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THE DISTANCE MATRIX OF A BIDIRECTED TREE*

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Abstract. A bidirected tree is a tree in which each edge is replaced by two arcs in either direction. Formulas are obtained for the determinant and the inverse of a bidirected tree, generalizing well-known formulas in the literature.

Key words. Tree, Distance matrix, Laplacian matrix, Determinant, Block matrix.

AMS subject classifications. 05C50, 15A15.

1. Introduction. We refer to [4], [8] for basic definitions and terminology in graph theory. A *tree* is a simple connected graph without any circuit. We consider trees in which each edge is replaced by two arcs in either direction. In this paper, such trees are called *bidirected trees*.

We now introduce some notation. Let $\mathbf{e}, \mathbf{0}$ be the column vectors consisting of all ones and all zeros, respectively, of the appropriate order. Let $J = \mathbf{e}\mathbf{e}^t$ be the matrix of all ones. For a tree T on n vertices, let d_i be the degree of the i-th vertex and let $\mathbf{d} = (d_1, d_2, \ldots, d_n)^t$, $\delta = 2\mathbf{e} - \mathbf{d}$ and $\mathbf{z} = \mathbf{d} - \mathbf{e}$. Note that $\delta + \mathbf{z} = \mathbf{e}$.

Let T be a tree on n vertices. The distance matrix of a tree T is a $n \times n$ matrix D with $D_{ij} = k$, if the path from the vertex i to the vertex j is of length k; and $D_{ii} = 0$. The *Laplacian matrix,* L, of a tree T is defined by $L = \text{diag}(\mathbf{d}) - A$, where A is the adjacency matrix of T.

The distance matrix of a tree is extensively investigated in the literature. The classical result concerns the determinant of the matrix D (see Graham and Pollak [7]), which asserts that if T is any tree on n vertices then $\det(D) = (-1)^{n-1}(n-1)2^{n-2}$. Thus, $\det(D)$ is a function dependent only on n, the number of vertices of the tree. The formula for the inverse of the matrix D was obtained in a subsequent article by Graham and Lovász [6] who showed that $D^{-1} = \dfrac{(\mathbf{e}-\mathbf{z})(\mathbf{e}-\mathbf{z})^t}{2(n-1)} - \dfrac{L}{2}$. This result was

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extended to a weighted tree in [1]. A q-analogue of the distance matrix was considered in [2]. In this paper, we extend the result of Graham and Lovász by considering the distance matrix for a bidirected tree, denoted $\mathcal{D} = (\mathcal{D}_{ij})$.

2. Preliminaries. Let T be a tree on n vertices. Replace each undirected edge $f_i = \{u, v\}$ of T with two arcs (oppositely oriented edges) $e_i = (u, v)$ and $e'_i = (v, u)$. Let $u_i > 0$ and $v_i > 0$ be the weights of the arcs e_i and e'_i, respectively. We call the resulting graph a *bidirected tree* $\mathcal{T}$ with the underlying tree structure T. The distance $\mathcal{D}_{ij}$ from i to j is defined as the sum of the weights of the arcs in the unique directed path from i to j. Thus if $\mathcal{D}_{ij} = \sum\limits_{i \in A} u_i + \sum\limits_{j \in B} v_j$, then $\mathcal{D}_{ji} = \sum\limits_{i \in A} v_i + \sum\limits_{j \in B} u_j$. Note that the diagonal entries of the matrix $\mathcal{D}$ are zero and in general the matrix $\mathcal{D}$ is not a symmetric matrix. We are interested in extending the definition of a Laplacian to the bidirected trees. The *Laplacian matrix* $\mathcal{L} = (\mathcal{L}_{kl})$ of a bidirected tree $\mathcal{T}$ with the underlying tree structure T is defined by

$$\mathcal{L}_{k,l} = \begin{cases} 0 & \text{if } \{k,l\} \notin T \\ -\frac{1}{u_i + v_i} & \text{if } f_i = \{k,l\} \in T \\ \sum\limits_{f_i \sim k} \frac{1}{u_i+v_i} & \text{if } k = l, \end{cases}$$

where $e_i \sim k$ means that k is an endvertex of e_i. Notice that, in view of the Gersgorin disc theorem, the matrix $\mathcal{L}$ is a positive semidefinite matrix. For the sake of convenience, we write $w_t = u_t + v_t$. Then, the distance matrix $\mathcal{D}$ and the Laplacian matrix $\mathcal{L}$ of the bidirected tree $\mathcal{T}$ (shown in Figure 2.1) are given by

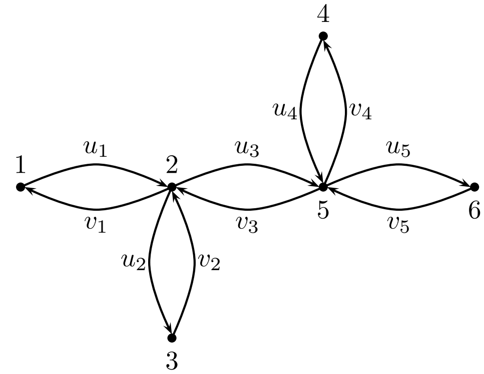

FIG. 2.1. *A bidirected Tree on* 6 *vertices*

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$$\mathcal{D} = \begin{bmatrix} 0 & u_1 & u_1+u_2 & u_1+u_3+v_4 & u_1+u_3 & u_1+u_3+u_5 \\ v_1 & 0 & u_2 & u_3+v_4 & u_3 & u_3+u_5 \\ v_1+v_2 & v_2 & 0 & v_2+u_3+v_4 & v_2+u_3 & v_2+u_3+u_5 \\ v_1+v_3+u_4 & v_3+u_4 & u_2+v_3+u_4 & 0 & u_4 & u_4+u_5 \\ v_1+v_3 & v_3 & u_2+v_3 & v_4 & 0 & u_5 \\ v_1+v_3+v_5 & v_3+v_5 & u_2+v_3+v_5 & v_4+v_5 & v_5 & 0 \end{bmatrix},$$

