_1\mathbf{x}_1 + x_2\mathbf{x}_2 + x_3\dot{\mathbf{x}}_3)^2)P^* = (\sigma(x_1)\mathbf{e}_1) \bullet (\sigma(x_2)\mathbf{e}_2) + \mathbf{e}_3^2 + P\Phi((x_1\bar{x}_3\mathbf{x}_1 + \dot{\mathbf{x}}_3)^2)P^* + P\Phi((x_2\bar{x}_3\mathbf{x}_2 + \dot{\mathbf{x}}_3)^2)P^*$$

has nonzero entry at position (1,2). Consequently, $\Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2, \dot{\mathbf{x}}_3 \rangle^{(2)}} = \Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle^{(2)}}$ preserves rank-one as claimed.

By Lemma 3.3 applied at $r = 3$, there exists a σ' -semilinear map $g' : \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle \rightarrow \mathbb{G}^m$ such that $\Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle^{(2)}} = \mathcal{P}(g')$. By Lemma 3.4, $g'|_{\langle \mathbf{x}_1, \mathbf{x}_2 \rangle} = t \cdot g$ for some nonzero t . Hence, if we σ -semilinearly extend g by $g(\mathbf{x}_3) := \bar{t}g'(\mathbf{x}_3)$ then $\Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle^{(2)}} = \mathcal{P}(g)$.

Case 2. Assume now that $\text{Im } \Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle^{(2)}} = \text{Im } \Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2 \rangle^{(2)}}$. We claim that in this case there exists $\ddot{\mathbf{x}}_3 \in \{\mathbf{x}_3, \mathbf{x}_1 + \mathbf{x}_3, \iota\mathbf{x}_1 + \mathbf{x}_3, \bar{\iota}\mathbf{x}_1 + \mathbf{x}_3\}$ such that $\Phi(\ddot{\mathbf{x}}_3^2) \in \{g(\mathbf{x}_1)^2, g(\mathbf{x}_2)^2\}$. Otherwise $\Phi(\ddot{\mathbf{x}}_3^2) \in \{0, (g(\mathbf{x}_1) + g(\mathbf{x}_2))^2, (\iota g(\mathbf{x}_1) + g(\mathbf{x}_2))^2, (\bar{\iota}g(\mathbf{x}_1) + g(\mathbf{x}_2))^2\}$ for all such $\ddot{\mathbf{x}}_3$, so if an invertible matrix P satisfies $Pg(\mathbf{x}_1) = \mathbf{e}_1$ and $Pg(\mathbf{x}_2) = \mathbf{e}_2$ then

$$(3.3) \quad 0 = P\Phi(0)P^* = P\Phi(\mathbf{x}_1^2 + \mathbf{x}_2^2 + (\mathbf{x}_1 + \mathbf{x}_3)^2 + (\iota\mathbf{x}_1 + \mathbf{x}_3)^2 + (\bar{\iota}\mathbf{x}_1 + \mathbf{x}_3)^2)P^*$$

$$(3.4) \quad = E_{11} + A + B + C + D,$$

where $A, B, C, D \in \{0\} \cup \{E_{11} + xE_{12} + \bar{x}E_{21} + E_{22} \mid x = 1, \iota, \bar{\iota}\}$. However, this is not possible, since the matrix in (3.4) has a nonzero diagonal for any choice of matrices A, B, C, D . Hence, $\Phi(\ddot{\mathbf{x}}_3^2) \in \{g(\mathbf{x}_1)^2, g(\mathbf{x}_2)^2\}$ for some $\ddot{\mathbf{x}}_3$. Moreover, we may assume that

$$(3.5) \quad \Phi(\ddot{\mathbf{x}}_3^2) = g(\mathbf{x}_1)^2$$

(otherwise permute the indices 1 and 2 from the beginning). Now, if in (3.3) we replace \mathbf{x}_3 by $\ddot{\mathbf{x}}_3$ then we deduce that $\Phi((\mathbf{x}_1 + \ddot{\mathbf{x}}_3)^2) + \Phi((\iota\mathbf{x}_1 + \ddot{\mathbf{x}}_3)^2) + \Phi((\bar{\iota}\mathbf{x}_1 + \ddot{\mathbf{x}}_3)^2) = 0$. Since all three matrices are in $\text{Im } \Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle^{(2)}} = \langle g(\mathbf{x}_1), g(\mathbf{x}_2) \rangle^{(2)}$ and are of rank ≤ 1 , this is possible only if there exists $x \in \{1, \iota, \bar{\iota}\}$ such that

$$(3.6) \quad \Phi((x\mathbf{x}_1 + \ddot{\mathbf{x}}_3)^2) = 0 \quad \text{and} \quad \Phi((\iota x\mathbf{x}_1 + \ddot{\mathbf{x}}_3)^2) = \Phi((\bar{\iota}x\mathbf{x}_1 + \ddot{\mathbf{x}}_3)^2).$$

Since $\text{rk } \Phi(\mathbf{x}_2^2 + \ddot{\mathbf{x}}_3^2) = 2$, Lemma 3.2 implies that the set $\{\Phi((\mathbf{x}_2 + \ddot{\mathbf{x}}_3)^2), \Phi((\iota\mathbf{x}_2 + \ddot{\mathbf{x}}_3)^2), \Phi((\bar{\iota}\mathbf{x}_2 + \ddot{\mathbf{x}}_3)^2)\}$ equals $\{(g(\mathbf{x}_1) + g(\mathbf{x}_2))^2, (\iota g(\mathbf{x}_1) + g(\mathbf{x}_2))^2, (\bar{\iota}g(\mathbf{x}_1) + g(\mathbf{x}_2))^2\}$ or $\{0, g(\mathbf{x}_1)^2, g(\mathbf{x}_2)^2\}$. Actually, the last is not an option, since for $y \neq 0$, the matrix

