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ON THE KREĬN-LANGER INTEGRAL REPRESENTATION OF GENERALIZED NEVANLINNA FUNCTIONS*

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Abstract. The Kreĭn-Langer integral representation of a matrix-valued generalized Nevanlinna function arises in problems of spectral theory and interpolation. A version of this formula which is suitable for such problems, and a corresponding Stieltjes inversion formula, are derived. Some classes of generalized Nevanlinna functions which are defined in terms of behavior at infinity are characterized in terms of their integral representations.

Key words. Generalized Nevanlinna function, Matrix-valued function, Integral representation, Stieltjes inversion formula, Negative squares.

AMS subject classifications. Primary: 47A56. Secondary: 30D99, 46C20.

1. Introduction. The Kreĭn-Langer integral representation of a generalized Nevanlinna function extends the classical Nevanlinna representation of a function which is analytic and has nonnegative imaginary part in the upper half-plane. The representation is due to Kreĭn and Langer [5] in the scalar case and Daho and Langer [2] for matrix-valued functions. In this paper, we take [2] as our starting point and write the result in an alternative form which is closer to the scalar case. It is this form that we shall use in another place to solve spectral and interpolation problems from the viewpoint of operator identities [10, 11]. We also derive a Stieltjes inversion formula and characterize some subclasses.

Throughout the paper we fix a positive integer m. For any nonnegative integer $\varkappa$, the **generalized Nevanlinna class $\mathbf{N}_\varkappa$** is the set of meromorphic $m \times m$ matrix-valued functions $v(z)$ on the union $\mathbb{C}_+ \cup \mathbb{C}_-$ of the upper and lower half-planes such that $v(\bar{z})^* = v(z)$ and the kernel

$$K(z,\zeta) = \frac{v(z) - v(\zeta)^*}{z - \bar{\zeta}}$$

has $\varkappa$ negative squares. The latter condition means that for any finite set of points $z_1, \dots, z_n$ in the domain of analyticity of $v(z)$ and vectors $c_1, \dots, c_n$ in $\mathbb{C}^m$, the matrix

$$[c_k^* K(z_j, z_k) c_j]_{j,k=1}^n$$

has at most $\varkappa$ negative eigenvalues, and at least one such matrix has exactly $\varkappa$ negative eigenvalues (counting multiplicity). A generalized Nevanlinna function has at most a finite number of nonreal poles [2, 5].

It is well known that $\mathbf{N}_0$ reduces to the classical Nevanlinna class. The Nevanlinna representation of a function $v(z)$ in $\mathbf{N}_0$ has the form

$$v(z) = A + B\,z + \int_{-\infty}^{\infty} \left[\frac{1}{t-z} - \frac{t}{1+t^2} \right] d\tau(t), \tag{1.1}$$

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where $\tau(t)$ is a nondecreasing $m \times m$ matrix-valued function such that the integral

$$\int_{-\infty}^{\infty} \frac{d\tau(t)}{1+t^2}$$

is convergent and $A = A^*$ and $B \geq 0$ are constant matrices. The Stieltjes inversion formula

$$\tau(b) - \tau(a) = \lim_{y \downarrow 0} \frac{1}{\pi} \int_a^b \operatorname{Im} v(t+iy)\, dt \tag{1.2}$$

recovers the increments of $\tau(t)$ for any points a, b of continuity of $\tau(t)$.

The Kreĭn-Langer representation (2.1) generalizes (1.1) to $\mathbf{N}_\varkappa$ for any $\varkappa \geq 0$. In place of the nondecreasing function $\tau(t)$ on $(-\infty, \infty)$ in (1.1), in (2.1) we use a function $\tau(t)$ which is nondecreasing on each of the open subintervals of $(-\infty, \infty)$ determined by a finite number $\alpha_1, \dots, \alpha_r$ of real points. The Stieltjes inversion formula (1.2) holds on each of these subintervals. We deduce this formula in matrix form, namely,

$$\int_a^b f(t)\, d\tau(t)\, g(t) = \lim_{y \downarrow 0} \frac{1}{\pi} \int_a^b f(t) \left[\operatorname{Im} v(t+iy)\right] g(t)\, dt\,, \tag{1.3}$$

where $f(t)$ and $g(t)$ are any continuous matrix-valued functions on $[a, b]$ of compatible orders and a, b are points of continuity of $\tau(t)$.

In the case of scalar-valued functions ($m = 1$), it is a simple matter to rewrite the original form of the Kreĭn-Langer representation [5] in the form (2.1). In the matrix case, this is less clear, and in Section 2 we give an explicit proof based on a fundamental result in Daho and Langer [2]. The Stieltjes inversion formula is derived in a matrix version in Section 3. In Section 4 we characterize some subclasses of $\mathbf{N}_\varkappa$ in terms of the representation (2.1). These classes arise naturally in applications, which will appear separately, that generalize results of A. L. Sakhnovich [9] and our previous papers [7, 8].

Integrals that appear in the paper are interpreted in the Stieltjes sense. Let $\tau(t)$ be a nondecreasing $m \times m$ matrix-valued function on a closed and bounded interval $[a, b]$, and let $f(t)$ and $g(t)$ be continuous matrix-valued functions of orders $p \times m$ and $m \times q$ on the interval. We define

$$\int_a^b f(t)\, [d\tau(t)]\, g(t) = \lim \sum f(t_k^*)\, [\tau(t_k) - \tau(t_{k-1})]\, g(t_k^*),$$

where the t_k are the division points of a finite partition of $[a, b]$, t_k^* is a point in the k-th subinterval, and the limit is taken as the mesh of the partition tends to zero. The definition is extended to arbitrary bounded or unbounded intervals Δ, closed or not, by approximation. Elementary properties of the integral are assumed. We only note the estimate

$$\left\| \int_\Delta f(t)\, [d\tau(t)]\, g(t) \right\| \leq \int_\Delta \|f(t)\|\, \|g(t)\|\, d\left[\operatorname{tr} \tau(t)\right], \tag{1.4}$$

where $\operatorname{tr} M$ is the trace of a square matrix M. Integrals of the type

$$\int_\Delta f_0(t)\, d\tau(t),$$

where $f_0(t)$ is scalar valued, are included by writing $f(t) = f_0(t) I_m$ and $g(t) = I_m$.

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2. The Kreĭn-Langer representation. Our version of the Kreĭn and Langer generalization of (1.1) takes the following form.

