Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 14, pp. 51-58, September 2005

ELA
www.math.technion.ac.il/iic/ela

MATRICES AND GRAPHS IN EUCLIDEAN GEOMETRY*

MIROSLAV FIEDLER†

Abstract. Some examples of the interplay between matrix theory, graph theory and n-dimensional Euclidean geometry are presented. In particular, qualitative properties of interior angles in simplices are completely characterized. For right simplices, a relationship between the tree of legs and the circumscribed Steiner ellipsoids is proved.

Key words. Euclidean space, Gram matrix, Biorthogonal bases, Simplex, Interior angle, Steiner circumscribed ellipsoid, Right simplex.

AMS subject classifications. 05C50, 52A40.

1. Gram matrices. As is well known, the Gram matrix $G(S)$ of an ordered system $S = (a_1, \ldots, a_m)$ of vectors in a Euclidean n-space E_n is the matrix $G(S) = [\langle a_i, a_j \rangle]$, where $\langle u, v \rangle$ means the inner product of the vectors u and v.

Gram matrices form a natural link between positive semidefinite matrices and systems of vectors in a Euclidean space because of the following:

THEOREM 1.1. *Every positive semidefinite matrix is a Gram matrix of some system of vectors S in some Euclidean space. The rank of the matrix $G(S)$ is equal to the dimension of the smallest Euclidean space containing S.*

In addition, every linear relationship among the vectors in S is reflected in the same *linear relationship among the rows of $G(S)$, and conversely.*

Let us present an example.

THEOREM 1.2. *Let $A = [a_{ij}]$ be a positive semidefinite matrix with row sums zero. Then*

$$2 \max_i \sqrt{a_{ii}} \leq \sum_i \sqrt{a_{ii}}.$$

Proof. Let n be the order of A. By Theorem 1.1, there exist in some Euclidean space (its dimension is at least the rank of A) vectors $u_1, \ldots, u_n$, such that $\langle u_i, u_j \rangle = a_{ij}$ for $i, j = 1, \ldots, n$. Since the row sums of A are zero, the vectors u_i satisfy $\sum_i u_i = 0$.

Thus for every $k \in \{1, \ldots, n\}$, the lengths satisfy

$$\begin{aligned} |u_k| &= |\sum_{j \neq k} u_j| \\ &\leq \sum_{j \neq k} |u_j| \end{aligned}$$

* Received by the editors 27 September 2004. Accepted for publication 11 September 2005. Handling Editor: Bryan L. Shader.

†Academy of Sciences of the Czech Republic, Institute of Computer Science, Pod vodárenskou věží 2, 182 07 Praha 7, and Mathematics Institute, Žitná 25, 115 67 Praha 1, Czech Republic (fiedler@math.cas.cz). Supported by grant GAAV A1030302.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 14, pp. 51-58, September 2005

so that

$$2|u_k| \le \sum_j |u_j|.$$

Therefore,

$$2 \max_i |u_i| \le \sum_i |u_i|.$$

Since $a_{ii} = |u_i|^2$, the result follows. □

The following is well known:

THEOREM 1.3. *If $A = [a_{ij}]$ and $B = [b_{ij}]$ are positive definite matrices of the same oder, then their* Hadamard product $A \circ B = [a_{ij}b_{ij}]$ *is also positive definite.*

If A and B are positive semidefinite, then $A \circ B$ is positive semidefinite as well.

No direct geometric consequences of this result are known. We intend to mention a special one.

If A is a positive definite matrix, then, by Theorem 1.3, $A \circ A^{-1}$ is also positive definite. One can even prove:

THEOREM 1.4. ([4]) *If A is positive definite, then $A \circ A^{-1} - I$ is positive semidefinite and its row sums are equal to zero. If $A = [a_{ij}]$, $A^{-1} = [\alpha_{ij}]$, then*

$$a_{ii}\alpha_{ii} \ge 1 \text{ for all } i \tag{1.1}$$

and

$$2 \max_i \sqrt{a_{ii}\alpha_{ii} - 1} \le \sum_i \sqrt{a_{ii}\alpha_{ii} - 1}. \tag{1.2}$$