and

$$\mathcal{L} = \begin{bmatrix} \frac{1}{w_1} & -\frac{1}{w_1} & 0 & 0 & 0 & 0 \\ -\frac{1}{w_1} & \frac{1}{w_1}+\frac{1}{w_2}+\frac{1}{w_3} & -\frac{1}{w_2} & -\frac{1}{w_3} & 0 & 0 \\ 0 & -\frac{1}{w_2} & \frac{1}{w_2} & 0 & 0 & 0 \\ 0 & -\frac{1}{w_3} & 0 & \frac{1}{w_4} & -\frac{1}{w_4} & 0 \\ 0 & 0 & 0 & -\frac{1}{w_4} & \frac{1}{w_3}+\frac{1}{w_4}+\frac{1}{w_5} & -\frac{1}{w_5} \\ 0 & 0 & 0 & 0 & -\frac{1}{w_5} & \frac{1}{w_5} \end{bmatrix}.$$

Observe that if $u_i = v_i = 1$ for all i, then the matrices $\mathcal{D}$ and $\mathcal{L}$ reduce to the matrices D and $\frac{1}{2}L$, respectively.

We now introduce some further notation. Let $\mathcal{T}$ be a bidirected tree on n vertices. Let $\tilde{T}$ be a spanning tree of $\mathcal{T}$. Thus, $\tilde{T}$ is obtained from $\mathcal{T}$ by choosing one arc and hence $\mathcal{T}$ has 2^{n-1} spanning trees. Let us denote the *indegree* and the *outdegree* of the vertex v in $\tilde{T}$ by $\text{In}_{\tilde{T}}(v)$ and $\text{Out}_{\tilde{T}}(v)$, respectively. Consider the vectors $\mathbf{z}_1$ and $\mathbf{z}_2$ defined by

$$\mathbf{z}_1(i) = (-1)^n \sum_{\tilde{T}} \big[\text{In}_{\tilde{T}}(i) - 1\big] w(\tilde{T}) \tag{2.1}$$

$$\mathbf{z}_2(i) = (-1)^n \sum_{\tilde{T}} \big[\text{Out}_{\tilde{T}}(i) - 1\big] w(\tilde{T}), \tag{2.2}$$

where $w(\tilde{T})$ is the product of the arc weights of $\tilde{T}$. For example, the vectors $\mathbf{z}_1$ and $\mathbf{z}_2$ for the bidirected tree T given in Figure 2.1 are

$$\mathbf{z}_1 = \begin{bmatrix} -u_1w_2w_3w_4w_5 \\ \left[-u_2u_3v_1 + u_1u_3v_2 + u_1u_2v_3 + 2u_1v_2v_3 + v_1v_2v_3\right] w_4w_5 \\ -v_2w_1w_3w_4w_5 \\ -u_4w_1w_2w_3w_5 \\ w_1w_2\left[u_3u_4u_5 - u_5v_3v_4 + 2u_3u_4v_5 + u_4v_3v_5 + u_3v_4v_5\right] \\ -v_5w_1w_2w_3w_4 \end{bmatrix}$$

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and

$$\mathbf{z}_2 = \begin{bmatrix} -v_1w_2w_3w_4w_5 \\ [u_1u_2u_3 + 2u_2u_3v_1 + u_3v_1v_2 + u_2v_1v_3 - u_1v_2v_3]\,w_4w_5 \\ -u_2w_1w_3w_4w_5 \\ -v_4w_1w_2w_3w_5 \\ w_1w_2\,[u_4u5v_3 + u_3u_5v_4 + 2u_5v_3v_4 - u_3u_4v_5 + v_3v_4v_5] \\ -u_5w_1w_2w_3w_4 \end{bmatrix}.$$

Note that taking $u_i = v_i = 1$ for all i, and putting $k = \mathrm{In}_{\mathcal{T}}(i)$, we see that

$$(-1)^n\mathbf{z}_1(i) = \sum_{\tilde{T}}\big[\mathrm{In}_{\tilde{T}}(i) - 1\big] = \sum_{r=0}^{k} 2^{n-k-1} \sum_{\substack{\tilde{T} \\ \mathrm{In}_{\tilde{T}(i)=r}}} \big[\mathrm{In}_{\tilde{T}}(i) - 1\big]$$

$$= \Big[\sum_{r=0}^{k}\binom{k}{r}(r-1)\Big]2^{n-1-k} = \big(k2^{k-1} - 2^k\big)2^{n-1-k} = 2^{n-2}(k-2),$$

so that $\mathbf{z}_1 = \mathbf{z}_2 = (-1)^{n-1}2^{n-2}(\mathbf{e} - \mathbf{z})$.