$$(3.7) \quad \begin{aligned} &\Phi((\mathbf{x}_1 + y\mathbf{x}_2 + \bar{x}\ddot{\mathbf{x}}_3)^2) = \\ &= \Phi(\mathbf{x}_1^2 + \mathbf{x}_2^2 + \ddot{\mathbf{x}}_3^2 + (\bar{y}\mathbf{x}_1 + \mathbf{x}_2)^2 + (x\mathbf{x}_1 + \ddot{\mathbf{x}}_3)^2 + (yx\mathbf{x}_2 + \ddot{\mathbf{x}}_3)^2) \\ &= \Phi(\mathbf{x}_1^2) + \Phi(\mathbf{x}_2^2) + \Phi(\ddot{\mathbf{x}}_3^2) + \Phi((\bar{y}\mathbf{x}_1 + \mathbf{x}_2)^2) + \Phi((x\mathbf{x}_1 + \ddot{\mathbf{x}}_3)^2) + \Phi((yx\mathbf{x}_2 + \ddot{\mathbf{x}}_3)^2) \\ &= g(\mathbf{x}_1)^2 + g(\mathbf{x}_2)^2 + g(\mathbf{x}_1)^2 + (\sigma(\bar{y})g(\mathbf{x}_1) + g(\mathbf{x}_2))^2 + 0 + \Phi((yx\mathbf{x}_2 + \ddot{\mathbf{x}}_3)^2) \\ &= g(\mathbf{x}_2)^2 + (\sigma(\bar{y})g(\mathbf{x}_1) + g(\mathbf{x}_2))^2 + \Phi((yx\mathbf{x}_2 + \ddot{\mathbf{x}}_3)^2) \end{aligned}$$

is of rank ≤ 1 and consequently $\Phi((yx\mathbf{x}_2 + \check{\mathbf{x}}_3)^2) \neq 0$. So, $\Phi((yx\mathbf{x}_2 + \check{\mathbf{x}}_3)^2) \in \{(g(\mathbf{x}_1) + g(\mathbf{x}_2))^2, (\iota g(\mathbf{x}_1) + g(\mathbf{x}_2))^2, (\bar{\iota}g(\mathbf{x}_1) + g(\mathbf{x}_2))^2\}$ for all nonzero y . Actually,

$$(3.8) \quad \Phi((yx\mathbf{x}_2 + \check{\mathbf{x}}_3)^2) = (\sigma(\bar{y})g(\mathbf{x}_1) + g(\mathbf{x}_2))^2$$

is the only possibility for the matrix in (3.7) to be of rank ≤ 1 . Consequently, if in (3.7) we replace x by ιx and $\bar{\iota}x$ respectively then we deduce that $\text{rk}(g(\mathbf{x}_2)^2 + (\sigma(\bar{y}z)g(\mathbf{x}_1) + g(\mathbf{x}_2))^2 + M) \leq 1$ for $z = \iota, \bar{\iota}$, where $M := \Phi((\iota x\mathbf{x}_1 + \check{\mathbf{x}}_3)^2) = \Phi((\bar{\iota}x\mathbf{x}_1 + \check{\mathbf{x}}_3)^2)$. The only such matrix M of rank ≤ 1 in $\text{Im } \Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle^{(2)}} = \langle g(\mathbf{x}_1), g(\mathbf{x}_2) \rangle^{(2)}$ is $g(\mathbf{x}_1)^2$. Therefore,

$$(3.9) \quad \Phi((\iota x\mathbf{x}_1 + \check{\mathbf{x}}_3)^2) = g(\mathbf{x}_1)^2 = \Phi((\bar{\iota}x\mathbf{x}_1 + \check{\mathbf{x}}_3)^2)$$

holds for x which is defined in (3.6). Now, we σ -semilinearly extend g on $\langle \mathbf{x}_1, \mathbf{x}_2, \check{\mathbf{x}}_3 \rangle = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$ by $g(\check{\mathbf{x}}_3) := \sigma(x)g(\mathbf{x}_1)$. Since matrices $(yx\mathbf{x}_1 + \check{\mathbf{x}}_3)^2$ and $(yx\mathbf{x}_2 + \check{\mathbf{x}}_3)^2$; ($y \in \mathbb{G}$), together with matrices from $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle^{(2)}$, additively span $\langle \mathbf{x}_1, \mathbf{x}_2, \check{\mathbf{x}}_3 \rangle^{(2)} = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle^{(2)}$, we infer from (3.5), (3.6), (3.8), and (3.9) that $\Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle^{(2)}} = \mathcal{P}(g)$. \square

3.2. Nonstandard maps. We proceed with lemmas related to nonstandard maps. Recall that given a matrix $H \in \mathcal{H}_n(\mathbb{G})$ its *hermitian variety* is defined by $V_H = \{\langle \mathbf{x} \rangle \mid \mathbf{x}^{\text{tr}} H \bar{\mathbf{x}} = 0, \mathbf{x} \neq 0\}$. Here, $\langle \mathbf{x} \rangle$ is the 1-dimensional space generated by the vector \mathbf{x} , i.e., a point in the projective space $PG(n-1, 2^2) := \{\langle \mathbf{x} \rangle \mid \mathbf{x} \in \mathbb{G}^n \setminus \{0\}\}$.

The next lemma follows immediately from Bose and Chakravarti [4, Theorem 8.1 and its Corollary]. We give an alternative proof for the sake of completeness.

LEMMA 3.6. *The number of points on variety V_H in $PG(n-1, 2^2)$ equals*

$$|V_H| = \frac{2^{2n-1} - 1 + (-1)^{\text{rk } H} \cdot 2^{2n-\text{rk } H-1}}{3}.$$

Proof. Choose an invertible $Q \in \mathcal{M}_n(\mathbb{G})$ such that $H = Q^{\text{tr}} D \bar{Q}$, where $D = \sum_{j=1}^r E_{jj}$ and $r = \text{rk } H$. Then the map $V_H \rightarrow V_D$ defined by $\langle \mathbf{x} \rangle \mapsto \langle Q\mathbf{x} \rangle$ is bijective, i.e., $|V_H| = |V_D|$. Since $x\bar{x}$ equals 1 for $x \neq 0$ and 0 otherwise, the number of $\mathbf{x} = (x_1, \dots, x_n)^{\text{tr}} \in \mathbb{G}^n$ with $0 = \mathbf{x}^{\text{tr}} D \bar{\mathbf{x}} = x_1 \bar{x}_1 + \dots + x_r \bar{x}_r$ equals $4^{n-r} \cdot \sum_{j=0}^t \binom{r}{2j} 3^{2j}$ where $2t \leq r < 2t+2$. Note that $\sum_{j=0}^t \binom{r}{2j} 3^{2j} = \sum_{k=0}^r \binom{r}{k} 3^k \frac{1+(-1)^k}{2} = \frac{1}{2}((3+1)^r + (-3+1)^r) = 2^{2r-1} + (-1)^r \cdot 2^{r-1}$. Since $|\mathbb{G} \setminus \{0\}| = 3$ and, by the definition of the variety, $\langle 0 \rangle \notin V_D$, it follows that

$$|V_H| = |V_D| = \frac{4^{n-r} \cdot (2^{2r-1} + (-1)^r \cdot 2^{r-1}) - 1}{3} = \frac{2^{2n-1} - 1 + (-1)^r \cdot 2^{2n-r-1}}{3}. \quad \square$$

The lemma below will be crucial to classify nonstandard maps.

