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MORE CALCULATIONS ON DETERMINANT EVALUATIONS*

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Abstract. The purpose of this article is to prove several evaluations of determinants of matrices, the entries of which are given by the recurrence $a_{i,j} = a_{i-1,j-1} + a_{i-1,j}$, $i, j \geq 2$, with various choices for the first row $a_{1,j}$ and first column $a_{i,1}$.

Key words. Determinant, Matrix factorization, Recurrence relation.

AMS subject classifications. 15A15, 11C20.

1. Introduction. Determinants have played a significant part in various areas in mathematics. For instance, they are quite useful in the analysis and solution of systems of linear equations. There are different perspectives on the study of determinants. One may notice several practical and effective instruments for calculating determinants in the nice survey articles [3] and [4].

Much attention has been paid to the symbolic evaluation of determinants of matrices, especially when their entries are given recursively (see Section 5.6 in [4]). Toward these matrices one usually introduces the first row and column as initial conditions for a recurrence relation used for constructing the other entries. Both this relation and the initial conditions play an important part in constructing the matrix, as well as in evaluating its determinant. Several relevant studies on evaluating determinants can be found in the literature; e.g., see [1, 2, 5, 7]. In this article we are interested in the sequences of determinants of matrices satisfying the recurrence relation $a_{i,j} = a_{i-1,j-1} + a_{i-1,j}$ for $2 \leq i, j \leq n$, with various choices for the first row $a_{1,j}$ and first column $a_{i,1}$.

To state our results we need to introduce some notation. The Fibonacci numbers $F(n)$ satisfy

$$\begin{cases} F(0) = 0,\ F(1) = 1 \\ F(n+2) = F(n+1) + F(n) \quad (n \geq 0). \end{cases}$$

Throughout this article, we also use the following notation:

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$$\omega^k(n) = (\omega_i^k)_{1\leq i\leq n} = (\underbrace{1,1,\ldots,1}_{k-\text{times}}, \underbrace{0,0,\ldots,0}_{(n-2k)-\text{times}}, \underbrace{1,1,\ldots,1}_{k-\text{times}}),$$

$$\varpi^k(n) = (\varpi_i^k)_{1\leq i\leq n} = (\underbrace{1,1,\ldots,1}_{(n-k)-\text{times}}, \underbrace{0,0,\ldots,0}_{k-\text{times}}),$$

$$\chi^k = (\chi_i^k)_{i\geq 1} = (\underbrace{a,a,\ldots,a}_{k-\text{times}}, 0,0,0,\ldots),$$

$$\vartheta^k = (\vartheta_i^k)_{i\geq 1} = (a, \underbrace{0,0,\ldots,0}_{k-\text{times}}, a, 0,0,0,\ldots).$$

Given a matrix A we denote by $\mathrm{R}_i(A)$ and $\mathrm{C}_j(A)$ the row i and the column j of A, respectively. We denote by $e_{i,j}$ the square matrix having 1 in the (i,j) position and 0 elsewhere. Now, it is easy to see that

$$e_{i,j}\cdot e_{k,l} = \delta_{jk} e_{i,l}. \tag{1.1}$$

In the case that the two matrices $e_{i,j}$ and A are to be multiplied together, we obtain

$$\mathrm{R}_k(e_{i,j}\cdot A) = \delta_{ki}\mathrm{R}_j(A) \quad \text{and} \quad \mathrm{C}_k(A\cdot e_{i,j}) = \delta_{kj}\mathrm{C}_i(A). \tag{1.2}$$

In fact, the matrix $e_{i,j}\cdot A$ is a matrix that all its rows except for the ith row are 0 and its ith row is the jth row of the matrix A. Similarly, the matrix $A\cdot e_{i,j}$ is a matrix that all its columns except for the jth column are 0 and its jth column is the ith column of the matrix A.

