

THE REFLEXIVE RE-NONNEGATIVE DEFINITE SOLUTION TO A QUATERNION MATRIX EQUATION*

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Abstract. In this paper a necessary and sufficient condition is established for the existence of the reflexive re-nonnegative definite solution to the quaternion matrix equation $AXA^* = B$, where $*$ stands for conjugate transpose. The expression of such solution to the matrix equation is also given. Furthermore, a necessary and sufficient condition is derived for the existence of the general re-nonnegative definite solution to the quaternion matrix equation $A_1X_1A_1^* + A_2X_2A_2^* = B$. The representation of such solution to the matrix equation is given.

Key words. Quaternion matrix equation, Reflexive matrix, Re-nonnegative definite matrix, Reflexive re-nonnegative definite matrix.

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1. Introduction. Throughout this paper, we denote the real number field by \mathbb{R} , the complex number field by \mathbb{C} , the real quaternion algebra by

$$\mathbb{H} = \{a_0 + a_1i + a_2j + a_3k \mid i^2 = j^2 = k^2 = ijk = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\},$$

the set of all $m \times n$ matrices over \mathbb{H} by $\mathbb{H}^{m \times n}$, the set of all $m \times n$ matrices in $\mathbb{H}^{m \times n}$ with rank r by $\mathbb{H}_r^{m \times n}$, the set of all $n \times n$ invertible matrices over \mathbb{H} by \mathbb{GL}_n . $\mathcal{R}(A)$, $\mathcal{N}(A)$, A^* , A^\dagger , $\dim \mathcal{R}(A)$, I_i and $\text{Re}[b]$ stand for the column right space, the left row space, the conjugate transpose, the Moore-Penrose inverse of $A \in \mathbb{H}^{m \times n}$, the dimension of $\mathcal{R}(A)$, an $i \times i$ identity matrix, and the real part of a quaternion b , respectively. By [1], for a quaternion matrix A , $\dim \mathcal{R}(A) = \dim \mathcal{N}(A)$, which is called the rank of A and denoted by $r(A)$. Moreover, A^{-*} denote the inverse matrix of A^* if A is invertible. Obviously, $A^{-*} = (A^{-1})^* = (A^*)^{-1}$. $\mathcal{H}_n = \{A \in \mathbb{H}^{n \times n} \mid A^* = A\}$. Because \mathbb{H} is not commutative, one cannot directly extend various results on \mathbb{C} to \mathbb{H} . General properties of matrices over \mathbb{H} can be found in [2].

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We consider the classical matrix equation

$$AXA^* = B. \quad (1.1)$$

The general Hermitian positive semidefinite solutions of (1.1) have been studied extensively for years. For instance, Khatri and Mitra [3], Baksalary [4], Groβ [5], Zhang and Cheng [6] derived the general Hermitian positive semidefinite solution to matrix equation (1.1), respectively. Moreover, Dai and Lancaster [7] studied the similar problem and emphasized the importance of (1.1) within the real setting. Liao and Bai [8] investigated the bisymmetric positive semidefinite solution to matrix equation (1.1) by studying the symmetric positive semidefinite solution of the matrix equation

$$A_1 X_1 A_1^* + A_2 X_2 A_2^* = B, \quad (1.2)$$

which was also investigated by Deng and Hu [9] over the real number field.

The definition of reflexive matrix can be found in [10]: A complex square matrix A is reflexive if $A = PAP$, where P is a Hermitian involution, i.e., $P^* = P$, $P^2 = I$. Obviously, a reflexive matrix is a generalization of a centrosymmetric matrix. The reflexive matrices with respect to a Hermitian involution matrix P have been widely used in engineering and scientific computations (see, for instance, [10]). In 1996, Wu and Cain [11] defined the re-nonnegative definite matrix: $A \in \mathbb{C}^{n \times n}$ is re-nonnegative definite if $\operatorname{Re}[x^* Ax] \geq 0$ for every nonzero vector $x \in \mathbb{C}^{n \times 1}$. However, to our knowledge, the reflexive re-nonnegative definite solution to (1.1) and the re-nonnegative definite solution to (1.2) have not been investigated yet so far. Motivated by the work mentioned above and keeping the interests and wide applications of quaternion matrices in view (e.g. [12]–[27]), we in this paper consider the reflexive re-nonnegative definite solution to (1.1) and re-nonnegative definite solution to (1.2) over \mathbb{H} .