Let $\mathcal{T}$ be a bidirected graph. Since each arc of a spanning tree $\tilde{T}$ contributes 1 to exactly one entry in $\mathrm{In}_{\tilde{T}}$, we have $\sum\limits_{i=1}^{n} \mathrm{In}_{\tilde{T}}(i) = n - 1$. Hence,

$$\begin{aligned} \mathbf{z}_1^t\mathbf{e} &= \sum_{i=1}^{n} \mathbf{z}_1(i) = \sum_{i=1}^{n}(-1)^n \sum_{\tilde{T}}\big[\mathrm{In}_{\tilde{T}}(i) - 1\big]w(\tilde{T}) \\ &= (-1)^n \sum_{\tilde{T}} w(\tilde{T}) \ \sum_{i=1}^{n}\big[\mathrm{In}_{\tilde{T}}(i) - 1\big] = (-1)^{n-1} \sum_{\tilde{T}} w(\tilde{T}) \\ &= (-1)^{n-1}\prod_{i=1}^{n-1} w_i. \end{aligned} \tag{2.3}$$

A similar reasoning implies that

$$\mathbf{z}_2^t\mathbf{e} = (-1)^{n-1}\prod_{i=1}^{n-1} w_i. \tag{2.4}$$

For a bidirected tree $\mathcal{T}$ on n vertices we define $w(\mathcal{T})$ as

$$w(\mathcal{T}) = \sum_{\tilde{T}} w(\tilde{T}) = \prod_{i=1}^{n-1} w_i = (-1)^{n-1}\mathbf{z}_1^t\mathbf{e} = (-1)^{n-1}\mathbf{z}_2^t\mathbf{e}.$$

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We use the convention that if T is a tree on a single vertex then $\mathbf{z}_1 = \mathbf{e} = \mathbf{z}_2$ and $w(T) = 1$. With this convention, for a bidirected forest $\mathcal{F}$ with the bidirected trees $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_k$ as components, the weight of $\mathcal{F}$ is defined as $w(\mathcal{F}) = \prod_{i=1}^{k} w(\mathcal{T}_i)$.

In the next section, we relate the matrices $\mathcal{D}^{-1}$ and $\mathcal{L}$ and also obtain some properties of the matrix $\mathcal{D}^{-1}$ with respect to minors. As corollaries, we obtain the results of Graham and Pollak [7]) on $\det(D)$ and that of Graham and Lovasz [6] on D^{-1}.

3. The main result. In this section, we extend certain results on distance matrices of trees to distance matrices of bidirected trees. Recall that a *pendant vertex* is a vertex of degree one. Denote by $G-v$ the graph obtained by deleting the vertex v and all arcs incident on it from G. By $\mathbf{e}_k$ we denote the vector with only one nonzero entry 1 which appears at the kth place.

Given any tree T on vertices $\{1, 2, \ldots, n\}$ we may view it as a rooted tree and hence there is a relabeling of the vertices so that for each $i > 1$ the vertex i is adjacent to only one vertex from $\{1, \ldots, i-1\}$. With such a labeling the vertex n is always a pendant vertex. Henceforth, unless stated otherwise, each bidirected tree will be assumed to have an underlying tree with such a labeling. Furthermore, for $i < j$, the weight of an arc $e_{j-1} = (i, j)$ will be assumed to be u_{j-1} and the weight of the arc $e'_{j-1} = (j, i)$ will be assumed to be v_{j-1}. If $\mathcal{T}$ is a bidirected tree by $\mathcal{T} - e_{j-1} - e'_{j-1}$ we denote the bidirected graph obtained by deleting the arcs (i, j) and (j, i) from $\mathcal{T}$.

We use the method of mathematical induction to prove our results. In the induction step, we start with a bidirected tree $\mathcal{T}'$ on $k+1$ vertices, where the pendant vertex $k+1$ is adjacent to the vertex r. We use the definition of the distance matrix of the bidirected tree $\mathcal{T} = \mathcal{T}' - \{k+1\}$ to get the distance matrix of $\mathcal{T}'$. Putting $\mathcal{D}' = \mathcal{D}(\mathcal{T}')$, $\mathcal{D} = \mathcal{D}(\mathcal{T})$, $\mathcal{L}' = \mathcal{L}(\mathcal{T}')$, $\mathcal{L} = \mathcal{L}(\mathcal{T})$, we see that

$$\mathcal{D}' = \begin{bmatrix} \mathcal{D} & u_k\mathbf{e} + \mathcal{D}\mathbf{e}_r \\ v_k\mathbf{e}^t + \mathbf{e}_r^t\mathcal{D} & \mathbf{0} \end{bmatrix}, \quad \mathcal{L}' = \begin{bmatrix} \mathcal{L} + \frac{1}{w_k}\mathbf{e}_r\mathbf{e}_r^t & -\frac{1}{w_k}\mathbf{e}_r \\ -\frac{1}{w_k}\mathbf{e}_r^t & \frac{1}{w_k} \end{bmatrix}. \tag{3.1}$$

Furthermore,

$$\begin{aligned}(-1)^{k+1}\mathbf{z}'_1(k+1) &= \sum_{\tilde{T}} \big[\mathrm{In}_{\tilde{T}}(k+1) - 1\big] w(\tilde{T}) \\ &= \sum_{(k+1,r)\in\tilde{T}} [-1] w(\tilde{T}) \\ &= w(\mathcal{T})\ (-v_k).\end{aligned}$$