2. Main Results.

THEOREM 2.1. *Let $\alpha = (\alpha_i)_{i\geq 1}$ be a given sequence and let $A = (a_{i,j})_{1\leq i,j\leq n}$ be the doubly indexed sequence given by the recurrence*

$$a_{i,j} = a_{i-1,j-1} + a_{i-1,j}, \quad 2\leq i,j\leq n, \tag{2.1}$$

and the initial conditions $a_{i,1} = \alpha_1 + (i-1)d$, $a_{1,j} = \alpha_j$, $1\leq i,j\leq n$. Then we have $A = L\cdot B$, where $L = (L_{i,j})_{1\leq i,j\leq n}$ is a lower triangular matrix given by the recurrence

$$L_{i,j} = L_{i-1,j-1} + L_{i-1,j}, \quad 2\leq i,j\leq n, \tag{2.2}$$

and the initial conditions $L_{1,1} = 1$, $L_{1,j} = 0$, $2\leq j\leq n$, and $L_{i,1} = 1$, $2\leq i\leq n$, and $B = (B_{i,j})_{1\leq i,j\leq n}$ is a matrix given by the recurrence

$$B_{i,j} = B_{i-1,j-1}, \quad 2\leq i,j\leq n, \tag{2.3}$$

and the initial conditions $B_{1,j} = \alpha_j$, $1\leq j\leq n$, $B_{2,1} = d$ and $B_{i,1} = 0$, $3\leq i\leq n$. Note that B is a Hessenberg-Töplitz matrix. In particular, $\det(A) = \det(B)$.

Proof. For the proof of the claimed factorization we compute the $(i,j)-$entry of $L\cdot B$, that is

$$(L\cdot B)_{i,j} = \sum_{k=1}^{n} L_{i,k}B_{k,j}.$$

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ELA
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In fact, so as to prove the theorem, we should establish

$$\mathrm{R}_1(L\cdot B)=\mathrm{R}_1(A)=(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n),$$

$$\mathrm{C}_1(L\cdot B)=\mathrm{C}_1(A)=(\alpha_1,\ \alpha_1+d,\ \ldots,\ \alpha_1+(n-1)d),$$

and

$$(L\cdot B)_{i,j}=(L\cdot B)_{i-1,j-1}+(L\cdot B)_{i-1,j}, \tag{2.4}$$

for $2\le i,j\le n$.

Let us do the required calculations. First, suppose that $i=1$. Then

$$(L\cdot B)_{1,j}=\sum_{k=1}^{n}L_{1,k}B_{k,j}=L_{1,1}B_{1,j}=\alpha_j,$$

and so $\mathrm{R}_1(L\cdot B)=\mathrm{R}_1(A)=(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n)$.

Next, we suppose that $j=1$, and we obtain

$$(L\cdot B)_{i,1}=\sum_{k=1}^{n}L_{i,k}B_{k,1}=L_{i,1}B_{1,1}+L_{i,2}B_{2,1}=\alpha_1+(i-1)d,$$

which implies that $\mathrm{C}_1(L\cdot B)=\mathrm{C}_1(A)=(\alpha_1,\ \alpha_1+d,\ \ldots,\ \alpha_1+(n-1)d)$.

Finally, we must establish Eq. (2.4). At the moment, let us assume that $2\le i,j\le n$. In this case we have

$$\begin{aligned}
(L\cdot B)_{i,j} &= \textstyle\sum_{k=1}^{n}L_{i,k}B_{k,j}\\
&= L_{i,1}B_{1,j}+\textstyle\sum_{k=2}^{n}L_{i,k}B_{k,j}\\
&= L_{i,1}B_{1,j}+\textstyle\sum_{k=2}^{n}(L_{i-1,k-1}+L_{i-1,k})B_{k,j}\quad\text{(by (2.2))}\\
&= L_{i,1}B_{1,j}+\textstyle\sum_{k=2}^{n}L_{i-1,k-1}B_{k,j}+\sum_{k=2}^{n}L_{i-1,k}B_{k,j}\\
&= L_{i,1}B_{1,j}+\textstyle\sum_{k=2}^{n}L_{i-1,k-1}B_{k-1,j-1}+\sum_{k=1}^{n}L_{i-1,k}B_{k,j}-L_{i-1,1}B_{1,j}\\
&\quad \text{(by (2.3))}\\
&= (L_{i,1}-L_{i-1,1})B_{1,j}+\textstyle\sum_{k=1}^{n}L_{i-1,k}B_{k,j-1}+\sum_{k=1}^{n}L_{i-1,k}B_{k,j}\\
&\quad \text{(it should be noticed that } L_{i-1,n}=0)\\
&= (L\cdot B)_{i-1,j-1}+(L\cdot B)_{i-1,j},
\end{aligned}$$

which is Eq. (2.4). Our proof is thus complete. □

An interesting corollary to Theorem 2.1 is the following.