LEMMA 3.7. *Suppose matrices $H, G \in \mathcal{H}_n(\mathbb{G})$ are both nonzero. If the union $V_H \cup V_G$ contains all points of $PG(n-1, 2^2)$, i.e., $\mathbf{x}^{\text{tr}} H \bar{\mathbf{x}} \mathbf{x}^{\text{tr}} G \bar{\mathbf{x}} = 0$ for all vectors $\mathbf{x} \in \mathbb{G}^n$ then there exist an invertible matrix Q such that $H = Q(E_{11} + E_{22})Q^*$ and $G = Q(E_{12} + E_{21})Q^*$.*

Proof. Let \mathbf{x} be an arbitrary vector. Then,

$$(\mathbf{x}^{\text{tr}}(H + G)\bar{\mathbf{x}})^2 = (\mathbf{x}^{\text{tr}} H \bar{\mathbf{x}})^2 + (\mathbf{x}^{\text{tr}} G \bar{\mathbf{x}})^2.$$

Hence, $V_H \cap V_G \subseteq V_{H+G}$. Now, if $\langle \mathbf{x} \rangle \in V_{H+G}$ then, since $\text{char } \mathbb{G} = 2$, $\mathbf{x}^{\text{tr}} H \bar{\mathbf{x}} = \mathbf{x}^{\text{tr}} G \bar{\mathbf{x}}$. Consequently, $(\mathbf{x}^{\text{tr}} H \bar{\mathbf{x}})^2 = (\mathbf{x}^{\text{tr}} G \bar{\mathbf{x}})^2 = \mathbf{x}^{\text{tr}} H \bar{\mathbf{x}} \mathbf{x}^{\text{tr}} G \bar{\mathbf{x}} = 0$, i.e., $\langle \mathbf{x} \rangle \in V_H \cap V_G$. Therefore $V_H \cap V_G = V_{H+G}$ which further implies

$$(3.10) \quad |PG(n-1, 2^2)| + |V_{H+G}| = |V_H \cup V_G| + |V_H \cap V_G| = |V_H| + |V_G|.$$

Here, $|PG(n-1, 2^2)|$ denotes the number of all 1-dimensional subspaces in \mathbb{G}^n . Hence, $|PG(n-1, 2^2)| = (|\mathbb{G}|^n - 1) / (|\mathbb{G}| - 1) = (2^{2n} - 1) / 3$. We obtain the numbers $|V_H|$, $|V_G|$, and $|V_{H+G}|$ from Lemma 3.6. Then, routine calculations derive from (3.10) the equation

$$(3.11) \quad 1 + \frac{(-1)^{\text{rk}(H+G)}}{2^{\text{rk}(H+G)}} = \frac{(-1)^{\text{rk } H}}{2^{\text{rk } H}} + \frac{(-1)^{\text{rk } G}}{2^{\text{rk } G}}.$$

Since $\text{rk } H, \text{rk } G \geq 1$ by the assumption, the right side of (3.11) is ≤ 1 . Hence, $\text{rk}(H + G)$ is odd and in particular ≥ 1 . By rearranging equation (3.11) such that the second summand on the left side goes to the right side and the first or the second summand on the right side goes to the left side, we deduce that $\text{rk } H$ and $\text{rk } G$ are even and in particular ≥ 2 . Consequently, $1 = 1/2^{\text{rk } H} + 1/2^{\text{rk } G} + 1/2^{\text{rk}(H+G)}$ and this equation is satisfied precisely when the rank-values are minimal, i.e., $\text{rk } H = 2 = \text{rk } G$ and $\text{rk}(H + G) = 1$. Therefore, there exists an invertible matrix Q_1 such that $H = Q_1(E_{11} + E_{22})Q_1^*$. Write $G = Q_1(M_1 + M_2)Q_1^*$ where $\text{rk } M_1 = 1 = \text{rk } M_2$ and $\text{rk}(M_1 + M_2) = 2$. Assume erroneously that $M_1 \notin \mathcal{H}_2(\mathbb{G}) \oplus 0_{n-2}$. Then, by Lemma 3.1, $\text{rk}(E_{11} + E_{22} + M_1) = 3$. Consequently, $1 = \text{rk}(H + G) = \text{rk}(E_{11} + E_{22} + M_1 + M_2) \geq \text{rk}(E_{11} + E_{22} + M_1) - \text{rk}(M_2) = 2$, a contradiction. Hence, $M_1 \in \mathcal{H}_2(\mathbb{G}) \oplus 0_{n-2}$. The same argument implies that $M_2 \in \mathcal{H}_2(\mathbb{G}) \oplus 0_{n-2}$. Since $\text{rk}(E_{11} + E_{22} + (M_1 + M_2)) = 1$ and $\text{rk}(M_1 + M_2) = 2$ we easily check that $M_1 + M_2 = xE_{12} + \bar{x}E_{21}$ for some $x \in \{1, \iota, \bar{\iota}\}$. We now replace Q_1 by $Q := Q_1 \text{diag}(1, \bar{x}, 1, \dots, 1)$. Since $x\bar{x} = 1$, it follows that $H = Q(E_{11} + E_{22})Q^*$ and $G = Q(E_{12} + E_{21})Q^*$. \square

REMARK 3.8. If $(\mathbb{K}, \hat{\cdot})$ is any field with nonidentical involution, but distinct from \mathbb{G} , then the identity $\mathbf{x}^{\text{tr}} H \hat{\mathbf{x}} \mathbf{x}^{\text{tr}} G \hat{\mathbf{x}} = 0$ implies that at least one of hermitian matrices $H = \hat{H}^{\text{tr}}$ or $G = \hat{G}^{\text{tr}}$ must be zero. To see this we define an additive map on $\mathcal{H}_n(\mathbb{K})$, using the notations from Theorem 2.1, by $\Phi : X \mapsto \mathfrak{s}(H \circ X)E_{11} + \mathfrak{s}(G \circ X)E_{22}$. It is rank-one nonincreasing since $\mathfrak{s}(H \circ \alpha \mathbf{x} \hat{\mathbf{x}}^{\text{tr}}) \cdot \mathfrak{s}(G \circ \alpha \mathbf{x} \hat{\mathbf{x}}^{\text{tr}}) = \alpha^2 \cdot \mathbf{x}^{\text{tr}} H \hat{\mathbf{x}} \cdot \mathbf{x}^{\text{tr}} G \hat{\mathbf{x}} = 0$.

