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THE (WEAKLY) SIGN SYMMETRIC P-MATRIX COMPLETION PROBLEMS*

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Abstract. In this paper it is shown that a partial sign symmetric P-matrix, whose digraph of specified entries is a symmetric n-cycle with $n \geq 6$, can be completed to a sign symmetric P-matrix. The analogous completion property is also established for a partial weakly sign symmetric P-matrix and for a partial weakly sign symmetric P_0-matrix. Patterns of entries for 4×4 matrices are classified as to whether or not a partial (weakly) sign symmetric P- or weakly sign symmetric P_0-matrix specifying the pattern must have completion to the same type of matrix. The relationship between the weakly sign symmetric P- and sign symmetric P-matrix completion problems is also examined.

Key words. Matrix Completion, P-matrix, Weakly sign symmetric P-matrix, n-Cycle.

AMS subject classifications. 15A48, 05C50.

1. Introduction. A *partial matrix* is a rectangular array in which some entries are specified while others are free to be chosen. A *completion* of a partial matrix is a specific choice of values for the unspecified entries. A *pattern* for $n \times n$ matrices is a list of positions of an $n \times n$ matrix, that is, a subset of $\mathcal{N} \times \mathcal{N}$ where $\mathcal{N} = \{1, \ldots, n\}$. A partial matrix *specifies a pattern* if its specified entries lie exactly in those positions listed in the pattern. In this paper a pattern is assumed to contain all diagonal positions.

Matrix completion problems have been studied for many classes of matrices. We use most of the definitions and notation from [2]. One slight distinction: here the term "symmetric" is used for a pattern with the property that $(j,\ i)$ is in the pattern whenever $(i,\ j)$ is in the pattern; such patterns were called "positionally symmetric" in [2].

An $n \times n$ matrix is called a P_0*-matrix* (P*-matrix*) if all its principal minors are nonnegative (positive). We shall use P-(P_0-)matrix to mean P-matrix or P_0-matrix. A matrix, A, is *sign symmetric* if for each $i, j \in \mathcal{N}$, either $a_{ij} = 0 = a_{ji}$ or $a_{ij}a_{ji} > 0$. The matrix is *weakly sign symmetric* if $a_{ij}a_{ji} \geq 0$. We shall use (weakly) sign symmetric to mean weakly sign symmetric or sign symmetric. A *partial* P_0*-matrix* (*partial* P*-matrix*) is a partial matrix in which all fully specified principal submatrices are P_0-matrices (P-matrices). Similarly, a *partial (weakly) sign symmetric matrix* is a partial matrix in which all fully specified principal submatrices are (weakly) sign

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symmetric.

We shall use the following properties of (weakly) sign symmetric P-(P_0-)matrices. Let A be a (weakly) sign symmetric P-(P_0-)matrix. If D is a positive diagonal matrix then DA is a (weakly) sign symmetric P-(P_0-)matrix. If D is a nonsingular diagonal matrix then $D^{-1}AD$ is a (weakly) sign symmetric P-(P_0-)matrix. If P is a permutation matrix, then $P^{-1}AP$ is a (weakly) sign symmetric P-(P_0-)matrix.

Throughout the paper we will usually denote the entries of a partial matrix as follows: d_i denotes a (specified) diagonal entry, a_{ij} a specified off-diagonal entry, and x_{ij} an unspecified off-diagonal entry, $1 \leq i, j \leq n$. In addition, c_{ij} may be used to denote the value assigned to the unspecified entry x_{ij} during the process of completing a partial matrix. In the case of a symmetrically placed pair, a_{ij} and x_{ji}, in a partial matrix, the specified entry a_{ij} shall be referred to as the *specified twin*. The other member of the pair, x_{ij}, shall be referred to as the *unspecified twin*.

Digraphs are used to study matrices (nonzero digraphs) and patterns (pattern-digraphs), when all diagonal entries are nonzero in a matrix or all diagonal positions are present in a pattern. For a pattern Q for $n \times n$ matrices that contains all diagonal positions, the digraph of Q (pattern-digraph) is the digraph having as vertex set $\mathcal{N}$ and, as arcs the ordered pairs $(i, j) \in Q$, where $i \neq j$. When diagonal positions are omitted or diagonal entries of a matrix can be zero, it is sometimes necessary to use L-digraphs (digraphs that include loops). For a fully specified matrix A, the nonzero-L-digraph of A is the L-digraph having vertex set $\mathcal{N}$ and, as arcs, the ordered pairs (i, j). where $a_{ij} \neq 0$. The reader is referred to [2] for formal definitions of most graph theory terms.

A *path* in the digraph or L-digraph $G = (V_G, E_G)$ is a sequence of vertices $v_1, v_2, \ldots, v_{k-1}, v_k$ in V_G such that for $i = 1, \ldots, k-1$, the arc $(v_i, v_{i+1}) \in E_G$ and all vertices are distinct except possibly $v_1 = v_k$. A cycle is a path in which $v_1 = v_k$. If the partial matrix A specifies a digraph G that includes the k-cycle $\Gamma = v_1, v_2, \ldots, v_k, v_1$ then the *k-cycle product of Γ* in A is $a_{v_1v_2}a_{v_2v_3}\ldots a_{v_{k-1}v_k}a_{v_kv_1}$.

In Section 2 we present theorems that relate weakly sign symmetric P-, sign symmetric P-, and weakly sign symmetric P_0-completions. In Section 3 we consider the symmetric n-cycle, and demonstrate that it has weakly sign symmetric P_0-completion for $n \geq 6$. We apply the results of Section 2 to show that such a cycle has (weakly) sign symmetric P-completion, answering a question of [3]. In Section 4 of this paper we completely classify all digraphs up to order 4 as to (weakly) sign symmetric P-completion and weakly sign symmetric P_0-completion.

2. Relationship Theorems. If X and Y are classes of matrices with $X \subseteq Y$, in general it is not possible to conclude either that a pattern that has Y-completion must have X-completion (because the completion to a Y-matrix may not be an X-matrix) or that a pattern that has X-completion must have Y-completion (because there may be a partial Y-matrix that is not a partial X-matrix). However, in cases where there is a natural relationship between the classes X and Y, it is sometimes possible to conclude that certain (or all) patterns that have Y-completion have X-completion or vice versa (cf. [7]). In this section we present relationship theorems that are relevant to the three separate matrix completion problems we are studying,

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that is, the weakly sign symmetric P-, sign symmetric P- and weakly sign symmetric P_0-matrix completion problems.