COROLLARY 2.2. *In Theorem* 2.1*, if* $\alpha=\omega^k(n)$*,* $d=1$ *and* $D(n)=\det(A)$*, then*

$$D(n)=(-1)^{k+1}D(n-k-1),\qquad (n>3k). \tag{2.5}$$

Furthermore, we have

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ELA
http://math.technion.ac.il/iic/ela

(1) *if* $k = 2$, *then*

$$D(n) = \begin{cases} 2 \ (resp. \ -2) & if \quad n \overset{6}{\equiv} 0 \ (resp. \ 3) \\ 0 & if \quad n \overset{6}{\equiv} 1, 4 \\ 1 \ (resp. \ -1) & if \quad n \overset{6}{\equiv} 2 \ (resp. \ 5) \end{cases}$$

(2) *if* $k \geq 3$ *is even, then*

$$D(n) = \begin{cases} 1 & if \quad n \overset{2k+2}{\equiv} 0, 1, \\ -1 & if \quad n \overset{2k+2}{\equiv} k+1, k+2, \\ 0 & otherwise. \end{cases}$$

Also, if $k \geq 3$ *is odd, then*

$$D(n) = \begin{cases} 1 & if \quad n \overset{k+1}{\equiv} 0, 1 \\ 0 & otherwise. \end{cases}$$

Proof. Using Theorem 2.1, $D(n) = \det(B)$ where $B = (B_{i,j})_{1 \leq i,j \leq n}$ is a matrix given by the recurrence

$$B_{i,j} = B_{i-1,j-1}, \quad 2 \leq i, j \leq n,$$

and the initial conditions $B_{1,j} = \omega_j^k$, $1 \leq j \leq n$, $B_{2,1} = 1$ and $B_{i,1} = 0$, $3 \leq i \leq n$. To obtain the result we thus need to compute $\det(B)$. Put $B = B^k(n)$.

We claim that

$$B = U \cdot \tilde{B} \cdot L,$$

where the matrices $U = (U_{i,j})_{1 \leq i,j \leq n}$, $\tilde{B} = (\tilde{B}_{i,j})_{1 \leq i,j \leq n}$, and $L = (L_{i,j})_{1 \leq i,j \leq n}$ are defined as follows:

$$U_{i,j} = \begin{cases} 1 & \text{if} \quad i = j, \\ -1 & \text{if} \quad j = i + n - 2k, \ 1 \leq i \leq k, \\ 0 & \quad \text{otherwise}, \end{cases}$$

$$\tilde{B}_{i,j} = \begin{cases} 1 & \text{if} \quad 1 \leq i \leq k \ \text{ and } \ n - 2k + i - 1 \leq j \leq n - k + i - 1, \\ 0 & \text{if} \quad (i,j) = (n-k, n-k-1), \\ B_{i,j} & \quad \text{otherwise}, \end{cases}$$

$$L_{i,j} = \begin{cases} 1 & \text{if} \quad i = j \ \text{ or } \ (i,j) = (n-1, n-k-1), \\ -1 & \text{if} \quad (i,j) = (n, n-k-1), \\ 0 & \quad \text{otherwise}. \end{cases}$$

It is obvious that, the matrices U and L are the upper triangular matrix and lower triangular one, respectively, with 1's on their diagonals. In addition, we can restate the matrices U and L as follows:

$$U = I - \sum_{i=1}^{k} e_{i,n-2k+i} \quad \text{ and } \quad L = I + e_{n-1,n-k+1} - e_{n,n-k-1}.$$

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ELA
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Moreover, we can partition the matrix $\tilde{B}$ in this way

$$\tilde{B} = \left[\begin{array}{c|c} B_1 & * \\ \hline 0 & B_2 \end{array}\right],$$

where $B_1 = B^k(n-k-1)$, and

$$B_2 = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & 1 & \dots & 1 & 1 \\ 0 & 0 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 \end{bmatrix}.$$

Since $\det(B_2) = (-1)^{k+1}$, it is obvious that the claimed factorization of B immediately implies the validity of Eq. (2.5).