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Also

$$\begin{aligned}(-1)^{k+1}\mathbf{z}_1'(r) &= \sum_{\tilde{T}} \big[\mathrm{In}_{\tilde{T}}(r) - 1\big] w(\tilde{T}) \\ &= \sum_{(r,k+1)\in\tilde{T}} \big[\mathrm{In}_{\tilde{T}}(r) - 1\big] w(\tilde{T}) + \sum_{(k+1,r)\in\tilde{T}} \big[\mathrm{In}_{\tilde{T}}(r) - 1\big] w(\tilde{T}) \\ &= (-1)^k \mathbf{z}_1(r) u_k + \Big[(-1)^k \mathbf{z}_1(r) v_k + w(\mathcal{T}) v_k\Big],\end{aligned}$$

and for $i \neq k+1, r$, we have,

$$\begin{aligned}\mathbf{z}_1'(i) &= (-1)^{k+1} \sum_{\tilde{T}} \big[\mathrm{In}_{\tilde{T}}(i) - 1\big] w(\tilde{T}) \\ &= (-1)^{k+1} \sum_{(r,k+1)\in\tilde{T}} \big[\mathrm{In}_{\tilde{T}}(i) - 1\big] w(\tilde{T}) + (-1)^{k+1} \sum_{(k+1,r)\in\tilde{T}} \big[\mathrm{In}_{\tilde{T}}(i) - 1\big] w(\tilde{T}) \\ &= -\mathbf{z}_1(i) u_k - \mathbf{z}_1(i) v_k \\ &= -z_1(i) w_k.\end{aligned}$$

Thus we have

$$\mathbf{z}_1' = \begin{bmatrix} -w_k \mathbf{z}_1 + (-1)^{k+1} w(\mathcal{T})\ v_k \mathbf{e}_r \\ (-1)^{k+1} w(\mathcal{T})\ (-v_k) \end{bmatrix}. \tag{3.2}$$

Similarly we have

$$\mathbf{z}_2' = \begin{bmatrix} -w_k \mathbf{z}_2 + (-1)^{k+1} w(\mathcal{T})\ u_k \mathbf{e}_r \\ (-1)^{k+1} w(\mathcal{T})\ (-u_k) \end{bmatrix}. \tag{3.3}$$

Note that these two equations provide an efficient way of computing the vectors $\mathbf{z}_1$ and $\mathbf{z}_2$ for a bidirected tree. Combined with the next theorem they give an efficient way to compute $\mathcal{D}^{-1}$. We shall use our previous observations are in the proof of the next theorem.

THEOREM 3.1. *Let $\mathcal{D}$ be the distance matrix of a bidirected tree on n vertices where the pendant vertex n is adjacent to r. Then*

$$\det(\mathcal{D}) = (-1)^{n-1} \sum_{i=1}^{n-1} u_i v_i w(\mathcal{T} - e_i - e_i') \tag{3.4}$$

$$\mathcal{D}\mathbf{z}_1 = \det(\mathcal{D})\mathbf{e}, \quad \mathbf{z}_2^t \mathcal{D} = \det(\mathcal{D})\mathbf{e}^t, \quad \textit{and} \tag{3.5}$$

$$\mathcal{D}^{-1} = -\mathcal{L} - (-1)^n \frac{\mathbf{z}_1 \mathbf{z}_2^t}{\det(\mathcal{D}) w(\mathcal{T})}. \tag{3.6}$$

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Proof. We prove the theorem by induction on the number of vertices of any bidirected tree. So, as the first step, let $n = 2$. In this case, the matrices $\mathcal{D}, \mathcal{L}, \mathbf{z}_1$ and $\mathbf{z}_2^t$ are respectively,

$$\mathcal{D} = \begin{bmatrix} 0 & u_1 \\ v_1 & 0 \end{bmatrix}, \ \mathcal{L} = \begin{bmatrix} \frac{1}{w_1} & -\frac{1}{w_1} \\ -\frac{1}{w_1} & \frac{1}{w_1} \end{bmatrix}, \mathbf{z}_1 = -\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \quad \text{and} \quad \mathbf{z}_2 = -\begin{bmatrix} v_1 \\ u_1 \end{bmatrix}.$$

As $w(\mathcal{T} - e_1 - e_1') = 1$, $\det(\mathcal{D}) = -u_1v_1 = (-1)^{2-1}u_1v_1w(\mathcal{T} - e_1 - e_1')$, $\mathcal{D}\ \mathbf{z}_1 = \det(\mathcal{D})\ \mathbf{e}$ and $\mathbf{z}_2^t\ \mathcal{D} = \det(\mathcal{D})\ \mathbf{e}^t$. Thus (3.5) is true for $n = 2$. Also, for $n = 2$, the right hand side of (3.6) reduces to

$$\begin{aligned} -\mathcal{L} - \frac{\mathbf{z}_1\mathbf{z}_2^t}{\det(\mathcal{D})w(\mathcal{T})} &= -\begin{bmatrix} \frac{1}{w_1} & -\frac{1}{w_1} \\ -\frac{1}{w_1} & \frac{1}{w_1} \end{bmatrix} - \frac{1}{-w_1u_1v_1}\begin{bmatrix} u_1v_1 & u_1^2 \\ v_1^2 & u_1v_1 \end{bmatrix} \\ &= -\begin{bmatrix} \frac{1}{w_1} & -\frac{1}{w_1} \\ -\frac{1}{w_1} & \frac{1}{w_1} \end{bmatrix} + \begin{bmatrix} \frac{1}{w_1} & \frac{u_1}{v_1w_1} \\ \frac{v_1}{u_1w_1} & \frac{1}{w_1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{1}{v_1} \\ \frac{1}{u_1} & 0 \end{bmatrix} = \mathcal{D}^{-1} \end{aligned}$$

