Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 272-279, November 2003

ELA
http://math.technion.ac.il/iic/ela

ON GENERALIZED HERMITE MATRIX POLYNOMIALS*

K.A.M. SAYYED†, M.S. METWALLY †, AND R.S. BATAHAN‡

Abstract. In this paper a new generalization of the Hermite matrix polynomials is given. An explicit representation and an expansion of the matrix exponential in a series of these matrix polynomials is obtained. Some recurrence relations, in particular the three terms recurrence relation, are given for these matrix polynomials. It is proved that the generalized Hermite matrix polynomials satisfy a matrix differential equation.

Key words. Generalized Hermite matrix polynomials, Three terms recurrence relation, Hermite matrix differential equation.

AMS subject classifications. 33C25, 15A60.

1. Introduction. It is well known that special matrix functions appear in statistics, Lie group theory and number theory [1, 8, 14, 16]. Herz [7] defined special matrix functions through Laplace and inverse Laplace transforms. In the two last decades, matrix polynomials have become more important and some results in the theory of classical orthogonal polynomials have been extended to orthogonal matrix polynomials see for instance [5, 6, 9, 13, 15] and the references therein. In [10], the Laguerre and Hermite matrix polynomials are introduced as examples of right orthogonal matrix polynomial sequences for appropriate right matrix moment functionals of integral type. Hermite matrix polynomials have been introduced and studied in [11, 12] for matrices in $\mathbb{C}^{N\times N}$ whose eigenvalues are all situated in the right open half-plane. Moreover, some properties of the Hermite matrix polynomials are given in [2, 3].

Our main aim in this paper is to consider a new generalization of the Hermite matrix polynomials. The structure of this paper is the following. In section 2, we introduce the generalized Hermite matrix polynomials and an explicit representation is given. We expand the matrix exponential in a series of the generalized Hermite matrix polynomials. Section 3 deals with some recurrence relations in particular the three terms recurrence relation for these matrix polynomials. Furthermore, we prove that the generalized Hermite matrix polynomials satisfy a matrix differential equation.

Throughout this paper, for a matrix A in $\mathbb{C}^{N\times N}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane and A is a matrix in $\mathbb{C}^{N\times N}$ with $\sigma(A) \subset \Omega$, then from the properties of the matrix functional calculus [4, p. 558], it follows that:

$$f(A)g(A) = g(A)f(A). \tag{1.1}$$

If D_0 is the complex plane cut along the negative real axis and $\log(z)$ denoting the principle logarithm of z, then $z^{1/2}$ represents $\exp(\frac{1}{2}\log(z))$. If A is a matrix in

*Received by the editors on 25 January 2003. Accepted for publication on 23 October 2003. Handling Editor: Peter Lancaster.

†Department of Mathematics, Faculty of Science, Assiut University, 71516, Assiut, Egypt.
‡Department of Mathematics, Hadhramout University, Mukalla, Yemen (rbatahan@aun.edu.eg).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 272-279, November 2003

$\mathbb{C}^{N\times N}$ with $\sigma(A) \subset D_0$, then $A^{1/2} = \sqrt{A}$ denotes the image by $z^{1/2}$ of the matrix functional calculus acting on the matrix A.

Let A be a matrix in $\mathbb{C}^{N\times N}$ such that

$$\text{Re}(\mu) > 0 \quad \text{for} \quad \text{every} \quad \text{eigenvalue} \quad \mu \in \sigma(A). \tag{1.2}$$

Then the nth Hermite matrix polynomials $H_n(x, A)$ is defined by [11, p. 25]

$$H_n(x, A) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k}{k!(n-2k)!} (x\sqrt{2A})^{n-2k} \quad ; n \geq 0, \tag{1.3}$$

and satisfies the three terms recurrence relationship

$$H_n(x, A) = Ix\sqrt{2A}H_{n-1}(x, A) - 2(n-1)H_{n-2}(x, A); n \geq 1; \tag{1.4}$$

$$H_{-1}(x, A) = 0, \quad H_0(x, A) = I,$$

where I is the unit matrix in $\mathbb{C}^{N\times N}$.