THEOREM 2.1. *Let $v(z)$ be an $m \times m$ matrix-valued meromorphic function such that $v(\bar{z})^* = v(z)$ on $\mathbb{C}_+ \cup \mathbb{C}_-$. A necessary and sufficient condition that $v(z)$ belong to some class $\mathbf{N}_\varkappa$ is that it can be written in the form*

$$
\begin{aligned}
v(z) = &\int_{-\infty}^{\infty} \left[\frac{1}{t-z} - \sum_{j=0}^{r} S_j(t,z) \right] d\tau(t) \\
&+ R_0(z) - \sum_{j=1}^{r} R_j\left(\frac{1}{z-\alpha_j}\right) \\
&- \sum_{k=1}^{s} \left[M_k\left(\frac{1}{z-\beta_k}\right) + M_k\left(\frac{1}{\bar{z}-\beta_k}\right)^* \right],
\end{aligned}
\tag{2.1}
$$

where $\alpha_1, \dots, \alpha_r \in (-\infty, \infty)$ and $\beta_1, \dots, \beta_s \in \mathbb{C}_+$ are distinct numbers, and

(1°) *the real line is a union of sets $\Delta_0, \Delta_1, \dots, \Delta_r$ such that $\Delta_1, \dots, \Delta_r$ are bounded open intervals containing $\alpha_1, \dots, \alpha_r$ and having disjoint closures, Δ_0 is their complement, and*

$$
\begin{aligned}
\frac{1}{t-z} - S_j(t,z) &= \frac{1}{t-z}\left(\frac{t-\alpha_j}{z-\alpha_j}\right)^{2\rho_j} \quad \text{on } \Delta_j, \quad j = 1, \dots, r, \\
\frac{1}{t-z} - S_0(t,z) &= \frac{1+tz}{t-z}\,\frac{(1+z^2)^{\rho_0}}{(1+t^2)^{\rho_0+1}} \quad \text{on } \Delta_0,
\end{aligned}
$$

for some positive integers $\rho_1, \dots, \rho_r$ and a nonnegative integer ρ_0;

(2°) *$\tau(t)$ is an $m \times m$ matrix-valued function which is nondecreasing on each of the $r+1$ open intervals of the real line determined by the points $\alpha_1, \dots, \alpha_r$ such that the integral*

$$
\int_{-\infty}^{\infty} \frac{(t-\alpha_1)^{2\rho_1} \cdots (t-\alpha_r)^{2\rho_r}}{(1+t^2)^{\rho_1+\cdots+\rho_r}}\,\frac{d\tau(t)}{(1+t^2)^{\rho_0+1}}
$$

converges;

(3°) *for each $j = 0, 1, \dots, r$, $R_j(z)$ is a polynomial of degree at most $2\rho_j + 1$, having selfadjoint $m \times m$ matrix coefficients, such that if a term of maximum degree $C_j z^{2\rho_j+1}$ is present then $C_j \ge 0$, and $R_1(0) = \cdots = R_r(0) = 0$;*

(4°) *for each $k = 1, \dots, s$, $M_k(z)$ is a polynomial $\not\equiv 0$ with $m \times m$ matrix coefficients such that $M_k(0) = 0$.*

The proof shows that $\Delta_0, \Delta_1, \dots, \Delta_r$ can be chosen arbitrarily so long as the conditions in (1°) are met. If $v(z) \in \mathbf{N}_\varkappa$, $\rho_0, \rho_1, \dots, \rho_r$ can be chosen such that $\rho_0 + \rho_1 + \cdots + \rho_r \le \varkappa$. Then if $\varkappa = 0$, $r = s = 0$, $\rho_0 = 0$, and (2.1) reduces to (1.1).

The convergence terms in (2.1) are given explicitly by

$$
\begin{aligned}
S_j(t,z) &= -\sum_{p=0}^{2\rho_j-1} \frac{(t-\alpha_j)^p}{(z-\alpha_j)^{p+1}}\,\chi_{\Delta_j}(t), \qquad j = 1, \dots, r, \\
S_0(t,z) &= \left\{ (t+z)\sum_{p=0}^{\rho_0-1} \frac{(1+z^2)^p}{(1+t^2)^{p+1}} + t\,\frac{(1+z^2)^{\rho_0}}{(1+t^2)^{\rho_0+1}} \right\} \chi_{\Delta_0}(t).
\end{aligned}
$$

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THEOREM 2.2 (Daho and Langer [2], Proposition 2.1). *Let $v(z)$ be an $m \times m$ matrix-valued function in $\mathbf{N}_\varkappa$. If $v(z)$ is holomorphic at $z_0 = iy_0$, $y_0 > 0$, then*

$$v(z) = \frac{1}{\prod_{j=1}^q (z-\gamma_j)^{\rho_j}(z-\bar\gamma_j)^{\rho_j}} \cdot \cdot \left\{ (z^2+y_0^2)^\rho \int_{-\infty}^{\infty} \frac{tz+y_0^2}{t-z}\, d\sigma(t) + \sum_{\ell=0}^{2\rho+1} B_\ell\, z^\ell \right\}, \tag{2.2}$$

where

(1) *$q, \rho, \rho_1, \dots \rho_q$ are nonnegative integers such that $\rho_1 + \cdots + \rho_q \le \rho \le \varkappa$;*
(2) *$\gamma_1, \dots, \gamma_q$ are distinct points in $\mathbb{C}_+ \cup (-\infty, \infty)$, $\gamma_j \ne iy_0$ for all j, and the nonreal points among $\gamma_1, \dots, \gamma_q$ coincide with the poles of $v(z)$ in $\mathbb{C}_+$;*
(3) *$\sigma(t)$ is a bounded nondecreasing $m \times m$ matrix-valued function on the real line;*
(4) *$B_0, \dots, B_{2\rho+1}$ are selfadjoint $m \times m$ matrices with $B_{2\rho+1} \ge 0$.*

The converse is also true and proved in [2, p. 280]. In fact, the argument there shows that

$$v(z) = \frac{1}{q(z)q(\bar z)^*} \left\{ p(z)p(\bar z)^* v_0(z) + R(z) \right\} \tag{2.3}$$

belongs to some class $\mathbf{N}_\varkappa$ whenever $v_0(z)$ belongs to $\mathbf{N}_0$, $p(z)$ and $q(z)$ are polynomials, and $R(z)$ is a polynomial with selfadjoint $m \times m$ matrix coefficients. Factorizations of the form (2.3) play an important role in the theory and have been studied in a series of recent papers, including, for example, [3, 4, 6]. Such factorizations yield an exact description of the number of negative squares, which is lacking in our approach.