Proof. Since

$$\begin{bmatrix} A & I \\ I & A^{-1} \end{bmatrix}$$

is positive semidefinite (of rank n if A has order n), the same holds for

$$\begin{bmatrix} A^{-1} & I \\ I & A \end{bmatrix}.$$

Therefore, their Hadamard product

$$\begin{bmatrix} A \circ A^{-1} & I \\ I & A \circ A^{-1} \end{bmatrix}$$

is also positive semidefinite which implies, after a little thought, that all eigenvalues of $A \circ A^{-1}$ are at least one and $A \circ A^{-1} - I$ is positive semidefinite. It is easily checked that for any nonsingular matrix A, $(A^T \circ A^{-1})e = e$, where e is the column vector of all ones. This concludes the proof of the first part. The second part follows then by applying Theorem 1.2. □

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 14, pp. 51-58, September 2005

Observe that both conditions in Theorem 1.4 involve the diagonal entries of both matrices A and A^{-1} only. It was shown ([5]) that the *best possible* condition between the diagonal entries of A and A^{-1} is (in addition to (1.1))

$$2\max_i(\sqrt{a_{ii}\alpha_{ii}}-1) \le \sum_i(\sqrt{a_{ii}\alpha_{ii}}-1).$$

This last result has interesting geometrical consequences:

THEOREM 1.5. ([5]) *Let the vectors $u_1,\dots u_n$, $v_1,\dots,v_n$ form a pair of biorthogonal bases (i.e., the inner product of u_i and v_j is the Kronecker delta δ_{ij}) in a Euclidean n-space E_n. Then for the lengths,*

$$|u_i||v_i| \ge 1, \quad i=1,\dots,n,$$

$$2\max_i(|u_i||v_i|-1) \le \sum_i(|u_i||v_i|-1).$$

Conversely, if nonnegative numbers $\alpha_1,\dots,\alpha_n$, $\beta_1,\dots,\beta_n$ satisfy

$$\alpha_i\beta_i \ge 1, \quad i=1,\dots,n,$$

$$2\max_i(\alpha_i\beta_i-1) \le \sum_i(\alpha_i\beta_i-1),$$

then there exists in E_n a pair of biorthogonal bases u_i, v_j, such that

$$|u_i|=\alpha_i, |v_i|=\beta_i, \quad i=1,\dots,n.$$

We recall here somewhat modified conditions from [5] when equality is attained.

THEOREM 1.6. *Let $A=[a_{ij}]$ be an $n\times n$ positive definite matrix, $n\ge 2$, let $A^{-1}=[\alpha_{ij}]$. Then the following are equivalent:*

1.

$$\sqrt{a_{nn}\alpha_{nn}}-1=\sum_{i=1}^{n-1}(\sqrt{a_{ii}\alpha_{ii}}-1).$$

2.

$$\frac{a_{ij}}{\sqrt{a_{ii}}\sqrt{a_{jj}}}=\frac{\alpha_{ij}}{\sqrt{\alpha_{ii}}\sqrt{\alpha_{jj}}}, \quad i,j=1,\dots,n-1,$$

$$\frac{a_{in}}{\sqrt{a_{ii}}\sqrt{a_{nn}}}=-\frac{\alpha_{in}}{\sqrt{\alpha_{ii}}\sqrt{\alpha_{nn}}}, \quad i=1,\dots,n-1.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 14, pp. 51-58, September 2005

ELA
www.math.technion.ac.il/iic/ela

3. *A is diagonally similar to*

$$C = \begin{bmatrix} I_1 + \omega c c^T & c \\ c^T & 1 + \omega c^T c \end{bmatrix},$$

where c is a real vector with $n-1$ coordinates and

$$\omega = \frac{\sqrt{1 + c^T c} - 1}{c^T c}$$

if $c \neq 0$; if $c = 0$, $\omega = 0$.

2. Euclidean simplices. An *n-simplex* in E_n is a generalization of the triangle in the plane and the tetrahedron in the three-dimensional space. It is determined by its $n+1$ vertices, say, $A_1, \dots, A_{n+1}$, has $\binom{n+1}{2}$ edges A_iA_j, $i \neq j$, $(n-1)$-dimensional faces ω_i (opposite A_i), etc.

We denote by ϕ_{ij} the (dihedral) interior angles between ω_i and ω_j.