According to [11], we have

$$\exp(xt\sqrt{2A} - t^2 I) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x, A) t^n. \tag{1.5}$$

Also, we recall that if $A(k, n)$ and $B(k, n)$ are matrices in $\mathbb{C}^{N\times N}$ for $n \geq 0$ and $k \geq 0$, then it follows that [2]:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} A(k, n-2k), \tag{1.6}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k, n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k, n-k). \tag{1.7}$$

For m is a positive integer, similarly to (1.6) one can find

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]} A(k, n-mk) \quad ; n > m. \tag{1.8}$$

2. Definition of generalized Hermite matrix polynomials. In this section, we introduce a new matrix polynomial which represents a generalization of the Hermite matrix polynomials as given by the relation (1.5). Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfies (1.2). For n = 0,1,2,. . ., $\lambda \in \mathbb{R}^+$ and m is a positive integer, we define the generalized Hermite matrix polynomials by

$$F(x, t) = \exp(\lambda(xt\sqrt{2A} - t^m I)) = \sum_{n=0}^{\infty} H_{n,m}^{\lambda}(x, A) t^n. \tag{2.1}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 272-279, November 2003

ELA
http://math.technion.ac.il/iic/ela

Since

$$\begin{aligned}\exp(\lambda(xt\sqrt{2A}-t^mI)) &= \exp(\lambda(xt\sqrt{2A}))\exp(-\lambda(t^mI))\\ &= \sum_{n=0}^{\infty}\frac{\lambda^n(x\sqrt{2A})^n}{n!}t^n\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^k}{k!}t^{mk}I\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^{n+k}(\sqrt{2A})^n}{k!n!}x^nt^{n+mk},\end{aligned}$$

then by using (1.8) we have

$$\exp(\lambda(xt\sqrt{2A}-t^mI)) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{(-1)^k\lambda^{n-(m-1)k}(\sqrt{2A})^{n-mk}}{k!(n-mk)!}x^{n-mk}t^n.$$

Thus, we obtain an explicit representation for the generalized Hermite matrix polynomials in the form:

$$H^{\lambda}_{n,m}(x,A) = \lambda^n\sum_{k=0}^{[n/m]}\frac{(-1)^k(\sqrt{2A})^{n-mk}}{\lambda^{(m-1)k}k!(n-mk)!}x^{n-mk}. \tag{2.2}$$

For simplicity we denote $H_{n,m}(x,A)$ for the generalized Hermite matrix polynomials when $\lambda = 1$. It should be observed that, in view of the explicit representation (2.2), the generalized Hermite matrix polynomials $H_{n,2}(x,A)$ reduces to the Hermite matrix polynomials $H_n(x,A)/n!$ as given in (1.3).

Note that

$$(\sqrt{2A})^{-1}\frac{d}{dx}\exp(xt\sqrt{2A}) = t\exp(xt\sqrt{2A}),$$

and hence

$$[(\sqrt{2A})^{-1}\frac{d}{dx}]^n\exp(xt\sqrt{2A}) = t^n\exp(xt\sqrt{2A}).$$

Thus

$$\begin{aligned}\exp(-(\sqrt{2A})^{-m}\frac{d^m}{dx^m})\exp(xt\sqrt{2A}) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}[(\sqrt{2A})^{-1}\frac{d}{dx}]^{mn}\exp(xt\sqrt{2A})\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^{mn}\exp(xt\sqrt{2A})\\ &= \exp(xt\sqrt{2A}-t^mI).\end{aligned}$$

Therefore, by (2.1), we have

$$\exp(-(\sqrt{2A})^{-m}\frac{d^m}{dx^m})\exp(xt\sqrt{2A}) = \sum_{n=0}^{\infty}H_{n,m}(x,A)t^n,$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 272-279, November 2003

ELA
http://math.technion.ac.il/iic/ela

which by expanding in powers of t becomes

$$\exp(-(\sqrt{2A})^{-m}\frac{d^m}{dx^m})\sum_{n=0}^{\infty}\frac{x^n}{n!}(\sqrt{2A})^n t^n=\sum_{n=0}^{\infty}H_{n,m}(x,A)t^n.$$

