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A NOTE ON THE CP-RANK OF MATRICES GENERATED BY SOULES MATRICES*

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Abstract. It is proved that a nonnegative matrix generated by a Soules matrix is a completely positive matrix with cp-rank equal to the rank.

Key words. Completely positive matrices, cp-rank, Soules matrices.

AMS subject classifications. 15A23, 15F48.

1. Introduction.

An $n \times n$ matrix A is *completely positive* if it can be represented as a product $A = BB^T$ where B is a nonnegative matrix. The *cp-rank* of a completely positive matrix A is the minimal k for which there exists a nonnegative $n \times k$ matrix B satisfying $A = BB^T$. A lot of work has been done on completely positive matrices [2], but the two basic problems are still unsolved: Determining which nonnegative positive semidefinite matrices are completely positive, and computing the cp-ranks of completely positive matrices. As for the second problem, some bounds are known: By the definition, for any completely positive matrix

$$\operatorname{rank} A \leq \text{cp-rank}\, A.$$

However, the cp-rank may be much larger than the rank. It is known that the cp-rank of a rank r completely positive matrix may be as big as $r(r+1)/2 - 1$ (but not bigger) [1, 4]. The equality $\text{cp-rank}\, A = \operatorname{rank} A$ is known to hold for completely positive matrices of rank at most 2, or of order at most 3×3, and for completely positive matrices with certain zero patterns [2, 5].

An $n \times n$ matrix R is a *Soules matrix* if it is an orthogonal matrix whose first column is positive, and for every nonnegative diagonal matrix D with nonincreasing diagonal elements the matrix RDR^T is nonnegative. Such matrices were introduced in [6], where Soules constructed for any positive vector x an orthogonal matrix R with these properties and with x as its first column. In [3], Elsner, Nabben and Neumann described the structure of *all* such orthogonal matrices (and named them after Soules). A matrix of the form RDR^T, where R is a Soules matrix and $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \ldots d_n \geq 0$, will be called here a *nonnegative matrix generated by a Soules matrix.*

OBSERVATION 1.1. *Any $n \times n$ nonnegative matrix A generated by a Soules matrix is completely positive, and* $\text{cp-rank}\, A \leq n$.

Proof. Let $A = RDR^T$, where R is a Soules matrix and $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ is a nonnegative diagonal matrix with nonincreasing diagonal elements. Then the matrix $B = R\sqrt{D}R^T$, where $\sqrt{D} = \operatorname{diag}(\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_n})$, is also an $n \times n$ nonnegative matrix generated by a Soules matrix, and $A = BB^T$. ☐

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In this note we show that the cp-rank of a nonnegative matrix generated by a Soules matrix is actually equal to the rank. The proof is based on the structure of Soules matrices described in [3]. For the description we need some notations (similar, but not identical, to those used in [3]). The set $\{1, 2, \ldots, n\}$ will be denoted by $\langle n \rangle$. For any $x \in \mathbb{R}^n$ and a set of indices $K \subseteq \langle n \rangle$ let $x_K = [y_i]$ be the n-vector defined by

$$y_i = \begin{cases} x_i & i \in K, \\ 0 & i \notin K. \end{cases}$$

Now consider n partitions $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_n$ of $\langle n \rangle$, such that $\mathcal{N}_l$ is a partition of $\langle n \rangle$ into l disjoint subsets, $N(l,1), N(l,2), \ldots, N(l,l)$ and $\mathcal{N}_{l+1}$ is obtained from $\mathcal{N}_l$ by choosing $1 \le k(l) \le l$ such that $|N(l,k(l))| \ge 2$, and then splitting $N(l,k(l))$ into two disjoint subsets, $N(l+1,k(l)), N(l+1,k(l)+1)$. The rest of the partition sets of $\mathcal{N}_{l+1}$ are the same as those of $\mathcal{N}_l$. That is, for $1 \le i < k(l)$, $N(l+1,i) = N(l,i)$, and for $k(l)+1 < i \le l+1$, $N(l+1,i) = N(l,i-1)$. We refer to such a sequence of partitions a sequence of *S-partitions* of $\langle n \rangle$. Finally, $x_{N(l,j)}$ will be abbreviated as $x_{(l,j)}$, and the j-th column of the matrix A will be denoted by A_j.

The description in [3] of Soules matrices can be stated as follows: An $n \times n$ matrix R is a Soules matrix if and only if there exists a positive unit vector x and a sequence of S-partitions $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_n$ of $\langle n \rangle$ such that

$$R_1 = x, \quad \text{and for } j = 2, \ldots, n \quad R_j = \alpha_j x_{(j,k(j-1))} - \beta_j x_{(j,k(j-1)+1)} \;, \tag{1.1}$$

where α_j and β_j are the unique nonnegative numbers satisfying

$$\alpha_j^2 \|x_{(j,k(j-1))}\|^2 + \beta_j^2 \|x_{(j,k(j-1)+1)}\|^2 = 1 \tag{1.2}$$

and

$$\alpha_j \|x_{(j,k(j-1))}\|^2 - \beta_j \|x_{(j,k(j-1)+1)}\|^2 = 0 \tag{1.3}$$

Observe that (1.2) guarantees that the norm of R_j is 1, and (1.3) guarantees that the column R_j is orthogonal to all its predecessors, so α_j and β_j are the unique nonnegative numbers making the matrix R defined in (1.1) orthogonal.