However, if H and G are both nonzero then Φ does not fit the forms obtained in [28, Main Theorem], a contradiction.

The next lemma classifies additive functionals on $\mathcal{H}_n(\mathbb{G})$.

LEMMA 3.9. *Let $f : \mathcal{H}_n(\mathbb{G}) \rightarrow \mathbb{Z}_2$ be an additive map. Then there exists $H \in \mathcal{H}_n(\mathbb{G})$ such that $f(X) = \mathfrak{s}(H \circ X)$ for all $X \in \mathcal{H}_n(\mathbb{G})$.*

Proof. It is easy to see that any additive map $c : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ is of the form $c(x) = hx$ for some $h \in \mathbb{Z}_2$. Similarly, any additive map $c : \mathbb{G} \rightarrow \mathbb{Z}_2$ is of the form $c(x) = hx + \overline{hx}$ for some $h \in \mathbb{G}$. Hence, there exist $h_{jj} \in \mathbb{Z}_2$ and $h_{jk} \in \mathbb{G}$ such that $f(X) = \sum_{j=1}^n h_{jj}[X]_{jj} + \sum_{j < k} h_{jk}[X]_{jk} + \overline{h_{jk}[X]_{jk}}$, i.e., $f(X) = \mathfrak{s}(H \circ X)$, where H is defined by $[H]_{jk} := h_{jk}$ for $j \leq k$ and $[H]_{jk} := \overline{h_{kj}}$ for $j \geq k$. \square

3.3. Proof of the main result. We are now ready to prove Theorem 2.1.

Proof. First we prove the “if” part. It is obvious that the maps of the forms (i)–(ii) do not increase rank-one. If Φ is of the form (iii) then, for any rank-one hermitian matrix \mathbf{x}^2 , the upper-left 2×2 minor of $P^{-1}\Phi(\mathbf{x}^2)(P^{-1})^*$ equals

$$([Q\mathbf{x}^2Q^*]_{11} + [Q\mathbf{x}^2Q^*]_{22})([Q\mathbf{x}^2Q^*]_{12} + [Q\mathbf{x}^2Q^*]_{21}) = (z_1\bar{z}_1 + z_2\bar{z}_2)(z_1\bar{z}_2 + \bar{z}_1z_2),$$

where $(Q\mathbf{x})^{\text{tr}} = (z_1, \dots, z_n)$. If z_1 and z_2 are both nonzero then $z_1\bar{z}_1 + z_2\bar{z}_2 = 1+1 = 0$. Otherwise, $z_1\bar{z}_2 + \bar{z}_1z_2 = 0$. Consequently, $\text{rk } \Phi(\mathbf{x}^2) \leq 1$.

To prove the “only if” part let Φ be an additive rank-one nonincreasing map. We separate two cases. The maps from the first case turn out to be nonstandard, while those from the second case turn out to be standard.

Case 1. Suppose that for any triple of rank-one matrices M_1, M_2, M_3 their images $\Phi(M_1), \Phi(M_2), \Phi(M_3)$ are linearly dependent over \mathbb{Z}_2 .

Then, since rank-one matrices additively span $\mathcal{H}_n(\mathbb{G})$, we can easily see that $\text{Im } \Phi = \{0, \Phi(M_1), \Phi(M_2), \Phi(M_1) + \Phi(M_2)\}$ for some M_1 and M_2 . If matrices $\Phi(M_1)$ and $\Phi(M_2)$ are equal, or some of them is zero, then obviously $\Phi(X) = f(X)M$ for some additive functional f and rank-one matrix M . Consequently, we deduce by Lemma 3.9 that Φ is of the form (ii). If $\Phi(M_1)$ and $\Phi(M_2)$ are nonzero and distinct then $\text{rk } (\Phi(M_1) + \Phi(M_2)) = 2$, so there exists an invertible P such that $\Phi(M_i) = PE_{ii}P^*$ ($i = 1, 2$). Then, $\Phi(X) = P(h(X)E_{11} + g(X)E_{22})P^*$ for some nonzero additive functionals $h, g : \mathcal{H}_n(\mathbb{G}) \rightarrow \mathbb{Z}_2$. By Lemma 3.9, $h(X) = \mathfrak{s}(H \circ X)$ and $g(X) = \mathfrak{s}(G \circ X)$ for some nonzero hermitian matrices H and G . Since $\text{rk } \Phi(\mathbf{x}^2) \leq 1$ for all \mathbf{x} , it follows that $\mathfrak{s}(H \circ \mathbf{x}^2) \cdot \mathfrak{s}(G \circ \mathbf{x}^2) = 0$. This equation can be rewritten as $\mathbf{x}^{\text{tr}}H\bar{\mathbf{x}}\mathbf{x}^{\text{tr}}G\bar{\mathbf{x}} = 0$. Hence, we can invoke Lemma 3.7 to deduce that $H = Q(E_{11} + E_{22})Q^*$ and $G = Q(E_{12} + E_{21})Q^*$ for some invertible matrix Q . Note that, for arbitrary matrices X, Y, A, B of appropriate dimensions, the scalars $\mathfrak{s}((AYB) \circ X)$ and $\mathfrak{s}((A^{\text{tr}}XB^{\text{tr}}) \circ Y)$ equal. Con-

sequently, $h(X) = \mathfrak{s}(H \circ X) = [Q^{\text{tr}}X\overline{Q}]_{11} + [Q^{\text{tr}}X\overline{Q}]_{22}$ and $g(X) = \mathfrak{s}(G \circ X) = [Q^{\text{tr}}X\overline{Q}]_{12} + [Q^{\text{tr}}X\overline{Q}]_{21}$, that is, Φ is of the form (iii).

