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# BOUNDS FOR THE SPECTRAL RADIUS OF BLOCK H-MATRICES*

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**Abstract.** Simple upper bounds for the spectral radius of an H-matrix and a block H-matrix are presented. They represent an improvement over the bounds in [T.Z. Huang, R.S. Ran, A simple estimation for the spectral radius of (block) H-matrices, Journal of Computational Applied Mathematics, 177 (2005), pp. 455–459].

**Key words.** Spectral radius, H-matrices, Block H-matrices.

**AMS subject classifications.** 65F10, 65F15.

**1. Introduction.** H-matrices have important applications in many fields, such as numerical analysis, control theory, and mathematical physics. Recently, Huang and Ran [5] have presented a simple upper bound for the spectral radius of (block) H-matrices. In this paper, we give some new upper bounds.

A square complex or real matrix $A$ is called an *H-matrix* if there exists a square positive diagonal matrix $X$ such that $AX$ is strictly diagonally dominant (SDD) [5]. Let $\mathbb{C}^{n,n}(\mathbb{R}^{n,n})$ denote the set of $n\times n$ complex (real) matrices. If $A=[a_{ij}]\in\mathbb{C}^{n,n}$, we write $|A|=[|a_{ij}|]$, where $|a_{ij}|$ is the modulus of $a_{ij}$. We denote by $\rho(A)$ the spectral radius of $A$, which is just the radius of the smallest disc centered at the origin in the complex plane that includes all the eigenvalues of $A$ (see [4, Def. 1.1.4]).

Throughout the paper, we let $\|\cdot\|$ denote a consistent family of norms on matrices of all sizes, which satisfies the following four axioms:

(1) $\|A\|\geq 0$, and $\|A\|=0$ if and only if $A=0$;
(2) $\|cA\|=|c|\,\|A\|$ for all complex scalars $c$;
(3) $\|A+B\|\leq\|A\|+\|B\|$, where $A$ and $B$ are in the same size; and
(4) $\|AB\|\leq\|A\|\,\|B\|$ provided that $AB$ is defined.

Axioms (1) and (4) ensure that $\|I\|\geq 1$, where $I$ is the identity matrix. For example, the Frobenius norm $\|\cdot\|_F$, 1-norm $\|\cdot\|_1$, and $\infty$-norm $\|\cdot\|_\infty$ (see e.g., [4, Chap. 5]) are all consistent families of norms.

Let $A=[a_{ij}]\in\mathbb{C}^{n,n}$ be partitioned in the following form

$$A=\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k}\\ A_{21} & A_{22} & \cdots & A_{2k}\\ \cdots & \cdots & \cdots & \cdots\\ A_{k1} & A_{k2} & \cdots & A_{kk}\end{pmatrix} \tag{1.1}$$

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in which $A_{ij} \in \mathbb{C}^{n_i,n_j}$ and $\sum_{i=1}^{k} n_i = n$. If each diagonal block $A_{ii}$ is nonsingular and

$$\left\|A_{ii}^{-1}\right\|^{-1} > \sum\nolimits_{j\neq i} \|A_{ij}\| \quad \text{for all} \quad i = 1, 2, \ldots, k,$$

then $A$ is said to be *block strictly diagonally dominant with respect to* $\|\cdot\|$ (BSDD) [3]; if there exist positive numbers $x_1, x_2, \ldots, x_k$ such that

$$x_i\left\|A_{ii}^{-1}\right\|^{-1} > \sum\nolimits_{j\neq i} x_j \|A_{ij}\| \quad \text{for all} \;\; i = 1, 2, \ldots, k,$$

then $A$ is said to be *block H-matrix with respect to* $\|\cdot\|$ [6].

