Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 191-200, June 2006

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UPPER BOUNDS ON CERTAIN FUNCTIONALS DEFINED ON GROUPS OF LINEAR OPERATORS*

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Abstract. The problem of estimating certain functionals defined on a group of linear operators generating a group induced cone (GIC) ordering is studied. A result of Berman and Plemmons [Math. Inequal. Appl., 2(1):149–152, 1998] is extended from the sum function to Schur-convex functions. It is shown that the problem has a closed connection with both Schur type inequality and weak group majorization. Some applications are given for matrices.

Key words. Group majorization, GIC ordering, Normal decomposition system, Cone preordering, Schur type inequality, Schur-convex function, Eigenvalues, Singular values.

AMS subject classifications. 06F20, 39B62, 15A48, 15A18.

1. Introduction. Berman and Plemmons [2] proved that the functional

$$f(U) = \sum_{i=1}^{n} \max_j \{(U^T M_j U)_{ii}\} \tag{1.1}$$

over all $n \times n$ orthogonal matrices U is maximized by an orthogonal matrix Q which simultaneously diagonalizes the symmetric matrices M_j, $j = 1, \dots, k$. An analogous result holds for Hermitian matrices [2, Section 3].

In the present paper we study a similar problem for a general linear space endowed with the structure of normal decomposition (ND) system (to be defined below). Also, we replace the sum function in (1.1) by an increasing function with respect to certain vector orderings (see Section 2). Some applications are given for matrices in Section 3. A further extension to weak group majorization is discussed in Section 4.

2. Results. Let V be a finite-dimensional real linear space equipped with an inner product $\langle \cdot, \cdot \rangle$. By $O(V)$ we denote the orthogonal group acting on V. Let G be a closed subgroup of $O(V)$. The *group majorization* induced by G, abbreviated as G-majorization and written as $\preceq_G$, is the preordering on V defined by

$$y \preceq_G x \;\text{ iff }\; y \in \operatorname{conv} Gx,$$

where $\operatorname{conv} Gx$ denotes the convex hull of the orbit $Gx := \{gx : g \in G\}$ (see [18]).

Let $(\cdot)_\downarrow : V \to V$ be a G-invariant map, that is $(gx)_\downarrow = x_\downarrow$ for any $x \in V$ and $g \in G$. We say that $(V, G, (\cdot)_\downarrow)$ is a *normal decomposition (ND) system* (see [10, 11]) if

(A1) for any $x \in V$ there exists $g \in G$ satisfying $x = gx_\downarrow$,
(A2) $\max_{g \in G} \langle x, gy \rangle = \langle x_\downarrow, y_\downarrow \rangle$ for all $x, y \in V$.

*Received by the editors 27 November 2005. Accepted for publication 13 June 2006. Handling Editor: Shmuel Friedland.

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A publication of the International Linear Algebra Society
Volume 15, pp. 191-200, June 2006

In this event, it can be deduced (see [10, Thm 2.4]) that the set $D = V_\downarrow$, the range of $(\cdot)_\downarrow$, is a closed convex cone. Under axioms (A1)-(A2), the group majorization $\preceq_G$ is said to be a *group induced cone (GIC) ordering* [4, 5, 6, 18]. The operator $(\cdot)_\downarrow$ is called a *normal map.* See [4, 5, 6, 10, 11] for examples of ND systems and GIC orderings. See also Section 3.

Axiom (A1) generalizes the Spectral Theorem for Hermitian (or real symmetric) matrices as well as the Singular Value Decomposition Theorem for complex (or real) matrices [4]. On the other hand, (A2) can be viewed as an extension of the fundamental von Neumann trace result on singular values [4] and of the analogous result of Miranda and Thompson [14, 15]. This makes it possible to apply results for a general ND system in matrix theory, convex analysis, optimization, etc.