Proof of Theorem 2.1, *necessity*. Assume that $v(z) \in \mathbf{N}_\varkappa$. We first reduce to the case where $v(z)$ is holomorphic on $\mathbb{C}_+ \cup \mathbb{C}_-$. Since $v(z)$ has at most a finite number of nonreal poles and $v(z) = v(\bar z)^*$, if there are nonreal poles we can write $v(z) = v_1(z) + v_2(z)$, where $v_1(z)$ is holomorphic on $\mathbb{C}_+ \cup \mathbb{C}_-$ and

$$v_2(z) = -\sum_{k=1}^{s} \left[M_k\left(\frac{1}{z-\beta_k}\right) + M_k\left(\frac{1}{\bar z - \beta_k}\right)^* \right]$$

for some polynomials $M_k(z) = \sum_{\ell=1}^{\sigma_k} M_{k\ell}\, z^\ell$, $k = 1, \dots, s$, as in ($4°$). A straightforward calculation yields

$$\frac{v_2(z) - v_2(\zeta)^*}{z - \bar\zeta} = \sum_{k=1}^{s} B_k(\zeta)^* \begin{bmatrix} 0 & H_k \\ H_k^* & 0 \end{bmatrix} B_k(z), \tag{2.4}$$

where for each $k = 1, \dots, s$,

$$B_k(\zeta)^* = \left[\frac{1}{\bar\zeta - \beta_k} \ \cdots \ \frac{1}{(\bar\zeta-\beta_k)^{\sigma_k}} \ \frac{1}{\bar\zeta - \bar\beta_k} \ \cdots \ \frac{1}{(\bar\zeta - \bar\beta_k)^{\sigma_k}} \right],$$

$$H_k = \begin{bmatrix} M_{k1} & M_{k2} & \cdots & M_{k,\sigma_k-1} & M_{k,\sigma_k} \\ M_{k2} & M_{k3} & \cdots & M_{k,\sigma_k} & 0 \\ & & \cdots & & \\ M_{k,\sigma_k} & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

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ELA
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We deduce that $v_1(z) \in \mathbf{N}_{\varkappa_1}$ and $v_2(z) \in \mathbf{N}_{\varkappa_2}$ where $\varkappa_1 + \varkappa_2 = \varkappa$; for example, this follows by reproducing kernel methods as in [1, Section 1.5].

Thus we may assume that $v(z)$ is holomorphic on $\mathbb{C}_+ \cup \mathbb{C}_-$. Represent $v(z)$ in the form (2.2) with $z_0 = i$. The numbers $\gamma_1, \dots, \gamma_q$ in (2.2) are then all real, and we denote them $\alpha_1, \dots, \alpha_r$. The associated nonnegative integers $\rho_1, \dots, \rho_r$ may be presumed to be nonzero, since otherwise the corresponding terms play no role and can be omitted. Thus

$$v(z) = \varphi(z) \int_{-\infty}^{\infty} \frac{1 + tz}{t - z}\, d\sigma(t) + \frac{\sum_{\ell=0}^{2\rho+1} B_\ell\, z^\ell}{\prod_{j=1}^{r} (z - \alpha_j)^{2\rho_j}}\,, \tag{2.5}$$

where

$$\varphi(z) = \frac{(z^2+1)^\rho}{\prod_{j=1}^{r} (z - \alpha_j)^{2\rho_j}}\,. \tag{2.6}$$

We show that if ρ_0 is defined by $\rho_0 + \rho_1 + \cdots + \rho_r = \rho$, then it is possible to rewrite (2.5) in the form (2.1). To do this, we prove that

$$v(z) \sim \int_{-\infty}^{\infty} \left[\frac{1}{t - z} - \sum_{j=0}^{r} S_j(t, z) \right] d\tau(t)\,, \tag{2.7}$$

where $F_1(z) \sim F_2(z)$ means that

$$F_1(z) = F_2(z) + R_0(z) - \sum_{j=1}^{r} R_j \left(\frac{1}{z - \alpha_j} \right)$$

for some polynomials $R_0(z), \dots, R_r(z)$ as in (3°). The relation $\sim$ is transitive but not reflexive: $F_1(z) \sim F_2(z)$ and $F_2(z) \sim F_3(z)$ imply $F_1(z) \sim F_3(z)$, but $F_1(z) \sim F_2(z)$ does not have the same meaning as $F_2(z) \sim F_1(z)$.

We first show that a jump $\sigma_p = \sigma(\alpha_p + 0) - \sigma(\alpha_p - 0)$ at one of the points $\alpha_1, \dots, \alpha_r$ in the integral part of (2.5) produces a contribution

$$\varphi(z)\, \frac{1 + \alpha_p z}{\alpha_p - z}\, \sigma_p \sim 0. \tag{2.8}$$

In fact, by a partial fraction decomposition,

$$\varphi(z)\, \frac{1 + \alpha_p z}{\alpha_p - z}\, \sigma_p = T_0(z) - \sum_{j=1}^{r} T_j \left(\frac{1}{z - \alpha_j} \right),$$

where for $j = 0, \dots, r$, $j \neq p$, $T_j(z)$ is a polynomial of degree at most $2\rho_j$ and $T_p(z)$ has degree $2\rho_p + 1$ with leading coefficient

$$\begin{aligned}
&\lim_{z \to \alpha_p} \left[-(z - \alpha_p)^{2\rho_p + 1} \varphi(z)\, \frac{1 + \alpha_p z}{\alpha_p - z}\, \sigma_p \right] \\
&\qquad = \lim_{z \to \alpha_p} (z - \alpha_p)^{2\rho_p + 1} \frac{(1 + z^2)^\rho}{\prod_{j=1}^{r} (z - \alpha_j)^{2\rho_j}}\, \frac{1 + \alpha_p z}{z - \alpha_p}\, \sigma_p \\
&\qquad = \frac{(1 + \alpha_p^2)^{\rho+1}}{\prod_{\substack{j=1 \\ j \neq p}}^{r} (\alpha_p - \alpha_j)^{2\rho_j}}\, \sigma_p \\
&\qquad \geq 0\,.
\end{aligned}$$

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ELA
http://math.technion.ac.il/iic/ela

In a similar way, $\sum_{\ell=0}^{2\rho+1} B_\ell\, z^\ell \big/ \prod_{j=1}^r (z-\alpha_j)^{2\rho_j} \sim 0$ because $B_{2\rho+1} \geq 0$. Therefore by (2.5),

$$v(z) \sim \varphi(z) \int_{-\infty}^{\infty} \frac{1+tz}{t-z}\, d\sigma(t)\,, \tag{2.9}$$

where without loss of generality we may assume that $\sigma(t)$ is continuous at the points $\alpha_1, \dots, \alpha_r$.

