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SUBDIRECT SUMS OF DOUBLY DIAGONALLY DOMINANT MATRICES*

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Abstract. The problem of when the k-subdirect sum of a doubly diagonally dominant matrix (DDD matrix) is also a DDD matrix is studied. Some sufficient conditions are given. The same situation is analyzed for diagonally dominant matrices and strictly diagonally dominant matrices. Additionally, some conditions are also derived when card(S)>card(S_1) which was not studied by Bru, Pedroche and Szyld [*Electron. J. Linear Algebra*, 15:201-209, 2006]. Examples are given to illustrate the conditions presented.

Key words. Subdirect sum, Doubly diagonally dominant matrices, H-matrices, Overlapping blocks.

AMS subject classifications. 15A48.

1. Introduction. The concept of k-subdirect sums of square matrices was introduced by Fallat and Johnson [4], where many of their properties were analyzed. They showed that the subdirect sum of positive definite matrices, or of symmetric M-matrices, is a positive definite matrix or symmetric M-matrix, respectively. Subdirect sums of matrices are generalizations of the usual sum of matrices and arise naturally in several contexts. For example, subdirect sums arise in matrix completion problems, overlapping subdomains in domain decomposition methods, and global stiffness matrices in finite elements (see [1, 2, 4]).

In [1], Bru, Pedroche and Szyld have given sufficient conditions such that the subdirect sum of two nonsingular M-matrices is a nonsingular M-matrix. Also in [3], they also showed that k-subdirect sum of S-strictly diagonally dominant matrices, which is a subclass of H-matrices, is an S-strictly diagonally dominant matrix (S-SDD matrix). Sufficient conditions are also given.

In this paper, we will give sufficient conditions such that the k-subdirect sum of matrices belongs to the same class. We show this for certain strictly diagonally dominant (SDD) matrices and for doubly diagonally dominant (DDD) matrices which were introduced in [5]; see also [6] for further properties and analysis. Furthermore, we also discuss the conditions such that the subdirect sum of S-SDD matrices is in the class of S-SDD matrices (for a fixed set S) when card(S)>card(S_1) which was not studied in [3]. Notice, the sets S and S_1 appear in section 2.

2. Subdirect sums. Let A and B be two square matrices of order m_1 and m_2, respectively, and let k be an integer such that $1 \le k \le \min\{m_1, m_2\}$. Let A and B be

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partitioned into 2×2 blocks as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{2.1}$$

where A_{22} and B_{11} are square matrices of order k. Following [4], we call the following square matrix of order $N = m_1 + m_2 - k$,

$$C = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} + B_{11} & B_{12} \\ 0 & B_{21} & B_{22} \end{bmatrix}$$

the k-subdirect sum of A and B and we denote it by $C = A \oplus_k B$.

In order to more explicitly express each element of C in terms of the ones of A and B, we can write C as follows,

$$C = \begin{bmatrix} \overbrace{a_{11} \quad \cdots}^{S_1} & \overbrace{a_{1p} \quad \cdots \quad a_{1,m_1}}^{S_2} & \overbrace{\cdots \quad 0}^{S_3} \\ \vdots & \vdots \qquad\qquad \vdots & \vdots \\ a_{p1} \quad \cdots & a_{pp} + b_{11} \quad \cdots \quad a_{p,m_1} + b_{1,m_1-t} & \cdots \quad b_{p,N-t} \\ \vdots & \vdots \qquad\qquad \vdots & \vdots \\ a_{m_1,1} \quad \cdots & a_{m_1,p} + b_{m_1-t,1} \quad \cdots \quad a_{m_1,m_1} + b_{m_1-t,m_1-t} & \cdots \quad b_{m_1-t,N-t} \\ \vdots & \vdots \qquad\qquad \vdots & \vdots \\ 0 \quad \cdots & b_{N-t,1} \quad \cdots \quad b_{N-t,m_1-t} & \cdots \quad b_{N-t,N-t} \end{bmatrix}$$

$$\begin{aligned} S_1 &= \{1, 2, \ldots, m_1 - k\}, \\ S_2 &= \{m_1 - k + 1, m_1 - k + 2, \ldots, m_1\}, \\ S_3 &= \{m_1 + 1, m_1 + 2, \ldots, N\}. \end{aligned} \tag{2.2}$$

$$c_{ij} = \begin{cases} a_{ij} \text{ or } 0, & i \in S_1,\ j \in S_1 \cup S_2 \cup S_3, \\ a_{ij} \text{ or } a_{ij} + b_{i-t,j-t} \text{ or } b_{i-t,j-t}, & i \in S_2,\ j \in S_1 \cup S_2 \cup S_3, \\ b_{i-t,j-t} \text{ or } 0, & i \in S_3,\ j \in S_1 \cup S_2 \cup S_3, \end{cases} \tag{2.3}$$

$$N = m_1 + m_2 - k, \quad t = m_1 - k, \quad p = t + 1.$$

In general, the subdirect sum of two DDD matrices is not always a DDD matrix. We show this in the following example.