Identification of the coefficients of t^n in both sides gives a new representation for the generalized Hermite matrix polynomials for $\lambda=1$ in the form:

$$H_{n,m}(x,A)=\frac{1}{n!}\exp(-(\sqrt{2A})^{-m}\frac{d^m}{dx^m})\,(\sqrt{2A})^n x^n. \tag{2.3}$$

For $m=2$, the expression (2.3) gives another representation for the Hermite matrix polynomials in the form:

$$H_n(x,A)=\exp(-(\sqrt{2A})^{-2}\frac{d^2}{dx^2})\,(\sqrt{2A})^n x^n.$$

Let B be a matrix in $\mathbb{C}^{N\times N}$ satisfies the spectral property

$$|\mathrm{Re}(\mu)|>|\mathrm{Im}(\mu)| \quad \text{for all} \quad \mu\in\sigma(B). \tag{2.4}$$

Suppose that $A=\frac{1}{2}B^2$. In view of the spectral mapping theorem [4] it is easy to find that $\sigma(A)=\{\frac{1}{2}b^2 : b\in\sigma(B)\}$ and by (2.4) we have

$$\mathrm{Re}(\frac{1}{2}b^2)=\frac{1}{2}[(\mathrm{Re}(b))^2-(\mathrm{Im}(b))^2]>0, \quad b\in\sigma(B).$$

That is, A is a positive stable matrix. In (2.1), putting $t=1$ and $B=\sqrt{2A}$ gives

$$\exp(\lambda(xB-I))=\sum_{n=0}^{\infty}H_{n,m}^{\lambda}(x,\frac{1}{2}B^2).$$

Therefore, for the matrix B satisfies (2.4), an expansion of $\exp(\lambda Bx)$ in a series of the generalized Hermite matrix polynomials is obtained in the form:

$$\exp(\lambda xB)=\exp(\lambda)\sum_{n=0}^{\infty}H_{n,m}^{\lambda}(x,\frac{1}{2}B^2), \quad -\infty<x<\infty. \tag{2.5}$$

3. Recurrence relations. In this section the three terms recurrence relation is carried out on the generalized Hermite matrix polynomials. At first, we obtain the following:

THEOREM 3.1. *The generalized Hermite matrix polynomials satisfy the following relations:*

$$D^k H_{n,m}^{\lambda}(x,A)=(\lambda\sqrt{2A})^k H_{n-k,m}^{\lambda}(x,A); \tag{3.1}$$

$$n\sqrt{2A}H_{n,m}^{\lambda}(x,A)=x\sqrt{2A}DH_{n,m}^{\lambda}(x,A)-mDH_{n-m+1,m}^{\lambda}(x,A); \tag{3.2}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 272-279, November 2003

ELA
http://math.technion.ac.il/iic/ela

(3.3) $$\frac{x^n}{n!}I = (\sqrt{2A})^{-n}\sum_{k=0}^{[n/m]}\frac{1}{k!}H_{n-m,m}(x,A);$$

(3.4) $$u^n H_{n,m}(x,A) = \sum_{k=0}^{[n/m]}\frac{(1-u^m)^k}{k!}H_{n-mk,m}(x,A),$$

where D=d/dx.

Proof. Differentiating (2.1) with respect to x yields

(3.5) $$\lambda t\sqrt{2A}\exp(\lambda(xt\sqrt{2A}-t^m I)) = \sum_{n=1}^{\infty} DH_{n-mk,m}(x,A)t^n.$$

By (2.1) and (3.5) we have

$$\lambda\sqrt{2A}\sum_{n=0}^{\infty}H^{\lambda}_{n,m}(x,A)t^{n+1} = \sum_{n=1}^{\infty}DH^{\lambda}_{n,m}(x,A)t^n.$$

Since $DH^{\lambda}_{0,m}(x,A) = 0$, then for $n \geq 1$ one obtains

(3.6) $$DH^{\lambda}_{n,m}(x,A) = \lambda\sqrt{2A}H^{\lambda}_{n-1,m}(x,A).$$

Iteration (3.6), for $0 \leq k \leq n$ gives (3.1).