In matrix theory, completion problems are now very popular. Let us mention one for the lengths of edges in an n-simplex. What are necessary and sufficient conditions for the lengths of some set of segments $\{l_{ij}, (i,j) \in S\}$ that they can serve as lengths of edges A_iA_j of some n-simplex? (For instance, strict triangle inequalities have to hold for every "closed" triplet, and a strict "polygonal inequality" for every closed polygon.)

Similarly as in matrix theory, graphs play an important role here.

THEOREM 2.1. ([3]). *A set of assigned lengths $\{l_{ij}\}$ can serve as the set of lengths of edges A_iA_j of an n-simplex if and only if it is the set of lengths of edges of such simplex all interior angles of which opposite to missing edges are right.*

The following characterization of the lengths of *all* edges of the simplex is worth mentioning:

THEOREM 2.2. (Menger, Schoenberg, see [1]). *The numbers e_{ij} can serve as* squares *of the lengths of edges between A_i and A_j, if and only if $e_{ii} = 0$ for all i, and*

$$\sum_{i,j} e_{ij} x_i x_j < 0, \quad \text{whenever} \quad \sum_i x_i = 0.$$

This is equivalent to the condition that the matrix

$$M_0 = \begin{bmatrix} 0 & e^T \\ e & M \end{bmatrix}, \tag{2.1}$$

where e is the column vector of all ones and $M = [e_{ij}]$ (the *Menger matrix*) is *elliptic*, i.e. has one eigenvalue positive and the remaining negative.

The following crucial fact is presented without proof and will be used in the sequel.

THEOREM 2.3. ([3], [7]) *The inverse of the matrix M_0 is then a $(-\frac{1}{2})$-multiple of the matrix*

$$Q_0 = \begin{bmatrix} q_{00} & q_0^T \\ q_0 & Q \end{bmatrix}, \tag{2.2}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 14, pp. 51-58, September 2005

ELA
www.math.technion.ac.il/iic/ela

where the matrix Q is the Gram matrix of the outer normals in the above mentioned n-simplex (in a certain way normalized in order that the sum of the normals be zero).

As is well known, there is an intimate relationship between the irreducibility of a symmetric matrix and the connectedness of the corresponding (undirected) graph:

THEOREM 2.4. *Let $A = [a_{ik}]$ be an $n \times n$ symmetric matrix, $G = (N, E)$ its graph, where $N = \{1, \dots, n\}$and E the set of unordered pairs (i, k), $i \neq k$, for which $a_{ik} \neq 0$. Then A is irreducible (i.e. for no simultaneous permutation of rows and columns having the form $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ with non-void A_1 and A_2) if and only if the graph G is connected.*

A real matrix A can always be (uniquely) written as $A = A^+ + A^-$, where A^+ contains the positive entries of A, having zeros elsewhere, and A^- contains all negative entries of A, having zeros elsewhere. Let us call A^+ (respectively, A^-) the *positive* (respectively,*negative*) *part* of A.

THEOREM 2.5. ([2], [7]) *Suppose A is a real symmetric positive semidefinite $n \times n$ matrix with rank $n - 1$ and such that $Ae = 0$ (e as above). Then the negative part of A is irreducible.*

Conversely, if C is a nonpositive irreducible symmetric $n \times n$ matrix with zero diagonal entries, then there exists a real symmetric positive semidefinite matrix A with rank $n - 1$, such that its negative part is C. In addition, the positive part can have any symmetric zero - nonzero structure.

Proof. Suppose A^- is reducible, $A = [a_{ik}]$; then $N = \{1, \dots, n\}$ can be decomposed, i. e. $N = N_1 \cup N_2$, $N_1 \cap N_2 = \emptyset$, $N_1 \neq \emptyset$, $N_2 \neq \emptyset$, in such a way that

$$a_{ik} \geq 0 \text{ for } i \in N_1, k \in N_2.$$

Since $Ae = 0$, we have

$$\begin{aligned} 0 &= \sum_{i \in N_1, k \in N} a_{ik} \\ &= \sum_{i \in N_1, k \in N_1} a_{ik} + \sum_{i \in N_1, k \in N_2} a_{ik}. \end{aligned}$$

Thus both -nonnegative- terms are zero, we obtain a contradiction with the positive definiteness of A since the first summand is the value (Ay, y) for the non-zero vector $y = (y_i)$, $y_i = 1$ if $i \in N_1$, $y_i = 0$ otherwise.