Example 2.1. The matrices

$$A = \begin{bmatrix} 1 & \vdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ -2 & \vdots & 4 & 0 \\ -1 & \vdots & 0 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -2 & \vdots & -1 \\ -1 & 2 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \vdots & 4 \end{bmatrix},$$

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are two DDD matrices, but

$$C = A \oplus_2 B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & 7 & -2 & -1 \\ -1 & -1 & 6 & 0 \\ 0 & 1 & 1 & 4 \end{bmatrix}$$

is not a DDD matrix.

3. Subdirect sums of doubly diagonally dominant matrices. We now formally introduce some notations and definitions which can be found in [5], [6] and [9].

Let $A = (a_{ij}) \in C^{n,n}, n \geq 2$. For $i = 1, 2, \ldots, n$, we define

$$R_i(A) = \sum_{j \neq i, j=1}^{n} |a_{ij}|.$$

Recall that A is called (row) diagonally dominant if for all $i = 1, 2, \ldots, n$,

$$|a_{ii}| \geq R_i(A). \tag{3.1}$$

If the inequalities in (3.1) hold strictly for all i, we say that A is strictly diagonally dominant.

Given any nonempty set of indices $S \subseteq N = \{1, 2, \ldots, n\}$, we denote its complement in N by $\bar{S} = N \backslash S$. We have

$$R_i(A) = R_i^S(A) + R_i^{\bar{S}}(A), \;\; R_i^S(A) = \sum_{j \neq i, j \in S} |a_{ij}|, \; \forall i = 1, 2, \ldots, n.$$

Definition 3.1. The matrix $A = (a_{ij}) \in C^{n,n}$ is doubly diagonally dominant if

$$|a_{ii}||a_{jj}| \geq R_i(A) R_j(A), \quad i, j = 1, 2, \ldots, n, \quad i \neq j \;. \tag{3.2}$$

Furthermore, if the inequality in (3.2) is strict for all distinct $i, j = 1, 2, \ldots, n$, we say A is strictly doubly diagonally dominant.

Definition 3.2. Let the matrix $A = (a_{ij}) \in C^{n,n}, n \geq 2$, and given any nonempty proper subset S of N. Then A is an S-strictly diagonally dominant matrix if the following two conditions hold:

$$\begin{cases} 1) \; |a_{ii}| > R_i^S(A) \; \textit{for at least one } i \in S, \\ 2) \; (|a_{ii}| - R_i^S(A))(|a_{jj}| - R_j^{\bar{S}}(A)) > R_i^{\bar{S}}(A) R_j^S(A) \; \textit{for all } i \in S, j \in \bar{S}. \end{cases}$$

It was shown in [5] that SDD matrices are contained into DDD matrices. In addition, if A is doubly diagonally dominant, there exists at most one index i_0 such that $|a_{i_0 i_0}| < R_{i_0}(A)$. It is easy to show that a doubly diagonally dominant matrix is not necessary an S-strictly diagonally dominant matrix. We show this in the following example.

Example 3.3. The following matrix:

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$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -0.5 & 1 \end{bmatrix}$$

is a DDD matrix, but is not an S-SDD matrix for any subset S of $\{1,2,3\}$.

We now give the following theorem, which presents sufficient condition such that $C = A \oplus_k B$ is a DDD matrix.

THEOREM 3.4. *Let A and B be square matrices of order m_1 and m_2 partitioned as in* (2.1)*, respectively, and k be an integer such that $1 \le k \le \min(m_1, m_2)$ which defines the sets S_1, S_2, S_3 as in* (2.2)*. We assume that A is doubly diagonally dominant with*

$$\max_{l \in S_2 \cup S_3} \frac{R_{l-t}(B)}{|b_{l-t,l-t}|} R_{i_0}(A) \le |a_{i_0 i_0}| < R_{i_0}(A)$$

for some i_0, and B is strictly diagonally dominant. If all diagonal entries of A_{22} and B_{11} are positive (or all negative), then the k-subdirect sum $C = A \oplus_k B$ is doubly diagonally dominant.

Proof. Without loss of generality, we can assume $a_{11} < R_1(A)$ and $a_{ii} \ge R_i(A)$, $i = 2, 3, \ldots, m_1$.