The paper is organized as follows. In Section 2, we first present a criterion that a 3×3 partitioned quaternion matrix is re-nonnegative definite. Then we establish a criterion that a quaternion matrix is reflexive re-nonnegative definite. In Section 3, we establish a necessary and sufficient condition for the existence of re-nonnegative definite solution to (1.2) over \mathbb{H} as well as an expression of the general solution. In Section 4, based on the results obtained in Section 3, we present a necessary and sufficient condition for the existence and the expression of the reflexive re-nonnegative definite solution to (1.1) over \mathbb{H} . In closing this paper, we in Section 5 give a conclusion and a further research topic related to this paper.

2. Reflexive re-nonnegative definite quaternion matrix. In this section, we first present a criterion that a 3×3 partitioned quaternion matrix is re-nonnegative definite, then derive a criterion that a quaternion matrix is reflexive re-nonnegative definite.

Throughout, the set of all $n \times n$ reflexive quaternion matrices with a Hermitian involution $P \in \mathbb{H}^{n \times n}$ is denoted by $\mathbb{RH}^{n \times n}(P)$, i.e.,

$$\mathbb{RH}^{n \times n}(P) = \{A \in \mathbb{H}^{n \times n} \mid A = PAP\}.$$

DEFINITION 2.1. Let $A \in \mathbb{H}^{n \times n}$. Then

- (1) $A \in \mathcal{H}_n$ is said to be nonnegative definite, abbreviated nnd, if $x^*Ax \geq 0$ for any nonzero vector $x \in \mathbb{H}^{n \times 1}$. The set of all $n \times n$ nnd matrices is denoted by \mathbb{SP}_n .
- (2) A is re-nonnegative definite, abbreviated rennd, if $\text{Re}[x^*Ax] \geq 0$ for every nonzero vector $x \in \mathbb{H}^{n \times 1}$. The set of all rennd matrices in $\mathbb{H}^{n \times n}$ is denoted by \mathbb{SP}_n^* .
- (3) A is called reflexive re-nonnegative definite matrix if $A \in \mathbb{RH}^{n \times n}(P) \cap \mathbb{SP}_n^*$. The set of all $n \times n$ reflexive re-nonnegative definite matrices is denoted by $\mathbb{RSP}_n^*(P)$.

For any quaternion matrix A , it can be uniquely expressed as

$$A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) \stackrel{\text{def}}{=} \mathcal{H}(A) + \mathcal{S}(A).$$

It is easy to prove the following.

LEMMA 2.2. Let $Q \in \mathbb{GL}_n, A \in \mathbb{H}^{n \times n}$. Then

- (1) $A \in \mathbb{SP}_n^*$ if and only if $\mathcal{H}(A) \in \mathbb{SP}_n$.
- (2) $A \in \mathbb{SP}_n \iff Q^*AQ \in \mathbb{SP}_n$.
- (3) $A \in \mathbb{SP}_n^* \iff Q^*AQ \in \mathbb{SP}_n^*$.
- (4) $A \in \mathbb{RSP}_n^*(P) \iff Q^*AQ \in \mathbb{RSP}_n^*(P)$.

The following lemma is due to Albert [28] which can be generalized to \mathbb{H} .

LEMMA 2.3. Let $A \in \mathbb{H}^{n \times n}$ be partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \in \mathcal{H}_n$$

where $A_{ii} \in \mathcal{H}_{n_i}$ ($n_1 + n_2 = n$). Then the following statements are equivalent:

- (1) $A \in \mathbb{SP}_n$.
- (2) $r(A_{11}) = r(A_{11}, A_{12})$, A_{11} and $A_{22} - A_{12}^*A_{11}^\dagger A_{12}$ are nnd.
- (3) $r(A_{22}) = r(A_{12}^*, A_{22})$, A_{22} and $A_{11} - A_{12}A_{22}^\dagger A_{12}^*$ are nnd.

The following lemma follows from Lemma 2.2 and Lemma 2.3.