THEOREM 2.1 ([7]). *Any pattern that has weakly sign symmetric P_0-completion has weakly sign symmetric P-completion.*

LEMMA 2.2. *Let Q be a pattern that has weakly sign symmetric P-completion, where for any partial weakly sign symmetric P-matrix specifying Q, there is a weakly sign symmetric P-completion in which zero is assigned to any unspecified twin whose specified twin is zero. Then Q has sign symmetric P-completion.*

Proof. Let A be a partial sign symmetric matrix specifying Q. The matrix A is a partial weakly sign symmetric matrix specifying Q, and so can be completed to a weakly sign symmetric P-matrix $B = [b_{ij}]$ in a way that zero is assigned to any unspecified twin whose specified twin is zero. The only reason B might not be a sign symmetric P-matrix is if some b_{ij} is nonzero and b_{ji} is zero. In this case, either both entries were unspecified in A, or b_{ij} was specified nonzero in A while the unspecified twin x_{ji} was assigned zero. Since there are only finitely many principal minors of B and these are continuous functions of the entries of B, we can slightly perturb zero entries that were originally unspecified while maintaining all principal minors positive. This converts B into a sign symmetric P-matrix that completes A. □

COROLLARY 2.3. *Any symmetric pattern that has weakly sign symmetric P-completion has sign symmetric P-completion.*

LEMMA 2.4. *Any asymmetric pattern that has sign symmetric P-completion has weakly sign symmetric P-completion.*

Proof. Since the pattern is asymmetric, any partial weakly sign symmetric P-matrix is a sign symmetric P-matrix, and so can be completed to a sign symmetric P-matrix, which is necessarily a weakly sign symmetric P-matrix. □

QUESTION 2.5. *In all cases examined in this paper, a pattern has weakly sign symmetric P-completion if and only if it has sign symmetric P-completion. Is this always the case?*

3. Symmetric n-cycles. The completability of symmetric n-cycles was first studied in [3] for various classes of matrices, including the classes of (weakly) sign symmetric P- and (weakly) sign symmetric P_0-matrices. It is shown in Example 3.4 of [3] that for any $n > 3$, the symmetric n-cycle does not have sign symmetric P_0-completion, because the partial sign symmetric P_0-matrix

$$A = \begin{bmatrix} 1 & 1 & ? & \cdots & ? & -1 \\ 1 & 1 & 1 & \cdots & ? & ? \\ ? & 1 & 1 & \cdots & ? & ? \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ ? & ? & ? & \cdots & 1 & 1 \\ -1 & ? & ? & \cdots & 1 & 1 \end{bmatrix}$$

cannot be completed to a sign symmetric P_0-matrix. In Theorem 4.1 of [3], it is asserted that this example also shows that for any $n > 3$, the symmetric $n-$cycle does not have weakly sign symmetric P_0-completion. However, the proof in Example 3.4 of [3] requires sign symmetry; weak sign symmetry is not enough, as shown in Example 3.1 below. In fact, Theorem 4.1 of [3] applies only to sign symmetric P_0-

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and $P_{0,1}$-matrices, not weakly sign symmetric P_0- and $P_{0,1}$-matrices, as established below in Theorem 3.7.

EXAMPLE 3.1. *The partial weakly sign symmetric P_0-matrix, A, can be completed to a weakly sign symmetric P_0-matrix, $\widehat{A}$, where*

$$A = \begin{bmatrix} 1 & 1 & ? & ? & ? & -1 \\ 1 & 1 & 1 & ? & ? & ? \\ ? & 1 & 1 & 1 & ? & ? \\ ? & ? & 1 & 1 & 1 & ? \\ ? & ? & ? & 1 & 1 & 1 \\ -1 & ? & ? & ? & 1 & 1 \end{bmatrix} \text{ and } \widehat{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

It is a straightforward computation to verify that all principal minors of $\widehat{A}$ are either zero or one.

However, as noted in Example 3.3 of [3], the symmetric 4-cycle does not have (weakly) sign symmetric P-(P_0)-completion, and our Example 3.2 shows the symmetric 5-cycle also lacks completion.

EXAMPLE 3.2. *The symmetric 5-cycle does not have (weakly) sign symmetric P- or P_0-completion because the partial sign symmetric P-matrix*

$$A = \begin{bmatrix} 1 & 1 & x_{13} & z_{14} & -0.99 \\ 0.99 & 1 & 1 & x_{24} & z_{25} \\ x_{31} & 0.99 & 1 & 1 & x_{35} \\ z_{41} & x_{42} & 0.99 & 1 & 1 \\ -1 & z_{52} & x_{53} & 0.99 & 1 \end{bmatrix}$$

cannot be completed to a weakly sign symmetric P_0-matrix (here x_{ij} and z_{ij} are both used to denote unspecified entries).

Proof. Suppose that there is a weakly sign symmetric P_0-completion of A. Table 3.1 lists selected principal minors that must be nonnegative, giving conditions on the variables x_{ij} and z_{ij}.

TABLE 3.1

α	Det$A(\alpha)$
{1,2,3}	$-0.98 + 0.9801x_{13} + x_{31} - x_{13}x_{31}$
{2,3,4}	$-0.98 + 0.9801x_{24} + x_{42} - x_{24}x_{42}$
{3,4,5}	$-0.98 + 0.9801x_{35} + x_{53} - x_{35}x_{53}$
{1,4,5}	$-0.98 - z_{14} - 0.9801z_{41} - z_{14}z_{41}$
{1,2,5}	$-0.98 - z_{25} - 0.9801z_{52} - z_{25}z_{52}$
{1,3,4}	$0.01 - x_{13}x_{31} + 0.99x_{31}z_{14} + x_{13}z_{41} - z_{14}z_{41}$
{1,2,4}	$0.01 - x_{24}x_{42} + 0.99x_{42}z_{14} + x_{24}z_{41} - z_{14}z_{41}$
{1,3,5}	$0.01 - x_{13}x_{31} - x_{13}x_{35} - 0.99x_{31}x_{53} - x_{35}x_{53}$
{2,3,5}	$0.01 - x_{35}x_{53} + 0.99x_{53}z_{25} + x_{35}z_{52} - z_{25}z_{52}$
{2,4,5}	$0.01 - x_{24}x_{42} + 0.99x_{42}z_{25} + x_{24}z_{52} - z_{25}z_{52}$