The notion of ND system was introduced by Lewis (see [10, pp. 928-929] and [11, pp. 817-818]). The corresponding theory of group induced cone orderings was developed by Eaton and Perlman [7], Eaton [4, 5, 6], Giovagnoli and Wynn [8], Steerneman [18] and Niezgoda [16, 17].

For an ND system $(V, G, (\cdot)_\downarrow)$ with D being the range of $(\cdot)_\downarrow$, it can be shown that $x_\downarrow$ is the unique element of the set $D \cap Gx$. Denote

$$W := \mathit{span}\, D \text{ and } H := \{h = g_{|W} : g \in G,\ gW = W\}.$$

It is known (see [16, Thm 3.2], [16, p. 14], [18, Thm 4.1] and [9]) that the following statements (i)-(iv) are mutually equivalent:

(i) The group majorizations $\preceq_G$ and $\preceq_H$ are equivalent on W, that is

$$y \preceq_G x \text{ iff } y \preceq_H x \quad \text{for all } x, y \in W.$$

(ii) $(W, H, (\cdot)_\downarrow)$ is an ND system (with the inherited normal map).

(iii) Schur type inequality holds, that is

$$Px \preceq_H x_\downarrow \quad \text{for all } x \in V, \tag{2.1}$$

where P is the orthogonal projection from the space V onto the space W.

(iv) H is a finite reflection group.

Let $C \subset W$ be a convex cone. The *cone preordering* $\preceq_C$ induced by C is defined as follows:

$$y \preceq_C x \text{ iff } x - y \in C$$

for $x, y \in W$. A function $\varphi : W \to \mathbb{R}$ is said to be *C-increasing* (resp. *H-increasing*) if $y \preceq_C x$ (resp. $y \preceq_H x$) implies $\varphi(y) \le \varphi(x)$ for $x, y \in W$. The function φ is said to be *CH-increasing* if it is both C-increasing and H-increasing. We say that C has *max property* if for any vectors $a_1, \ldots, a_k \in W$ there exists a maximal vector $\max\limits_j a_j$ with respect to $\preceq_C$. We call a linear operator $L : V \to V$ *C-positive* if $LC \subset C$. Observe that the C-positivity of L implies $Ly \preceq_C Lx$ whenever $y \preceq_C x$.

THEOREM 2.1. *Let $(V, G, (\cdot)_\downarrow)$ be an ND system satisfying any of the above equivalent conditions (i)-(iv). Assume $C \subset W$ is an H-invariant convex cone. Let $\varphi : W \to \mathbb{R}$ be a CH-increasing function.*

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(I) *Let $w, w_1, \ldots, w_k$ be vectors in W. If $w_j \preceq_C w$ for $j = 1, \ldots, k$, then for each $g \in G$*

$$Pgw_j \preceq_C Pgw \preceq_H w \quad \text{for } j = 1, \ldots, k, \tag{2.2}$$

$$\max_j \varphi(Pgw_j) \le \varphi(Pgw) \le \varphi(w). \tag{2.3}$$

(II) *If, in addition, C has max property, then for each $g \in G$*

$$\max_j Pgw_j \preceq_C Pgw \preceq_H w, \tag{2.4}$$

$$\varphi(\max_j Pgw_j) \le \varphi(Pgw) \le \varphi(w). \tag{2.5}$$

(III) *In particular, if C has max property and $w = \max_j w_j$, then for each $g \in G$*

$$\varphi(\max_j Pgw_j) \le \varphi(\max_j w_j), \tag{2.6}$$

i.e., the functional

$$f(g) := \varphi(\max_j Pgw_j), \quad g \in G, \tag{2.7}$$

is maximized by $g = id$, the identity operator on V.

Proof. Fix arbitrarily $g \in G$. We shall prove that the linear operator Pg is C-positive. Denote $D = V_\downarrow$. Recall that $\{x_\downarrow\} = D \cap Gx$ for $x \in V$. Using (2.1) and the G-invariance of the normal map $(\cdot)_\downarrow : V \to D$, we obtain $Pgz \preceq_H (gz)_\downarrow = z_\downarrow$ for $z \in W$.