Case 2. Suppose lastly that there exist three rank-one matrices M_1, M_2, M_3 such that $\Phi(M_1), \Phi(M_2), \Phi(M_3)$ are linearly independent over \mathbb{Z}_2 .

Assume erroneously that any such triple satisfies

$$(3.12) \quad \text{rk}(M_1 + M_2 + M_3) = 3.$$

Denote one of these triples by $\mathbf{x}_1^2, \mathbf{x}_2^2, \mathbf{x}_3^2$, that is, $\text{rk}(\mathbf{x}_1^2 + \mathbf{x}_2^2 + \mathbf{x}_3^2) = 3$, while matrices $\Phi(\mathbf{x}_j^2) =: \mathbf{y}_j^2$ ($j = 1, 2, 3$) are linearly independent over \mathbb{Z}_2 . Note that linear independence implies

$$(3.13) \quad \text{rk}(\Phi(\mathbf{x}_j^2) + \Phi(\mathbf{x}_k^2)) = 2 \quad (1 \leq j < k \leq 3)$$

as well as

$$(3.14) \quad \text{rk}(\Phi(\mathbf{x}_1^2) + \Phi(\mathbf{x}_2^2) + \Phi(\mathbf{x}_3^2)) \geq 2.$$

To clarify the last inequality consider first the case when $\mathbf{y}_3^2 \notin \langle \mathbf{y}_1, \mathbf{y}_2 \rangle^{(2)}$. Then the matrix in (3.14) is of rank three by Lemma 3.1. Otherwise $\mathbf{y}_3^2 \in \langle \mathbf{y}_1, \mathbf{y}_2 \rangle^{(2)}$ and (3.13) implies that $\mathbf{y}_3^2 \notin \{0, \mathbf{y}_1^2, \mathbf{y}_2^2\}$. Consequently, $\mathbf{y}_3^2 \in \{(\mathbf{y}_1 + \mathbf{y}_2)^2, (\imath\mathbf{y}_1 + \mathbf{y}_2)^2, (\overline{\imath}\mathbf{y}_1 + \mathbf{y}_2)^2\}$, so the matrix in (3.14) is of rank two. Now, equation (3.13) together with Lemma 3.2 implies that the set $\{\Phi((\mathbf{x}_j + \mathbf{x}_k)^2), \Phi((\imath\mathbf{x}_j + \mathbf{x}_k)^2), \Phi((\overline{\imath}\mathbf{x}_j + \mathbf{x}_k)^2)\}$ equals $\{(\mathbf{y}_j + \mathbf{y}_k)^2, (\imath\mathbf{y}_j + \mathbf{y}_k)^2, (\overline{\imath}\mathbf{y}_j + \mathbf{y}_k)^2\}$ or $\{0, \mathbf{y}_j^2, \mathbf{y}_k^2\}$ for all $1 \leq j < k \leq 3$. However, the first is not possible by the assumption (3.12), since matrices $(\mathbf{y}_j + \mathbf{y}_k)^2, (\imath\mathbf{y}_j + \mathbf{y}_k)^2, (\overline{\imath}\mathbf{y}_j + \mathbf{y}_k)^2$ are linearly independent, while $\text{rk}((\mathbf{x}_j + \mathbf{x}_k)^2 + (\imath\mathbf{x}_j + \mathbf{x}_k)^2 + (\overline{\imath}\mathbf{x}_j + \mathbf{x}_k)^2) = 2$. Hence, as a set, $\{\Phi((\mathbf{x}_j + \mathbf{x}_k)^2), \Phi((\imath\mathbf{x}_j + \mathbf{x}_k)^2), \Phi((\overline{\imath}\mathbf{x}_j + \mathbf{x}_k)^2)\} = \{0, \mathbf{y}_j^2, \mathbf{y}_k^2\}$. Now, choose nonzero $y, z \in \mathbb{G}$ such that $\Phi((y\mathbf{x}_1 + \mathbf{x}_3)^2) = 0 = \Phi((z\mathbf{x}_2 + \mathbf{x}_3)^2)$. The matrix

$$\begin{aligned} \Phi((y\mathbf{x}_1 + z\mathbf{x}_2 + \mathbf{x}_3)^2) &= \\ &= \Phi(\mathbf{x}_1^2 + \mathbf{x}_2^2 + \mathbf{x}_3^2 + (y\mathbf{x}_1 + \mathbf{x}_3)^2 + (z\mathbf{x}_2 + \mathbf{x}_3)^2 + (y\overline{z}\mathbf{x}_1 + \mathbf{x}_2)^2) \\ &= \Phi(\mathbf{x}_1^2) + \Phi(\mathbf{x}_2^2) + \Phi(\mathbf{x}_3^2) + \Phi((y\overline{z}\mathbf{x}_1 + \mathbf{x}_2)^2) \end{aligned}$$

should be of rank ≤ 1 , however $\Phi((y\overline{z}\mathbf{x}_1 + \mathbf{x}_2)^2) \in \{0, \Phi(\mathbf{x}_1^2), \Phi(\mathbf{x}_2^2)\}$, so the above matrix is of rank ≥ 2 by (3.13) and (3.14). This is a contradiction, so the assumption (3.12) was wrong.