Case 1: $\forall i,\ l \in S_1,\ j \in S_1 \cup S_2 \cup S_3$, using equation (2.3) we obtain

$$|c_{ij}| = |a_{ij}|,\ R_i(C) = R_i(A),$$

since A is doubly diagonally dominant, we obtain

$$|c_{ii}||c_{ll}| = |a_{ii}||a_{ll}| \ge R_i(A)R_l(A) = R_i(C)R_l(C).$$

Case 2: $\forall i \in S_1,\ l \in S_2,\ j \in S_1 \cup S_2 \cup S_3$, from equation (2.3), we have the following relations:

$$|c_{ii}| = |a_{ii}|,\ R_i(C) = R_i(A),$$

$$\begin{aligned} R_l(C) &= \sum_{j \in S_1} |a_{lj}| + \sum_{j \in S_2, j \ne l} |a_{lj} + b_{l-t,j-t}| + \sum_{j \in S_3} |b_{l-t,j-t}| \\ &\le \sum_{j \in S_1} |a_{lj}| + \sum_{j \in S_2, j \ne l} |a_{lj}| + \sum_{j \in S_2, j \ne l} |b_{l-t,j-t}| + \sum_{j \in S_3} |b_{l-t,j-t}| \\ &= R_l(A) + R_{l-t}(B). \end{aligned}$$

Since all diagonal entries of A_{22} and B_{11} are positive (or all negative), we obtain

$$|c_{ll}| = |a_{ll} + b_{l-t,l-t}| = |a_{ll}| + |b_{l-t,l-t}|.$$

If $i = 1$, we have

$$\begin{aligned} |c_{11}||c_{ll}| &= |a_{11}|(|a_{ll} + b_{l-t,l-t}|) \\ &= |a_{11}||a_{ll}| + |a_{11}||b_{l-t,l-t}| \\ &\ge R_1(A)R_l(A) + |b_{l-t,l-t}| \max_{l \in S_2 \cup S_3} \tfrac{R_{l-t}(B)}{|b_{l-t,l-t}|} R_1(A) \\ &\ge R_1(A)R_l(A) + |b_{l-t,l-t}| \tfrac{R_{l-t}(B)}{|b_{l-t,l-t}|} R_1(A) \\ &= R_1(A)(R_l(A) + R_{l-t}(B)) \\ &\ge R_1(C)R_l(C). \end{aligned}$$

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If $i = 2, \ldots, m_1 - k$, then we can write

$$\begin{aligned} |c_{ii}||c_{ll}| &= |a_{ii}|(|a_{ll} + b_{l-t,l-t}|) \\ &\geq R_i(A)(R_l(A) + R_{l-t}(B)) \\ &\geq R_i(C)R_l(C). \end{aligned}$$

Case 3: $\forall i \in S_1,\ l \in S_3,\ j \in S_1 \cup S_2 \cup S_3$, we get

$$|c_{ij}| = |a_{ij}|,\ R_i(C) = R_i(A),$$

$$|c_{lj}| = |b_{l-t,j-t}|,\ R_l(C) = R_{l-t}(B).$$

Like the proof of Case 2, let $i = 1$, we conclude

$$\begin{aligned} |c_{11}||c_{ll}| &= |a_{11}||b_{l-t,l-t}| \\ &\geq \max_{l \in S_2 \cup S_3} \tfrac{R_{l-t}(B)}{|b_{l-t,l-t}|} R_1(A)|b_{l-t,l-t}| \\ &\geq R_1(A)\tfrac{R_{l-t}(B)}{|b_{l-t,l-t}|}|b_{l-t,l-t}| = R_1(C)R_l(C). \end{aligned}$$

If $i = 2, \ldots, m_1 - k$, obviously, the result still holds.

Case 4: $\forall i \in S_2,\ l \in S_2,\ j \in S_1 \cup S_2 \cup S_3$, using equation (2.3), it follows that

$$|c_{ii}| = |a_{ii} + b_{i-t,i-t}| = |a_{ii}| + |b_{i-t,i-t}|,$$

$$R_i(C) = \sum_{i \neq j} |c_{ij}| \leq R_i(A) + R_{i-t}(B).$$

Note that $(|a_{ii}| \geq R_i(A))$ since A is a DDD matrix and $(|b_{i-t,i-t}| > R_{i-t}(B))$ since B is an SDD matrix, thus we can write

$$\begin{aligned} |c_{ii}||c_{ll}| &= |a_{ii} + b_{i-t,i-t}||a_{ll} + b_{l-t,l-t}| \\ &> (R_i(A) + R_{i-t}(B))(R_l(A) + R_{l-t}(B)) \\ &\geq R_i(C)R_l(C). \end{aligned}$$

For the rest of cases $i \in S_2,\ l \in S_3,\ j \in S_1 \cup S_2 \cup S_3$ and $i \in S_3,\ l \in S_3$, $j \in S_1 \cup S_2 \cup S_3$, the proofs are similar to the proofs of the Case 1 and Case 2.