Hence (3.6) holds for $n = 2$. We now assume that the equalities in (3.4), (3.5) and (3.6) are true for $n = k$. Let $n = k + 1$ and $\mathcal{T}'$ be a bidirected tree on $k + 1$ vertices. Put $\mathcal{T} = \mathcal{T}' - \{k + 1\}$. To establish the first equality (3.5) we need to show that

$$\det(D') = (-1)^k \sum_{i=1}^{k} u_iv_iw(\mathcal{T}' - e_i - e_i').$$

As $\mathcal{D}$ is invertible, using (3.1), the induction hypothesis and (2.3), we have

$$\begin{aligned} \det(\mathcal{D}') &= \det(\mathcal{D})\big[0 - (v_k\mathbf{e}^t + \mathbf{e}_r^t\mathcal{D})\mathcal{D}^{-1}(u_k\mathbf{e} + \mathcal{D}\mathbf{e}_r)\big] && (3.7) \\ &= -\det(\mathcal{D})\big[u_kv_k\mathbf{e}^t\mathcal{D}^{-1}\mathbf{e} + v_k\mathbf{e}^t\mathbf{e}_r + u_k\mathbf{e}_r^t\mathbf{e} + \mathbf{e}_r^t\mathcal{D}\mathbf{e}_r\big] \\ &= -\det(\mathcal{D})\big[u_kv_k\frac{\mathbf{e}^t\mathbf{z}_1}{\det(\mathcal{D})} + v_k + u_k\big] \\ &= (-1)^k u_kv_kw(\mathcal{T}) - w_k\det(\mathcal{D}) && (3.8) \\ &= (-1)^k u_kv_kw(\mathcal{T}) + (-1)^k w_k \sum_{i=1}^{k-1} u_iv_iw(\mathcal{T} - e_i - e_i') \\ &= (-1)^k \Big[u_kv_kw(\mathcal{T}' - e_k - e_k') + \sum_{i=1}^{k-1} u_iv_iw(\mathcal{T}' - e_i - e_i')\Big] \\ &= (-1)^k \sum_{i=1}^{k} u_iv_iw(\mathcal{T}' - e_i - e_i'). \end{aligned}$$

Hence the first equality holds for $n = k + 1$.

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To prove the second equality we need to show that

$$\mathcal{D}'\mathbf{z}_1' = \det(\mathcal{D}')\mathbf{e}, \quad \mathbf{z}_2'^t\mathcal{D}' = \det(\mathcal{D}')\mathbf{e}^t.$$

Using the expressions given in (3.1) and (3.2) we have

$$\mathcal{D}'\mathbf{z}_1' = \begin{bmatrix} \mathcal{D} & u_k\mathbf{e} + \mathcal{D}\mathbf{e}_r \\ v_k\mathbf{e}^t + \mathbf{e}_r^t\mathcal{D} & \mathbf{0} \end{bmatrix} \begin{bmatrix} -w_k\mathbf{z}_1 + (-1)^{k+1}w(\mathcal{T})\ v_k\mathbf{e}_r \\ (-1)^k w(\mathcal{T})\ v_k \end{bmatrix}.$$

The first block of the vector $\mathcal{D}'\mathbf{z}_1'$ reduces to

$$-w_k\mathcal{D}\mathbf{z}_1 + (-1)^k u_k v_k w(\mathcal{T})\mathbf{e}.$$

Substituting $\det(\mathcal{D})\mathbf{e}$ for $\mathcal{D}\mathbf{z}_1$ and using (3.8),

$$\text{the first block of } \mathcal{D}'\mathbf{z}_1' = \det(\mathcal{D}')\mathbf{e}. \tag{3.9}$$

The second block of the vector $\mathcal{D}'\mathbf{z}_1'$ reduces to

$$-v_k w_k\mathbf{e}^t\mathbf{z}_1 - w_k\mathbf{e}_r^t\mathcal{D}\mathbf{z}_1 + (-1)^{k+1}v_k w(\mathcal{T})\big(v_k\mathbf{e}^t\mathbf{e}_r + \mathbf{e}_r^t\mathcal{D}\mathbf{e}_r\big).$$

Now using the equality $\mathbf{e}_r^t\mathcal{D}\mathbf{e}_r = 0$, the equations (2.3), (3.4) and (3.8), we have

$$\text{the second block of } \mathcal{D}'\mathbf{z}_1' = \det(\mathcal{D}'). \tag{3.10}$$

A similar reasoning gives that $\mathbf{z}_2'^t\mathcal{D}' = \det(\mathcal{D}')\mathbf{e}^t$. Hence the second equality is established for $n = k+1$.

We now prove that the matrix $\mathcal{D}'^{-1}$ is indeed given by (3.6). As $\det(\mathcal{D}') \neq 0$, put $W = 0 - (v_k\mathbf{e}^t + \mathbf{e}_r^t\mathcal{D})\mathcal{D}^{-1}(u_k\mathbf{e} + \mathcal{D}\mathbf{e}_r)$. From (3.7), it follows that

$$W^{-1} = \frac{\det\mathcal{D}}{\det(\mathcal{D}')}. \tag{3.11}$$

Let $\mathcal{D}'^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$. Since $\mathcal{D}' = \begin{bmatrix} \mathcal{D} & u_k\mathbf{e} + \mathcal{D}\mathbf{e}_r \\ v_k\mathbf{e}^t + \mathbf{e}_r^t\mathcal{D} & \mathbf{0} \end{bmatrix}$, it is straightforward to see that