On the other hand, employing condition (A1) for the ND systems $(V, G, (\cdot)_\downarrow)$ and $(W, H, (\cdot)_\downarrow)$, for each $z \in W$ we get $z = g_0 z_\downarrow$ and $z = h_0 d$ for some $g_0 \in G$, $h_0 \in H$ and $d \in D$. By [16, Lemma 2.1], we derive $d = z_\downarrow$. Therefore $z = h_0 z_\downarrow$. In consequence, $Hz = Hz_\downarrow$ for $z \in W$. It now follows that $Pgz \in \operatorname{conv} Hz_\downarrow = \operatorname{conv} Hz$, since $Pgz \preceq_H z_\downarrow$. Hence

$$Pgz \preceq_H z \text{ for } z \in W. \tag{2.8}$$

Since C is H-invariant, we obtain $Hz \in C$ and $\operatorname{conv} Hz \subset C$ for $z \in C$. Therefore $Pgz \in C$ for $z \in C$, since $Pgz \in \operatorname{conv} Hz$ by (2.8). This yields the C-positivity of Pg, as required.

(I). Since $w_j \preceq_C w$ for $j = 1, \ldots, k$, we get $Pgw_j \preceq_C Pgw$ by the C-positivity of Pg. Additionally, $Pgw \preceq_H w$ by (2.8). This completes the proof of (2.2).

Applying (2.2) and the fact that φ is CH-increasing, one obtains $\varphi(Pgw_j) \le \varphi(Pgw) \le \varphi(w)$ for $j = 1, \ldots, k$. This proves (2.3).

(II). Suppose that C has max property. Then there exists $\max_j Pgw_j$. Using (2.2) we derive (2.4). Moreover, (2.4) implies (2.5), because φ is CH-increasing.

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A publication of the International Linear Algebra Society
Volume 15, pp. 191-200, June 2006

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(III). Substituting $w := \max_j w_j$ into (2.5) yields (2.6). Since $w_j \in W$ and $Pw_j = w_j$, (2.6) means that the functional defined by (2.7) is maximized by the identity operator id on V. □

COROLLARY 2.2. *Let $(V, G, (\cdot)_\downarrow)$ be an ND system satisfying any of the equivalent conditions (i)-(iv). Assume $C \subset W$ is an H-invariant convex cone having max property. Suppose that $\varphi : W \to \mathbb{R}$ is a CH-increasing function.*

Let $v_1, \ldots, v_k$ be vectors in V such that $g_0v_1, \ldots, g_0v_k \in W$ for some $g_0 \in G$. Then g_0 maximizes the functional

$$f(g) := \varphi(\max_j Pgv_j), \quad g \in G.$$

Proof. Apply Theorem 2.1, part (III), for $w_j := g_0v_j \in W$, $j = 1, \ldots, k$, and $w := \max_j g_0vj$. □

Theorem 2.1, part (I), can be modified. See Theorem 3.2 for application of the result below.

COROLLARY 2.3. *Under the hypotheses of Theorem 2.1, assume that φ is H-increasing on some set W_0 (in place of W) such that $W_\downarrow \subset W_0 \subset W$. Suppose that there exists a subgroup $H_0 \subset H$ satisfying (i) for each $h_0 \in H_0$ there exists $g_0 \in G$ such that $h_0P = Pg_0$, and (ii) for each $w \in W$ there exists an $h_0 \in H_0$ such that $h_0w \in W_0$.*

Then

$$\max_j \varphi(Pg_0gw_j) \le \varphi(Pg_0gw) \le \varphi(w_\downarrow), \tag{2.9}$$

where $h_0Pgw = Pg_0gw \in W_0$.