For the proof of the claim, we observe that

$$\begin{aligned} U \cdot \tilde{B} \cdot L &= (I - \textstyle\sum_{i=1}^k e_{i,n-2k+i}) \cdot \tilde{B} \cdot (I + e_{n-1,n-k-1} - e_{n,n-k-1}) \\ &= (\tilde{B} - \textstyle\sum_{i=1}^k e_{i,n-2k+i} \cdot \tilde{B}) \cdot (I + e_{n-1,n-k-1} - e_{n,n-k-1}) \\ &= (B - e_{n-1,n-k-1} + e_{k,n-k-1}) \cdot (I + e_{n-1,n-k-1} - e_{n,n-k-1}) \\ &= B + B \cdot e_{n-1,n-k-1} - B \cdot e_{n,n-k-1} - e_{n-1,n-k-1} + e_{k,n-k-1} \\ &\quad \text{(by (1.1))} \end{aligned}$$

.

Thus, it is enough to show that $B \cdot e_{n-1,n-k-1} - B \cdot e_{n,n-k-1} - e_{n-1,n-k-1} + e_{k,n-k-1} = 0$. To see this, from Eq. (1.2), we obtain

$$\mathrm{C}_l(B \cdot e_{n-1,n-k-1}) = \delta_{l,n-k-1}\mathrm{C}_{n-1}(B) \quad \text{and} \quad \mathrm{C}_l(B \cdot e_{n,n-k-1}) = \delta_{l,n-k-1}\mathrm{C}_n(B),$$

for $l = 1, 2, \dots, n$. Hence, by the structure of B, it follows that

$$B \cdot e_{n-1,n-k-1} - B \cdot e_{n,n-k-1} = e_{n-1,n-k-1} - e_{k,n-k-1}.$$

The rest of the proof is simple and left to the reader. □

Second Proof of Corollary 2.2. Here, we compute $\det(B)$ directly. We will assume that $k > 1$, since the case $k = 1$ is easy. Put $R_i = \mathrm{R}_i(B)$ and $C_j = \mathrm{C}_j(B)$. We first subtract C_1 from C_2 through C_k and from C_{n-k+2} through C_n. Then we subtract R_2 from R_1. So this leaves us with a matrix

$$B_1 = \begin{bmatrix} 0 & 0 & \dots & 0 & \boxed{-1} & \dots & \boxed{1} & 0 & \dots & 0 \\ 1 & 0 & & & & \dots & & & & 0 \\ & & & & & & & & & \\ & & & & & * & & & & \end{bmatrix},$$

where the boxed entries $\boxed{-1}$ and $\boxed{1}$ are in positions $(1, k+1)$ and $(1, n-k+1)$ respectively, and the $*$ part is unchanged from B. The goal now is to push the

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ELA
http://math.technion.ac.il/iic/ela

entries $\boxed{1}$ and $\boxed{-1}$ all the way to position $(1, n)$, with no other changes in the matrix. Let us first deal with $\boxed{1}$. Unless $k = 2$, we can get rid of it by replacing $R_1 \to R_1 + R_{n-k+1} - R_{n-k+2}$. If $k = 2$, we make $R_1 \to R_1 - R_n$ and we then have -1 at $(1, n)$. Next we treat $\boxed{-1}$. We can move it $k+1$ positions to the right by doing $R_1 \to R_1 + R_{k+2} - R_{k+3}$. We can repeatedly move it to the right $k+1$ places by adding and subtracting the appropriate consecutive rows. This is done until we are at $(1, r)$ for $r > n - k$. There are three cases:

(i) If $n \equiv 0 \pmod{k+1}$, we bring the $\boxed{-1}$ to $(1, n)$.

(ii) If $n \equiv 1 \pmod{k+1}$, then $\boxed{-1}$ is brought to $(1, n-1)$ and then $R_1 \to R_1 + R_n$ brings it to $(1, n)$ by changing sign.