In the case $i_0 \in S_2$ the conclusion still holds and the proofs are similar to the cases above. Therefore the details of the case are omitted. ◻

Corollary 3.5. *Let A_1 be a DDD matrix with existing only i_0 such that $\max_{l_1 \in S_2 \cup S_3} \frac{R_{l_1-t}(A_i)}{|a_{l_1-t,l_1-t}|} R_{i_0}(A_1) \leq |a_{i_0 i_0}| < R_{i_0}(A_1),\ i = 2, 3, \ldots,$ and let $A_2,\ A_3, \ldots,$ be SDD matrices. If the diagonals of A_i in the overlapping have the same sign pattern, then $C = (A_1 \oplus_{k_1} A_2) \oplus_{k_2} A_3 \oplus \cdots$ is a DDD matrix.*

Remark 3.6. Since $A_2, A_3, \ldots$ are SDD we have that their diagonals are nonzero. Since the overlapping blocks of A_2 and A_1 have the same sign pattern, this implies that overlapping block of A_1 has its diagonal nonzero.

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THEOREM 3.7. *Let A and B be matrices of order m_1 and m_2 partitioned as in* (2.1)*, respectively, and k be an integer such that $1 \leq k \leq \min(m_1, m_2)$, which defines the sets S_1, S_2, S_3 as in* (2.2)*. Let A be diagonally dominant and B be strictly diagonally dominant. If all diagonal entries of A_{22} and B_{11} are positive (or all negative), then the k-subdirect sum $C = A \oplus_k B$ is doubly diagonally dominant.*

Proof. We only prove the following cases.

Case 1: $\forall i,\ l \in S_1,\ j \in S_1 \cup S_2 \cup S_3$ from equation (2.3), we obtain $|c_{ij}| = |a_{ij}|,\ R_i(C) = R_i(A)$, so

$$|c_{ii}||c_{ll}| = |a_{ii}||a_{ll}| \geq R_i(A)R_l(A) = R_i(C)R_l(C).$$

Case 2: $\forall i \in S_2,\ l \in S_2,\ j \in S_1 \cup S_2 \cup S_3$, using equation (2.3), it follows that

$$|c_{il}| = |a_{il} + b_{i-t,l-t}|,$$

$$\begin{aligned} R_i(C) = \sum_{i \neq j} |c_{ij}| &= \sum_{j \in S_1} |a_{ij}| + \sum_{i \neq j, j \in S_2} |a_{ij} + b_{i-t,j-t}| + \sum_{j \in S_3} |b_{i-t,j-t}| \\ &\leq R_i(A) + R_{i-t}(B). \end{aligned}$$

Then we can write

$$\begin{aligned} |c_{ii}||c_{ll}| &= |a_{ii} + b_{i-t,i-t}||a_{ll} + b_{l-t,l-t}| \\ &> (R_i(A) + R_{i-t}(B))(R_l(A) + R_{l-t}(B)) \\ &\geq R_i(C)R_l(C). \end{aligned}$$

Finally, $\forall i \in S_1,\ l \in S_3,\ j \in S_1 \cup S_2 \cup S_3$; $\forall i \in S_2,\ l \in S_3,\ j \in S_1 \cup S_2 \cup S_3$; and $\forall i \in S_3,\ l \in S_3,\ j \in S_1 \cup S_2 \cup S_3$; the proofs are similar to the cases above. □

COROLLARY 3.8. *Successive k-subdirect sums of the form $(A_1 \oplus_{k_1} A_2) \oplus_{k_2} A_3 \oplus \cdots$ is doubly diagonally dominant if the diagonals of A_i in the overlapping have the same sign pattern, where A_1 is diagonally dominant and $A_2,\ A_3, \ldots$, are strictly diagonally dominant.*

EXAMPLE 3.9. In this example, A is doubly diagonally dominant with

$$\max_{j \in S_2 \cup S_3} \frac{R_{j-t}(B)}{|b_{j-t,j-t}|} R_{i_0}(A) > |a_{i_0 i_0}| \text{ for all } i_0 = 1, 2, 3, 4,$$

and B is strictly diagonally dominant, but the subdirect sum C is not doubly diagonally dominant. Let

$$A = \begin{bmatrix} 1.0 & -0.3 & \vdots & -0.4 & -0.5 \\ -0.8 & 1.6 & \vdots & 0.2 & -0.3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -0.2 & -0.4 & \vdots & 1.2 & -0.4 \\ -0.1 & -0.9 & \vdots & -0.5 & 2.0 \end{bmatrix},\ B = \begin{bmatrix} 2.0 & -0.4 & \vdots & -0.5 & -0.6 \\ -0.6 & 2.3 & \vdots & -0.8 & -0.8 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -0.5 & -0.1 & \vdots & 2.4 & -0.9 \\ -0.5 & -0.8 & \vdots & -0.2 & 2.9 \end{bmatrix}.$$

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However,

$$C = A \oplus_2 B = \begin{bmatrix} 1.0 & -0.3 & -0.4 & -0.5 & 0 & 0 \\ -0.8 & 1.6 & 0.2 & -0.3 & 0 & 0 \\ -0.2 & -0.4 & 3.2 & -0.8 & -0.5 & -0.6 \\ -0.1 & -0.9 & -1.1 & 4.3 & -0.8 & -0.8 \\ 0 & 0 & -0.5 & -0.1 & 2.4 & -0.9 \\ 0 & 0 & -0.5 & -0.8 & -0.2 & 2.9 \end{bmatrix}$$

is not a DDD matrix, since $a_{11} = 1 < \frac{R_2(B)}{b_{22}} R_1(A) = \frac{22}{23} \times \frac{6}{5} \approx 1.15$.