Proof. Applying (i)-(ii) we take an $h_0 \in H_0$ and $g_0 \in G$ such that $h_0Pgw \in W_0$ and $h_0P = Pg_0$. Since C is H-invariant, $H_0 \subset H$ and $\preceq_H$ is H-invariant, it follows from (2.3) that $h_0Pgw_j \preceq_C h_0Pgw$ and $h_0Pgw \preceq_H w_\downarrow$ with $w_\downarrow \in W_\downarrow \subset W_0$. Therefore (2.9) holds. □

3. Applications. In this section we interpret the results of Section 2 in matrix setting. We consider two special cases. The first leads to a result generalizing a theorem of Berman and Plemmons [2] (see Corollary 3.1).

To do this, we set

$V :=$ the linear space $\mathbb{S}_n$ of $n \times n$ real symmetric matrices,
$G :=$ the group of operators of the form $X \to UXU^T$, $X \in V$, where U runs over the group $\mathbb{O}_n$ of $n \times n$ orthogonal matrices,
$X_\downarrow :=$ diag $\lambda(X)$ for $X \in \mathbb{S}_n$, where $\lambda(X)$ denotes the vector of the *eigenvalues* of X arranged in decreasing order on the main diagonal, and

diag x stands for the diagonal matrix with $x \in \mathbb{R}^n$. We adopt the convention that the members of $\mathbb{R}^n$ are row n-vectors. Then $(V, G, (\cdot)_\downarrow)$ is an ND system by virtue of the Spectral Theorem and Theobald's trace inequality on eigenvalues (see [4, 12, 19]).

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A publication of the International Linear Algebra Society
Volume 15, pp. 191-200, June 2006

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Also, it is well known that $D = V_{\downarrow}$ is the convex cone of $n \times n$ real diagonal matrices with decreasingly ordered diagonal entries.

In addition, $(W, H, (\cdot)_{\downarrow})$ is an ND system for

$W :=$ the linear space of $n \times n$ diagonal matrices,
$H :=$ the group of operators of the form $X \to UXU^T$, $X \in W$, where U varies over the group $\mathbb{P}_n$ of $n \times n$ permutation matrices.

The orthoprojector P from V onto W is given by

$$P(X) = \operatorname{diag} \Delta(X) \quad \text{for } X \in \mathbb{S}_n, \tag{3.1}$$

where the symbol $\Delta(\cdot)$ means "the diagonal of".

Conditions (i)-(iv) of Section 2 are fulfilled (see [16, Example 4.1] for details). In particular, inequality (2.1) takes the form of the *classical Schur inequality*:

$$\Delta(UXU^T) \preceq_m \lambda(X) \quad \text{for } X \in S_n \text{ and } U \in O_n. \tag{3.2}$$

Here the relation $\preceq_m$ is the *ordinary majorization* on $\mathbb{R}^n$ defined as follows (see [13, p. 7]). Given two vectors $a, b \in \mathbb{R}^n$, we say that *a majorizes b* if

$$\sum_{i=1}^{m} b_{[i]} \leq \sum_{i=1}^{m} a_{[i]} \text{ for } m = 1, \dots, n, \tag{3.3}$$

with equality for $m = n$. By $c_{[i]}$ we denote the ith largest entry of a vector $c = (c_1, \dots, c_n) \in \mathbb{R}^n$. A function $\psi : \mathbb{R}^n \to \mathbb{R}$ is said to be *Schur-convex* if $\psi(b) \leq \psi(a)$ whenever $b \preceq_m a$ for $a, b \in \mathbb{R}^n$ (cf. [13, p. 54]).