LEMMA 2.4. Let $A \in \mathbb{H}^{n \times n}$ be partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where $A_{ii} \in \mathbb{H}^{n_i \times n_i}$ ($n_1 + n_2 = n$). Then the following statements are equivalent:

- (1) $A \in \mathbb{SP}_n^*$.

(2) $r(A_{11}^* + A_{11}) = r(A_{11}^* + A_{11}, A_{12} + A_{21}^*)$, A_{11} and $A_{22} - \frac{1}{2}(A_{12}^* + A_{21})(A_{11} + A_{11}^*)^\dagger(A_{12} + A_{21}^*)$ are rennd.

(3) $r(A_{22} + A_{22}^*) = r(A_{12}^* + A_{21}, A_{22} + A_{22}^*)$, A_{22} and $A_{11} - \frac{1}{2}(A_{12} + A_{21}^*)(A_{22} + A_{22}^*)^\dagger(A_{12}^* + A_{21})$ are rennd.

The case of complex matrices of Lemma 2.4 can also be found in [11].

Now we present a criterion that a 3×3 partitioned Hermitian quaternion matrix is nnd.

LEMMA 2.5. *Let*

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^* & A_{22} & X_{23} \\ A_{13}^* & X_{23}^* & A_{33} \end{pmatrix} \in \mathcal{H}_n$$

where $A_{11} \in \mathcal{H}_{r_1}$, $A_{22} \in \mathcal{H}_{r_2}$, $A_{33} \in \mathcal{H}_{n-r_1-r_2}$. Then there exists $X_{23} \in \mathbb{H}^{r_2 \times (n-r_1-r_2)}$ such that $A \in \mathbb{SP}_n$ if and only if the following

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \stackrel{\text{def}}{=} A_1, \begin{pmatrix} A_{11} & A_{13} \\ A_{13}^* & A_{33} \end{pmatrix} \stackrel{\text{def}}{=} A_2$$

are all nnd.

Proof. If there exists a quaternion matrix X_{23} such that $A \in \mathbb{SP}_n$, then by Lemma 2.3, A_1 is nnd. Clearly, $A_2 \in \mathcal{H}_{n-r_2}$ by $A \in \mathcal{H}_n$. For an arbitrary nonzero column vector $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$, it follows from $A \in \mathbb{SP}_n$ that

$$\alpha^* A_2 \alpha = \begin{pmatrix} \alpha_1^* & 0^* & \alpha_2^* \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^* & A_{22} & X_{23} \\ A_{13}^* & X_{23}^* & A_{33} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ 0 \\ \alpha_2 \end{pmatrix} \geq 0.$$

Therefore, A_2 is also nnd.

Conversely, if A_1 and A_2 are all nnd, then A_{11} , $A_{22} - A_{12}^* A_{11}^\dagger A_{12}$, $A_{33} - A_{13}^* A_{11}^\dagger A_{13}$ are all nnd by Lemma 2.3. Taking $X_{23} = A_{12}^* A_{11}^\dagger A_{13}$ and

$$P = \begin{pmatrix} I_{r_1} & -A_{11}^\dagger A_{12} & -A_{11}^\dagger A_{13} \\ 0 & I_{r_2} & 0 \\ 0 & 0 & I_{n-r_1-r_2} \end{pmatrix}$$

yields that

$$P^* A P = \text{diag}(A_{11}, A_{22} - A_{12}^* A_{11}^\dagger A_{12}, A_{33} - A_{13}^* A_{11}^\dagger A_{13}).$$

Hence by Lemma 2.2 and Lemma 2.3, A is nnd. \square

Based on the above lemma, we now present a criterion that a 3×3 partitioned quaternion matrix is rennd. The result plays an important role in constructing a rennd matrix.

THEOREM 2.6. *Assume that*

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & X_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \in \mathbb{H}^{n \times n}$$

where $A_{11} \in \mathbb{H}^{r_1 \times r_1}$, $A_{22} \in \mathbb{H}^{r_2 \times r_2}$, $A_{33} \in \mathbb{H}^{(n-r_1-r_2) \times (n-r_1-r_2)}$. Then there exists $X_{23} \in \mathbb{H}^{r_2 \times (n-r_1-r_2)}$ such that $A \in \mathbb{SP}_n^*$ if and only if the following

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \stackrel{\text{def}}{=} A_1, \begin{pmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{pmatrix} \stackrel{\text{def}}{=} A_2$$

are all rennd.