Since $x_{ij}x_{ji}$ and $z_{ij}z_{ji} \geq 0$, from the first five principal minors in Table 3.1 it is clear that x_{13}, x_{31}, x_{24}, x_{42}, x_{35}, $x_{53} \geq 0$ and z_{14}, z_{41}, z_{25}, $z_{52} \leq 0$. Furthermore: at least one of x_{13}, $x_{31} > 0.49$, at least one of $x_{24}, x_{42} > 0.49$, at least one of x_{35},

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$x_{53} > 0.49$, at least one of $|z_{14}|$, $|z_{41}| > 0.49$, and at least one of $|z_{25}|$, $|z_{52}| > 0.49$.

From $\mathrm{Det}A(\{1,3,4\}) \geq 0$ taken in conjunction with the signs of the entries, each of the terms $x_{13}x_{31}$, $x_{31}z_{14}$, $x_{13}z_{41}$, $z_{14}z_{41}$ must be less in absolute value than 0.0102.

Thus, one of two things must happen:

(1a) $x_{13} \geq 0.49$, $x_{31} \leq 0.021$, $|z_{41}| \leq 0.021$ and $|z_{14}| \geq 0.49$,

(1b) $x_{31} \geq 0.49$, $x_{13} \leq 0.021$, $|z_{14}| \leq 0.021$ and $|z_{41}| \geq 0.49$.

Similarly, from $\mathrm{Det}A(\{1,2,4\})$, one of two things must happen:

(2a) $x_{24} \geq 0.49$, $x_{42} \leq 0.021$, $|z_{41}| \leq 0.021$ and $|z_{14}| \geq 0.49$,

(2b) $x_{42} \geq 0.49$, $x_{24} \leq 0.021$, $|z_{14}| \leq 0.021$ and $|z_{41}| \geq 0.49$.

From $\mathrm{Det}A(\{1,3,5\})$, one of two things must happen:

(3a) $x_{13} \geq 0.49$, $x_{35} \leq 0.021$, $x_{53} \geq 0.49$ and $x_{31} \leq 0.021$,

(3b) $x_{31} \geq 0.49$, $x_{53} \leq 0.021$, $x_{13} \geq 0.49$ and $x_{35} \leq 0.021$.

From $\mathrm{Det}A(\{2,3,5\})$, one of two things must happen:

(4a) $x_{35} \geq 0.49$, $x_{53} \leq 0.021$, $|z_{52}| \leq 0.021$ and $|z_{25}| \geq 0.49$,

(4b) $x_{53} \geq 0.49$, $x_{35} \leq 0.021$, $|z_{25}| \leq 0.021$ and $|z_{52}| \geq 0.49$.

From $\mathrm{Det}A(\{2,4,5\})$, one of two things must happen:

(5a) $x_{24} \geq 0.49$, $x_{42} \leq 0.021$, $|z_{52}| \leq 0.021$ and $|z_{25}| \geq 0.49$,

(5b) $x_{42} \geq 0.49$, $x_{24} \leq 0.021$, $|z_{25}| \leq 0.021$ and $|z_{52}| \geq 0.49$.

Consider Case 1a, $x_{13} \geq 0.49$, $x_{31} \leq 0.021$, $|z_{41}| \leq 0.021$ and $|z_{14}| \geq 0.49$.

Then by examining z_{14}, we must have Case 2a, $x_{24} \geq 0.49$, $x_{42} \leq 0.021$, $|z_{41}| \leq 0.021$ and $|z_{14}| \geq 0.49$. By examining x_{24}, we must have Case 5a, $x_{24} \geq 0.49$, $x_{42} \leq 0.021$, $|z_{52}| \leq 0.021$ and $|z_{25}| \geq 0.49$.

By examining x_{13}, we must have Case 3a, $x_{13} \geq 0.49$, $x_{35} \leq 0.021$, $x_{53} \geq 0.49$ and $x_{31} \leq 0.021$ and by examining x_{35}, we must have Case 4b, $x_{53} \geq 0.49$, $x_{35} \leq 0.021$, $|z_{25}| \leq 0.021$ and $|z_{52}| \geq 0.49$, which contradicts the previously established restrictions on $|z_{25}|$ and $|z_{52}|$.

Case 1b is similar. □

We will now establish the main theorem of this section, namely, the symmetric 6-cycle does have weakly sign symmetric P_0-completion, through a series of lemmas.

LEMMA 3.3. *Let A be a partial weakly sign symmetric P_0-matrix, whose digraph of specified entries is a symmetric n-cycle with $n > 4$, such that neither n-cycle product is negative. Then A can be completed to a weakly sign symmetric P_0-matrix.*

Proof. Use a permutation similarity to renumber the vertices of the graph as desired. We may assume that the symmetric $n-$cycle specified by A is $1, 2, \ldots, n, 1$. Index arithmetic is mod n.

Case 1: At least one n-cycle product is positive, or there is a k such that $a_{k,k+1} = 0 = a_{k+1,k}$. In the latter situation, use a permutation similarity to get $a_{n1} = 0 = a_{1n}$.

Since A is weakly sign symmetric, $a_{k,k+1}$ and $a_{k+1,k}$ cannot have opposite signs, although one can be 0 and the other either sign. For $k = 2, \ldots, n$, define $s_k = 1$ unless $a_{k-1,k}$ or $a_{k,k-1} < 0$, in which case $s_k = -1$. Then $s_k^2 = 1$, $s_k a_{k,k-1} \geq 0$, and $s_k a_{k-1,k} \geq 0$. Define the diagonal matrix $D = \mathrm{diag}(1, s_2, s_2s_3, \ldots, s_2s_3\ldots s_n)$. Note $D = D^{-1}$.