It is known that $\preceq_m$ is the group majorization on $\mathbb{R}^n$ induced by the permutation group $\mathbb{P}_n$ (see [4, p. 16]). There is a closed connection between the H-majorization $\preceq_H$ on W and the ordinary majorization $\preceq_m$ on $\mathbb{R}^n$. Namely,

$$\operatorname{diag} y \preceq_H \operatorname{diag} x \quad \text{iff} \quad y \preceq_m x \quad \text{for } x, y \in \mathbb{R}^n. \tag{3.4}$$

To see this, employ the formula

$$U(\operatorname{diag} x)U^T = \operatorname{diag}(xU^T) \quad \text{for } U \in \mathbb{P}_n \text{ and } x \in \mathbb{R}^n. \tag{3.5}$$

Applying Corollary 2.2 for

$C :=$ the convex cone of $n \times n$ diagonal matrices with nonnegative diagonal entries,
$\preceq_C :=$ the entrywise ordering on W,
$\max :=$ the maximum operator with respect to the entrywise ordering $\preceq_C$ on W,

we get

Corollary 3.1. *Let M_j, $j = 1, \dots, k$, be a collection of pairwise commuting $n \times n$ symmetric matrices, and let U_0 be an $n \times n$ orthogonal matrix such that the similarity operator $g_0 := U_0(\cdot)U_0^T$ simultaneously diagonalizes the M_j.*

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A publication of the International Linear Algebra Society
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Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be an $\mathbb{R}^n_+$-increasing function. If ψ is Schur-convex then U_0 maximizes the functional

$$f(U) := \psi(\max_j \Delta(UM_jU^T))$$

over all $n \times n$ orthogonal matrices U.

Proof. Apply (3.1)-(3.5) and Corollary 2.2 for the function $\varphi(X) := \psi(x)$, where $X := \operatorname{diag} x$ for $x \in \mathbb{R}^n$. □

The above result extends the mentioned theorem of Berman and Plemmons [2]. To see this, use Corollary 3.1 for the function $\psi(x) := \sum_{i=1}^{n} x_i$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

We are now interested in using Theorem 2.1 and Corollary 2.3 to obtain a result for singular values of matrices in $\mathbb{M}_n$ (see Theorem 3.2) under the action of the group G_0 of orthogonal equivalences. However, there are no G_0-invariant convex cones in $\mathbb{M}_n$ (except $\mathbb{M}_n$ and $\{0\}$). To avoid this difficulty, we use embedding of the space $\mathbb{M}_n$ in $\mathbb{M}_{2n}$. Namely, let V be the linear space of all $2n \times 2n$ matrices of the form

$$\begin{bmatrix} \alpha I & X \\ X^T & \alpha I \end{bmatrix},$$

where X is an $n \times n$ real matrix, I is the $n \times n$ identity matrix and α is a real number. Put G to be the group of all linear maps g from V to V of the form

$$\begin{aligned} \begin{bmatrix} \alpha I & X \\ X^T & \alpha I \end{bmatrix} \to \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} \alpha I & X \\ X^T & \alpha I \end{bmatrix} \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}^T \\ = \begin{bmatrix} \alpha I & U_1 X U_2^T \\ (U_1 X U_2^T)^T & \alpha I \end{bmatrix}, \end{aligned}$$

where U_1 and U_2 run over the group of $n \times n$ orthogonal matrices. Then the Singular Value Decomposition Theorem and the von Neumann trace inequality for $n \times n$ matrices imply that $(V, G, (\cdot)_\downarrow)$ is an ND system for the normal map $(\cdot)_\downarrow$ defined on V by

$$\begin{bmatrix} \alpha I & X \\ X^T & \alpha I \end{bmatrix}_\downarrow := \begin{bmatrix} \alpha I & \operatorname{diag} s(X) \\ \operatorname{diag} s(X) & \alpha I \end{bmatrix},$$

where $s(X) := (s_1(X), \ldots, s_n(X))$ is the n-vector of the *singular values* of X arranged in decreasing order (cf. [1, p. 106], [4, pp. 17-18]). That is, the numbers $s_1(X) \geq \ldots \geq s_n(X)$ are the nonnegative square roots of the eigenvalues of the positive semi-definite matrix $X^T X$. The range of the normal map $(\cdot)_\downarrow$ is the convex cone

$$D = \left\{ \begin{bmatrix} \alpha I & \operatorname{diag} x \\ \operatorname{diag} x & \alpha I \end{bmatrix} : \alpha \in \mathbb{R},\, x \in \mathbb{R}^n_{+\downarrow} \right\},$$

where $\mathbb{R}^n_{+\downarrow}$ is the set of nonnegative real n-vectors with decreasingly ordered entries.