Example 3.10. Let A be doubly diagonally dominant with

$$\max_{j \in S_2 \cup S_3} \frac{R_{j-t}(B)}{|b_{j-t,j-t}|} R_{i_0}(A) \leq |a_{i_0 i_0}| \text{ for some } i_0,$$

and B be strictly diagonally dominant,

$$A = \begin{bmatrix} 0.4 & \vdots & 0.1 & 0.4 \\ \cdots & \cdots & \cdots & \cdots \\ 0.2 & \vdots & 0.5 & -0.2 \\ 0.5 & \vdots & -1 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 1.0 & -0.4 & \vdots & -0.4 \\ -0.2 & 1.2 & \vdots & -0.4 \\ \cdots & \cdots & \cdots & \cdots \\ -0.4 & -0.6 & \vdots & 1.4 \end{bmatrix}.$$

Then

$$C = A \oplus_2 B = \begin{bmatrix} 0.4 & 0.1 & 0.4 & 0 \\ 0.2 & 1.5 & -0.6 & -0.4 \\ 0.5 & -1.2 & 3.2 & -0.4 \\ 0 & -0.4 & -0.6 & 1.4 \end{bmatrix}$$

is a DDD matrix, since $a_{11} = 0.4 = \frac{R_1(B)}{b_{11}} R_1(A) = \frac{4}{5} \times \frac{1}{2} = 0.4$.

As Bru, Pedroche and Szyld studied the result in section 4 of [3], we continue to consider A and B to be principal submatrices of a given doubly diagonally dominant matrix such that they have a common block with all positive (or negative) diagonal entries. This situation, as well as a more general case outlined in Theorem 3.11 later, appears in many variants of additive Schwarz preconditioning (see [2, 7, 8]). Let

$$H = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \tag{3.3}$$

be doubly diagonally dominant with H_{22} a square matrix of order $k \geq 1$ and let

$$A = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \; B = \begin{bmatrix} H_{22} & H_{23} \\ H_{32} & H_{33} \end{bmatrix} \tag{3.4}$$

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be of order $m_1 \times m_1$ and $m_2 \times m_2$, respectively. The k-subdirect sum of A and B is thus given by

$$C = A \oplus_k B = \begin{bmatrix} H_{11} & H_{12} & O \\ H_{21} & 2H_{22} & H_{23} \\ O & H_{32} & H_{33} \end{bmatrix}. \tag{3.5}$$

We now have the following theorem which shows that C is a DDD matrix.

THEOREM 3.11. *Let H be a DDD matrix partitioned as in* (3.3), *and let A and B be two overlapping principal submatrices given by* (3.4). *Then the k-subdirect sum $C = A \oplus_k B$ is a DDD matrix.*

Proof. For any $i \in S_1,\ l \in S_2,\ j \in S_1 \cup S_2 \cup S_3$, from equation (3.5), we have

$$|c_{ii}| = |a_{ii}| = |h_{ii}|,$$

$$R_i(C) = R_i(A) \leq R_i(H) \text{ (if } H_{13} = 0\text{, equality holds)},$$

$$|c_{ll}| = |a_{ll}| + |b_{l-t,l-t}| = 2|a_{ll}| = 2|h_{ll}|, \text{ and}$$

$$\begin{aligned} R_l(C) &= \sum_{j\in S_1} |a_{lj}| + \sum_{j\in S_2, j\neq l} |a_{lj} + b_{l-t,j-t}| + \sum_{j\in S_3} |b_{l-t,j-t}| \\ &= \sum_{j\in S_1} |h_{lj}| + 2\sum_{j\in S_2, j\neq l} |h_{lj}| + \sum_{j\in S_3} |h_{lj}| \\ &< 2R_l(H). \end{aligned}$$

Using now that H is doubly diagonally dominant, we can get that

$$\begin{aligned} |c_{ii}||c_{ll}| &= |a_{ii}||2a_{ll}| \\ &= 2|h_{ii}||h_{ll}| \\ &\geq 2R_i(H)R_l(H) \\ &> R_i(C)R_l(C). \end{aligned}$$

The proofs of other cases are analogous. ∎

EXAMPLE 3.12. Let the following DDD matrix H be partitioned as

$$H = \left[\begin{array}{c|ccc|c} 1.0 & 0.6 & -0.5 & & -0.1 \\ \hline 0.7 & 2.9 & -0.3 & & -0.5 \\ -0.5 & -0.1 & 2.4 & & -0.9 \\ \hline -0.5 & -0.7 & -0.7 & & 2.3 \end{array}\right],$$