Proof. Put $B_{11} = A_{11} + A_{11}^*$, $B_{12} = A_{21}^* + A_{12}$, $B_{13} = A_{13} + A_{31}^*$, $B_{22} = A_{22} + A_{22}^*$, $B_{23} = X_{23} + A_{32}^*$, $B_{33} = A_{33} + A_{33}^*$. Then

$$2\mathcal{H}(A_1) = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix}, 2\mathcal{H}(A_2) = \begin{pmatrix} B_{11} & B_{13} \\ B_{13}^* & B_{33} \end{pmatrix}$$

$$2\mathcal{H}(A) = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{12}^* & B_{22} & B_{23} \\ B_{13}^* & B_{23}^* & B_{33} \end{pmatrix}.$$

Hence the theorem follows immediately from Lemma 2.2 and Lemma 2.5. \square

Now we turn our attention to establish a criterion that a quaternion matrix is reflexive rennd.

For $A \in \mathbb{H}^{n \times n}$, a quaternion λ is said to be a right (or left) eigenvalue of A if $Ax = x\lambda$ (or $Ax = \lambda x$) for some nonzero vector $x \in \mathbb{H}^{n \times 1}$. In this paper, we only use the right eigenvalue (simply say eigenvalue). It can be verified easily that the eigenvalues of an involution $P \in \mathbb{H}^{n \times n}$ are 1 and -1 .

In [29], a practical method to represent an involutory quaternion matrix was given as follows.

LEMMA 2.7. (Theorem 2.1 in [29]) *Suppose that $P \in \mathbb{H}^{n \times n}$ is a nontrivial involution and $K = \begin{pmatrix} I_n + P \\ I_n \end{pmatrix}$, then we have the following:*

(i) K can be reduced into $\begin{pmatrix} N & 0 \\ \Phi & M \end{pmatrix}$, where N is a full column rank matrix of size

$n \times r$ and $r = r(I_n + P)$, by applying a sequence of elementary column operations on K .

(ii) Put $T = \begin{pmatrix} N & M \end{pmatrix}$, then

$$P = T \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} T^{-1}. \quad (2.1)$$

By Lemma 2.7, $\mathcal{R}(N)$ is the eigenspace corresponding to the eigenvalue 1 of P and $\mathcal{R}(M)$ the eigenspace corresponding to the eigenvalue -1 of P . Since P is Hermitian, it can be orthogonally diagonalized. To see this, we perform the Gram-Schmidt process to the columns of N and M , respectively. Suppose that the corresponding orthonormal column vectors are as follows:

$$V = \{\alpha_1, \dots, \alpha_r\}, W = \{\beta_1, \dots, \beta_{n-r}\}$$

It is clear that $PV = V$, $PW = -W$ and $U = \begin{pmatrix} V & W \end{pmatrix}$ is unitary. Hence it follows from $P^* = P$ that

$$P = U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^*. \quad (2.2)$$

Therefore, similar to Theorem 2.2 in [29], we get the following lemma.

LEMMA 2.8. *Let $A \in \mathbb{H}^{n \times n}$. Then $A \in \mathbb{RH}^{n \times n}(P)$ if and only if A can be expressed as*

$$A = U \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} U^* \quad (2.3)$$

where $A_1 \in \mathbb{H}^{r \times r}$, $A_2 \in \mathbb{H}^{(n-r) \times (n-r)}$, U is defined as (2.2).

At the end of the section, we present a criterion that a quaternion matrix is reflexive rennd.

THEOREM 2.9. *Let $A \in \mathbb{H}^{n \times n}$. Then $A \in \mathbb{RSP}_n^*(P)$ if and only if*

$$A = U \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} U^* \quad (2.4)$$

where $A_1 \in \mathbb{SP}_r^*$, $A_2 \in \mathbb{SP}_{n-r}^*$ and U is defined as (2.2).

Proof. If $A \in \mathbb{RSP}_n^*(P)$, then $A \in \mathbb{RH}^{n \times n}(P)$. Hence by Lemma 2.8, A can be expressed as (2.4) where $A_1 \in \mathbb{H}^{r \times r}$, $A_2 \in \mathbb{H}^{(n-r) \times (n-r)}$. It follows from $A \in \mathbb{RSP}_n^*(P)$, Lemma 2.2 and Lemma 2.4 that $A_1 \in \mathbb{SP}_r^*$, $A_2 \in \mathbb{SP}_{n-r}^*$.