THEOREM 1.1. ([5]) *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$. If $A$ is an H-matrix, then*

$$\rho(A) < 2 \max_i |a_{ii}|. \tag{1.2}$$

THEOREM 1.2. ([5]) *Let $A \in \mathbb{C}^{n,n}$ be partitioned as in (1.1). Let $\|\cdot\|$ be a consistent family of norms. If $A$ is a block H-matrix with respect to $\|\cdot\|$, then*

$$\rho(A) < \max_i \left\{ \|A_{ii}\| + \left\|A_{ii}^{-1}\right\|^{-1} \right\}. \tag{1.3}$$

**2. Main results.** In this section, we present some new bounds for the spectral radius of an H-matrix and a block H-matrix, respectively. We need the following two lemmas.

LEMMA 2.1. ([4, Thm 8.1.18]) *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$. Then $\rho(A) \le \rho(|A|)$.*

LEMMA 2.2. ([1]) *Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$ be a nonnegative matrix. Then*

$$\rho(A) \le \max_{i\neq j} \frac{1}{2} \left\{ a_{ii} + a_{jj} + \left[ \left(a_{ii} - a_{jj}\right)^2 + 4 \sum\nolimits_{k\neq i} a_{ik} \sum\nolimits_{k\neq j} a_{jk} \right]^{\frac{1}{2}} \right\}.$$

The following is one of the main results of this paper.

THEOREM 2.3. *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be an H-matrix. Then*

$$\rho(A) < \max_{i\neq j} \left(|a_{ii}| + |a_{jj}|\right) \le 2 \max_i |a_{ii}|. \tag{2.1}$$

*Proof.* Let $X = diag(x_1, x_2, \ldots, x_n)$ be a square positive diagonal matrix such that $AX$ is SDD. Then $X^{-1}AX$ is also SDD, i.e.,

$$|a_{ii}| = |X^{-1}AX|_{ii} > \sum_{j\neq i} |X^{-1}AX|_{ij} = \sum_{j\neq i} \frac{|a_{ij}|x_j}{x_i} \quad \text{for all} \quad i = 1, 2, \ldots, n.$$

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The spectral radii of $A$ and $X^{-1}AX$ are equal since the two matrices are similar. Lemma 2.1 and Lemma 2.2 ensure that

$$\begin{aligned}\rho(A) &= \rho(X^{-1}AX) \leq \rho(|X^{-1}AX|)\\ &\leq \max_{i\neq j} \frac{1}{2}\left\{|a_{ii}| + |a_{jj}| + \left[(|a_{ii}| - |a_{jj}|)^2 + 4\sum_{k\neq i}\frac{|a_{ik}|x_k}{x_i}\sum_{k\neq j}\frac{|a_{jk}|x_k}{x_j}\right]^{\frac{1}{2}}\right\}\\ &< \max_{i\neq j}\frac{1}{2}\left\{|a_{ii}| + |a_{jj}| + \left[(|a_{ii}| - |a_{jj}|)^2 + 4|a_{ii}||a_{jj}|\right]^{\frac{1}{2}}\right\}\\ &= \max_{i\neq j}(|a_{ii}| + |a_{jj}|) \leq 2\max_i |a_{ii}|. \qquad \square\end{aligned}$$

We now consider block H-matrices. The following lemma was stated in [2] but the proof offered there is not correct.

LEMMA 2.4. ([2]) *Let $A = [A_{ij}] \in \mathbb{R}^{n,n}$ be a nonnegative block matrix of the form (1.1). Let $B = [\|A_{ij}\|]$, where $\|\cdot\|$ is a consistent family of norms. Then*

$$\rho(A) \leq \rho(B). \tag{2.2}$$

*Proof.* First we assume that $A$ is a positive matrix. By Perron's Theorem [4, Thm 8.2.11], $\rho(A)$ is an eigenvalue of $A$ corresponding to a positive eigenvector $x$, i.e.,

$$Ax = \rho(A)x, \quad x > 0.$$

Partition $x^T = (x_1^T, \ldots, x_k^T)$, where each $x_i \in \mathbb{R}^{n_i}$, $i = 1, \ldots, k$. Let $z_i = \|x_i\|$. Define $z := (z_1, \ldots, z_k)^T \in \mathbb{R}^k$, so $z > 0$ and for all $1 \leq i \leq k$,

$$\sum\nolimits_{j=1}^{k} A_{ij}x_j = \rho(A)x_i,$$

which implies

$$\rho(A)z_i = \rho(A)\,\|x_i\| = \left\|\sum\nolimits_{j=1}^{k} A_{ij}x_j\right\| \leq \sum\nolimits_{j=1}^{k}\|A_{ij}\|\,\|x_j\| = \sum\nolimits_{j=1}^{k}\|A_{ij}\|z_j.$$