To prove the converse, let C be given. Define a diagonal matrix D_0 such that $(D_0 + C)e = 0$ and set $D_0 + C = Q$. Then Q is a symmetric irreducible M-matrix of rank $n-1$, thus positive semidefinite. Choose now any symmetric nonnegative matrix, A_1, with zero entries in all nonzero off-diagonal positions of C, and otherwise arbitrary zero - nonzero pattern. Let D_1 be the diagonal matrix for which $(D_1 + A_1)e = 0$.

Since all principal minors of Q of order less than n are positive, there exists an $\varepsilon > 0$ such that the matrix

$$A = Q + \varepsilon(D_1 + A_1)$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 14, pp. 51-58, September 2005

ELA
www.math.technion.ac.il/iic/ela

will still have this property and, in addition, the row-sums of A are zero. The matrix A clearly satisfies all conditions prescribed. □

We can now formulate an important geometrical application:

THEOREM 2.6. ([2]) *Let us color each edge* A_iA_j *of an* n*-simplex with vertices* $A_1, \ldots, A_{n+1}$ *by one of the following three colors:*

red*, if the opposite interior angle* ϕ_{ij} *is* **acute***;*

blue*, if the opposite interior angle* ϕ_{ij} *is* **obtuse***;*

white*, if the opposite interior angle* ϕ_{ij} *is* **right***.*

Then, the set of red edges connects all the vertices of the simplex.

Conversely, if we color all edges of an n*-simplex by three colors red, blue and white in such a way that the red edges connect all vertices, then there exists such deformation of the simplex that opposite red edges there are acute, opposite blue edges obtuse and opposite white edges right interior angles.*

Proof. If we assign to the given simplex the matrices M_0 and Q_0 from (2.1) and (2.2), then the matrix $Q = [q_{ij}]$ as Gram matrix of the outer normals has the property that $q_{ij} < 0$, $q_{ij} = 0$, or $q_{ij} > 0$, according to whether the interior angle ϕ_{ij} is acute, right, or obtuse. (Indeed, the angle spanned by outer normals is $\pi - \phi_{ij}$.) Theorem 2.5 applied to the matrix Q yields the result. □

This result has many consequences:

THEOREM 2.7. *Every* n*-simplex has at least* n *acute interior angles.*

THEOREM 2.8. *There are* n*-simplices which have exactly* n *acute interior angles and all the remaining* $\binom{n}{2}$ *interior angles right. The red edges span a tree over the vertices of the simplex.*

We called such simplices *right simplices* in [2].

3. Right simplices. We recall here the main properties of right simplices and add a new one.

THEOREM 3.1. ([2]). *The red edges (opposite acute angles) are mutually perpendicular.*

In accordance with the right triangle, we call these edges *legs*.

Hence:

THEOREM 3.2. *The* tree of legs *can be completed to an (n-dimensional) box (i.e., a rectangular parallelepiped); its center of symmetry is thus the circumcenter of the simplex.*

REMARK 3.3. A particularly interesting example is the *Schlaefli simplex* when the tree of legs is a path.

We recall here its main properties.

THEOREM 3.4. *Every face of a Schlaefli simplex is also a Schlaefli simplex. The Schlaefli simplex is the only simplex all 2-dimensional faces of which are right triangles. The circumcenter is the middle point of the longest edge.*

In the case of the right simplex, the matrix Q from (2.2) is an *acyclic* matrix in the sense of [6]. We can thus apply a theorem from [6] which describes sign-patterns of eigenvectors of such matrix in terms of those edges -so called *negative*- of the corresponding tree for which the coordinates of the eigenvector have different signs. In our terms, the assertion from [6] gives:

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 14, pp. 51-58, September 2005

ELA
www.math.technion.ac.il/iic/ela

THEOREM 3.5. *Let y be an eigenvector of Q. If all coordinates of y are different from zero, then the corresponding eigenvalue λ of Q is simple, and there are as many eigenvalues of Q smaller than λ as is the number of negative edges with respect to y.*