Differentiating (2.1) with respect to x and t we find

$$\partial F/\partial x = \lambda t\sqrt{2A}\exp(\lambda(xt\sqrt{2A}-t^m I)),$$

and

$$\partial F/\partial t = \lambda(x\sqrt{2A}-mt^{m-1}I)\exp(\lambda(xt\sqrt{2A}-t^m I)).$$

Therefore, $F(x,t)$ satisfies the partial matrix differential equation

$$(xI - mt^{m-1}(\sqrt{2A})^{-1})\partial F/\partial x - t\partial F/\partial t = 0,$$

which, by using (2.1), becomes

$$(xI - mt^{m-1}(\sqrt{2A})^{-1})\sum_{n=1}^{\infty}DH^{\lambda}_{n,m}(x,A)t^n - \sum_{n=1}^{\infty}nH^{\lambda}_{n,m}(x,A)t^n = 0,$$

or

$$\sum_{n=1}^{\infty}nH^{\lambda}_{n,m}(x,A)t^n = \sum_{n=1}^{\infty}xDH^{\lambda}_{n,m}(x,A)t^n - (\sqrt{2A})^{-1}\sum_{n=1}^{\infty}mDH^{\lambda}_{n,m}(x,A)t^{n+m-1}.$$

Since $H^{\lambda}_{n,m}(x,A) = (\lambda x\sqrt{2A})^n/n!$ for $0 \leq n \leq m-1$, then we get (3.2).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 272-279, November 2003

For $\lambda = 1$, (2.1) reduces to

$$\exp(xt\sqrt{2A} - t^m I) = \sum_{n=0}^{\infty} H_{n,m}(x,A)t^n.$$

Hence

$$\exp(xt\sqrt{2A}) = \sum_{k=0}^{\infty} \frac{t^{mk}}{k!} \sum_{n=0}^{\infty} H_{n,m}(x,A)t^n,$$

and by (1.8) we get

$$\sum_{n=0}^{\infty} \frac{(x\sqrt{2A})^n}{n!} t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} \frac{1}{k!} H_{n-mk,m}(x,A)t^n.$$

By equating of the coefficients of t^n one gets (3.3).

Since

$$\exp(xt\sqrt{2A} - t^m u^m I) = exp(xt\sqrt{2A} - t^m I) exp(t^m I - t^m u^m I),$$

then

$$\sum_{n=0}^{\infty} H_{n,m}(x,A)t^n u^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(1-u^m)^k t^{mk}}{k!} H_{n,m}(x,A)t^n$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} \frac{(1-u^m)^k}{k!} H_{n,m}(x,A)t^n.$$

which, by comparing the coefficients of t^n, we get (3.4). ☐

Now, inserting (3.6) in (3.2) yields

$$nH^{\lambda}_{n,m}(x,A) = \lambda x\sqrt{2A} H^{\lambda}_{n-1,m}(x,A) - m(\sqrt{2A})^{-1} DH^{\lambda}_{n-m+1,m}(x,A). \tag{3.7}$$

Replacing n by $n - m + 1$ in (3.6) gives

$$DH^{\lambda}_{n-m+1,m}(x,A) = \lambda\sqrt{2A} H^{\lambda}_{n-m,m}(x,A). \tag{3.8}$$

Substituting from (3.8) into (3.7) yields the three terms recurrence relation as given in the following theorem:

THEOREM 3.2. *The generalized Hermite matrix polynomials $H^{\lambda}_{n,m}(x,A)$, satisfy the three terms recurrence relation:*

$$nH^{\lambda}_{n,m}(x,A) = \lambda(x\sqrt{2A} H^{\lambda}_{n-1,m}(x,A) - mH^{\lambda}_{n-m,m}(x,A)),\ n \geq m, \tag{3.9}$$

with initial values $H^{\lambda}_{n,m}(x,A) = (\lambda x\sqrt{2A})^n/n!$, $0 \leq n \leq m-1$.