The system $(W, H, (\cdot)_\downarrow)$ is given by the space

$$W := \left\{ \begin{bmatrix} \alpha I & \operatorname{diag} x \\ \operatorname{diag} x & \alpha I \end{bmatrix} : \alpha \in \mathbb{R},\, x \in \mathbb{R}^n \right\},$$

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A publication of the International Linear Algebra Society
Volume 15, pp. 191-200, June 2006

and by the group H of linear operators from W to W of the type

$$\begin{bmatrix} \alpha I & \operatorname{diag} x \\ \operatorname{diag} x & \alpha I \end{bmatrix} \to \begin{bmatrix} \alpha I & U_1 \operatorname{diag} x\, U_2^T \\ U_1 \operatorname{diag} x\, U_2^T & \alpha I \end{bmatrix},$$

where U_1 and U_2 run over the group of $n \times n$ generalized permutation matrices (i.e., matrices with exactly one nonzero entry ± 1 in each column and each row) such that

$$U_1 \operatorname{diag} x\, U_2^T = \operatorname{diag}(\pm x_{i_1}, \ldots, \pm x_{i_n}) \tag{3.6}$$

for any choice of $\pm$ signs and of permutation $x_{i_1}, \ldots, x_{i_n}$ of the entries of x. Therefore

$$\begin{bmatrix} \beta I & \operatorname{diag} y \\ \operatorname{diag} y & \beta I \end{bmatrix} \preceq_H \begin{bmatrix} \alpha I & \operatorname{diag} x \\ \operatorname{diag} x & \alpha I \end{bmatrix} \quad \text{iff} \quad \beta = \alpha \quad \text{and} \quad y \preceq_{aw} x. \tag{3.7}$$

Here the relation $y \preceq_{aw} x$ means that (3.3) holds for $a := (|x_1|, \ldots, |x_n|)$ and $b := (|y_1|, \ldots, |y_n|)$ (cf. [4, p. 16]). Observe that

$$\begin{aligned} y \preceq_{aw} x \ \text{ iff } & \ (|y|_\downarrow, -|y|_\uparrow) \ \preceq_m \ (|x|_\downarrow, -|x|_\uparrow) \\ \text{iff } & \ (\alpha 1_n + |y|, \alpha 1_n - |y|) \preceq_m (\alpha 1_n + |x|, \alpha 1_n - |x|) \end{aligned} \tag{3.8}$$

(cf. [1, p. 107]), where $|x|$ denotes the vector of the moduli of the entries of x, and the vector $|x|_\downarrow$ (resp. $|x|_\uparrow$) consists of the entries of $|x|$ arranged decreasingly (resp. increasingly).

The orthoprojector P from V onto W is given by

$$P \begin{bmatrix} \alpha I & X \\ X^T & \alpha I \end{bmatrix} = \begin{bmatrix} \alpha I & \operatorname{diag} \Delta(X) \\ \operatorname{diag} \Delta(X) & \alpha I \end{bmatrix},$$

where $\Delta(X)$ stands for the diagonal of X.