(iii) Otherwise we bring $\boxed{-1}$ to $(1, r)$ for $n - k < r < n - 1$ and then we make it disappear by $R_1 \to R_1 + R_{r+1} - R_{r+2}$. At this point we have the matrix

$$B_2 = \begin{bmatrix} 0 & \dots & 0 & c \\ 1 & 0 & \dots & 0 \\ & & * & \end{bmatrix},$$

where $c = a_n + b_n$ for

$$a_n = \begin{cases} -1 & k = 2 \\ 0 & k > 2 \end{cases} \qquad \text{and} \qquad b_n = \begin{cases} -1 & n \equiv 0 \pmod{k+1} \\ 1 & n \equiv 1 \pmod{k+1} \\ 0 & \text{otherwise} \end{cases}.$$

Expanding $\det(B_2)$ by the first row we see that $\det(B) = \det(B_2) = (-1)^{n-1}c$, and the corollary follows from here immediately. $\square$

Corollary 2.3. *Suppose that in Theorem* 2.1, $D(n) = \det(A)$. *Then*

(i) If $\alpha = \chi^k$, *then*

$$D(n) = a\sum_{i=1}^{k}(-d)^{i-1}D(n-i), \qquad n \geq k+1.$$

In particular, in case that $k = 2$ *and* $a = 1$, *we have*

$$D(n) = \frac{1}{\eta}\Big[\Big(\frac{1+\eta}{2}\Big)^{n+1} - \Big(\frac{1-\eta}{2}\Big)^{n+1}\Big].$$

where $\eta = \sqrt{1-4d}$.

(ii) If $\alpha = \varpi^k(n)$ *and* $d = 1$, *then we have*

$$\det(A) = \begin{cases} (-1)^n & \text{if } n \equiv 0 \ (\text{mod } n-k+1), \\ (-1)^{n+1} & \text{if } n \equiv 1 \ (\text{mod } n-k+1), \\ 0 & \text{otherwise.} \end{cases}$$

(iii) If $\alpha = \vartheta^k$, *then*

$$D(n) = \begin{cases} a^n & \text{if } \ 1 \leq n \leq k+1, \\ a^n + (-1)^{n-1}ad^{n-1} & \text{if } \ n = k+2, \\ aD(n-1) + a(-d)^{k+1}D(n-k-2) & \text{if } \ n \geq k+3. \end{cases}$$

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ELA
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Proof. First of all, by Theorem 2.1, $D(n) = \det(B)$ where $B = (B_{i,j})_{1\leq i,j\leq n}$ is a matrix given by the recurrence

$$B_{i,j} = B_{i-1,j-1}, \quad 2 \leq i, j \leq n,$$

and the initial conditions $B_{1,j} = \alpha_j$, $1 \leq j \leq n$, $B_{2,1} = d$ and $B_{i,1} = 0$, $3 \leq i \leq n$.

(i). Suppose $\alpha = \chi^k$. If we expand the determinant $\det(B)$ along the first row, then the $(1, j)$ cofactor is of the form $\det(dI_{j-1} \oplus B_{n-j})$, where B_{n-j} equals the principal submatrix of B taking the first $n - j$ rows and columns. Now, we easily get the desired recursion, that is

$$D(n) = a\sum_{i=1}^{k}(-d)^{i-1}D(n-i), \qquad n \geq k+1.$$

Now, assume $k = 2$ and $a = 1$. In this case, we have

$$D(1) = 1, \quad D(2) = 1 - d, \quad \text{and} \quad D(n) = D(n-1) - dD(n-2), \quad n \geq 3.$$

To solve this recurrence relation, we see that its characteristic equation as follows

$$x^2 - x + d = 0,$$

and its characteristic roots are

$$x_1 = \frac{1-\eta}{2} \qquad \text{and} \qquad x_2 = \frac{1+\eta}{2},$$

where $\eta = \sqrt{1-4d}$. Therefore, the general solution is given by

$$D(n) = \lambda_1\Big(\frac{1-\eta}{2}\Big)^n + \lambda_2\Big(\frac{1+\eta}{2}\Big)^n, \tag{2.6}$$

where λ_1 and λ_2 are constants. The initial conditions $D(1) = 1$ and $D(2) = 1 - d$ imply that $\lambda_1 = -\frac{1}{\eta}(\frac{1-\eta}{2})$ and $\lambda_2 = \frac{1}{\eta}(\frac{1+\eta}{2})$, and so

$$D(n) = \frac{1}{\eta}\Big[\Big(\frac{1+\eta}{2}\Big)^{n+1} - \Big(\frac{1-\eta}{2}\Big)^{n+1}\Big].$$