$$A_{11} = \mathcal{D}^{-1} + \mathcal{D}^{-1}(u_k\mathbf{e} + \mathcal{D}\mathbf{e}_r)W^{-1}(v_k\mathbf{e}^t + \mathbf{e}_r^t\mathcal{D})\mathcal{D}^{-1}, \tag{3.12}$$

$$A_{12} = -\mathcal{D}^{-1}(u_k\mathbf{e} + \mathcal{D}\mathbf{e}_r)W^{-1}, \tag{3.13}$$

$$A_{21} = -W^{-1}(v_k\mathbf{e}^t + \mathbf{e}_r^t\mathcal{D})\mathcal{D}^{-1}, \tag{3.14}$$

$$A_{22} = W^{-1}. \tag{3.15}$$

Using (3.11) and the induction hypothesis, we have

$$\begin{aligned} A_{11} &= \mathcal{D}^{-1} + \frac{\det\mathcal{D}}{\det(\mathcal{D}')}\big(u_k\mathcal{D}^{-1}\mathbf{e} + \mathbf{e}_r\big)\big(v_k\mathbf{e}^t\mathcal{D}^{-1} + \mathbf{e}_r^t\big) \\ &= \mathcal{D}^{-1} + \frac{\det\mathcal{D}}{\det(\mathcal{D}')}\left(u_k\frac{\mathbf{z}_1}{\det(\mathcal{D})} + \mathbf{e}_r\right)\left(v_k\frac{\mathbf{z}_2^t}{\det(\mathcal{D})} + \mathbf{e}_r^t\right) \\ &= \mathcal{D}^{-1} + \frac{1}{\det(\mathcal{D}')}\left[\frac{u_k v_k}{\det(\mathcal{D})}\mathbf{z}_1\mathbf{z}_2^t + \big(u_k z_1\mathbf{e}_r^t + v_k\mathbf{e}_r\mathbf{z}_2^t\big) + \det(\mathcal{D})\mathbf{e}_r\mathbf{e}_r^t\right] \end{aligned} \tag{3.16}$$

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and

$$A_{12} = -\mathcal{D}^{-1}(u_k\mathbf{e} + \mathcal{D}\mathbf{e}_r)W^{-1} = -\frac{\det \mathcal{D}}{\det(\mathcal{D}')}\left[u_k\mathcal{D}^{-1}\mathbf{e} + \mathbf{e}_r\right] = -\frac{u_k\mathbf{z}_1 + \det(\mathcal{D})\mathbf{e}_r}{\det(\mathcal{D}')}. \tag{3.17}$$

Similarly

$$A_{21} = -\frac{v_k\mathbf{z}_2^t + \det(\mathcal{D})\mathbf{e}_r^t}{\det(\mathcal{D}')}. \tag{3.18}$$

We now determine the first and second blocks of the matrix

$$-\mathcal{L}' - (-1)^{k+1}\frac{\mathbf{z}_1'\mathbf{z}_2'^{\,t}}{\det(\mathcal{D}')w(\mathcal{T}')}. \tag{3.19}$$

Using Equations (3.1), (3.2), (3.3), (3.7), (3.11) and the induction hypothesis, the first block of (3.19) equals

$$-\left(\mathcal{L} + \frac{\mathbf{e}_r\mathbf{e}_r^t}{w_k}\right) + \frac{(-1)^k\left(w_k^2\mathbf{z}_1\mathbf{z}_2^t + u_kv_kw(\mathcal{T})^2\mathbf{e}_r\mathbf{e}_r^t\right) + u_kw_kw(\mathcal{T})\mathbf{z}_1\mathbf{e}_r^t + v_kw_kw(\mathcal{T})\mathbf{e}_r\mathbf{z}_2^t}{\det(\mathcal{D}')w_kw(\mathcal{T})}$$

$$\begin{aligned}
&= -\mathcal{L} + \frac{(-1)^kw_k\mathbf{z}_1\mathbf{z}_2^t}{\det(\mathcal{D}')w(\mathcal{T})} - \frac{\mathbf{e}_r\mathbf{e}_r^t}{w_k} + \frac{(-1)^ku_kv_kw(\mathcal{T})\mathbf{e}_r\mathbf{e}_r^t}{w_k\det(\mathcal{D}')} + \frac{u_k\mathbf{z}_1\mathbf{e}_r^t + v_k\mathbf{e}_r\mathbf{z}_2^t}{\det(\mathcal{D}')}\\
&= \mathcal{D}^{-1} + \frac{(-1)^k\mathbf{z}_1\mathbf{z}_2^t}{\det(\mathcal{D}')w(\mathcal{T})}\left[w_k + \frac{\det(\mathcal{D}')}{\det(\mathcal{D})}\right] - \frac{\mathbf{e}_r\mathbf{e}_r^t}{w_k} + \frac{\det(\mathcal{D}') + w_k\det(\mathcal{D})}{w_k\det(\mathcal{D}')}\mathbf{e}_r\mathbf{e}_r^t\\
&\qquad + \frac{u_k\mathbf{z}_1\mathbf{e}_r^t + v_k\mathbf{e}_r\mathbf{z}_2^t}{\det(\mathcal{D}')}\\
&= \mathcal{D}^{-1} + \frac{u_kv_k\mathbf{z}_1\mathbf{z}_2^t}{\det(\mathcal{D}')\det(\mathcal{D})} + \frac{\det(\mathcal{D})}{\det(\mathcal{D}')}\mathbf{e}_r\mathbf{e}_r^t + \frac{u_k\mathbf{z}_1\mathbf{e}_r^t + v_k\mathbf{e}_r\mathbf{z}_2^t}{\det(\mathcal{D}')}
\end{aligned} \tag{3.20}$$