Denote by $\mathbb{L}_{2n}$ the *Loewner cone* of all $2n \times 2n$ positive semidefinite matrices. Notice that $\mathbb{L}_{2n}$ is G-invariant. Define

$$C := \mathbb{L}_{2n} \cap W.$$

Evidently, C is an H-invariant convex cone. In addition, $\preceq_C$ is the cone preordering on W such that

$$\begin{bmatrix} \beta I & \operatorname{diag} y \\ \operatorname{diag} y & \beta I \end{bmatrix} \preceq_C \begin{bmatrix} \alpha I & \operatorname{diag} x \\ \operatorname{diag} x & \alpha I \end{bmatrix} \tag{3.9}$$

implies

$$(\beta 1_n + y, \beta 1_n - y)_\downarrow \quad \leq \quad (\alpha 1_n + x, \alpha 1_n - x)_\downarrow, \tag{3.10}$$

where $\leq$ is the entrywise ordering on $\mathbb{R}^{2n}$ and the n-vector 1_n consists of ones. To see this, note that the vectors in inequality (3.10) consist of the eigenvalues of the matrices in (3.9), respectively, (see [1, p. 105]), and use Weyl's monotonicity theorem [3, Cor. III.2.3, p. 63].

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Take H_0 to be the subgroup of H consisting of all operators of the form $h_0 = U_0(\cdot)I$, where $U_0 = \operatorname{diag}(\pm 1, \dots, \pm 1)$ is a sign change matrix.

With the above notation, and from Theorem 2.1 and Corollary 2.3, we obtain

THEOREM 3.2. *Let α_j and M_j for $j = 0, 1, \dots, k$ be, respectively, real numbers and $n \times n$ real matrices, and let*

$$\widetilde{M_j} := \begin{bmatrix} \alpha_j I & M_j \\ M_j^T & \alpha_j I \end{bmatrix}$$

be the corresponding matrices in V such that $\widetilde{M_j} \preceq_{\mathbb{L}_{2n}} \widetilde{M_0}$. Assume that there exists an operator g_0 in G simultaneously sending the $\widetilde{M_j}$ into W, that is

$$g_0\widetilde{M_j} = \begin{bmatrix} \alpha_j I & \operatorname{diag}\sigma(M_j) \\ \operatorname{diag}\sigma(M_j) & \alpha_j I \end{bmatrix} \quad \text{for } j = 0, 1, \dots, k,$$

where $\sigma(M_j) := (\pm s_{i_1}(M_j), \dots, \pm s_{i_n}(M_j))$ and $s_1(M_j) \geq \ldots \geq s_n(M_j) \geq 0$ are the singular values of M_j with any choice of $\pm$ signs and any permutation $i_1, \dots, i_n$ of $1, \dots, n$.

Let ψ be a real function which is Schur-convex and entrywise increasing on $\mathbb{R}^{2n}$. Then

$$\begin{aligned} \max_{1\leq j\leq k} \psi(\alpha_j 1_n + \Delta(U_1 M_j U_2^T)U_0, \alpha_j 1_n - \Delta(U_1 M_j U_2^T)U_0) \\ \leq \psi(\alpha_0 1_n + s(M_0), \alpha_0 1_n - s(M_0)) \end{aligned} \tag{3.11}$$

for any orthogonal $n \times n$ matrices U_1 and U_2, and for some $U_0 = \operatorname{diag}(\pm 1, \dots, \pm 1)$ such that $\Delta(U_1 M_0 U_2^T)U_0 = |\Delta(U_1 M_0 U_2^T)|$.

Proof. Consider the functions

$$\varphi(X) := \psi(\alpha 1_n + x, \alpha 1_n - x) \quad \text{and} \quad \widetilde{\varphi}(X) := \psi(\alpha 1_n + |x|, \alpha 1_n - |x|) \tag{3.12}$$

for $X = \begin{bmatrix} \alpha I & \operatorname{diag} x \\ \operatorname{diag} x & \alpha I \end{bmatrix}$ with $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. It is obvious that $\varphi(X) = \widetilde{\varphi}(X)$ for nonnegative x.

It follows that φ is C-increasing, because (3.9)-(3.10) are met, and ψ is Schur-convex, permutation-invariant and entrywise increasing. Using (3.6) and (3.12) we obtain that $\widetilde{\varphi}$ is H-invariant. Applying (3.7)-(3.8) and (3.12) we deduce that $\widetilde{\varphi}$ is H-increasing. Thus φ is H-increasing on $W_0 := \{X \in W : x \in \mathbb{R}^n_+\}$.