(ii). Assume $\alpha = \varpi^k(n)$ and $d = 1$. We put $n - k = p$. Similar arguments as in part (i) with k replaced by p, show that

$$D(1) = 1, \; D(2) = D(3) = \cdots = D(p) = 0,$$

and

$$D(n) = \sum_{i=1}^{p}(-1)^{i+1}D(n-i), \qquad n \geq p+1.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Hence, we observe that

$$\begin{aligned} D(m+p+1) &= \textstyle\sum_{i=1}^{p}(-1)^{i+1}D(m+p+1-i) \\ &= D(m+p)+\textstyle\sum_{i=2}^{p}(-1)^{i+1}D(m+p+1-i) \\ &= \textstyle\sum_{i=1}^{p}(-1)^{i+1}D(m+p-i)+\sum_{i=2}^{p}(-1)^{i+1}D(m+p+1-i) \\ &= (-1)^{p+1}D(m), \qquad (m\geq 1). \end{aligned} \tag{2.7}$$

Now, using the division algorithm we find integers r and s such that

$$n = s(p+1)+r, \quad \text{and} \quad 0 \leq r < p+1.$$

First, assume that $r = 0$. In this case, from Eq. (2.7) we obtain

$$D(n) = D(s(p+1)) = (-1)^{(s-1)(p+1)}D(p+1) = (-1)^{(s-1)(p+1)}(-1)^{(p+1)} = (-1)^n.$$

Next assume that $1 \leq r < p+1$. In this case, again by Eq. (2.7) it follows that

$$D(n) = D(s(p+1)+r) = (-1)^{s(p+1)}D(r) = \begin{cases} 0 & \text{if} \quad 1 < r < p+1 \\ (-1)^{n-1} & \text{if} \quad r = 1. \end{cases}$$

This completes the proof of part (ii).

(iii) The proof is similar to the previous parts. ☐

THEOREM 2.4. *Let $\alpha = (\alpha_i)_{i\geq 1}$ be a given sequence and let $\beta = (\beta_i)_{i\geq 1}$ be a sequence satisfying $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$ and the linear recursion*

$$\beta_i = \beta_{i-2} + \beta_{i-1}, \quad i \geq 3.$$

Let $(a_{i,j})_{i,j\geq 1}$ be the doubly indexed sequence given by the recurrence

$$a_{i,j} = a_{i-1,j-1} + a_{i-1,j}, \quad i,j \geq 2, \tag{2.8}$$

and the initial conditions $a_{i,1} = \beta_i$ and $a_{1,j} = \alpha_j$, where $i,j \geq 1$. Then

$$\det_{1\leq i,j\leq n}(a_{i,j}) = \begin{cases} \alpha_1 & \textit{if} \quad n = 1, \\ \alpha_1^2 - \alpha_2^2 + \alpha_1\alpha_2 & \textit{if} \quad n = 2, \\ (\alpha_1^2 - \alpha_2^2 + \alpha_1\alpha_2)(\alpha_1+\alpha_2-\alpha_3)^{n-2} & \textit{if} \quad n \geq 3. \end{cases}$$

Proof. For $n \leq 3$ the result is straightforward. Hence, we assume that $n \geq 4$. Let A denote the matrix $(a_{i,j})_{1\leq i,j\leq n}$. We claim that

$$A = L \cdot B,$$

where $L = (L_{i,j})_{1\leq i,j\leq n}$ is a lower triangular matrix by the recurrence

$$L_{i,j} = L_{i-1,j-1} + L_{i-1,j}, \quad i \geq 2, j \geq 3, \tag{2.9}$$

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ELA
http://math.technion.ac.il/iic/ela

and the initial conditions $L_{i,1} = F(i)$, $L_{1,i} = 0$, $i \geq 2$, and $L_{i,2} = F(i-1)$, $i \geq 2$, and where $B = (B_{i,j})_{1 \leq i,j \leq n}$ with

$$B_{i,j} = \begin{cases} \alpha_j & \text{if} \quad i = 1,\, j \geq 1, \\ \alpha_2 - \alpha_1 & \text{if} \quad i = 2, j = 1, \\ B_{2,j-1} + B_{2,j} - B_{1,j} & \text{if} \quad i = 3, j \geq 2, \\ B_{i-1,j-1} & \text{if} \quad i \neq 1, 3,\, j \geq 2, \\ 0 & \text{if} \quad i \geq 3, j = 1. \end{cases} \tag{2.10}$$