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where principal submatrices A, B are partitioned as

$$A = \begin{bmatrix} 1.0 & \vdots & 0.6 & -0.5 \\ \cdots & \cdots & \cdots & \cdots \\ 0.7 & \vdots & 2.9 & -0.3 \\ -0.5 & \vdots & -0.1 & 2.4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2.9 & -0.3 & \vdots & -0.5 \\ -0.1 & 2.4 & \vdots & -0.9 \\ \cdots & \cdots & \cdots & \cdots \\ -0.7 & -0.7 & \vdots & 2.3 \end{bmatrix}.$$

The 2-subdirect sum $C = A \oplus_2 B$,

$$C = \begin{bmatrix} 1.0 & 0.6 & -0.5 & 0 \\ 0.7 & 5.8 & -0.6 & -0.5 \\ -0.5 & -0.2 & 4.8 & -0.9 \\ 0 & -0.7 & -0.7 & 2.3 \end{bmatrix},$$

is a DDD matrix, according to Theorem 3.11.

Here, we consider consecutive principal submatrices defined by consecutive indices of the form $\{i, i+1, i+2, \ldots\}$.

COROLLARY 3.13. *Let H be a DDD matrix with all positive (or negative) diagonal entries. Let $A_i, i = 1, \ldots, p$ be consecutive principal submatrices of H of order $n_i \times n_i$. Then each of the k_i-subdirect sums C_i*

$$C_i = C_{i-1} \oplus_{k_i} A_{i+1}, i = 1, \ldots, p-1$$

is a DDD matrix. In particular,

$$C_{p-1} = A_1 \oplus_{k_1} A_2 \oplus_{k_2} \cdots \oplus_{k_p} A_p$$

is a DDD matrix, where $C_0 = A_1$ and $k_i < \min(n_i, n_{i+1})$.

Here, we discuss the conditions of the subdirect sum of S-strictly diagonally dominant matrices when card(S) >card(S_1).

THEOREM 3.14. *Let A and B be square matrices of order m_1 and m_2 partitioned as in* (2.1), *respectively, and k be an integer such that $1 \leq k \leq min(m_1, m_2)$, which defines the sets S_1, S_2, S_3 as in* (2.2). *Let A be S-strictly diagonally dominant with card(S)>card(S_1), S be a set of indices of the form $S = \{1, 2, \ldots\}$ and B be strictly diagonally dominant. If all diagonal entries of A_{22} and B_{11} are positive (or all negative) and*

$$|a_{ii}| - R_i(A) > R_{i-t}(B) - |b_{i-t,i-t}|, \quad \textit{for all } i \in S_2, \ t = m_1 - k, \tag{3.6}$$

then the k-subdirect sum $C = A \oplus_k B$ is S-strictly diagonally dominant.

Proof. We take $S_2 = S' \cup S''$, $S' \cap S'' = \emptyset$. Let the set $S = S_1 \cup S'$ (condition card(S)>card(S_1)), $\bar{S} = S'' \cup S_3$, and let $i \in S'$. We first prove the condition 1) of Definition 3.2 for C to be an S-SDD matrix. Observe that from (2.3) and the fact that A_{22} and B_{11} have positive diagonals (or both negative diagonals), we have

$$|c_{ii}| = |a_{ii} + b_{i-t,i-t}| = |a_{ii}| + |b_{i-t,i-t}|,$$

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$$\begin{aligned} R_i^{S_1\cup S'}(C) &= \sum_{l\neq i, l\in S_1\cup S'} |c_{il}| = \sum_{l\in S_1} |a_{il}| + \sum_{l\neq i, l\in S'} |a_{il}+b_{i-t,l-t}| \\ &\leq \sum_{l\neq i, l\in S_1\cup S'} |a_{il}| + \sum_{l\neq i, l\in S'} |b_{i-t,l-t}| = R_i^{S_1\cup S'}(A) + R_{i-t}^{S'}(B). \end{aligned}$$

Therefore we can write

$$\begin{aligned} |c_{ii}| - R_i^{S_1\cup S'}(C) &\geq |a_{ii}+b_{i-t,i-t}| - R_i^{S_1\cup S'}(A) - R_{i-t}^{S'}(B) \\ &= (|a_{ii}| - R_i^{S_1\cup S'}(A)) + (|b_{i-t,i-t}| - R_{i-t}^{S'}(B)) \end{aligned}$$

where we have used that $(|a_{ii}| - R_i^{S_1\cup S'}(A))$ is positive since A is S-strictly diagonally dominant. Note that $(|b_{i-t,i-t}| - R_{i-t}^{S'}(B))$ is also positive since B is strictly diagonally dominant, and thus we can obtain

$$|c_{ii}| - R_i^{S_1\cup S'}(C) > 0, \textit{for all } i\in S'.$$

Since $S = S_1\cup S'$ we have that condition 1) of Definition 3.2 holds.