Define $\tau(t)$ in the $r+1$ open intervals determined by $\alpha_1, \dots, \alpha_r$ so that

$$d\tau(t) = \varphi(t)\,(1+t^2)\,d\sigma(t).$$

Then (2°) holds by construction. In any way, choose $\Delta_0, \dots, \Delta_r$ as in (1°). By (2.9),

$$\begin{aligned} v(z) \sim \sum_{j=1}^r \frac{\varphi_j(z)}{(z-\alpha_j)^{2\rho_j}} \int_{\Delta_j} \frac{1+tz}{t-z}\, d\sigma(t) \\ + (1+z^2)^{\rho_0} \varphi_0(z) \int_{\Delta_0} \frac{1+tz}{t-z}\, d\sigma(t)\,, \end{aligned} \tag{2.10}$$

where

$$\varphi_j(z) = (z-\alpha_j)^{2\rho_j} \varphi(z) = \frac{(1+z^2)^\rho}{\prod_{\substack{\ell=1 \\ \ell\neq j}}^r (z-\alpha_\ell)^{2\rho_\ell}}, \qquad j = 1, \dots, r,$$

$$\varphi_0(z) = (1+z^2)^{-\rho_0} \varphi(z) = \frac{(1+z^2)^{\rho_1+\cdots+\rho_r}}{\prod_{\ell=1}^r (z-\alpha_\ell)^{2\rho_\ell}}.$$

A typical term in the first part of (2.10) is

$$\begin{aligned} &\frac{\varphi_j(z)}{(z-\alpha_j)^{2\rho_j}} \int_{\Delta_j} \frac{1+tz}{t-z}\, d\sigma(t) \\ &\quad = \frac{\varphi_j(z)}{(z-\alpha_j)^{2\rho_j}} \int_{\Delta_j} \left[\frac{1}{t-z} - \frac{t}{1+t^2}\right] (1+t^2)\, d\sigma(t) \\ &\quad \sim \frac{\varphi_j(z)}{(z-\alpha_j)^{2\rho_j}} \int_{\Delta_j} \frac{1}{t-z}\, (1+t^2)\, d\sigma(t) \\ &\quad = \frac{1}{(z-\alpha_j)^{2\rho_j}} \int_{\Delta_j} \frac{1}{t-z}\, \varphi_j(t)\, (1+t^2)\, d\sigma(t) \\ &\qquad - \frac{1}{(z-\alpha_j)^{2\rho_j}} \int_{\Delta_j} \frac{\varphi_j(t)-\varphi_j(z)}{t-z}\, (1+t^2)\, d\sigma(t) \\ &\quad \overset{(*)}{\sim} \frac{1}{(z-\alpha_j)^{2\rho_j}} \int_{\Delta_j} \frac{1}{t-z}\, \varphi_j(t)\, (1+t^2)\, d\sigma(t) \\ &\quad = \int_{\Delta_j} \frac{1}{t-z} \left(\frac{t-\alpha_j}{z-\alpha_j}\right)^{2\rho_j} \varphi(t)\,(1+t^2)\,d\sigma(t) \\ &\quad = \int_{\Delta_j} \left[\frac{1}{t-z} - S_j(t,z)\right] d\tau(t). \end{aligned}$$

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To justify $(*)$, form the partial fraction decomposition of $\varphi_j(z)$, and then use the identities

$$\frac{z^{p+1} - t^{p+1}}{z - t} = \sum_{\mu+\nu=p} t^\mu \, z^\nu \,,$$

$$\frac{1}{z-t}\left[\frac{1}{(z-\alpha_j)^p} - \frac{1}{(t-\alpha_j)^p}\right] = -\sum_{\mu+\nu=p+1} \frac{1}{(t-\alpha_j)^\mu}\,\frac{1}{(z-\alpha_j)^\nu}\,,$$

to see that

$$-\frac{1}{(z-\alpha_j)^{2\rho_j}} \int_{\Delta_j} \frac{\varphi_j(t) - \varphi_j(z)}{t - z}\,(1+t^2)\,d\sigma(t) \sim 0.$$

In a similar way, the last term in (2.10) is

$$\begin{aligned}
&(1+z^2)^{\rho_0}\varphi_0(z) \int_{\Delta_0} \frac{1+tz}{t-z}\,d\sigma(t) \\
&\qquad = (1+z^2)^{\rho_0} \int_{\Delta_0} \frac{1+tz}{t-z}\,\varphi_0(t) d\sigma(t) \\
&\qquad\qquad - (1+z^2)^{\rho_0} \int_{\Delta_0} (1+tz)\,\frac{\varphi_0(t)-\varphi_0(z)}{t-z}\,d\sigma(t) \\
&\qquad \sim (1+z^2)^{\rho_0} \int_{\Delta_0} \frac{1+tz}{t-z}\,\varphi_0(t) d\sigma(t) \\
&\qquad = \int_{\Delta_0} (1+z^2)^{\rho_0}\,\frac{1+tz}{t-z}\frac{1}{(1+t^2)^{\rho_0+1}}\,\varphi(t)\,(1+t^2)\,d\sigma(t) \\
&\qquad = \int_{\Delta_0} \left[\frac{1}{t-z} - S_0(t,z)\right]\,d\tau(t).
\end{aligned}$$

On combining these results, we obtain (2.7) and hence (2.1). $\square$

Proof of Theorem 2.1*, sufficiency.* Assume $v(z)$ is given by (2.1). Then for each $j = 1, \ldots, r$,

$$v_j(z) = \int_{\Delta_j} \left[\frac{1}{t-z} - S_j(t,z)\right] d\tau(t) = \int_{\Delta_j} \frac{1}{t-z}\left(\frac{t-\alpha_j}{z-\alpha_j}\right)^{2\rho_j} d\tau(t)$$

has the form (2.3) and hence represents a generalized Nevanlinna function. Similarly,

$$v_0(z) = \int_{\Delta_0} \left[\frac{1}{t-z} - S_0(t,z)\right] d\tau(t) = \int_{\Delta_0} \frac{(1+z^2)^{\rho_0}}{(1+t^2)^{\rho_0+1}}\,\frac{1+tz}{t-z}\,d\tau(t)$$

is a generalized Nevanlinna function. The discrete terms coincide with a rational function with selfadjoint $m \times m$ matrix values on the real axis, and these also have the form (2.3) (since in that formula we can take $v_0(z) \equiv 0$). Thus $v(z)$ belongs to some class $\mathbf{N}_\varkappa$. $\square$

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
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3. Stieltjes inversion formula. The Kreĭn-Langer representation (2.1) of a given generalized Nevanlinna function $v(z)$ is not unique because of the arbitrariness of the choice of sets $\Delta_0, \Delta_1, \dots, \Delta_r$. But this choice affects only the discrete parts, and the function $\tau(t)$ in (2.1) is unique and can be recovered from $v(z)$ by a Stieltjes inversion formula. We prove this in a matrix form.