and the second block of (3.19) equals

$$\begin{aligned}
&\frac{\mathbf{e}_r}{w_k} - \frac{u_kw_kw(\mathcal{T})\mathbf{z}_1 - (-1)^ku_kv_kw(\mathcal{T})^2\mathbf{e}_r}{\det(\mathcal{D}')w_kw(\mathcal{T})}\\
&= \frac{\mathbf{e}_r}{w_k} - \frac{u_k\mathbf{z}_1}{\det(\mathcal{D}')} - \frac{\mathbf{e}_r}{w_k\det(\mathcal{D}')}\left[\det(\mathcal{D}') + w_k\det(\mathcal{D})\right]\\
&= -\frac{u_k\mathbf{z}_1 + \det(\mathcal{D})\mathbf{e}_r}{\det(\mathcal{D}')} = -(u_kD^{-1}\mathbf{e} + \mathbf{e}_r)W^{-1}.
\end{aligned} \tag{3.21}$$

Showing that A_{21} is the (2,1)-block of (3.19) is similar. The (2,2)-block of (3.19) is

$$-\frac{1}{w_k} + \frac{(-1)^ku_kv_kw(\mathcal{T})^2}{\det(\mathcal{D}')w_kw(\mathcal{T})} = -\frac{1}{w_k} + \frac{\det(\mathcal{D}') + w_k\det(\mathcal{D})}{\det(\mathcal{D}')w_k} = W^{-1}.$$

Hence the third equality is established for $n = k + 1$ and the proof is complete using induction. ■

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4. Bidirected trees with two types of weights. Suppose T is a rooted tree with root r. Let u and v be two vertices of T. As we traverse the u-v path from u to v there exists a vertex, say w (which may be u itself), such that the path from u to v moves in the direction of r until it meets vertex w and then moves away from r. Let the lengths of the two paths u-w and w-v be ℓ_1 and ℓ_2, respectively. Also, let x and y be two constants. We define the distance between u and v as

$$\bar{D}(u,v) = \ell_1 y + \ell_2 x. \tag{4.1}$$

Clearly, when $x = y = 1$, this reduces to the usual distance between u and v. We illustrate this with the following example.

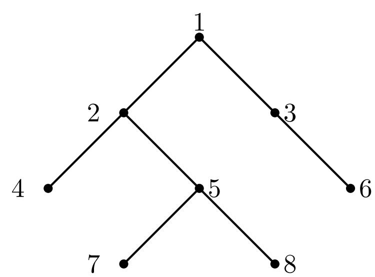

FIG. 4.1. *A rooted tree*

Consider the tree given in Figure 4.1. The distance matrix of the tree is as follows:

$$\bar{D} = \begin{bmatrix} 0 & x & x & 2x & 2x & 2x & 3x & 3x \\ y & 0 & x+y & x & x & 2x+y & 2x & 2x \\ y & x+y & 0 & 2x+y & 2x+y & x & 3x+y & 3x+y \\ 2y & y & x+2y & 0 & x+y & 2x+2y & 2x+y & 2x+y \\ 2y & y & x+2y & x+y & 0 & 2x+2y & x & x \\ 2y & x+2y & y & 2x+2y & 2x+2y & 0 & 3x+2y & 3x+2y \\ 3y & 2y & x+3y & x+2y & y & 2x+3y & 0 & x+y \\ 3y & 2y & x+3y & x+2y & y & 2x+3y & x+y & 0 \end{bmatrix}.$$

Observe that if we apply a similar labeling to T as in the previous section and consider the bidirected tree $\mathcal{T}$ with the underlying tree structure T, and use the weights $u_i = x \ \forall i$, $v_i = y \ \forall i$, then the distance matrix $\mathcal{D}$ of the bidirected tree is nothing but the distance matrix $\bar{D}$.

Henceforth a rooted tree is assumed to have the root 1 and the labeling as described earlier. Let u be a vertex of a rooted tree T. A vertex v is called a *child* of u if u and v are adjacent and u is on the v-1 path. Let us denote the number of children of u by $\text{ch}(u)$. With the notations defined above, we have the following result.

COROLLARY 4.1. *Let T be a rooted tree on n vertices and consider the distance*

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matrix $\bar{D}$. Also, let $\mathbf{z}_1$ and $\mathbf{z}_2$ be vectors of order n given by

$$(\mathbf{z}_1)_i = \begin{cases} (-1)^n\big((ch(i)-1)y-x\big)(x+y)^{n-2}, & \text{if } i=1, \\ (-1)^{n-1}y(x+y)^{n-2}, & \text{if } i \text{ is a pendant vertex}, \\ (-1)^n(ch(i)-1)y(x+y)^{n-2}, & \text{otherwise} \end{cases} \tag{4.2}$$

and

$$(\mathbf{z}_2)_i = \begin{cases} (-1)^n\big((ch(i)-1)x-y\big)(x+y)^{n-2}, & \text{if } i=r, \\ (-1)^{n-1}x(x+y)^{n-2}, & \text{if } i \text{ is a pendant vertex}, \\ (-1)^n(ch(i)-1)x(x+y)^{n-2}, & \text{otherwise.} \end{cases} \tag{4.3}$$

Then

$$\det(\mathcal{D}) = (-1)^{n-1}(n-1)xy(x+y)^{n-2},$$

and

$$\mathcal{D}^{-1} = -\frac{L}{x+y} + \frac{\mathbf{z}_1\mathbf{z}_2^t}{(n-1)xy(x+y)^{2n-3}},$$

where L is the usual Laplacian matrix.