We show DAD is nonnegative:

- $(DAD)_{k-1,k} = (s_2 \ldots s_{k-1})^2 s_k a_{k-1,k} \geq 0$, for $k = 2, \ldots, n$.
- $(DAD)_{k,k-1} = (s_2 \ldots s_{k-1})^2 s_k a_{k,k-1} \geq 0$, for $k = 2, \ldots, n$.

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- $(DAD)_{1n} = (s_2 \dots s_n)a_{1n} \geq 0$, because:
  If $a_{1n} = 0$ then $(DAD)_{1n} = 0$.
  If $a_{1n} \neq 0$ and $a_{n,n-1} \dots a_{32}a_{21}a_{1n} > 0$, then $(s_2 \dots s_n)a_{1n} > 0$.
  If $a_{1n} \neq 0$ and $a_{n,n-1} \dots a_{32}a_{21}a_{1n} = 0$, then $a_{12}a_{23} \dots a_{n-1,n}a_{n1} > 0$.
  Thus $(s_2 \dots s_n)a_{n1} > 0$, and so $(s_2 \dots s_n)a_{1n} > 0$.
- Similarly, $(DAD)_{n1} = (s_2 \dots s_n)a_{n1} \geq 0$.

Thus $B = DAD$ is a partial nonnegative P_0-matrix with the same pattern of entries as A.

Complete B to C by Theorem 3.2 of [2]. Then DCD will complete A to a weakly sign symmetric P_0-matrix.

Case 2: Both n-cycle products are 0 and for each k, $a_{k,k+1} \neq 0$ or $a_{k+1,k} \neq 0$.

Since both n-cycle products are 0, there exist distinct i and j such that $a_{i,i+1} = 0$ and $a_{j+1,j} = 0$. Without loss of generality, $i = n$. Let $\alpha = 1, \dots, j$ and $\beta = j+1, \dots, n$. Then $A(\alpha)$ and $A(\beta)$ are partial weakly sign symmetric P_0-matrices specifying block-clique patterns and so can be completed to weakly sign symmetric P_0-matrices $C(\alpha)$ and $C(\beta)$ [3]. Complete A by setting all other entries to 0.

The resulting matrix is weakly sign symmetric, and is also a P_0-matrix because it is a block triangular matrix with diagonal blocks $C(\alpha)$ and $C(\beta)$, [6]. □

LEMMA 3.4. *Let A be a partial weakly sign symmetric P_0-matrix, whose digraph of specified entries is a symmetric 6-cycle, such that one 6-cycle product is negative and $d_k = 0$ for some k. Then A can be completed to a weakly sign symmetric P_0-matrix.*

Proof. Without loss of generality we may assume $d_6 = 0$, $d_i = 1$ or 0 for $i = 1, \dots 5$, the 6-cycle is $1, 2, \dots, 6, 1$, and $a_{12}a_{23}a_{34}a_{45}a_{56}a_{61} < 0$. This implies $a_{16} = 0$ and $a_{65} = 0$. By use of a diagonal similarity, $a_{12} = a_{23} = a_{34} = a_{45} = a_{56} = 1$, and $a_{61} < 0$ (since a diagonal similarity does not affect cycle products). Complete A to the weakly sign symmetric matrix

$$\widehat{A} = \begin{bmatrix} d_1 & 1 & 0 & 0 & 0 & 0 \\ a_{21} & d_2 & 1 & 0 & 0 & 0 \\ a_{32}a_{21} & a_{32} & d_3 & 1 & 0 & 0 \\ a_{43}a_{32}a_{21} & a_{43}a_{32} & a_{43} & d_4 & 1 & 0 \\ a_{54}a_{43}a_{32}a_{21} & a_{54}a_{43}a_{32} & a_{54}a_{43} & a_{54} & d_5 & 1 \\ a_{61} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We establish $\widehat{A}$ is a P_0-matrix by examining the principal minors. First, we observe that $\text{Det}\widehat{A} = -a_{61} > 0$. Furthermore, all proper principal minors are zero except those minors in which row and column 6 are removed. When row and column 6 are removed from matrix A, the completion is precisely that used to complete a block-clique partial P_0-matrix. Thus $\widehat{A}$ is a P_0-matrix. □

LEMMA 3.5. *Let A be a partial weakly sign symmetric P_0-matrix, whose digraph of specified entries is a symmetric 6-cycle, such that $d_k \neq 0$ for all k, and where one 6-cycle product is negative and the other is zero. Then A can be completed to a weakly sign symmetric P_0-matrix.*

Proof. By left multiplication by a positive diagonal matrix, we may assume $d_k = 1$ for all k. Without loss of generality, the 6-cycle is $1, 2, \dots, 6, 1$, $a_{12}a_{23} \dots a_{56}a_{61}$ is negative and $a_{16} = 0$. Apply a diagonal similarity to make all superdiagonal entries

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equal to 1. Since cycle products are not affected by diagonal similarity, $a_{61} < 0$. Complete A to the weakly sign symmetric matrix

$$\widehat{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ a_{21} & 1 & 1 & 1 & 0 & 0 \\ 0 & a_{32} & 1 & 1 & 1 & 0 \\ 0 & 0 & a_{43} & 1 & 1 & 1 \\ 0 & 0 & 0 & a_{54} & 1 & 1 \\ a_{61} & 0 & 0 & 0 & a_{65} & 1 \end{bmatrix}.$$

We establish that $\widehat{A}$ is a P_0-matrix by computing all principal minors and writing them as sums of nonnegative terms. Since A is a partial P_0-matrix, all the original 2×2 minors $1-a_{k+1,k}, k = 1,\ldots,5$ are nonnegative. Since A is weakly sign symmetric, $a_{k+1,k} \geq 0$, for $k = 1,\ldots,5$, and $a_{61} < 0$ by hypothesis.