THEOREM 3.1. *Let $v(z)$ be an $m \times m$ matrix-valued function in $\mathbf{N}_\varkappa$ which is represented in the form* (2.1)*. Let $[a, b]$ be an interval whose endpoints are points of continuity of $\tau(t)$ and which does not contain any point $\alpha_1, \dots, \alpha_r$. Then*

$$\lim_{y \downarrow 0} \frac{1}{\pi} \int_a^b g(t)^* \left[\operatorname{Im} v(t+iy)\right] h(t)\, dt = \int_a^b g(t)^*\, d\tau(t)\, h(t) \tag{3.1}$$

for any continuous $\mathbb{C}^m$-valued functions $g(t)$ and $h(t)$ on $[a, b]$.

It follows that in (3.1), the vector-valued functions $g(t)$ and $h(t)$ can be replaced by any continuous matrix-valued functions of compatible orders.

Proof of Theorem 3.1 *in the case* $\varkappa = 0$. This case is known, but we include a proof for the sake of completeness.

Let $\mathfrak{S}_0$ be the set of $\mathbb{C}^m$-valued step functions on $[a, b]$ whose jumps occur at points of continuity of $\tau(t)$. The identity (3.1) holds if $g(t) = h(t) = \text{const.}$ by the classical Stieltjes inversion formula. By the polarization identity, it holds if $g(t)$ and $h(t)$ are possibly different constants. Hence by additivity, (3.1) holds for all functions $g(t)$ and $h(t)$ in $\mathfrak{S}_0$.

Fix $g(t)$ in $\mathfrak{S}_0$, and consider a continuous $\mathbb{C}^m$-valued function $h(t)$ on $[a, b]$. Since $h(t)$ is uniformly continuous on the interval, we may choose a sequence $h_k(t)$ in $\mathfrak{S}_0$ which converges uniformly to $h(t)$ on $[a, b]$. Write

$$\begin{aligned} &\frac{1}{\pi} \int_a^b g(t)^* \left[\operatorname{Im} v(t+iy)\right] h(t)\, dt - \int_a^b g(t)^* d\tau(t)\, h(t) \\ &\quad = \frac{1}{\pi} \int_a^b g(t)^* \left[\operatorname{Im} v(t+iy)\right] \left[h(t) - h_k(t)\right] dt \\ &\qquad + \left\{ \frac{1}{\pi} \int_a^b g(t)^* \left[\operatorname{Im} v(t+iy)\right] h_k(t)\, dt - \int_a^b g(t)^*\, d\tau(t)\, h_k(t) \right\} \\ &\qquad + \int_a^b g(t)^*\, d\tau(t) \left[h_k(t) - h(t)\right]. \\ &\quad = \text{Term } 1 + \text{Term } 2 + \text{Term } 3\,. \end{aligned} \tag{3.2}$$

Let $\varepsilon > 0$ be given. In Term 1,

$$\left| g(t)^* \left[\operatorname{Im} v(t+iy)\right] \left[h(t) - h_k(t)\right] \right| \le M \operatorname{tr} \left[\operatorname{Im} v(t+iy)\right] \cdot \|h(t) - h_k(t)\|\,,$$

where M is a bound for $g(t)$ on $[a, b]$. Hence for all sufficiently large k, Term 1 is less than $\varepsilon/3$ for all y such that $0 < y \le 1$. Using estimates of the type (1.4), we obtain

$$\lim_{k \to \infty} \int_a^b g(t)^*\, d\tau(t)\, h_k(t) = \int_a^b g(t)^*\, d\tau(t)\, h(t)\,.$$

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ELA
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Hence Term 3 is less than $\varepsilon/3$ for all sufficiently large k. Choose k large enough that the first and third terms are less than $\varepsilon/3$. Then with k fixed we can find $\eta > 0$ such that the middle term is less than $\varepsilon/3$ for $0 < y < \eta$. We obtain (3.1) when $g(t)$ is in $\mathfrak{S}_0$ and $h(t)$ is continuous.

A similar argument extends (3.1) to arbitrary continuous functions $g(t)$ and $h(t)$ on $[a, b]$. $\square$

Lemma 3.2. *For any $m \times m$ matrix-valued function $v(z)$ in $\mathbf{N}_0$, $yv(x + iy)$ is bounded in every rectangle $a \le x \le b$ and $0 < y \le 1$.*

Proof of Lemma 3.2. It is enough to consider the scalar case and

$$v(z) = \int_c^d \frac{d\mu(t)}{t - z}\,,$$

where $-\infty < c < a < b < d < \infty$ and $\mu(t)$ is nondecreasing on $[c, d]$. Then for $z = x + iy$,

$$y\, v(z) = \int_c^d \frac{(t - \bar z)y}{|t - z|^2}\, d\mu(t) = \int_c^d \left[\frac{(t - x)y}{(t - x)^2 + y^2} + \frac{iy^2}{(t - x)^2 + y^2} \right] d\mu(t),$$

so $|y\, v(z)| \le \int_c^d \left[\frac{1}{2} + 1\right] d\mu(t)$. $\square$

Proof of Theorem 3.1. Write (2.1) in the form

$$v(z) = \sum_{j=0}^{r} \int_{\Delta_j} \left[\frac{1}{t - z} - \sum_{j=0}^{r} S_j(t, z) \right] d\tau(t) + R_0(z) - \sum_{j=1}^{r} R_j\left(\frac{1}{z - \alpha_j}\right) - \sum_{k=1}^{s} \left[M_k\left(\frac{1}{z - \beta_k}\right) + M_k\left(\frac{1}{\bar z - \beta_k}\right)^* \right]. \tag{3.3}$$

Since changing $\Delta_0, \Delta_1, \ldots, \Delta_r$ only affects the discrete parts, and these parts have selfadjoint values on the $r + 1$ real intervals determined by $\alpha_1, \ldots, \alpha_r$, it is sufficient to prove the result when

$$v(z) = \int_{\Delta_j} \left[\frac{1}{t - z} - S_j(t, z) \right] d\tau(t) \tag{3.4}$$

and $[a, b]$ is contained in the interior of Δ_j for some $j = 0, 1, \ldots, r$.