Hence, there exists a triple M_1, M_2, M_3 such that $\Phi(M_1), \Phi(M_2), \Phi(M_3)$ are linearly independent over \mathbb{Z}_2 and $\text{rk}(M_1 + M_2 + M_3) = 2$. Note that linear independence implies that $\text{rk}\Phi(M_1 + M_2) = 2 = \text{rk}(M_1 + M_2)$. Similarly as above denote $M_j =: \mathbf{x}_j^2$ and $\Phi(M_j) =: \mathbf{y}_j^2$. Then we infer from Lemma 3.1 that $M_3 = \mathbf{x}_3^2 \in \langle \mathbf{x}_1, \mathbf{x}_2 \rangle^{(2)}$, i.e., $\mathbf{x}_3^2 \in \{(\mathbf{x}_1 + \mathbf{x}_2)^2, (\imath\mathbf{x}_1 + \mathbf{x}_2)^2, (\overline{\imath}\mathbf{x}_1 + \mathbf{x}_2)^2\}$. Since $\Phi(M_1) = \Phi(\mathbf{x}_1^2) = \mathbf{y}_1^2$, $\Phi(M_2) = \Phi(\mathbf{x}_2^2) = \mathbf{y}_2^2$, and $\Phi(M_3) = \Phi(\mathbf{x}_3^2)$ are linearly independent over \mathbb{Z}_2 , we deduce from Lemma 3.2 that the set $\{\Phi((\mathbf{x}_1 + \mathbf{x}_2)^2), \Phi((\imath\mathbf{x}_1 + \mathbf{x}_2)^2), \Phi((\overline{\imath}\mathbf{x}_1 + \mathbf{x}_2)^2)\} \ni \Phi(\mathbf{x}_3^2)$ equals

$\{(\mathbf{y}_1 + \mathbf{y}_2)^2, (i\mathbf{y}_1 + \mathbf{y}_2)^2, (\bar{i}\mathbf{y}_1 + \mathbf{y}_2)^2\}$. Therefore, $\Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2 \rangle^{(2)}}$ preserves rank-one, so Lemma 3.3, applied at $r = 2$, implies that $\Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2 \rangle^{(2)}} = \mathcal{P}(g)$ for some σ -semilinear map $g : \langle \mathbf{x}_1, \mathbf{x}_2 \rangle \rightarrow \mathbb{G}^m$. Now, extend vectors $\mathbf{x}_1, \mathbf{x}_2$ to a basis $\mathbf{x}_1, \mathbf{x}_2, \check{\mathbf{x}}_3, \dots, \check{\mathbf{x}}_n$ of \mathbb{G}^n . By Lemma 3.5, we can σ -semilinearly extend g on whole \mathbb{G}^n such that $\Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2, \check{\mathbf{x}}_j \rangle^{(2)}} = \mathcal{P}(g); j = 3, \dots, n$. Note that $\Phi(\check{\mathbf{x}}_j^2) \neq 0$ for at least one vector $\check{\mathbf{x}}_j \in \{\check{\mathbf{x}}_j, \mathbf{x}_1 + \check{\mathbf{x}}_j, \mathbf{x}_2 + \check{\mathbf{x}}_j, \mathbf{x}_1 + \mathbf{x}_2 + \check{\mathbf{x}}_j\}$. Otherwise we would deduce that $0 = \Phi(\mathbf{x}_1 \bullet \mathbf{x}_2) = g(\mathbf{x}_1) \bullet g(\mathbf{x}_2)$, which is a contradiction, since $g(\mathbf{x}_1)$ and $g(\mathbf{x}_2)$ are linearly independent. Clearly, $\mathbf{x}_1, \mathbf{x}_2, \check{\mathbf{x}}_3, \dots, \check{\mathbf{x}}_n$ is still a basis and $\langle \mathbf{x}_1, \mathbf{x}_2, \check{\mathbf{x}}_j \rangle = \langle \mathbf{x}_1, \mathbf{x}_2, \check{\mathbf{x}}_j \rangle$, i.e., $\Phi|_{\langle \mathbf{x}_1, \mathbf{x}_2, \check{\mathbf{x}}_j \rangle^{(2)}} = \mathcal{P}(g)$ holds for all $j = 3, \dots, n$. To complete the proof it only remains to see that $\Phi|_{\langle \check{\mathbf{x}}_j, \check{\mathbf{x}}_k \rangle^{(2)}} = \mathcal{P}(g)|_{\langle \check{\mathbf{x}}_j, \check{\mathbf{x}}_k \rangle^{(2)}}$, for then, $\Phi = \mathcal{P}(g)$, and therefore, $\Phi(X) = TX\sigma T^*$ for a matrix T , defined by $T\mathbf{e}_j := g(\mathbf{e}_j)$, as anticipated.

Now, fix $j, k \geq 3$ and choose $\mathbf{x} \in \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_1 + \mathbf{x}_2\}$ such that $g(\mathbf{x}), g(\check{\mathbf{x}}_j)$ as well as $g(\mathbf{x}), g(\check{\mathbf{x}}_k)$ are linearly independent. Then, $\text{rk } \Phi(\mathbf{x}^2 + \check{\mathbf{x}}_j^2) = 2$ so Lemma 3.5 applied on vectors $\mathbf{x}, \check{\mathbf{x}}_j, \check{\mathbf{x}}_k$ gives $\Phi|_{\langle \mathbf{x}, \check{\mathbf{x}}_j, \check{\mathbf{x}}_k \rangle^{(2)}} = \mathcal{P}(g')$, where σ -semilinear map g' is defined on $\langle \mathbf{x}, \check{\mathbf{x}}_j, \check{\mathbf{x}}_k \rangle$ and extends $g|_{\langle \mathbf{x}, \check{\mathbf{x}}_j \rangle}$. Since $\text{rk } \Phi(\mathbf{x}^2 + \check{\mathbf{x}}_k^2) = 2$, Lemma 3.4 implies that $g'|_{\langle \mathbf{x}, \check{\mathbf{x}}_k \rangle} = t \cdot g|_{\langle \mathbf{x}, \check{\mathbf{x}}_k \rangle}$. Obviously, $t = 1$. Hence, $g' = g|_{\langle \mathbf{x}, \check{\mathbf{x}}_j, \check{\mathbf{x}}_k \rangle}$, and so $\Phi|_{\langle \mathbf{x}, \check{\mathbf{x}}_j, \check{\mathbf{x}}_k \rangle^{(2)}} = \mathcal{P}(g') = \mathcal{P}(g)|_{\langle \mathbf{x}, \check{\mathbf{x}}_j, \check{\mathbf{x}}_k \rangle^{(2)}}$, as anticipated. \square

4. Applications. Here we recall some known applications of rank-one nonincreasing maps and extend them to arbitrary field. We also consider the preservers of hermitian varieties.