Since the inequality $\rho(A)z_i \leq \sum_{j=1}^{k}\|A_{ij}\|z_j$ holds for all $i = 1, \ldots, k$, we have

$$\rho(A)z \leq Bz.$$

Since $B$ is nonnegative and $z > 0$, we obtain $\rho(A) \leq \rho(B)$ [4, Cor. 8.1.29].

Next we show the inequality (2.2) holds for all nonnegative matrices $A$. For any given $\varepsilon > 0$, define $A(\varepsilon) := [a_{ij} + \varepsilon]$ and let $B(\varepsilon) := [\|A_{ij}(\varepsilon)\|]$. Since every $A_{ij}(\varepsilon)$ is positive, therefore $\rho(A(\varepsilon)) \leq \rho(B(\varepsilon))$. By the continuity of $\rho(\cdot)$, we have

$$\rho(A) = \lim_{\varepsilon\to 0}\rho(A(\varepsilon)) \leq \lim_{\varepsilon\to 0}\rho(B(\varepsilon)) = \rho(B). \qquad \square$$

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THEOREM 2.5. *Let $A \in \mathbb{C}^{n,n}$ be partitioned as in (1.1). Suppose $A$ is a block H-matrix with respect to a consistent family of norms $\|\cdot\|$. Then*

$$\rho(A) < \max_{i \neq j} \frac{1}{2} \left\{ \|A_{ii}\| + \|A_{jj}\| + \left[ (\|A_{ii}\| - \|A_{jj}\|)^2 + 4 \left\|A_{ii}^{-1}\right\|^{-1} \left\|A_{jj}^{-1}\right\|^{-1} \right]^{\frac{1}{2}} \right\} \tag{2.3}$$
$$\leq \max_{i \neq j} (\|A_{ii}\| + \|A_{jj}\|).$$

*Proof.* Let $x_1, x_2, \ldots, x_k$ be positive numbers such that

$$x_i \left\|A_{ii}^{-1}\right\|^{-1} > \sum\nolimits_{j \neq i} x_j \|A_{ij}\| \quad \text{for all} \quad i = 1, 2, \ldots, k.$$

Let $X = diag(x_1 I_{n_1}, x_2 I_{n_2}, \ldots, x_k I_{n_k})$. Then $AX$ is BSDD. Let

$$B = X^{-1}AX = \begin{pmatrix} A_{11} & \frac{x_2}{x_1} A_{11} & \cdots & \frac{x_n}{x_1} A_{1k} \\ \frac{x_1}{x_2} A_{21} & A_{22} & \cdots & \frac{x_n}{x_2} A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{x_1}{x_n} A_{k1} & \frac{x_2}{x_n} A_{k2} & \cdots & A_{kk} \end{pmatrix}.$$

Then $B = [B_{ij}]$ is also BSDD. Let $C = [\|B_{ij}\|] \in \mathbb{R}^{k,k}$. Then Lemma 2.2 and Lemma 2.4 ensure that

$$\rho(A) = \rho(B) \leq \rho(C)$$
$$\leq \max_{i \neq j} \frac{1}{2} \left\{ \|A_{ii}\| + \|A_{jj}\| + \left[ (\|A_{ii}\| - \|A_{jj}\|)^2 + 4 \sum_{k \neq i} \frac{\|A_{ik}\| x_k}{x_i} \sum_{k \neq j} \frac{\|A_{jk}\| x_k}{x_j} \right]^{\frac{1}{2}} \right\}$$
$$< \max_{i \neq j} \frac{1}{2} \left\{ \|A_{ii}\| + \|A_{jj}\| + \left[ (\|A_{ii}\| - \|A_{jj}\|)^2 + 4 \left\|A_{ii}^{-1}\right\|^{-1} \left\|A_{jj}^{-1}\right\|^{-1} \right]^{\frac{1}{2}} \right\}.$$