Suppose first that $j = 1, \ldots, r$. By (1°), $v(z) = v_0(z)/(z - \alpha_j)^{2\rho_j}$ where

$$v_0(z) = \int_{\Delta_j} \frac{d\tau_j(t)}{t - z}\,, \qquad d\tau_j(t) = (t - \alpha_j)^{2\rho_j}\, d\tau(t)\,.$$

Since a and b are points of continuity of $\tau(t)$, they are points of continuity of $\tau_j(t)$. Write

$$\begin{aligned} \operatorname{Im} v(z) &= \frac{1}{(z - \alpha_j)^{2\rho_j}} \operatorname{Im} v_0(z) + \frac{1}{2i} \left[\frac{1}{(z - \alpha_j)^{2\rho_j}} - \frac{1}{(\bar z - \alpha_j)^{2\rho_j}} \right] v_0(z)^* \\ &= \frac{1}{(z - \alpha_j)^{2\rho_j}} \operatorname{Im} v_0(z) + F(z). \end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
http://math.technion.ac.il/iic/ela

Since

$$\frac{1}{z-\bar z}\left[\frac{1}{(z-\alpha_j)^{2\rho_j}}-\frac{1}{(\bar z-\alpha_j)^{2\rho_j}}\right]=-\sum_{\mu+\nu=2\rho_j+1}\frac{1}{(z-\alpha_j)^{\mu}}\,\frac{1}{(\bar z-\alpha_j)^{\nu}}$$

and $y\,v_0(z)$ is bounded near the real axis by Lemma 3.2, $F(z)$ is bounded for $a \le x \le b$ and $0 < y \le 1$. Hence there is a constant $M > 0$ such that

$$|g(t)^* F(t+iy)h(t)| \le M$$

for $a \le t \le b$ and $0 < y \le 1$. Since $v_0(t+i0)^*$ exists a.e.,

$$\lim_{y\downarrow 0} g(t)^* F(t+iy)h(t) = 0$$

a.e. on $[a,b]$. Therefore

$$\begin{aligned}
&\lim_{y\downarrow 0}\ \frac{1}{\pi}\int_a^b g(t)^*\left[\operatorname{Im} v(t+iy)\right] h(t)\ dt\\
&\quad=\lim_{y\downarrow 0}\ \frac{1}{\pi}\int_a^b \frac{1}{(t+iy-\alpha_j)^{2\rho_j}}\ g(t)^*\left[\operatorname{Im} v_0(t+iy)\right] h(t)\ dt\\
&\quad=\lim_{y\downarrow 0}\left\{\frac{1}{\pi}\int_a^b \frac{1}{(t-\alpha_j)^{2\rho_j}}\ g(t)^*\left[\operatorname{Im} v_0(t+iy)\right] h(t)\ dt\right.\\
&\qquad\quad\left.+\ \frac{1}{\pi}\int_a^b\left[\frac{1}{(t+iy-\alpha_j)^{2\rho_j}}-\frac{1}{(t-\alpha_j)^{2\rho_j}}\right] g(t)^*\left[\operatorname{Im} v_0(t+iy)\right] h(t)\ dt\right\}\\
&\quad=\lim_{y\downarrow 0}\ \frac{1}{\pi}\int_a^b \frac{1}{(t-\alpha_j)^{2\rho_j}}\ g(t)^*\left[\operatorname{Im} v_0(t+iy)\right] h(t)\ dt\\
&\quad=\int_a^b \frac{1}{(t-\alpha_j)^{2\rho_j}}\ g(t)^*\, d\tau_j(t)\, h(t)\\
&\quad=\int_a^b g(t)^*\, d\tau(t)\, h(t)\,.
\end{aligned}$$

Here we justify the third equality by estimates similar to those used above. The fourth equality follows by the special case $\varkappa = 0$ of the theorem, which was proved above.

The case $j = 0$ is similar. In this case by (1°), $v(z) = (1+z^2)^{\rho_0} v_0(z)$, where

$$v_0(z)=\int_{\Delta_0}\left[\frac{1}{t-z}-\frac{t}{1+t^2}\right]\ d\tau_0(t)\,,$$

$$d\tau_0(t)=\frac{d\tau(t)}{(1+t^2)^{\rho_0}},\qquad \int_{\Delta_0}\frac{d\tau_0(t)}{1+t^2}<\infty\,,$$

and a and b are also points of continuity of $\tau_0(t)$. Write

$$\begin{aligned}
\operatorname{Im} v(z) &= (1+z^2)^{\rho_0}\ \operatorname{Im} v_0(z)+\frac{1}{2i}\ \left[(1+z^2)^{\rho_0}-(1+\bar z^2)^{\rho_0}\right]\ v_0(z)^*\\
&= (1+z^2)^{\rho_0}\ \operatorname{Im} v_0(z)+G(z)\,.
\end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
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Since

$$\frac{(1+z^2)^{\rho_0}-(1+\bar z^2)^{\rho_0}}{z^2-\bar z^2}=\sum_{\mu+\nu=\rho_0-1}(1+z^2)^{\mu}(1+\bar z^2)^{\nu}$$

and $y\,v_0(z)$ is bounded near the real axis by Lemma 3.2, the term $G(z)$ makes no contribution in the limit:

$$\begin{aligned}
&\lim_{y\downarrow 0}\frac{1}{\pi}\int_a^b g(t)^*\left[\operatorname{Im} v(t+iy)\right]h(t)\,dt\\
&\quad=\lim_{y\downarrow 0}\frac{1}{\pi}\int_a^b \left(1+(t+iy)^2\right)^{\rho_0} g(t)^*\left[\operatorname{Im} v_0(t+iy)\right]h(t)\,dt\\
&\quad=\lim_{y\downarrow 0}\left\{\frac{1}{\pi}\int_a^b (1+t^2)^{\rho_0}\; g(t)^*\left[\operatorname{Im} v_0(t+iy)\right]h(t)\,dt\right.\\
&\qquad\qquad\left.+\frac{1}{\pi}\int_a^b\left[\left(1+(t+iy)^2\right)^{\rho_0}-(1+t^2)^{\rho_0}\right]g(t)^*\left[\operatorname{Im} v_0(t+iy)\right]h(t)\,dt\right\}\\
&\quad=\lim_{y\downarrow 0}\frac{1}{\pi}\int_a^b (1+t^2)^{\rho_0}\; g(t)^*\left[\operatorname{Im} v_0(t+iy)\right]h(t)\,dt\\
&\quad=\int_a^b (1+t^2)^{\rho_0}\; g(t)^*\,d\tau_0(t)\,h(t)\\
&\quad=\int_a^b g(t)^*\,d\tau(t)\,h(t)\,.
\end{aligned}$$

Again we have used the case $\varkappa=0$ of the theorem which was proved above. The result follows. ☐

Corollary 3.3. *Let $v(z)$ be an $m\times m$ matrix-valued function in $\mathbf{N}_\varkappa$. Then for any Kreĭn-Langer representation* (2.1) *of $v(z)$,*

$$\tau(b)-\tau(a)=\lim_{y\downarrow 0}\frac{1}{\pi}\int_a^b \operatorname{Im} v(t+iy)\,dt \tag{3.5}$$

for every interval $[a,b]$ which does not contain any point $\alpha_1,\dots,\alpha_r$ and whose endpoints are points of continuity of $\tau(t)$.

We note a generalization of Lemma 3.2.