Conversely, by Lemmas 2.2, 2.4 and 2.8, it is easy to verify that the matrix A with the form of (2.4) is reflexive rennd, where $A_1 \in \mathbb{SP}_r^*$, $A_2 \in \mathbb{SP}_{n-r}^*$. \square

3. The re-nonnegative definite solution to (1.2). In this section, given $A_1 \in \mathbb{H}^{m \times r}$, $A_2 \in \mathbb{H}^{m \times (n-r)}$ and $B \in \mathbb{H}^{m \times m}$, we investigate the rennd solution to the matrix equation (1.2).

In order to investigate the rennd solution to (1.2), we recall the following lemma.

LEMMA 3.1. ([30]) *Let $A_1 \in \mathbb{H}_{r_{a_1}}^{m \times r}$, $A_2 \in \mathbb{H}_{r_{a_2}}^{m \times (n-r)}$. Then there exist $P \in \mathbb{GL}_m$, $Q \in \mathbb{GL}_r$, $T \in \mathbb{GL}_{n-r}$ such that*

$$PA_1Q = \begin{pmatrix} I_{r_{a_1}} & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} r_{a_1} \\ m - r_{a_1} \end{matrix}, \quad (3.1)$$

$$PA_2T = \begin{pmatrix} I_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_t \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} s \\ r_{a_1} - s \\ t \\ m - r_{a_1} - t \end{matrix} \quad (3.2)$$

$s \quad n - r - r_{a_2} \quad t$

where

$$r_{ab} = r(A_1, A_2), s = r_{a_1} + r_{a_2} - r_{ab}, t = r_{ab} - r_{a_1}.$$

In some way, Lemma 3.1 can be regarded as an extension of the generalized singular value decomposition (GSVD) on a complex matrix pair (see [31]-[33]). It is worth pointing out that there is a good collection of literature, [8] and [9] for example, on the GSVD approach for solving matrix equations.

Now we propose an algorithm for finding out P, Q and T in Lemma 3.1.

ALGORITHM 3.2. *Let*

$$K = \begin{pmatrix} I_m & A_1 & A_2 \\ 0 & 0 & I_{n-r} \\ 0 & I_r & 0 \end{pmatrix}.$$

Apply a sequence of elementary row operations on the submatrices $(I_m \ A_1 \ A_2)$ of K , and a sequence of elementary column operations on the submatrices $\begin{pmatrix} A_1 \\ 0 \\ I_r \end{pmatrix}$

and $\begin{pmatrix} A_2 \\ I_{n-r} \end{pmatrix}$ of K , respectively, till we obtain the following

$$\begin{pmatrix} P & \begin{pmatrix} I_{r_{a_1}} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} I_s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_t \\ 0 & 0 & 0 \end{pmatrix} \\ 0 & 0 & T \\ 0 & Q & 0 \end{pmatrix}.$$

Then P, Q and T are obtained.

Now we consider (1.2). Let

$$Q^{-1}X_1Q^{-*} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{matrix} r_{a_1} \\ r - r_{a_1} \\ r_{a_1} \\ r - r_{a_1} \end{matrix}, \quad (3.3)$$

$$T^{-1}X_2T^{-*} = \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{pmatrix} \begin{matrix} s \\ n - r - r_{a_2} \\ t \\ s \\ n - r - r_{a_2} \\ t \end{matrix}, \quad (3.4)$$

$$PBP^* = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{pmatrix} \begin{matrix} s \\ r_{a_1} - s \\ t \\ m - r_{a_1} - t \\ s \\ r_{a_1} - s \\ t \\ m - r_{a_1} - t \end{matrix}. \quad (3.5)$$

LEMMA 3.3. Let $A_1 \in \mathbb{H}^{m \times r}$, $A_2 \in \mathbb{H}^{m \times (n-r)}$ and $B \in \mathbb{H}^{m \times m}$. Then matrix equation (1.2) is consistent if and only if

$$B_{\alpha 4} = 0, B_{4\beta} = 0, \alpha, \beta = 1, 2, 3, 4; B_{32} = 0, B_{23} = 0. \quad (3.6)$$