Let \mathbb{K} be a field and $\hat{} : \mathbb{K} \rightarrow \mathbb{K}$ a nonidentical involution on it, i.e., $\hat{} \neq \text{id}$. Recall that the set $\mathcal{H}_n(\mathbb{K})$ of all hermitian matrices over $(\mathbb{K}, \hat{})$ consists of all those $n \times n$ matrices X with entries from \mathbb{K} that satisfy $X^* := \widehat{X}^{\text{tr}} = X$. The set $\mathcal{H}_n(\mathbb{K})$ is a vector space over the field $\mathbb{F} := \{x \in \mathbb{K} | \hat{x} = x\}$, which is the fixed field of the involution. A map $\Phi : \mathcal{H}_n(\mathbb{K}) \rightarrow \mathcal{H}_n(\mathbb{K})$ preserves rank-additivity if $\text{rk}(\Phi(X) + \Phi(Y)) = \text{rk } \Phi(X) + \text{rk } \Phi(Y)$ whenever $\text{rk}(X + Y) = \text{rk } X + \text{rk } Y$.

As far as we know, the importance of linear maps which preserve rank additivity was first emphasized in [2]. Later, linear maps which preserve rank additivity on matrices over general field with sufficiently many elements were classified in [11], with applications to order preserving linear bijections. The main idea was the reduction to singularity preservers. It was since observed in many papers that additive preservers of rank-additivity can also be characterized via the classification of rank-one nonincreasing maps (see e.g. [25, 34, 6, 29, 30, 23]). In the case of hermitian matrices we are now able to give a classification of additive rank-additivity preserving maps for any field \mathbb{K} . The main part of the proof for $n \geq 3$ is very similar to the proof of [35, Lemma 6].

THEOREM 4.1. *Let $(\mathbb{K}, \hat{})$ be a field with nonidentical involution and $n \geq 2$ an integer. An additive map $\Phi : \mathcal{H}_n(\mathbb{K}) \rightarrow \mathcal{H}_n(\mathbb{K})$ preserves rank-additivity if and only*

if it is of the form

$$(4.1) \quad \Phi(X) = \xi T X \sigma T^*,$$

where T is invertible, $\xi = \widehat{\xi}$ is a (possibly zero) scalar, and σ is a field homomorphism that commutes with $\widehat{\cdot}$, that is, $\sigma(\widehat{x}) = \widehat{\sigma(x)}$.

Proof. The “if” part is obvious. We prove the “only if” part.

Case 1. Let $n = 2$. We claim that Φ does not increase rank–one. Otherwise, $\text{rk } \Phi(X) = 2$ for some rank–one X . If $Y \notin \mathbb{F}X$ is of rank one then, as Φ preserves rank–additivity, $2 \geq \text{rk } \Phi(X + Y) = \text{rk } \Phi(X) + \text{rk } \Phi(Y) \geq \text{rk } \Phi(X) = 2$. Hence, $\Phi(Y) = 0$. Recall that a rank–one matrix X is of the form $X = \alpha W E_{11} W^* = \alpha W \mathbf{e}_1^2 W^*$ for some nonzero scalar α and an invertible matrix W .

Now, if $\text{char } \mathbb{K} \neq 2$ then $2X = 2\alpha W \mathbf{e}_1^2 W^* = -2\alpha W \mathbf{e}_2^2 W^* + \alpha W (\mathbf{e}_1 + \mathbf{e}_2)^2 W^* + \alpha W (\mathbf{e}_1 - \mathbf{e}_2)^2 W^*$ and we get a contradiction as $0 \neq 2\Phi(X) = \Phi(-2\alpha W \mathbf{e}_2^2 W^*) + \Phi(\alpha W (\mathbf{e}_1 + \mathbf{e}_2)^2 W^*) + \Phi(\alpha W (\mathbf{e}_1 - \mathbf{e}_2)^2 W^*) = 0 + 0 + 0 = 0$. If $\text{char } \mathbb{K} = 2$ then there exists $j \in \mathbb{K}$ such that $\widehat{j} = 1 + j$ (see e.g. [28, Lemma 2.2]). Hence, $X = \alpha W \mathbf{e}_2^2 W^* + \alpha W (\mathbf{e}_1 + \mathbf{e}_2)^2 W^* + \alpha W (\mathbf{e}_1 + j \mathbf{e}_2)^2 W^* + \alpha W (\mathbf{e}_1 + \widehat{j} \mathbf{e}_2)^2 W^*$ and we get a contradiction as before. Hence, $\text{rk } \Phi(X) \leq 1$ for any rank–one X .

If $\Phi \neq 0$ then $\Phi(X) \neq 0$ for some rank–one X and $\text{Im } \Phi$ must contain some matrix of rank > 1 . In fact, the opposite would imply $1 \geq \text{rk } \Phi(X + Y) = \text{rk } \Phi(X) + \text{rk } \Phi(Y) \geq \text{rk } \Phi(X) \geq 1$ for any $Y \notin \mathbb{F}X$ of rank one. So $\Phi(Y) = 0$ and we get a contradiction as above. Consequently, Φ is not of the form (ii) from [28, Main Theorem] or from Theorem 2.1, when $\mathbb{K} = \mathbb{C}$. In the case of this field Φ can not be of the form (iii) from Theorem 2.1 either, since such maps Ψ satisfy $\Psi(Q^{-1} E_{11} (Q^{-1})^*) + \Psi(Q^{-1} E_{22} (Q^{-1})^*) = 0$ and $\text{rk } \Psi(Q^{-1} E_{11} (Q^{-1})^*) = 1 = \Psi(Q^{-1} E_{22} (Q^{-1})^*)$, so they do not preserve rank–additivity. Therefore, Φ is of the form (i) from [28, Main Theorem]/Theorem 2.1, that is, Φ is as in (4.1). Since $\text{Im } \Phi$ contains a matrix of rank two, the matrix T must be invertible.

Case 2. Let $n \geq 3$. Pick any rank–one hermitian matrix X . As before, write it as $X = \alpha W \mathbf{e}_1^2 W^*$ for some invertible W , and define $X_i := \alpha W \mathbf{e}_i^2 W^*$ (then, $X_1 = X$). Clearly, $\text{rk } \sum_{i=1}^n X_i = \sum_{i=1}^n \text{rk } X_i$. Since Φ preserves rank–additivity, a simple induction argument gives

$$\sum_{i=1}^n \text{rk } \Phi(X_i) = \text{rk } \sum_{i=1}^n \Phi(X_i) \leq n.$$

Consequently, if it can be shown that $\text{rk } \Phi(X_i) = \text{rk } \Phi(X_j)$ for each i, j , then the above identity implies $\text{rk } \Phi(X) = \text{rk } \Phi(X_1) \leq 1$, and we could use [28, Main Theorem] and Theorem 2.1 again. The form (ii) of [28, Main Theorem]/Theorem 2.1 and the form (iii) of Theorem 2.1 are excluded as in Case 1. Moreover, if $\text{rk } \Phi(X) = 1$ for at least some X of rank one then $\text{rk } \sum_i \Phi(X_i) = n$. Thus, we can only have (i) of [28, Main Theorem]/Theorem 2.1 with T invertible.