Table 3.2

α	$\mathrm{Det}\widehat{A}(\alpha)$
$\{1,2,3,4,5\}$	$(1-a_{32})(1-a_{43})(1-a_{54})(1-a_{21}) + a_{32}a_{43}a_{54}(1-a_{21}) + a_{21}a_{32}a_{43}$
$\{1,2,3,4,6\}$	$(1-a_{43})(1-a_{21})(1-a_{32}) - a_{61}(1-a_{32}) + a_{43}a_{32}a_{21}$
$\{1,3,4,5,6\}$	$(1-a_{43})(1-a_{54})(1-a_{65}) - a_{61}(1-a_{54}) + a_{43}a_{54}a_{65}$
$\{2,3,4,5,6\}$	$(1-a_{32})(1-a_{43})(1-a_{54})(1-a_{65}) + a_{32}a_{43}a_{54}(1-a_{65}) + a_{65}a_{54}a_{43}$
$\{1,2,3,4,5,6\}$	$(1-a_{21})(1-a_{32})(1-a_{43})(1-a_{54})(1-a_{65}) + a_{21}a_{32}a_{43}(1-a_{65}) + a_{43}a_{54}a_{65}(1-a_{21}) + a_{32}a_{43}a_{54}(1-a_{65})(1-a_{21}) - a_{43}a_{61}$

Table 3.2 exhibits the factored forms that demonstrate the listed principal minors of $\widehat{A}$ are nonnegative. Each of the omitted principal minors of $\widehat{A}$ is either equal to 1, one of the original 2×2 minors, a product of two original minors, a product of three original minors, the sum of a nonnegative term and the product of three original minors, or $(1 - a_{k+1,k}) - a_{61}$. Thus $\widehat{A}$ is a P_0-matrix. □

Lemma 3.6. *Let A be a partial weakly sign symmetric P_0-matrix, whose digraph of specified entries is a symmetric 6-cycle, such that $d_k \neq 0$ for all k and both 6-cycle products are negative. Then A can be completed to a weakly sign symmetric P_0-matrix.*

Proof. We may assume that $d_k = 1$ for all k, the symmetric 6-cycle specified by A is $1, 2, \ldots, 6, 1$, with $|a_{12}a_{23}a_{34}a_{45}a_{56}a_{61}| \leq |a_{21}a_{32}a_{43}a_{54}a_{65}a_{16}|$ and $a_{61}a_{16} \leq a_{k,k+1}a_{k+1,k}$ for $k = 1,\ldots,5$, and all superdiagonal entries equal to one. Since cycle products are not affected by diagonal similarity, $a_{61} < 0$, $|a_{61}| \leq |a_{21}a_{32}a_{43}a_{54}a_{65}a_{16}|$, and $a_{16}a_{61} \leq a_{k+1,k}$. By weak sign symmetry and the fact that both cycle products are nonzero, $a_{16} < 0$ and $a_{k+1,k} > 0, k = 1,\ldots,5$.

To reduce the number of subscripts and make all symbols positive, we introduce

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the following notation: $a_k = a_{k,k-1}, b = -a_{61}, c = -a_{16}$. The specified 2×2 minors (original minors) are $1 - a_k,\ k = 2, \dots, 6$ and $1 - bc$, all of which are nonnegative. Complete A to the weakly sign symmetric matrix $\widehat{A}$ as shown:

$$\widehat{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & -c \\ a_2 & 1 & 1 & 1 & 0 & 0 \\ 0 & a_3 & 1 & 1 & 1 & 0 \\ 0 & 0 & a_4 & 1 & 1 & 1 \\ -b & 0 & 0 & a_5 & 1 & 1 \\ -b & -b & 0 & 0 & a_6 & 1 \end{bmatrix}.$$

We must show $\widehat{A}$ is a P_0-matrix. From the fact A is a partial P_0-matrix, the assumptions for this lemma, and the preprocessing we have:

$0 < a_k \le 1,$ for $k = 2, \dots, 6$; $bc \le a_k,$ for $k = 2, \dots, 6$; $b \le a_2a_3a_4a_5a_6c$.

Let $a_m = \min_{k=2,\dots,6} \{a_k\}$. Then $bc \le a_m$, so $b \le a_2a_3a_4a_5a_6c$ implies $b^2 \le a_2a_3a_4a_5a_6bc \le a_2a_3a_4a_5a_6a_m$, or $b \le \sqrt{a_2a_3a_4a_5a_6a_m} \le a_m$ (since $a_k \le 1$). Thus, also $b \le 1$ and $b \le a_k$ for $k = 2, \dots, 6$. Also $c \le \frac{a_m}{b}$ since $bc \le a_m$ and $\frac{b}{a_2a_3a_4a_5a_6} \le c$ since $b \le a_2a_3a_4a_5a_6c$.

Each of the 2×2 principal minors of $\widehat{A}$ is either equal to 1 or one of the original minors. Each of the 3×3 principal minors of $\widehat{A}$ is one of: an original minor, a product of two such original minors, or $1 - b$. Each of the 4×4 principal minors is one of: a product of two original minors, a sum of a product of two original minors and b, or a sum of a product of three original minors and a nonnegative term.

The 5×5 principal minors:

$\mathrm{Det}\widehat{A}(\{2,3,4,5,6\}) = (1 - a_3)(1 - a_4)(1 - a_5)(1 - a_6) + a_4a_5(a_3 + a_6 - a_3a_6) + b(1 - a_4 - a_5)$. The first term, $(1 - a_3)(1 - a_4)(1 - a_5)(1 - a_6) \ge 0$ since it is the product of original minors. We show $a_4a_5(a_3 + a_6 - a_3a_6) + b(1 - a_4 - a_5) \ge 0$. Note, $a_3 + a_6 - a_3a_6 = a_3 + a_6(1 - a_3) \ge a_m > 0$, so if $1 - a_4 - a_5 \ge 0$ we are done. If $1 - a_4 - a_5 < 0$, $b \le a_m$ so $a_4a_5(a_3 + a_6 - a_3a_6) + b(1 - a_4 - a_5) \ge a_4a_5(a_m) + a_m(1 - a_4 - a_5) \ge a_m(1 - a_4)(1 - a_5) \ge 0$.

$\mathrm{Det}\widehat{A}(\{1,3,4,5,6\}) = ((1 - a_4)(1 - bc) + b)(1 - a_5)(1 - a_6) + (a_4 - b)a_5a_6 + a_5a_6bc(1 - a_4) \ge 0$.