Proposition 3.4. *Let $v(z)$ be an $m\times m$ matrix-valued function in $\mathbf{N}_\varkappa$ with Kreĭn-Langer representation* (2.1)*. Let $[a,b]$ be a closed bounded interval not containing any point $\alpha_1,\dots,\alpha_r$. Then $yv(x+iy)$ is bounded for $a\le x\le b$ and $0<y\le 1$.*

Proof. As in the proof of Theorem 3.1 we can reduce to the case where $v(z)$ has the form (3.4) and $[a,b]$ is contained in the interior of Δ_j for some $j=0,1,\dots,r$. As in the argument there, the result follows from the special case $\varkappa=0$ proved in Lemma 3.2. ☐

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
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4. Characterization of some special classes. We characterize two subclasses of $\mathbf{N}_\varkappa$ which are determined by behavior at infinity in terms of their Kreĭn-Langer representations.

THEOREM 4.1. *Let $v(z)$ be an $m \times m$ matrix-valued function in $\mathbf{N}_\varkappa$. Assume that $v(iy)/y \to 0$ as $y \to \infty$. Then the part*

$$\int_{\Delta_0} \left[\frac{1}{t-z} - S_0(t,z) \right] d\tau(t) + R_0(z) \tag{4.1}$$

in the Kreĭn-Langer representation of $v(z)$ can be chosen with $\rho_0 = 0$ and $R_0(z)$ constant; that is, this part can be reduced to the form

$$\int_{\Delta_0} \left[\frac{1}{t-z} - \frac{t}{1+t^2} \right] d\tau(t) + C_0, \tag{4.2}$$

where $\int_{\Delta_0} d\tau(t)/(1+t^2)$ is convergent and C_0 is a constant selfadjoint $m \times m$ matrix. Conversely, if (4.1) *has the form* (4.2)*, then*

$$v(z)/z \to 0 \tag{4.3}$$

as $|z| \to \infty$ in any sector $|x| \le \delta\,|y|$, $\delta > 0$ $(z = x + iy)$.

Proof. Without loss of generality, in both the direct and converse statements we can assume that $v(z)$ has the form (4.1). In fact, consider any Kreĭn-Langer representation (2.1). Write this first in the form (3.3) and then

$$v(z) = v_1(z) + \int_{\Delta_0} \left[\frac{1}{t-z} - S_0(t,z) \right] d\tau(t) + R_0(z).$$

By examining the parts in $v_1(z)$ and using elementary estimates, we see that $z\,v_1(z) = \mathcal{O}(1)$ as $|z| \to \infty$ in any sector $|x| \le \delta\,|y|$, $\delta > 0$. Hence without loss of generality we may assume that $v(z)$ is given by (4.1). Then by (1°), we can write $v(z)$ in the form

$$\begin{aligned} v(z) &= (1+z^2)^{\rho_0} \int_{\Delta_0} \frac{1+tz}{t-z} \frac{d\tau(t)}{(1+t^2)^{\rho_0+1}} + R_0(z) \\ &= (1+z^2)^{\rho_0} \int_{\Delta_0} \frac{1+tz}{t-z}\, d\sigma(t) + R_0(z), \end{aligned} \tag{4.4}$$

where $\int_{\Delta_0} d\sigma(t)$ is convergent, $R_0(z) = \sum_{j=0}^{2\rho_0+1} C_j z^j$ has selfadjoint coefficients, and $C_{2\rho_0+1} \ge 0$.

Now assume that $v(iy)/y \to 0$ as $y \to \infty$. By (4.4),

$$\begin{aligned} v(iy) = &\Big\{ (1-y^2)^{\rho_0} \int_{\Delta_0} \frac{(1-y^2)t}{t^2+y^2}\, d\sigma(t) \\ &+ C_0 - C_2 y^2 + \cdots + (-1)^{\rho_0} C_{2\rho_0} y^{2\rho_0} \Big\} \\ &+ i \Big\{ (1-y^2)^{\rho_0} \int_{\Delta_0} \frac{y(1+t^2)}{t^2+y^2}\, d\sigma(t) \\ &+ C_1 y - C_3 y^3 + \cdots + (-1)^{\rho_0 - 1} C_{2\rho_0 - 1} y^{2\rho_0 - 1} \\ &+ (-1)^{\rho_0} C_{2\rho_0+1} y^{2\rho_0+1} \Big\}. \end{aligned} \tag{4.5}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
http://math.technion.ac.il/iic/ela

From the imaginary parts of (4.5), we get

$$
\begin{aligned}
(-1)^{\rho_0}\frac{\operatorname{Im}[v(iy)/y]}{y^{2\rho_0}} = \frac{(y^2-1)^{\rho_0}}{y^{2\rho_0}}\int_{\Delta_0}\frac{t^2+1}{t^2+y^2}\,d\sigma(t) \\
+ (-1)^{\rho_0}\frac{C_1 - C_3y^2 + \cdots + (-1)^{\rho_0-1}C_{2\rho_0-1}y^{2\rho_0-2}}{y^{2\rho_0}} \\
+ C_{2\rho_0+1}.
\end{aligned}
$$

Therefore because $v(iy)/y \to 0$ as $y \to \infty$, $C_{2\rho_0+1} = 0$. If already $\rho_0 = 0$, this shows that (4.1) has the form (4.2), and we are done.

Suppose $\rho_0 \geq 1$. Then we obtain

$$
\begin{aligned}
(-1)^{\rho_0}\frac{\operatorname{Im}[v(iy)/y]}{y^{2\rho_0-2}} = \frac{(y^2-1)^{\rho_0}}{y^{2\rho_0}}\int_{\Delta_0}\frac{y^2(t^2+1)}{t^2+y^2}\,d\sigma(t) \\
+ (-1)^{\rho_0}\frac{C_1y^2 - C_3y^4 + \cdots + (-1)^{\rho_0-1}C_{2\rho_0-1}y^{2\rho_0}}{y^{2\rho_0}}.
\end{aligned}
$$

Letting $y \to \infty$, we deduce that $\int_{\Delta_0}(1+t^2)\,d\sigma(t)$ is convergent and

$$
0 = \int_{\Delta_0}(1+t^2)\,d\sigma(t) - C_{2\rho_0-1}\,. \tag{4.6}
$$

Again by (4.5),

$$
\begin{aligned}
(-1)^{\rho_0+1}\frac{\operatorname{Re}[v(iy)/y]}{y^{2\rho_0-1}} = \frac{(y^2-1)^{\rho_0}}{y^{2\rho_0}}\int_{\Delta_0}\frac{y^2-1}{t^2+y^2}\,\frac{t}{1+t^2}\,(1+t^2)\,d\sigma(t) \\
+ (-1)^{\rho_0+1}\frac{C_0 - C_2y^2 + \cdots + (-1)^{\rho_0}C_{2\rho_0}y^{2\rho_0}}{y^{2\rho_0}}\,.
\end{aligned}
$$