Proof. Let $\mathcal{T}$ be the bidirected tree associated with T. As $\bar{D}$ is the same as $\mathcal{D}$ with $u_i = x$ and $v_i = y$, the assertion about the determinant follows easily from (3.4).

The vectors $\mathbf{z}_1$, $\mathbf{z}_2$ defined here are nothing but the vectors defined in (2.1) and (2.2). In order to see this note that let $\tilde{T}$ be a spanning tree of $\mathcal{T}$ and put $k = ch(1)$.

$$(-1)^n\mathbf{z}_1(1) = \sum_{\tilde{T}}\big[\mathrm{In}_{\tilde{T}}(1)-1\big]w(\tilde{T}) = \sum_{r=0}^{k}(x+y)^{n-1-k}\sum_{\substack{\tilde{T} \\ \mathrm{In}_{\tilde{T}}(1)=r}}\big[\mathrm{In}_{\tilde{T}}(1)-1\big]y^r x^{k-r}$$

$$= (x+y)^{n-1-k}\sum_{r=0}^{k}\binom{k}{r}(r-1)y^r x^{k-r} = (x+y)^{n-1-k}\Big[ky(x+y)^{k-1}-(x+y)^k\Big]$$

$$= (x+y)^{n-2}\big[(ch(1)-1)y-x\big].$$

If i is a pendant vertex, put $k = ch(i)$ and observe that

$$(-1)^n\mathbf{z}_1(i) = \sum_{\tilde{T}}\big[\mathrm{In}_{\tilde{T}}(i)-1\big]w(\tilde{T}) = -x(x+y)^{n-2}.$$

If i is any other vertex, then put $k = ch(i)$, and let p be the parent of i. We have

$$(-1)^n\mathbf{z}_1(i) = \sum_{\tilde{T}}\big[\mathrm{In}_{\tilde{T}}(i)-1\big]w(\tilde{T}) = \sum_{(i,p)\in\tilde{T}}\big[\mathrm{In}_{\tilde{T}}(i)-1\big]w(\tilde{T}) + \sum_{(p,i)\in\tilde{T}}\big[\mathrm{In}_{\tilde{T}}(i)-1\big]w(\tilde{T})$$

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$$= (x+y)^{n-3}y\big[(k-1)y - x\big] + kxy(x+y)^{n-3} = \big[ch(i)-1\big]y(x+y)^{n-2}.$$

The vector $\mathbf{z}_2$ may be verified similarly. Now the assertion about inverse of $\bar{D}$ follows from (3.6). ■

As a corollary, we obtain the result of Graham and Pollak [7] on $\det(D)$.

Corollary 4.2. *Let T be a tree on n vertices and let D be its distance matrix. Then* $\det(D) = (-1)^{n-1}(n-1)2^{n-2}$.

Proof. Let us denote by T the bidirected tree obtained from the given tree T. As observed earlier, the substitution of $u_i = v_i = 1$ for $1 \le i \le n-1$, reduces the matrix $\mathcal{D}$ to the distance matrix D. Under this condition, we have $w_i = u_i + v_i = 2$ and $w(\mathcal{T} - e_i - e_i') = 2^{n-2}$ for $1 \le i \le n-1$. Therefore

$$\det(D) = \det(\mathcal{D})\big|_{u_i=v_i=1} = (-1)^{n-1}\sum_{i=1}^{n-1} u_i v_i w(T-e_i)\big|_{u_i=v_i=1} = (-1)^{n-1}(n-1)2^{n-2}. ■$$

We now give a corollary to our result that gives a formula for D^{-1}. This result was also obtained by Graham and Lovasz (see [6]).

Corollary 4.3. *Let T be a tree on n vertices and let D be its distance matrix, L be its Laplacian matrix and let $\mathbf{z}$ and $\mathbf{e}$ be the vectors defined earlier. Then*

$$D^{-1} = \frac{(\mathbf{e}-\mathbf{z})(\mathbf{e}-\mathbf{z})^t}{2(n-1)} - \frac{L}{2}.$$

Proof. Let us denote by T the bidirected tree obtained from the given tree T. Observe that under the condition, $u_i = v_i = 1$, the matrix $\mathcal{D}$ reduces to D, the matrix $\mathcal{L}$ reduces to $\dfrac{L}{2}$ and $\mathbf{z}_1 = \mathbf{z}_2 = (-1)^{n-2}2^{n-2}(\mathbf{z}-\mathbf{e})$. So, we have

$$\begin{aligned} D^{-1} = \mathcal{D}^{-1}\big|_{u_i=v_i=1} &= -\mathcal{L} + (-1)^{n-1}\frac{\mathbf{z}_1\mathbf{z}_2^t}{\det(\mathcal{D})w(T)}\Big|_{u_i=v_i=1} \\ &= -\frac{L}{2} + \frac{2^{2n-4}(\mathbf{e}-\mathbf{z})(\mathbf{e}-\mathbf{z})^t}{(n-1)2^{n-2}2^{n-1}} \\ &= -\frac{L}{2} + \frac{(\mathbf{e}-\mathbf{z})(\mathbf{e}-\mathbf{z})^t}{2(n-1)}. \end{aligned}$$

Hence the required result follows. ■

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