In that case, the general solution of (1.2) can be expressed as

$$X_1 = Q \begin{pmatrix} \begin{pmatrix} B_{11} - Y_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} Q^*, \quad (3.7)$$

$$X_2 = T \begin{pmatrix} Y_{11} & Y_{12} & B_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ B_{31} & Y_{32} & B_{33} \end{pmatrix} T^*, \quad (3.8)$$

where $X_{12}, X_{21}, X_{22}; Y_{11}, Y_{12}, Y_{21}, Y_{22}, Y_{23}, Y_{32}$ are arbitrary quaternion matrices whose sizes are determined by (3.3) and (3.4).

Proof. Obviously, matrix equation (1.2) is equivalent to the following matrix equation

$$PA_1QQ^{-1}X_1Q^{-*}Q^*A_1^*P^* + PA_2TT^{-1}X_2T^{-*}T^*A_2^*P^* = PBP^*. \quad (3.9)$$

If (1.2) is consistent, then by (3.1)–(3.5) and (3.9), we have the following

$$\begin{pmatrix} X_{11} + \begin{pmatrix} Y_{11} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} Y_{13} & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} Y_{31} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} Y_{33} & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{pmatrix}$$

yielding

$$X_{11} = \begin{pmatrix} B_{11} - Y_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, Y_{13} = B_{13}, Y_{31} = B_{31}, Y_{33} = B_{33},$$

and (3.6) holds. Therefore X_1 and X_2 can be expressed as (3.7) and (3.8), respectively.

Conversely, if (3.6) holds, then it can be verified that the matrices X_1 and X_2 with the form (3.7) and (3.8), respectively, consist of a solution of (1.2). \square

THEOREM 3.4. Let $A_1 \in \mathbb{H}^{m \times r}$, $A_2 \in \mathbb{H}^{m \times (n-r)}$ and $B \in \mathbb{H}^{m \times m}$ be given. Put

$$C = \frac{1}{2}(B_{12} + B_{21}^*)(B_{22} + B_{22}^*)^\dagger(B_{12}^* + B_{21}), D = \frac{1}{2}(B_{13} + B_{31}^*)(B_{33} + B_{33}^*)^\dagger(B_{13}^* + B_{31}),$$

and $L = B_{11} - C - D$. Then matrix equation (1.2) has a solution $X_1 \in \mathbb{SP}_r^*$, $X_2 \in \mathbb{SP}_{n-r}^*$ if and only if (3.6) holds,

$$r(B_{22} + B_{22}^*, B_{21} + B_{12}^*) = r(B_{22} + B_{22}^*), r(B_{33} + B_{33}^*, B_{31} + B_{13}^*) = r(B_{33} + B_{33}^*), \quad (3.10)$$

and $L \in \mathbb{SP}_s^*$, $B_{22} \in \mathbb{SP}_{r_{a_1-s}}^*$, $B_{33} \in \mathbb{SP}_t^*$. In that case, the general solution $X_1 \in \mathbb{SP}_r^*$, $X_2 \in \mathbb{SP}_{n-r}^*$ of (1.2) can be expressed as the following, respectively,

$$X_1 = Q \begin{pmatrix} N & -X_{21}^* + (N + N^*)U_1 \\ X_{21} & F + \frac{1}{2}U_1^*(N^* + N)U_1 \end{pmatrix} Q^*, \quad (3.11)$$

$$X_2 = TMT^*, \quad (3.12)$$

with

$$N = \begin{pmatrix} E + C & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad (3.13)$$

$$M = \begin{pmatrix} G + D & -Y_{21}^* + (G + D + G^* + D^*)U_2 & B_{13} \\ Y_{21} & K + \frac{1}{2}U_2^*(G + D + G^* + D^*)U_2 & Y_{23} \\ B_{31} & Y_{32} & B_{33} \end{pmatrix}, \quad (3.14)$$

where $E, G \in \mathbb{SP}_r^*$ are arbitrary but satisfy

$$E + G = L; \quad (3.15)$$

$F \in \mathbb{SP}_{n-r_{a_1}}^*, K \in \mathbb{SP}_{n-r-r_{a_2}}^*; Y_{23} \in \{Y_{23} \in \mathbb{H}^{(n-r-r_{a_2}) \times t} \mid M \in \mathbb{SP}_{n-r}^*\},$
 $Y_{32} \in \mathbb{H}^{t \times (n-r-r_{a_2})}, U_1 \in \mathbb{H}^{r_{a_1} \times (n-r_{a_1})}, U_2 \in \mathbb{H}^{s \times (n-r-r_{a_2})}$ are all arbitrary.