Finally, we prove the following:

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 272-279, November 2003

ELA
http://math.technion.ac.il/iic/ela

THEOREM 3.3. *Suppose that A is a matrix in $\mathbb{C}^{N\times N}$ satisfying* (1.2)*. Then the generalized Hermite matrix polynomials $H^{\lambda}_{n,m}(x,A)$ are a solution of the differential equation of m-th order in the form:*

$$Y^{(m)} - m^{-1}\lambda^{m-1}(\sqrt{2A})^m(xY' - nY) = 0. \tag{3.10}$$

Proof. With the aid of the relations (3.1) and (3.9), we have

$$(\lambda\sqrt{2A})^m H^{\lambda}_{n-m,m}(x,A) - m^{-1}\lambda^m(\sqrt{2A})^{m+1}xH^{\lambda}_{n-1,m}(x,A))+$$

$$m^{-1}\lambda^{m-1}(\sqrt{2A})^m nH^{\lambda}_{n-m,m}(x,A))$$

$$= m^{-1}\lambda^{m-1}(\sqrt{2A})^m[m\lambda H^{\lambda}_{n-m,m}(x,A) - x\lambda\sqrt{2A}H^{\lambda}_{n-1,m}(x,A) + nH^{\lambda}_{n,m}(x,A)] = 0. \square$$

The differential equation (3.10) will be called the generalized Hermite matrix differential equation. For m=2 and $\lambda = 1$, the differential equation (3.10) gives the Hermite matrix differential equation in the form:

$$Y^{''} - A(xY' - nY) = 0.$$

Acknowledgments. The authors wish to express their gratitude to the unknown referee for several helpful suggestions.

REFERENCES

[1] A.G. Constantine and R.J. Muirhead. Partial differential equations for hypergeometric functions of two argument matrix. *J. Multivariate Anal.*, 3:332–338, 1972.
[2] E. Defez and L. Jódar. Some applications of the Hermite matrix polynomials series expansions. *J. Comp. Appl. Math.*, 99:105–117, 1998.
[3] E. Defez, M. Garcia-Honrubia and R.J. Villanueva. A procedure for computing the exponential of a matrix using Hermite matrix polynomials. *Far East J. Applied Mathematics*, 6(3):217–231, 2002.
[4] N. Dunford and J. Schwartz. *Linear Operators.* Vol. I, Interscience, New York, 1957.
[5] A.J. Durán. Markov's Theorem for orthogonal matrix polynomials. *Can. J. Math.*, 48:1180–1195, 1996.
[6] A.J. Durán and W. Van Assche. Orthogonal matrix polynomials and higher order recurrence relations. *Linear Algebra and its Applications*, 219:261–280, 1995.
[7] C.S. Herz. Bessel functions of matricial argument. *Ann. Math.*, 61:474–523, 1955.
[8] A.T. James. Special functions of matrix and single argument in statistics. in: *Theory and Applications of Special Functions*, R.A. Askey, Editor, Academic Press, pp. 497–520, 1975.
[9] L. Jódar, R. Company and E. Navarro. Laguerre matrix polynomials and system of second-order differential equations. *Appl. Num. Math.*, 15:53–63, 1994.
[10] L. Jódar, E. Defez and E. Ponsoda. Orthogonal matrix polynomials with respect to linear matrix moment functionals: Theory and applications. *J. Approx. Theory Appl.*, 12(1):96–115, 1996.
[11] L. Jódar and R. Company. Hermite matrix polynomials and second order matrix differential equations. *J. Approx. Theory Appl.*, 12(2):20–30, 1996.
[12] L. Jódar and E. Defez. On Hermite matrix polynomials and Hermite matrix function. *J. Approx. Theory Appl.*, 14(1):36–48, 1998.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 10, pp. 272-279, November 2003

ELA
http://math.technion.ac.il/iic/ela

[13] L. Jódar and J. Sastre. The growth of Laguerre matrix polynomials on bounded intervals. *Appl. Math. Lett.*, 13:21–26, 2000.

[14] R.J. Muirhead. Systems of partial differential equations for hypergeometric functions of matrix argument. *Ann. Math. Statist.*, 41:991–1001, 1970.

[15] A. Sinap and W. Van Assche. Orthogonal matrix polynomials and applications. *J. Comp. Appl. Math.*, 66:27–52, 1996.

[16] A. Terras. Special functions for the symmetric space of positive matrices. *SIAM J. Math. Anal.*, 16:620–640, 1985.