In order to see that condition 2) holds we first study the case $i\in S'$, $j\in S''$. Using equation (2.3) we obtain the following relations:

$$\begin{aligned} R_j^{S''\cup S_3}(C) &= \sum_{l\neq j, l\in S''\cup S_3} |c_{jl}| = \sum_{l\neq j, l\in S''} |a_{jl}+b_{j-t,l-t}| + \sum_{l\in S_3} |b_{j-t,l-t}| \\ &\leq \sum_{l\neq j, l\in S''} |a_{jl}| + \sum_{l\in S''\cup S_3} |b_{j-t,l-t}| \\ &= R_j^{S''}(A) + R_{j-t}^{S''\cup S_3}(B), \end{aligned}$$

$$\begin{aligned} R_j^{S_1\cup S'}(C) &= \sum_{l\in S_1\cup S'} |c_{jl}| = \sum_{l\in S_1} |a_{jl}| + \sum_{l\in S'} |a_{jl}+b_{j-t,l-t}| \\ &\leq R_j^{S_1\cup S'}(A) + R_{j-t}^{S'}(B), \end{aligned}$$

$$\begin{aligned} R_i^{S''\cup S_3}(C) &= \sum_{l\in S''\cup S_3} |c_{il}| = \sum_{l\in S''} |a_{il}+b_{i-t,l-t}| + \sum_{l\in S_3} |b_{i-t,l-t}| \\ &\leq R_i^{S''}(A) + R_{i-t}^{S''\cup S_3}(B), \end{aligned}$$

$$|c_{jj}| = |a_{jj}+b_{j-t,j-t}| = |a_{jj}| + |b_{j-t,j-t}|.$$

Therefore we can write

$$\begin{aligned} &(|c_{ii}| - R_i^{S_1\cup S'}(C))(|c_{jj}| - R_j^{S''\cup S_3}(C)) \\ &= (|a_{ii}+b_{i-t,i-t}| - R_i^{S_1\cup S'}(C))(|a_{jj}+b_{j-t,j-t}| - R_j^{S''\cup S_3}(C)). \end{aligned}$$

Now observe that:

$$|a_{ii}+b_{i-t,i-t}| - R_i^{S_1\cup S'}(C) \geq |a_{ii}| - R_i^{S_1\cup S'}(A) + |b_{i-t,i-t}| - R_{i-t}^{S'}(B), \tag{3.7}$$

$$|a_{jj}+b_{j-t,j-t}| - R_j^{S''\cup S_3}(C) \geq |a_{jj}| - R_j^{S''}(A) + |b_{j-t,j-t}| - R_{j-t}^{S''\cup S_3}(B), \tag{3.8}$$

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and from condition (3.6) we can write the inequality

$$|a_{ii}| - R_i(A) = |a_{ii}| - (R_i^{S_1\cup S'}(A) + R_i^{S''}(A)) > R_{i-t}(B) - |b_{i-t,i-t}|.$$

Therefore

$$|a_{ii}| - R_i^{S_1\cup S'}(A) > R_i^{S''}(A) + R_{i-t}(B) - |b_{i-t,i-t}|,$$

$$|a_{jj}| - R_j^{S''}(A) > R_j^{S_1\cup S'}(A) + R_{j-t}(B) - |b_{j-t,j-t}|,$$

$$R_{i-t}(B) = R_{i-t}^{S'}(B) + R_{i-t}^{S''\cup S_3}(B),$$

$$R_{j-t}(B) = R_{j-t}^{S'}(B) + R_{j-t}^{S''\cup S_3}(B),$$

which jointly with (3.7) and (3.8) lead to the inequality

$$\begin{aligned}&(|c_{ii}| - R_i^{S_1\cup S'}(C))(|c_{jj}| - R_j^{S''\cup S_3}(C))\\ &> (R_i^{S''}(A) + R_{i-t}^{S''\cup S_3}(B))(R_j^{S_1\cup S'}(A) + R_{j-t}^{S'}(B))\\ &\geq R_i^{S''\cup S_3}(C) R_j^{S_1\cup S'}(C).\end{aligned}$$

Therefore condition 2) of Definition 3.2 is fulfilled and the proof for the case $\forall i \in S',\ j \in S''$ is completed.