Proof. If matrix equation (1.2) has a solution $X_1 \in \mathbb{SP}_r^*, X_2 \in \mathbb{SP}_{n-r}^*$, then by Lemma 3.3, (3.6) holds and X_1, X_2 has the form of (3.7) and (3.8), respectively. Hence by Lemma 2.2,

$$\begin{pmatrix} Y_{11} & Y_{12} & B_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ B_{31} & Y_{32} & B_{33} \end{pmatrix} \in \mathbb{SP}_{n-r}^* \quad (3.16)$$

and

$$\begin{pmatrix} \begin{pmatrix} B_{11} - Y_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in \mathbb{SP}_r^*. \quad (3.17)$$

For (3.16), it follows from Theorem 2.6 that

$$\begin{pmatrix} Y_{11} & B_{13} \\ B_{31} & B_{33} \end{pmatrix} \in \mathbb{SP}_{s+t}^*, \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \in \mathbb{SP}_{n-r-r_{a_2}+s}^*.$$

By Lemma 2.4, $B_{33} \in \mathbb{SP}_t^*$ and the last equation of (3.10) holds. Moreover,

$$Y_{11} - D \stackrel{\text{def}}{=} G$$

is also rennd, i.e.,

$$Y_{11} = G + D \quad (3.18)$$

where G is rennd;

$$r(Y_{11} + Y_{11}^*, Y_{12} + Y_{21}^*) = r(Y_{11} + Y_{11}^*), \quad (3.19)$$

and

$$Y_{22} - \frac{1}{2}(Y_{21} + Y_{12}^*)(Y_{11} + Y_{11}^*)^\dagger(Y_{21}^* + Y_{12}) \stackrel{\text{def}}{=} K \quad (3.20)$$

is rennd.

It follows from (3.19) that the matrix equation $(Y_{11} + Y_{11}^*)X = Y_{12} + Y_{21}^*$ is consistent for X . Let U_2 is an any solution of the matrix equation. Then by (3.18),

$$Y_{12} = -Y_{21}^* + (G + D + G^* + D^*)U_2. \quad (3.21)$$

Hence by (3.20),

$$\begin{aligned} Y_{22} &= K + \frac{1}{2}U_2^*(Y_{11} + Y_{11}^*)(Y_{11} + Y_{11}^*)^\dagger(Y_{11} + Y_{11}^*)U_2 \\ &= K + \frac{1}{2}U_2^*(Y_{11} + Y_{11}^*)U_2 \\ &= K + \frac{1}{2}U_2^*(G + D + G^* + D^*)U_2. \end{aligned} \quad (3.22)$$

Accordingly, it follows from (3.18), (3.20), (3.21) and (3.22) that (3.16) can be expressed as (3.14) implying X_2 can be expressed as (3.12).

For (3.17), by Lemma 2.4,

$$\begin{pmatrix} B_{11} - Y_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \in \mathbb{S}\mathbb{P}_{r_{a_1}}^*. \quad (3.23)$$

Hence it follows from Lemma 2.4 that $B_{22} \in \mathbb{S}\mathbb{P}_{r_{a_1} - s}^*$ and the first equation of (3.10) holds, and

$$B_{11} - Y_{11} - C \stackrel{\text{def}}{=} E \quad (3.24)$$

is rennd. It is implies that (3.23) can be expressed as (3.13). Therefore, (3.15) follows from (3.18) and $L \in \mathbb{S}\mathbb{P}_s^*$ from E and G is rennd. By (3.17) and Lemma 2.4,

$$r(N + N^*, X_{12} + X_{21}^*) = r(N + N^*) \quad (3.25)$$

and

$$X_{22} - \frac{1}{2}(X_{21} + X_{12}^*)(N + N^*)^\dagger(X_{21}^* + X_{12}) \stackrel{\text{def}}{=} F \quad (3.26)$$

is rennd. It follows from (3.25) that the matrix equation $(N + N^*)X = X_{12} + X_{21}^*$ is solvable for X . Assume that U_1 is an any solution of the matrix equation. By (3.26), we have that

$$X_{12} = -X_{21}^* + (N + N^*)U_1 \quad (3.27)$$

and

$$X_{22} = F + \frac{1}{2}U_1^*(N^* + N)U_1.$$

Consequently, X_1 can be expressed as (3.11).