For instance, when $n = 4$ the matrix B is hence given by

$$B = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_2 - \alpha_1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 0 & \alpha_1 + \alpha_2 - \alpha_3 & \alpha_2 + \alpha_3 - \alpha_4 \\ 0 & 0 & 0 & \alpha_1 + \alpha_2 - \alpha_3 \end{bmatrix}.$$

Notice that, by the structure of matrices L and B, for every $i \geq 2$, it follows that

$$L_{i,3} - L_{i-1,3} = L_{i-1,2}, \quad L_{i,2} = L_{i-1,1}, \quad B_{2,i} = B_{1,i-1}. \tag{2.11}$$

By expanding, in terms of the first column, we easily see that

$$\det(B) = \begin{cases} \alpha_1 & \text{if} \quad n = 1, \\ \alpha_1^2 - \alpha_2^2 + \alpha_1\alpha_2 & \text{if} \quad n = 2, \\ (\alpha_1^2 - \alpha_2^2 + \alpha_1\alpha_2)(\alpha_1 + \alpha_2 - \alpha_3)^{n-2} & \text{if} \quad n \geq 3, \end{cases}$$

and so the claimed factorization of A immediately implies the validity of the theorem.

As before, the proof of the claim requires again some calculations. In fact, it suffices to show that $\mathrm{R}_1(L \cdot B) = \mathrm{R}_1(A)$, $\mathrm{C}_1(L \cdot B) = \mathrm{C}_1(A)$ and

$$(L \cdot B)_{i,j} = (L \cdot B)_{i-1,j-1} + (L \cdot B)_{i-1,j}, \tag{2.12}$$

for $2 \leq i, j \leq n$.

First, suppose that $i = 1$. Then

$$(L \cdot B)_{1,j} = \sum_{k=1}^{n} L_{1,k} B_{k,j} = L_{1,1} B_{1,j} = \alpha_j,$$

and so $\mathrm{R}_1(L \cdot B) = \mathrm{R}_1(A) = (\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n)$.

Next, we assume that $i \geq 2$ and $j = 1$. In this case we have

$$\begin{aligned} (L \cdot B)_{i,1} &= \sum_{k=1}^{n} L_{i,k} B_{k,1} = L_{i,1} B_{1,1} + L_{i,2} B_{2,1} \\ &= F(i)\alpha_1 + F(i-1)(\alpha_2 - \alpha_1) = F(i-2)\alpha_1 + F(i-1)\alpha_2, \end{aligned}$$

and we get $\mathrm{C}_1(L \cdot B) = \mathrm{C}_1(A) = (\alpha_1,\ \alpha_2,\ \ldots,\ F(n-2)\alpha_1 + F(n-1)\alpha_2)$.

Finally, we must establish Eq. (2.12). Therefore, we assume that $2 \leq i, j \leq n$. In this case we have