$\mathrm{Det}\widehat{A}(\{1,2,4,5,6\}) = (1 - a_2)(1 - a_5)(1 - a_6 - bc) + b(1 - a_6)(1 - bc) + a_6bc(1 - b) \ge ((1 - a_2)(1 - a_5)(1 - a_6 - bc)) + a_6bc(1 - b)$. Also, $b \le a_2$, so $1 - b \ge 1 - a_2 \ge (1 - a_2)(1 - a_5)$. Thus $((1 - a_2)(1 - a_5)(1 - a_6 - bc)) + a_6bc(1 - b) \ge ((1 - a_2)(1 - a_5)(1 - a_6 - bc)) + a_6bc(1 - a_2)(1 - a_5) = (1 - a_2)(1 - a_5)(1 - a_6)(1 - bc) \ge 0$.

The computation of $\mathrm{Det}\widehat{A}(\{1,2,3,5,6\})$ is similar to that of $\mathrm{Det}\widehat{A}(\{1,2,4,5,6\})$. Other corresponding pairs of minors are $\mathrm{Det}\widehat{A}(\{1,2,3,4,6\})$ with $\mathrm{Det}\widehat{A}(\{1,3,4,5,6\})$ and $\mathrm{Det}\widehat{A}(\{1,2,3,4,5\})$ with $\mathrm{Det}\widehat{A}(\{2,3,4,5,6\})$.

The determinant of the whole 6×6 matrix, $\mathrm{Det}\widehat{A}$, can be expressed as:

$$(1 - a_4)(1 - a_2)(1 - a_5)(1 - a_3)(1 - a_6) \tag{3.1}$$

$$+ (1 - a_4)a_6(1 - a_3)bc \tag{3.2}$$

$$+ a_4a_2a_3(1 - a_6)(1 - a_5) \tag{3.3}$$

$$+ a_4a_5a_6(1 - a_2)(1 - a_3) \tag{3.4}$$

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$$\begin{aligned}
&(3.5) && + a_4a_3a_5(1-a_2a_6)\\
&(3.6) && + b(1-a_2)(1-a_5)\\
&(3.7) && + b(1-a_3)(1-a_6)\\
&(3.8) && + a_2a_3a_4a_5a_6c - b\\
&(3.9) && + b(1-bc)(1-a_4)\\
&(3.10) && + b^2 + (a_3 - a_3a_5 - a_3a_4)bc\\
&(3.11) && - (1-a_4)(1-a_2)(1-a_5)bc
\end{aligned}$$

Note that all terms are clearly nonnegative except (10) and (11). Note that $\mathrm{Det}\widehat{A}$ is a linear function of c, so it is sufficient to show $\mathrm{Det}\widehat{A} \geq 0$ for $c_{\min} = \frac{b}{a_2a_3a_4a_5a_6}$ and for $c_{\max} = \frac{a_m}{b}$.

If $c = c_{\max} = \frac{a_m}{b}$:

We rewrite (8): $a_2a_3a_4a_5a_6c - b = (\frac{1}{b})(a_2a_3a_4a_5a_6a_m - b^2)$. And (10):

$$\begin{aligned}
&(b^2 - a_2a_3a_4a_5a_6a_m) + a_2a_3a_4a_5a_6a_m + (a_3 - a_3a_5 - a_3a_4)a_m\\
&= (b^2 - a_2a_3a_4a_5a_6a_m) + a_3a_m(a_2a_4a_5a_6 + 1 - a_5 - a_4 + a_4a_5 - a_4a_5)\\
&= (b^2 - a_2a_3a_4a_5a_6a_m) + a_3a_m(1 - a_5 - a_4 + a_4a_5) - a_3a_4a_5a_m(1-a_2a_6)\\
&= (b^2 - a_2a_3a_4a_5a_6a_m) + a_3a_m(1-a_5)(1-a_4) - a_3a_4a_5a_m(1-a_2a_6).
\end{aligned}$$

We now add (8), (10), and (5):

$$\begin{aligned}
&(\tfrac{1}{b})(a_2a_3a_4a_5a_6a_m - b^2) + (b^2 - a_2a_3a_4a_5a_6a_m) + a_3a_m(1-a_5)(1-a_4)\\
&-a_3a_4a_5a_m(1-a_2a_6) + a_4a_3a_5(1-a_2a_6) = (\tfrac{1}{b} - 1)(a_2a_3a_4a_5a_6a_m - b^2)\\
&+a_3a_m(1-a_5)(1-a_4) + a_3a_4a_5(1-a_m)(1-a_2a_6) \geq a_3a_m(1-a_5)(1-a_4)\\
&= a_3(1-a_4)(1-a_5)bc.
\end{aligned}$$

If $c = c_{\min} = \frac{b}{a_2a_3a_4a_5a_6}$, we rewrite (10):

$$\begin{aligned}
&b^2 + \tfrac{(a_3 - a_3a_5 - a_3a_4)b^2}{a_2a_3a_4a_5a_6}\\
&= \tfrac{b^2}{a_2a_3a_4a_5a_6}(a_2a_3a_4a_5a_6 + a_3 - a_3a_5 - a_3a_4)\\
&= \tfrac{a_3b^2}{a_2a_3a_4a_5a_6}(a_2a_4a_5a_6 + 1 - a_5 - a_4 + a_4a_5 - a_4a_5)\\
&= \tfrac{a_3b^2}{a_2a_3a_4a_5a_6}(1-a_5)(1-a_4) - \tfrac{b^2}{a_2a_3a_4a_5a_6}a_3a_4a_5(1-a_2a_6).
\end{aligned}$$

We add (10) and (5):

$$\begin{aligned}
&\tfrac{a_3b^2}{a_2a_3a_4a_5a_6}(1-a_5)(1-a_4) - \tfrac{b^2}{a_2a_3a_4a_5a_6}a_3a_4a_5(1-a_2a_6) + a_4a_3a_5(1-a_2a_6)\\
&= \tfrac{a_3b^2}{a_2a_3a_4a_5a_6}(1-a_5)(1-a_4) + (1 - \tfrac{b^2}{a_2a_3a_4a_5a_6})a_3a_4a_5(1-a_2a_6)\\
&\geq \tfrac{a_3b^2}{a_2a_3a_4a_5a_6}(1-a_5)(1-a_4)\\
&= a_3bc(1-a_5)(1-a_4).
\end{aligned}$$