Conversely, suppose that (3.6), (3.10) and (3.15) hold, where $L \in \mathbb{S}\mathbb{P}_s^*$, $B_{22} \in \mathbb{S}\mathbb{P}_{r_{a_1}-s}^*$, $B_{33} \in \mathbb{S}\mathbb{P}_t^*$. It can be verified that M, N are all rennd by Theorem 2.6, Lemma 2.4. And the matrices with the form of (3.11) and (3.12) are all rennd by Lemma 2.2 and Lemma 2.4. It is easy to verify that the matrices X_1 and X_2 with the form of (3.11) and (3.12), respectively, are the solution of matrix equation (1.2). \square

REMARK 3.5. Setting A_2 vanish in Theorem 3.4, we can get the corresponding results on the rennd solution to matrix equation (1.1) over \mathbb{H} .

4. The reflexive re-nonnegative definite solution to (1.1). In this section, we consider the reflexive rennd solution to the matrix equation (1.1), where $A \in \mathbb{H}^{m \times n}$, $B \in \mathbb{H}^{m \times m}$ are given and $X \in \mathbb{R}\mathbb{S}\mathbb{P}_n^*(P)$ is unknown.

By Theorem 2.9, we can assume that

$$X = U \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} U^* \quad (4.1)$$

where U is defined as (2.2) and $X_1 \in \mathbb{S}\mathbb{P}_r^*$, $X_2 \in \mathbb{S}\mathbb{P}_{n-r}^*$.

Suppose that

$$AU = (A_1, A_2) \quad (4.2)$$

where $A_1 \in \mathbb{H}^{m \times r}$, $A_2 \in \mathbb{H}^{m \times (n-r)}$. Then (1.1) has a solution $X \in \mathbb{R}\mathbb{S}\mathbb{P}_n^*(P)$ if and only if matrix equation (1.2) has a rennd solution X_1, X_2 . By Theorem 3.4, we immediately get the following.

THEOREM 4.1. *Suppose that $A \in \mathbb{H}^{m \times n}$, $B \in \mathbb{H}^{m \times m}$ are given and C, D, L are the same as in Theorem 3.4, then matrix equation (1.1) has a solution $X \in \mathbb{R}\mathbb{S}\mathbb{P}_n^*(P)$ if and only if (3.6), (3.10) and (3.15) hold, $L \in \mathbb{S}\mathbb{P}_s^*$, $B_{22} \in \mathbb{S}\mathbb{P}_{r_{a_1}-s}^*$, $B_{33} \in \mathbb{S}\mathbb{P}_t^*$. In that case, the general solution $X \in \mathbb{R}\mathbb{S}\mathbb{P}_n^*(P)$ can be expressed as (4.1) where X_1 and X_2 are as the same as (3.11) and (3.12), respectively.*

5. Conclusions. In this paper we have presented the criteria that a 3×3 partitioned quaternion matrix is re-nonnegative definite and a quaternion matrix is reflexive re-nonnegative definite. A necessary and sufficient condition for the existence and the expression of re-nonnegative definite solution to the matrix equation (1.2) over \mathbb{H} have been established. Using Theorem 3.4, we have established a necessary and sufficient condition for the existence and the expression of the reflexive re-nonnegative definite solution to matrix equation (1.1) over \mathbb{H} . In closing this paper, we propose the following two open problems which are related to this paper:

(1) How do we investigate the least-square re-nonnegative definite solution to (1.2)

and the least-square reflexive re-nonnegative definite solution to (1.1) over \mathbb{H} ?

(2) How do we investigate the maximal and minimal ranks of the general reflexive re-nonnegative definite solution to the matrix equation (1.1) over \mathbb{H} ?

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