Since $\rho_0 \geq 1$, it follows that

$$
0 = \int_{\Delta_0} t\,d\sigma(t) - C_{2\rho_0}\,. \tag{4.7}
$$

We can now write (4.4) in the form

$$
\begin{aligned}
v(z) &= (1+z^2)^{\rho_0-1}\int_{\Delta_0}\frac{1+tz}{t-z}\left[(1+z^2)-(1+t^2)+(1+t^2)\right]d\sigma(t) + R_0(z) \\
&= (1+z^2)^{\rho_0-1}\int_{\Delta_0}\frac{1+tz}{t-z}\,(1+t^2)\,d\sigma(t) \\
&\quad - (1+z^2)^{\rho_0-1}\int_{\Delta_0}(1+tz)(t+z)\,d\sigma(t) + R_0(z) \\
&= (1+z^2)^{\rho_0-1}\int_{\Delta_0}\frac{1+tz}{t-z}\,d\tilde{\sigma}(t) + \widetilde{R}_0(z)\,,
\end{aligned}
$$

where

$$
\begin{aligned}
d\tilde{\sigma}(t) &= (1+t^2)\,d\sigma(t), \\
\widetilde{R}_0(z) &= R_0(z) - (1+z^2)^{\rho_0-1}\int_{\Delta_0}(1+tz)(t+z)\,d\sigma(t).
\end{aligned}
$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 11, pp. 1-15, January 2004

By (4.6) and (4.7),

$$
\begin{aligned}
\widetilde{R}_0(z) &= R_0(z) - (1+z^2)^{\rho_0 - 1}\left[\int_{\Delta_0} z(1+t^2)\,d\sigma(t)\right.\\
&\qquad \left. + \int_{\Delta_0} (1+z^2)t\,d\sigma(t)\right]\\
&= C_0 + C_1 z + \cdots + C_{2\rho_0-1}z^{2\rho_0-1} + C_{2\rho_0}z^{2\rho_0}\\
&\qquad - z(1+z^2)^{\rho_0-1}C_{2\rho_0-1} - (1+z^2)^{\rho_0}C_{2\rho_0}\\
&= \sum_{j=0}^{2\rho_0-1} \widetilde{C}_j z^j\,,
\end{aligned}
$$

where $\widetilde{C}_0, \dots, \widetilde{C}_{2\rho_0-1}$ are selfadjoint matrices and $\widetilde{C}_{2\rho_0-1} \geq 0$; in fact,

$$\widetilde{C}_{2\rho_0-1} = C_{2\rho_0-1} - C_{2\rho_0-1} = 0\,.$$

By repeating this argument if necessary, we obtain a representation (4.1) with $\rho_0 = 0$ and $R_0(z)$ constant. This completes the proof of the direct part of the theorem.

Conversely, suppose $v(z)$ has the form (4.2), and so

$$v(z) = \int_{\Delta_0} \frac{1+tz}{t-z}\,\frac{d\tau(t)}{1+t^2} + C_0,$$

where C_0 is a constant selfadjoint matrix. Using the elementary estimate

$$\left|\frac{t}{t-x-iy}\right| \leq \sqrt{1+\delta^2}, \qquad |x| \leq \delta\,|y|, \tag{4.8}$$

we obtain (4.3) as $|z| \to \infty$ in any sector $|x| \leq \delta\,|y|$, $\delta > 0$. ☐

Theorem 4.2. *Let $v(z)$ be an $m \times m$ matrix-valued function in $\mathbf{N}_\varkappa$. If $yv(iy) = \mathcal{O}(1)$ as $y \to \infty$, the part* (4.1) *in the Kreĭn-Langer representation can be reduced to the form*

$$\int_{\Delta_0} \frac{d\sigma(t)}{t-z}\,, \tag{4.9}$$

where $\int_{\Delta_0} d\sigma(t)$ is convergent. Conversely, if (4.1) *has the form* (4.9), *then*

$$z\,v(z) = \mathcal{O}(1) \tag{4.10}$$

as $|z| \to \infty$ in any sector $|x| \leq \delta\,|y|$, $\delta > 0$ $(z = x+iy)$.

Proof. Let $yv(iy) = \mathcal{O}(1)$ as $y \to \infty$. As in the proof of Theorem 4.1, we may assume that $v(z)$ is given by (4.1). Then by Theorem 4.1, we can further assume that

$$v(z) = \int_{\Delta_0} \left[\frac{1}{t-z} - \frac{t}{1+t^2}\right] d\tau(t) + C_0\,,$$

where $\int_{\Delta_0} d\tau(t)/(1+t^2)$ is convergent and C_0 is a constant selfadjoint $m \times m$ matrix. Equivalently,

$$v(z) = \int_{\Delta_0} \frac{1+tz}{t-z}\,d\tilde{\sigma}(t) + C_0\,,$$

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where $\int_{\Delta_0} d\tilde{\sigma}(t) = \int_{\Delta_0} d\tau(t)/(1+t^2)$ is convergent. Since

$$y \operatorname{Im} v(iy) = \int_{\Delta_0} \frac{y^2(1+t^2)}{t^2+y^2}\, d\tilde{\sigma}(t)$$

is $\mathcal{O}(1)$ as $y \to \infty$, $\int_{\Delta_0} (1+t^2)\, d\tilde{\sigma}(t)$ is convergent. Hence

$$\begin{aligned} v(z) &= \int_{\Delta_0} \frac{1+tz}{t-z}\, \frac{1}{1+t^2}\, (1+t^2)\, d\tilde{\sigma}(t) + C_0 \\ &= \int_{\Delta_0} \left[\frac{1}{t-z} - \frac{t}{1+t^2} \right] d\sigma(t) + C_0 \\ &= \int_{\Delta_0} \frac{d\sigma(t)}{t-z} + \widetilde{C}_0\,, \end{aligned}$$

where $\int_{\Delta_0} d\sigma(t) = \int_{\Delta_0} (1+t^2)\, d\tilde{\sigma}(t)$ is convergent and

$$\widetilde{C}_0 = C_0 - \int_{\Delta_0} \frac{t}{1+t^2}\, d\sigma(t).$$

The condition $yv(iy) = \mathcal{O}(1)$ as $y \to \infty$ implies that $\widetilde{C}_0 = 0$, and the direct part follows.

Conversely, if the part (4.1) in the Kreĭn-Langer representation of $v(z)$ has the form (4.9), then

$$z\, v(z) = \int_{\Delta_0} \frac{z}{t-z}\, d\sigma(t) = \int_{\Delta_0} \left[\frac{t}{t-z} - 1 \right] d\sigma(t)\,,$$

and we obtain (4.10) by (4.8). ∎

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