We have that

$$Pg\widetilde{M_j} = \begin{bmatrix} \alpha_j I & \operatorname{diag}\Delta(U_1 M_j U_2^T) \\ \operatorname{diag}\Delta(U_1 M_j U_2^T) & \alpha_j I \end{bmatrix}$$

for any $g = g(U_1, U_2) \in G$. Using Corollary 2.3 and (2.9) with $g_0 = h_0 = U_0(\cdot)I$, one obtains

$$\begin{aligned} \max_{1\leq j\leq k} \varphi(&\begin{bmatrix} \alpha_j I & \operatorname{diag}\Delta(U_1 M_j U_2^T)U_0 \\ \operatorname{diag}\Delta(U_1 M_j U_2^T)U_0 & \alpha_j I \end{bmatrix}) \\ \leq \quad \varphi(&\begin{bmatrix} \alpha_0 I & \operatorname{diag} s(M_0) \\ \operatorname{diag} s(M_0) & \alpha_0 I \end{bmatrix}), \end{aligned}$$

which is equivalent to (3.11). ∎

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 191-200, June 2006

4. Concluding remarks. A central role in the proof of Theorem 2.1 is played by Schur type inequality (2.1) and by the double inequalities $Pgw_j \preceq_C Pgw \preceq_H w$ with the mediate vector Pgw. Such kind of a relation is known in the literature as *weak group majorization* (see [8, p. 120]).

To be more precise, assume $\prec\!\!\prec$ is a preordering on W which is *H-compactible*, i.e., $\prec\!\!\prec$ is invariant under finite convex combinations of elements of H:

$$y \prec\!\!\prec x \quad \text{implies} \quad \sum_i \alpha_i h_i y \prec\!\!\prec \sum_i \alpha_i h_i x$$

for $h_i \in H$ and $\alpha_i > 0$ with $\sum_i \alpha_i = 1$. In particular, if $\prec\!\!\prec$ is a cone preordering induced by an H-invariant convex cone, then $\prec\!\!\prec$ is H-compactible (see [8, p. 120]). Given vectors $x, y \in V$, we say that y is *weakly H-majorized* by x (in symbol, $y \preceq_{H,w} x$), if there exists $z \in W$ such that $y \prec\!\!\prec z$ and $z \preceq_H x$.

It is clear that if a function $\varphi : W \to \mathbb{R}$ is both $\prec\!\!\prec$-increasing and $\preceq_H$-increasing, then $\varphi(y) \le \varphi(x)$ whenever $y \preceq_{H,w} x$. In particular, this is valid if φ is $\prec\!\!\prec$-increasing, H-invariant and convex [8, Thm 3].

Summarizing, our results can be extended to an H-compactible ordering $\prec\!\!\prec$ instead of the cone preordering $\preceq_C$, provided the linear operator Pg is $\prec\!\!\prec$-isotone for each $g \in G$. Then Theorem 2.1 remains true for any $\preceq_{H,w}$-increasing function $\varphi : W \to \mathbb{R}$.

Also, further generalizations are possible for H-stochastic operators L in place of Pg. Recall that a linear operator $L : W \to W$ is said to be *H-stochastic* if $Lx \preceq_H x$ for $x \in W$ (cf. [17, Thm 3.3]). Namely, if $w_j \prec\!\!\prec w$ then $\varphi(Lw_j) \le \varphi(Lw) \le \varphi(w)$ provided L is both $\prec\!\!\prec$-isotone and H-doubly stochastic, and, in addition, φ is both $\prec\!\!\prec$-increasing and H-increasing. Moreover, $\varphi(Lw_j) \le \varphi(w)$ and $\max_j \varphi(Lw_j) \le \varphi(w)$ for any $\preceq_{H,w}$-increasing function φ.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 15, pp. 191-200, June 2006

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