Thus, for $c = c_{\min}$ or $c_{\max}$

$$\begin{aligned}
(8) + (10) + (5) + (11) &\geq a_3bc(1-a_5)(1-a_4) - (1-a_4)(1-a_2)(1-a_5)bc\\
&= (1-a_4)(1-a_5)(a_3 + a_2 - 1)bc.
\end{aligned}$$

The sum (2)+(8)+(10)+(5)+(11) is greater than or equal to:

$$\begin{aligned}
&(1-a_4)a_6(1-a_3)bc + (1-a_4)(1-a_5)(a_3 + a_2 - 1)bc\\
&= -(1-a_4)bc(1 - a_2 - a_3 - a_5 - a_6 + a_3a_5 + a_3a_6 + a_5a_2)\\
&= (1-a_4)bc[-(1-a_2)(1-a_3)(1-a_5)(1-a_6) + a_2a_3(1-a_5)
\end{aligned}$$

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$$+a_2a_6(1-a_3)+a_5a_6(1-a_2)(1-a_3)]$$
$$\geq -(1-a_4)(1-a_2)(1-a_3)(1-a_5)(1-a_6)bc.$$

Thus, $\mathrm{Det}\widehat{A} \geq (1)+(2)+(8)+(10)+(5)+(11)$
$\geq (1-a_4)(1-a_2)(1-a_3)(1-a_5)(1-a_6) - (1-a_4)(1-a_2)(1-a_3)(1-a_5)(1-a_6)bc$
$= (1-a_4)(1-a_2)(1-a_3)(1-a_5)(1-a_6)(1-bc) \geq 0.$ ☐

The following theorem is a consequence of Lemmas 3.3 through 3.6.

THEOREM 3.7. *A pattern whose digraph is a symmetric* 6*-cycle has weakly sign symmetric* P_0*-completion.*

There are several immediate consequences to Theorem 3.7.

COROLLARY 3.8. *Any pattern whose digraph is a symmetric* 6*-cycle has (weakly) sign symmetric* P*-completion.*

Proof. The result for weakly sign symmetric P-matrices follows from Theorems 3.7 and 2.1. The result for sign symmetric P-matrices follows from the result for weakly sign symmetric P-matrices and Corollary 2.3. ☐

COROLLARY 3.9. *A pattern whose digraph is a symmetric* n*-cycle has (weakly) sign symmetric* P*-completion if and only if* $n \neq 4$ *and* $n \neq 5$*.*

Proof. A partial matrix, whose digraph of specified entries is a symmetric 2-cycle or 3-cycle, is complete. Example 3.3 of [3] shows that there is a partial sign symmetric P-matrix, whose digraph of specified entries is a symmetric 4-cycle, that cannot be completed to a weakly sign symmetric P-matrix. Our Example 3.2 establishes the same noncompletion result for $n = 5$. The proof of completion for $n \geq 6$ is by induction on n. The base case is provided by Corollary 3.8. For the inductive step we let A be a (weakly) sign symmetric P-matrix specifying a symmetric n-cycle. We may assume that the symmetric n-cycle is $1, 2, \ldots, n, 1$, and either $a_{12} = a_{21} = a_{23} = a_{32} = \ldots = a_{1n} = a_{n1} = 0$ or $a_{12} \neq 0$ (by permutation similarity). In the former case, setting all unspecified entries to 0 produces a positive diagonal matrix, which is certainly a (weakly) sign symmetric P-matrix. In the latter case we may apply the proof of the inductive step of Lemma 3.5 of [3] as noted in that paper. ☐

Corollary 3.9 completely answers the question about the completability of the symmetric n-cycle for sign symmetric and weakly sign symmetric P-matrices. The former problem was described as difficult in [3]. Note that "partial sign symmetric P-matrix, the graph of whose entries is an n-cycle" in [3] is equivalent to our "partial sign symmetric P-matrix, whose digraph of specified entries is a symmetric n-cycle." Theorem 3.10 below completely answers the analogous question for weakly sign symmetric P_0-matrices.

THEOREM 3.10. *A pattern whose digraph is a symmetric* n*-cycle has weakly sign symmetric* P_0*-completion if and only if* $n \neq 4$ *and* $n \neq 5$*.*

Proof. The cases $n = 2, 3, 4, 5$ are the same as in Corollary 3.9 and again the proof for $n \geq 6$ is by induction on n. Theorem 3.7 supplies the case $n = 6$. Assume true for $n-1$. Let A be an $n \times n$ partial weakly sign symmetric P_0-matrix specifying the symmetric n-cycle $1, 2, \ldots, n, 1$. By multiplication by a positive diagonal matrix we may assume that each diagonal entry of A is either 1 or 0.

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Case 1: There exists an index i such that $d_i = d_{i+1} = 1$ and at least one of $a_{i,i+1}$ and $a_{i+1,i}$ is nonzero. Renumber so that $d_1 = d_2 = 1$ and $a_{12} \neq 0$. Then A may be completed to a weakly sign symmetric P_0-matrix $\widehat{A}$ as in Lemma 3.5 of [3].

Case 2: The matrix does not satisfy the conditions of Case 1 and there exists an index i such that $d_i = d_{i+1} = 1$. Necessarily $a_{i,i+1} = a_{i+1,i} = 0$. Renumber so that $d_1 = d_2 = 1$ (and $a_{12} = a_{21} = 0$). Either $a_{n1} = 0$ or $a_{1n} = 0$, because if $d_n = 0$, then $\mathrm{Det}A(\{1,n\}) = -a_{n1}a_{1n}$, and if $d_n = 1$, $a_{n1} = 0$ and $a_{1n} = 0$, since Case 1 does not apply. The digraph of the pattern specified by $A(\{2,\ldots,n\})$ is block-clique, so it can be completed to a weakly sign symmetric P_0-matrix [3]. Set the remaining entries to 0 to obtain a completion $\widehat{A}$. This completion has either the first row of zeros (if $a_{1n} = 0$) or the first column of zeros (if $a_{n1} = 0$), and thus is a nonnegative block triangular matrix, both of whose diagonal blocks ($d_1 = 1$ and the completion of $A(\{2,\ldots,n\})$) are P_0-matrices. Thus $\widehat{A}$ is a weakly sign symmetric P_0-matrix (cf. [6, 3.3]).