For the case $\forall i \in S',\ j \in S_3$, we have from equation (2.3) that

$$|c_{ii}| = |a_{ii} + b_{i-t,i-t}|,\ |c_{jj}| = |b_{j-t,j-t}|,$$

$$R_j^{S''\cup S_3}(C) = \sum_{j\neq l, l\in S''\cup S_3} |c_{jl}| = \sum_{j\neq l, l\in S''\cup S_3} |b_{j-t,l-t}| = R_{j-t}^{S''\cup S_3}(B),$$

$$R_j^{S_1\cup S'}(C) = \sum_{l\in S_1\cup S'} |c_{jl}| = \sum_{l\in S_1\cup S'} |b_{j-t,l-t}| = R_{j-t}^{S'}(B).$$

Note that B is an SDD matrix, then we have

$$\begin{aligned}&(|c_{ii}| - R_i^{S_1\cup S'}(C))(|c_{jj}| - R_j^{S''\cup S_3}(C))\\ &> (|a_{ii}| - R_i^{S_1\cup S'}(A) + |b_{i-t,i-t}| - R_{i-t}^{S'}(B)) R_{j-t}^{S'}(B)\\ &> R_i^{S''\cup S_3}(C) R_j^{S_1\cup S'}(C).\end{aligned}$$

Therefore condition 2) is fulfilled. Condition 1) also holds, since as before, we have

$$|c_{ii}| - R_i^{S_1\cup S'}(C) \geq |a_{ii} + b_{i-t,i-t}| - R_i^{S_1\cup S'}(A) - R_{i-t}^{S'}(B) > 0.$$

The proof for the rest of cases $\forall i \in S_1, j \in S''$ and $\forall i \in S_1, j \in S_3$ are already included in the paper of Bru, Pedroche, and Szyld [3]. ☐

Example 3.15. In this example we show a matrix A that is an S-SDD matrix with card(S)>card(S_1), a matrix B that is an SDD matrix, and such that the 3-subdirect sum C is an S-SDD matrix. Let

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$$A=\begin{bmatrix} 1.0 & \vdots & -0.3 & -0.4 & -0.5 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0.9 & \vdots & 1.6 & -0.4 & -0.7 \\ -0.1 & \vdots & -0.4 & 1.3 & -0.4 \\ -0.1 & \vdots & -0.9 & -0.1 & 2.0 \end{bmatrix},\ B=\begin{bmatrix} 2.0 & 0.2 & -0.3 & \vdots & -0.1 \\ 0.8 & 2.9 & -0.2 & \vdots & -0.5 \\ -0.5 & -0.1 & 2.4 & \vdots & -0.9 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -0.6 & -0.1 & -0.1 & \vdots & 2.3 \end{bmatrix}.$$

Here we have $S_1=\{1\}$, $S_2=\{2,3,4\}$, $S_3=\{5\}$. It is easy to show that A is an S-SDD matrix, with $S=\{1,2\}$, and therefore $\text{card}(S)>\text{card}(S_1)$. Since B is an SDD matrix and

$$|a_{ii}|-R_i(A)>R_{i-t}(B)-|b_{i-t,i-t}|, for\ all\ i\in S_2,$$

with $t=m_1-k=4-3=1$ and $\bar{S}=\{3,4,5\}$, we have that the 3-subdirect sum

$$C=A\oplus_3 B=\begin{bmatrix} 1.0 & \vdots & -0.3 & -0.4 & -0.5 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0.9 & \vdots & 3.6 & -0.2 & -1.0 & \vdots & -0.1 \\ -0.1 & \vdots & 0.4 & 4.2 & -0.6 & \vdots & -0.5 \\ -0.1 & \vdots & -1.4 & -0.2 & 4.4 & \vdots & -0.9 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \vdots & -0.6 & -0.1 & -0.1 & \vdots & 2.3 \end{bmatrix}$$

is a $\{1,2\}$-SDD matrix, according to Theorem 3.14.

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REFERENCES

[1] R. Bru, F. Pedroche, and D.B. Szyld. Subdirect sums of nonsingular M-matrices and of their inverse. *Electron. J. Linear Algebra,* 13:162–174, 2005.
[2] R. Bru, F. Pedroche, and D.B. Szyld. Additive Schwarz iterations for markov chains. *SIAM J. Matrix Anal. Appl.,* 27:445–458, 2005.
[3] R. Bru, F. Pedroche, and D.B. Szyld. Subdirect sums of S-strictly diagonally dominant matrices. *Electron. J. Linear Algebra*, 15:201–209, 2006.
[4] S.M. Fallat and C.R. Johnson. Sub-direct sums and positivity classes of matrices. *Linear Algebra Appl.,* 288:149–173, 1999.
[5] B. Li and M.J. Tsatsomeros. Doubly diagonally dominant matrices. *Linear Algebra Appl.,* 261:221–235, 1997.
[6] R.A. Horn and C.R. Johnson. *Matrix Analysis.* Cambridge University Press, New York, 1985.
[7] A. Frommer and D.B. Szyld. Weighted max norms, splittings, and overlapping additive Schwarz iterations. *Numer. Math.,* 83:259-278, 1999.
[8] B.F. Smith, P.E. Bjorstad, and W.D. Gropp. *Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations.* Cambridge University Press, Cambridge, 1996.
[9] L. Cvetković. H-matrix theory vs. eigenvalue localization. *Numer. Algorithms,* 42:229-245, 2006.