2. The cp-rank of Completely Positive Matrices Generated by Soules Matrices. We prove the following theorem:

THEOREM 2.1. *Let A be an $n \times n$ nonnegative matrix generated by a Soules matrix, then* cp-rank A = rank A.

Proof. The matrix A is of the form $A = RDR^T$, where R is an $n \times n$ Soules matrix, and $D = \operatorname{diag}(d_1, d_2, \ldots, d_r, 0, \ldots, 0)$, $d_1 \ge d_2 \ge \ldots \ge d_r > 0$. As mentioned in Observation 1.1, A is completely positive. We will show that $A = BB^T$ where B is an $n \times r$ nonnegative matrix. Let

$$\tilde{R} = (R_1 \ldots R_r), \quad \tilde{D} = \operatorname{diag}(d_1, d_2, \ldots, d_r).$$

Then $A = \tilde{R}\tilde{D}\tilde{R}^T$. Since complete positivity and cp-rank are not affected by simultaneous permutation of rows and columns, we may assume for convenience that each

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of the partitions sets $N(r,i)$ consists of consecutive integers and $N(r,i+1)$ consists of subsequent consecutive integers, that is

$$N(r,1) = \{1,\dots,l_1\}, N(r,2) = \{l_1+1,\dots,l_2\},\dots,N(r,r) = \{l_{r-1}+1,\dots,n\}.$$

Thus for $j = 1,\dots,r$, $x_{(r,j)}$ is of the form

$$x_{(r,j)} = \begin{bmatrix} 0 \\ \bar{x}_{(r,j)} \\ 0 \end{bmatrix},$$

where $\bar{x}_{(r,j)}$ is the vector in $\mathbb{R}^{l_j - l_{j-1}}$ consisting of the nonzero entries in $x_{(r,j)}$. For simplicity we denote $v^i = \bar{x}_{(r,i)}$, $i = 1,\dots,r$. Then

$$\tilde{R} = \begin{bmatrix} v^1 & \gamma_{12}v^1 & \gamma_{13}v^1 & \dots & \gamma_{1r}v^1 \\ v^2 & \gamma_{22}v^2 & \gamma_{23}v^2 & \dots & \gamma_{2r}v^2 \\ \vdots & \vdots & \vdots & & \vdots \\ v^r & \gamma_{r2}v^r & \gamma_{r3}v^r & \dots & \gamma_{rr}v^r \end{bmatrix},$$

where for $i = 1,\dots,r$ and $j = 2,\dots,r$

$$\gamma_{ij} = \begin{cases} \alpha_i & \text{if } N(r,i) \subseteq N(j,k(j-1)) \\ -\beta_i & \text{if } N(r,i) \subseteq N(j,k(j-1)+1) \\ 0 & \text{otherwise.} \end{cases}$$

Now let S be the $n \times r$ nonnegative matrix

$$S = \begin{bmatrix} \frac{v^1}{\|v^1\|} & 0 & 0 & 0 \\ 0 & \frac{v^2}{\|v^2\|} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \frac{v^r}{\|v^r\|} \end{bmatrix}$$

and let C be the $r \times r$ matrix

$$C = \begin{bmatrix} \|v^1\| & \gamma_{12}\|v^1\| & \gamma_{13}\|v^1\| & \dots & \gamma_{1r}\|v^1\| \\ \|v^2\| & \gamma_{22}\|v^2\| & \gamma_{23}\|v^2\| & \dots & \gamma_{2r}\|v^2\| \\ \vdots & \vdots & \vdots & & \vdots \\ \|v^r\| & \gamma_{r2}\|v^r\| & \gamma_{r3}\|v^r\| & \dots & \gamma_{rr}\|v^r\| \end{bmatrix}.$$

Since the columns of $\tilde{R}$ are orthonormal, so are the columns of C, so C is an orthogonal matrix. The first column of C is the positive vector

$$v = \begin{bmatrix} \|v^1\| \\ \|v^2\| \\ \vdots \\ \|v^r\| \end{bmatrix}.$$

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The r partitions $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_r$ of $\langle n\rangle$ induce a sequence $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_r$ of S-partitions of $\langle r\rangle$ defined by:

$$M(l,j) = \{i \in \langle r\rangle \,|\, N(r,i) \subseteq N(l,j)\}.$$

In particular, observe that

$$N(l,k(l)) = N(l+1,k(l)) \cup N(l+1,k(l)+1)$$

implies that

$$M(l,k(l)) = M(l+1,k(l)) \cup M(l+1,k(l)+1),$$

and that for $j = 2, \ldots, r$

$$C_j = \alpha_j v_{(j,k(j-1))} - \beta_j v_{(j,k(j-1)+1)},$$

where

$$v_{(l,i)} = v_{M(l,i)}.$$

Hence C is an $r \times r$ Soules matrix. By Observation 1.1, $C\tilde{D}C^T$ is a completely positive matrix with cp-rank at most r. That is, $C\tilde{D}C^T = \tilde{B}\tilde{B}^T$ for some nonnegative $r \times r$ matrix $\tilde{B}$. But $\tilde{R} = SC$ and $A = \tilde{R}\tilde{D}\tilde{R}^T$, hence $A = BB^T$, where $B = S\tilde{B}$ is a nonnegative $n \times r$ matrix. (By the proof of Observation 1.1, we may take $B = SC\sqrt{\tilde{D}}C^T$). ☐

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