It hence remains to see that $\text{rk } \Phi(X_i) = \text{rk } \Phi(X_j)$. Let i, j, k be all distinct. Denote $A_1 := X_i = \alpha W \mathbf{e}_i^2 W^*$, $B_1 := \alpha W (\mathbf{e}_j^2 + \mathbf{e}_i \bullet \mathbf{e}_j + \mathbf{e}_j \bullet \mathbf{e}_i) W^*$, $A_2 := \alpha W (\mathbf{e}_j \bullet \mathbf{e}_k) W^*$, and $B_2 := \alpha W (\mathbf{e}_i + \mathbf{e}_j)^2 W^*$. Then $A_1 + B_1 = A_2 + B_2$, $\text{rk}(A_1 + B_1) = \text{rk } A_1 + \text{rk } B_1$, and $\text{rk}(A_2 + B_2) = \text{rk } A_2 + \text{rk } B_2$. Hence,

$$(4.2) \quad \text{rk } \Phi(A_1) + \text{rk } \Phi(B_1) = \text{rk } \Phi(A_2) + \text{rk } \Phi(B_2).$$

If $A'_1 := A_1$, $B'_1 := -A_2$, $A'_2 = B_2$, and $B'_2 := -B_1$ then we deduce that

$$(4.3) \quad \text{rk } \Phi(A'_1) + \text{rk } \Phi(B'_1) = \text{rk } \Phi(A'_2) + \text{rk } \Phi(B'_2)$$

in the same way as (4.2). Since $\Phi(-Z) = -\Phi(Z)$ and $\text{rk}(-\Phi(Z)) = \text{rk } \Phi(Z)$, we infer from equations (4.2)–(4.3) that $\text{rk } \Phi(A_1) = \text{rk } \Phi(B_2)$, i.e., $\text{rk } \Phi(X_i) = \text{rk } \alpha W (\mathbf{e}_i + \mathbf{e}_j)^2 W^*$. If we permute the indices i and j then we deduce that $\text{rk } \Phi(X_i) = \text{rk } \Phi(X_j)$. \square

REMARK 4.2. Let $(\mathbb{K}, \hat{})$ be a field with nonidentical involution and let $n \geq 2$. It can be shown that a nonzero additive map $\Phi : \mathcal{H}_n(\mathbb{K}) \rightarrow \mathcal{H}_n(\mathbb{K})$ preserves the Jordan triple product $X \cdot Y := XYX$ (i.e., $\Phi(XYX) = \Phi(X)\Phi(Y)\Phi(X)$) if and only if it is of the form (4.1), where T satisfies $\xi T^* T = \pm I$. Here, I denotes the identity matrix. To see this, follow the arguments in the proof of [34, Theorem 4.1] and use Theorem 4.1.

If \mathbb{K} is a finite field with nonidentical involution $\hat{}$ then the cardinality of \mathbb{K} is a square, i.e., $|\mathbb{K}| = q^2$ and the involution is given by $\hat{x} := x^q$ (see e.g. [7, Proof of Theorem 2]), where q is a power of a prime. Given a matrix $H \in \mathcal{H}_n(\mathbb{K})$ a hermitian variety V_H is defined as in the case $\mathbb{K} = \mathbb{C}$.

THEOREM 4.3. *Let $(\mathbb{K}, \hat{})$ be a finite field with a nonidentical involution and $n \geq 2$. Then the following three assertions are equivalent for an additive map $\Phi : \mathcal{H}_n(\mathbb{K}) \rightarrow \mathcal{H}_n(\mathbb{K})$.*

- (a) $|V_{\Phi(X)}| = |V_X|$ for all X ,
- (b) $|V_{\Phi(X)}| \leq |V_X|$ for all X of rank one,
- (c) $\Phi(X) = TX^\sigma T^*$ for all X .

In (c), T is invertible and σ is a nonzero field homomorphism that commutes with $\hat{}$.

Proof. Clearly, (a) implies (b). To see that (c) implies (a) note that \mathbb{K} is finite, so σ is invertible. Hence, $\langle \mathbf{x} \rangle \mapsto \langle (T^{-1})^{\text{tr}} \mathbf{x}^\sigma \rangle$ is a bijection from V_X to $V_{\Phi(X)}$, that is, $|V_X| = |V_{\Phi(X)}|$ for all X . It remains to prove that (b) implies (c). Let $|\mathbb{K}| = q^2$. It follows immediately from [4, Theorem 8.1 and its Corollary] that $\text{rk } X = r$ implies

$$v(r) := |V_X| = \frac{q^{2n-1} + (-1)^r (q-1) q^{2n-r-1} - 1}{q^2 - 1}$$

(this is a generalization of Lemma 3.6). Note that

$$(4.4) \quad \begin{aligned} v(0) &> v(2) > v(4) > \dots > v\left(n - \frac{1 - (-1)^n}{2}\right) > \\ &> v\left(n - \frac{1 + (-1)^n}{2}\right) > \dots > v(5) > v(3) > v(1). \end{aligned}$$

Now, if $|V_{\Phi(X)}| \leq |V_X|$ for X of rank one then it follows from (4.4) that $\text{rk } \Phi(X) = 1$. Consequently, any Z and W with $\text{rk}(Z - W) = 1$ satisfy $\text{rk}(\Phi(Z) - \Phi(W)) = \text{rk } \Phi(Z - W) = 1$. By [27, Theorem 3.1], Φ is of the form $\Phi(X) = TX^\sigma T^* + Y$, where T and σ are as in (c), while $Y \in \mathcal{H}_n(\mathbb{K})$. However, $0 = \Phi(0) = Y$ since Φ is additive. \square

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