To find the geometric interpretation of the eigenvalues and eigenvectors of Q, let us remind that for every n-simplex, there exists a distinguished circumscribed ellipsoid (circumscribed means that it contains all vertices of the simplex), the so called *Steined circumscribed ellipsoid*) which has its center in the centroid and the tangent hyperplane at every vertex is parallel to the opposite face. In fact, it is the affine image of the circumscribed hypersphere in the affine transformation of a regular n-simplex into the given simplex. It is easily seen that the equation of the Steiner circumscribed ellipsoid S in barycentric coordinates with respect to the simplex is

$$(\sum_i x_i)^2 - \sum_i x_i^2 = 0.$$

Indeed, the tangent hyperplane at the vertex A_k is then

$$\sum_{i\neq k} x_i = 0$$

and the polar of the centroid $(1, \ldots, 1)$ is $\sum_i x_i = 0$, thus the hyperplane at infinity. To find the directions of the axes, we use the fact that the pole with respect to Q of a hyperplane containing the centroid is the orthogonal direction to that hyperplane. Let thus $\sum_i \alpha_i x_i = 0$ be the equation of a hyperplane α and let $\sum_i \alpha_i = 0$. The orthogonal direction

$$d_i = \sum_j q_{ij}\alpha_j \tag{3.1}$$

is the axis direction if and only if its polar with respect to S

$$\sum_i d_i \sum_i x_i - \sum_i d_i x_i = 0$$

coincides with the hyperplane α:

$$\sum_j d_j - d_i = \xi\alpha_i,$$

or, since $\sum_j d_j = 0$,

$$-d_i = \xi\alpha_i.$$

By (3.1),

$$\sum_j q_{ij}\alpha_j = \lambda\alpha_i$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 14, pp. 51-58, September 2005

ELA
www.math.technion.ac.il/iic/ela

for

$$\lambda = -\xi.$$

Thus, the column vector $(\alpha_1, \ldots, \alpha_{n+1})^T$ is an eigenvector of Q. Suppose now that (p, q) is an edge of the tree and that $\alpha_p\alpha_q < 0$. Then the leg between A_p and A_q contains an intersection point with the hyperplane α as its interior point. We now use the result proved in [7]: The halfaxis of the Steiner ellipsoid corresponding to an eigenvalue λ of Q (and hence corresponding to the hyperplane α) is proportional to the reciprocal of λ.

We obtain:

Theorem 3.6. *Let Σ be a right n-simplex. Let α be such hyperplane orthogonal to an axis of the Steiner circumscribed ellipsoid of Σ which does not contain any vertex of Σ. Then the corresponding halfaxis is the kth largest if and only if α intersects the tree of the legs of Σ in exactly k points. In addition, there is no other halfaxis of the Steiner circumscribed ellipsoid having the same length.*

Remark 3.7. In fact, it follows by a continuity argument that one can allow the hyperplane α to contain a vertex of Σ in which exactly two legs meet. This vertex is then counted as a single intersection with the tree of legs.

REFERENCES

[1] L. M. Blumenthal. *Theory and Applications of Distance Geometry.* Clarendon Press, Oxford, 1953.
[2] M. Fiedler. Über qualitative Winkeleigenschaften der Simplexe. *Czechoslovak Math. J.*, 7(82):463–478, 1957.
[3] M. Fiedler. Über zyklische n-Simplexe und konjugierte Raumvielecke. *Comm. Math. Univ. Carol.* 2,2:3–26, 1961.
[4] M. Fiedler. Über eine Ungleichung für positiv definite Matrizen. *Math. Nachrichten*, 23:197–199, 1961.
[5] M. Fiedler. Relations between the diagonal elements of two mutually inverse positive definite matrices. *Czechoslovak Math. J.*, 14(89):39–51, 1964.
[6] M. Fiedler. Eigenvectors of acyclic matrices. *Czechoslovak Math. J.*, 25(100):607–618, 1975.
[7] M. Fiedler. A geometric approach to the Laplacian matrix of a graph. In: *Combinatorial and Graph-Theoretical Problems in Linear Algebra* (R.A. Brualdi, S. Friedland, V. Klee, Editors), Springer, New York, 1993, pp. 73-98.