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$$
\begin{aligned}
(L\cdot B)_{i,j} &= \textstyle\sum_{k=1}^{n} L_{i,k}B_{k,j} \\
&= \textstyle\sum_{k=1}^{3} L_{i,k}B_{k,j} + \sum_{k=4}^{n} L_{i,k}B_{k,j} \\
&= \textstyle\sum_{k=1}^{3} L_{i,k}B_{k,j} + \sum_{k=4}^{n} (L_{i-1,k-1}+L_{i-1,k})B_{k,j} \quad \text{(by (2.9))} \\
&= \textstyle\sum_{k=1}^{3} L_{i,k}B_{k,j} + \sum_{k=4}^{n} L_{i-1,k-1}B_{k,j} + \sum_{k=4}^{n} L_{i-1,k}B_{k,j} \\
&= \textstyle\sum_{k=1}^{3} L_{i,k}B_{k,j} + \sum_{k=4}^{n} L_{i-1,k-1}B_{k-1,j-1} + \sum_{k=4}^{n} L_{i-1,k}B_{k,j} \\
&\quad \text{(by (2.10))} \\
&= \textstyle\sum_{k=1}^{3} L_{i,k}B_{k,j} + \sum_{k=3}^{n} L_{i-1,k}B_{k,j-1} + \sum_{k=4}^{n} L_{i-1,k}B_{k,j} \\
&\quad \text{(it should be noticed that } L_{i-1,n}=0) \\
&= (L\cdot B)_{i-1,j-1} + (L\cdot B)_{i-1,j} + \textstyle\sum_{k=1}^{3} L_{i,k}B_{k,j} \\
&\quad -\textstyle\sum_{k=1}^{2} L_{i-1,k}B_{k,j-1} - \sum_{k=1}^{3} L_{i-1,k}B_{k,j}, \\
&= (L\cdot B)_{i-1,j-1} + (L\cdot B)_{i-1,j} + (L_{i,1}-L_{i-1,1})B_{1,j} + (L_{i-1,2} \\
&\quad +L_{i-1,3})B_{3,j} - L_{i-1,2}B_{2,j-1} - L_{i-1,2}B_{2,j} - L_{i-1,3}B_{3,j}, \\
&\quad \text{(by (2.11))} \\
&= (L\cdot B)_{i-1,j-1} + (L\cdot B)_{i-1,j} + L_{i-1,2}(B_{1,j}+B_{3,j}-B_{2,j-1}-B_{2,j}), \\
&\quad \text{(because } L_{i,1}-L_{i-1,1}=L_{i-2,1}=L_{i-1,2}) \\
&= (L\cdot B)_{i-1,j-1} + (L\cdot B)_{i-1,j} \quad \text{(by (2.10))},
\end{aligned}
$$

which is Eq. (2.12), and the proof is completed. □

As an immediate consequence of Theorem 2.4, we have the following corollary.

COROLLARY 2.5. *In Theorem* 2.4, *if* $\alpha = \omega^k(n)$ *with* $k \geq 2$ *and* $\beta = (\beta_i)_{1\leq i\leq n} = (1, 1, 2, \ldots, F(n))$, *then*

$$
\det_{1\leq i,j\leq n}(a_{i,j}) = \begin{cases} 1 & \text{if} \quad k=2,\ n=4, \\ 2^{n-2} & \text{if} \quad k=2,\ n\geq 5, \\ 1 & \text{if} \quad k\geq 3,\ n\geq 2k. \end{cases}
$$

THEOREM 2.6. *Let* $(a_{i,j})_{1\leq i,j\leq n}$ *be the doubly indexed sequence given by the recurrence*

$$
a_{i,j} = a_{i-1,j-1} + a_{i-1,j}, \quad i,j \geq 2, \tag{2.13}
$$

and the initial conditions $a_{i,1} = F(i)$ *and* $a_{1,j} = \omega_j^1$, *where* $1 \leq i, j \leq n$. *Then*

$$
\det_{1\leq i,j\leq n}(a_{i,j}) = \begin{cases} 2 & \text{if} \quad n \ \text{even}, \\ 0 & \text{if} \quad n \ \text{odd}; \end{cases}
$$

where $n \geq 3$.

Proof. Let us denote the matrix $(a_{i,j})_{1\leq i,j\leq n}$ by A. We claim that

$$
A = L \cdot B,
$$

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where $L = (L_{i,j})_{1 \le i,j \le n}$ is a matrix by the recurrence

$$L_{i,j} = L_{i-1,j-1} + L_{i-1,j}, \quad i \ge 2, j \ge 3, \tag{2.14}$$

and the initial conditions $L_{i,1} = F(i)$, $L_{1,i} = 0$, and $L_{i,2} = F(i-1)$, $i \ge 2$, and where $B = (B_{i,j})_{1 \le i,j \le n}$ with

$$B_{i,j} = \begin{cases} \omega_j^1 & \text{if} \quad i = 1,\, j \ge 1, \\ B_{2,j-1} + B_{2,j} - B_{1,j} & \text{if} \quad i = 3, j \ge 2, \\ B_{i-1,j-1} & \text{if} \quad i \ne 1, 3,\, j \ge 2, \\ 0 & \text{if} \quad i \ge 2, j = 1. \end{cases} \tag{2.15}$$

The proof of the claim is quite similar to the proof of Theorem 2.4 and we leave it to the reader. Notice that, the matrix L is a lower triangular one with 1's on the diagonal, and hence $\det(A) = \det(B)$. Through a short computation one gets $\det(B) = 1 + (-1)^n$, and so the theorem follows now immediately. ◻

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