Moreover, we have $1 \leq \|I\| = \left\|A_{ii} A_{ii}^{-1}\right\| \leq \|A_{ii}\| \left\|A_{ii}^{-1}\right\|$, so $\left\|A_{ii}^{-1}\right\|^{-1} \leq \|A_{ii}\|$ and

$$\|A_{ii}\| + \|A_{jj}\| + \left[ (\|A_{ii}\| - \|A_{jj}\|)^2 + 4 \left\|A_{ii}^{-1}\right\|^{-1} \left\|A_{jj}^{-1}\right\|^{-1} \right]^{\frac{1}{2}}$$
$$\leq \|A_{ii}\| + \|A_{jj}\| + \left[ (\|A_{ii}\| - \|A_{jj}\|)^2 + 4 \|A_{ii}\| \|A_{jj}\| \right]^{\frac{1}{2}}$$
$$= 2 (\|A_{ii}\| + \|A_{jj}\|). \qquad \square$$

REMARK 2.6. Without loss of generality, for given $i \neq j$, assume that

$$\|A_{ii}\| + \left\|A_{ii}^{-1}\right\|^{-1} \geq \|A_{jj}\| + \left\|A_{jj}^{-1}\right\|^{-1}.$$

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Then

$$\|A_{ii}\| + \|A_{jj}\| + \left[(\|A_{ii}\| - \|A_{jj}\|)^2 + 4\left\|A_{ii}^{-1}\right\|^{-1}\left\|A_{jj}^{-1}\right\|^{-1}\right]^{\frac{1}{2}}$$
$$\le \|A_{ii}\| + \|A_{jj}\| + \left[(\|A_{ii}\| - \|A_{jj}\|)^2 + 4\left\|A_{ii}^{-1}\right\|^{-1}\left(\|A_{ii}\| + \left\|A_{ii}^{-1}\right\|^{-1} - \|A_{jj}\|\right)\right]^{\frac{1}{2}}$$
$$= \|A_{ii}\| + \|A_{jj}\| + \left[\left(\|A_{ii}\| - \|A_{jj}\| + 2\left\|A_{ii}^{-1}\right\|^{-1}\right)^2\right]^{\frac{1}{2}}$$
$$= \|A_{ii}\| + \|A_{jj}\| + \left|\left(\|A_{ii}\| + \left\|A_{ii}^{-1}\right\|^{-1} - \|A_{jj}\|\right) + \left\|A_{ii}^{-1}\right\|^{-1}\right|$$
$$= \|A_{ii}\| + \|A_{jj}\| + \left(\|A_{ii}\| - \|A_{jj}\| + 2\left\|A_{ii}^{-1}\right\|^{-1}\right)$$
$$= 2\left(\|A_{ii}\| + \left\|A_{ii}^{-1}\right\|^{-1}\right) \le 2\max_i\left(\|A_{ii}\| + \left\|A_{ii}^{-1}\right\|^{-1}\right).$$

Hence, the first bound in (2.3) is at least as good as the bound (1.2).

**Example.** Consider the block matrix

$$A = \begin{bmatrix} 4 & -2 & \vdots & 1.5 & 0.5 \\ -2 & 6 & \vdots & 1 & -0.5 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & \vdots & 1 & 0 \\ 0.5 & 0.5 & \vdots & 0 & 1 \end{bmatrix}$$

and the norms $\|\cdot\|_\infty$. Then $A$ is a block H-matrix with spectral radius 7.2152. The bound in Theorem 1.2 is $\rho(A) \le 10.5$. The bound in Theorem 2.5 is $\rho(A) \le 8.34$.

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