Case 3: There does not exist an index i such that $d_i = d_{i+1} = 1$. Then for each i, $\mathrm{Det}A(\{i,i+1\}) = -a_{i,i+1}a_{i+1,i}$, so $a_{i,i+1} = 0$ or $a_{i+1,i} = 0$. This implies at least one of the two n-cycle products is zero.

Subcase A: If both n-cycle products are zero, then use the zero completion $\widehat{A}_0$ of A. From the nonzero-L-digraph of $\widehat{A}_0$ we see that $\mathrm{Det}\widehat{A}_0 = 0$ because there are no cycles of length greater than one (cf. Lemmas 1.2 and 1.3 of [2]).

For the remaining subcases, one n-cycle product is nonzero. Without loss of generality, the cycle product of $1, 2, \ldots, n, 1$ is nonzero. By use of a diagonal similarity we may assume $a_{12} = a_{23} = \ldots = a_{n-1,n} = 1$.

Subcase B: a_{n1} is positive. Then A is a nonnegative partial P_0-matrix and can be completed to a nonnegative (hence weakly sign symmetric) matrix [2].

Subcase C: a_{n1} is negative and n is even. Use the zero completion $\widehat{A}_0$ of A. The only cycles in the nonzero L-digraph of $\widehat{A}_0$ are the n-cycle and perhaps some loops (1-cycles). The signed product of the n-cycle is positive since n is even and $a_{n1} < 0$. Thus the determinant of any principal minor is nonnegative.

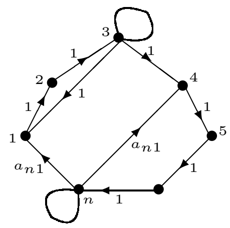

FIG. 3.1. *Nonzero-L-digraph of the completion of A for Case 3D*

Subcase D: a_{n1} is negative and n is odd. Complete A to $\widehat{A}$ by choosing $a_{31} = 1$ and $a_{n4} = a_{n1}$, and set all other unspecified entries to 0. The nonzero-L-digraph $\widehat{G}$ of $\widehat{A}$ contains the n-cycle $1, 2, \ldots, n, 1$, the 3-cycle $1, 2, 3, 1$, the $(n-3)$-cycle $4, 5, \ldots, n-1, n, 4$, and possibly some loops. See Figure 3.1. The signed cycle products for the

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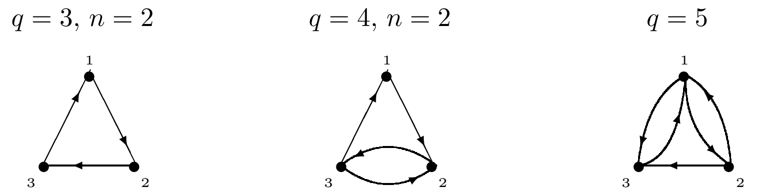
n-cycle, 3-cycle and $(n-3)$-cycle are $a_{n1}, 1$, and $-a_{n1}$ because n is odd so $n-3$ is even. There are exactly two permutation L-digraphs in $\widehat{G}$, one having arc set the n-cycle, and one having arc set the 3-cycle and the $(n-3)$-cycle. The cycle products are equal and the signs are opposite, so $\mathrm{Det}\widehat{A} = 0$. The nonzero L-digraph of any principal submatrix that is not the whole matrix cannot contain the n-cycle and thus has no negative signed products in its determinant. Hence any principal minor is nonnegative and $\widehat{A}$ is a weakly sign symmetric P_0-matrix. □

4. Classification of digraphs of order ≤ 4 regarding (weakly) sign symmetric P-completion and weakly sign symmetric P_0-completion. Any digraph referred to in this section is identified as in [5], where q is the number of edges and n is the diagram number.

LEMMA 4.1. *All patterns for 2×2 matrices have (weakly) sign symmetric P- and weakly sign symmetric P_0-completion. A pattern for 3×3 matrices has (weakly) sign symmetric P- and weakly sign symmetric P_0-completion if and only if its digraph does not contain a 3-cycle or is complete.*

Proof. Any partial weakly sign symmetric P-(P_0-)matrix specifying any one of the order two digraphs or one of the order three digraphs $q = 0$; $q = 1$; $q = 2$, $n =$ 1-4; $q = 3$, $n = 1$, 3-4; $q = 4$, $n =$ 3-4 may be completed to a weakly sign symmetric P-(P_0-)matrix by replacing each unspecified entry with a zero. These digraphs have sign symmetric P-completion by Lemma 2.2.

A partial (weakly) sign symmetric P-(P_0-) matrix specifying $q = 4, n = 1$ may be completed to a (weakly) sign symmetric P-(P_0-) matrix because it is block-clique [3]. Digraph $q = 6$ is complete.



$$\begin{bmatrix} 1 & 1 & x \\ y & 1 & 1 \\ -2 & z & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & x \\ y & 1 & 1 \\ -2 & \frac{1}{2} & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & -\frac{1}{2} \\ \frac{1}{2} & 1 & 1 \\ -2 & z & 1 \end{bmatrix}$$

FIG. 4.1. *Digraphs that do not have completion*

The example matrices in Figure 4.1 clearly show that the digraphs $q = 3$, $n = 2$; $q = 4$, $n = 2$; and $q = 5$ have neither weakly sign symmetric P-completion, sign symmetric P-completion, nor weakly sign symmetric P_0-completion. □

For sign symmetric P-matrices, the classification of digraphs of order 3, and the result in Lemma 4.2 also appear in [8].

LEMMA 4.2. *A 4×4 matrix satisfying the pattern with digraph $q = 7$, $n = 2$; $q = 4$, $n = 16$; $q = 5$, $n = 7$; $q = 6$, $n = 4, 7$ has (weakly) sign symmetric P-completion.*

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Proof. Consider the digraph in Figure 4.2.

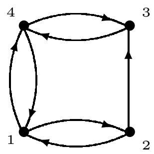

FIG. 4.2. $q = 7$, $n = 2$

Let $A = \begin{bmatrix} 1 & a_{12} & x_{13} & a_{14} \\ a_{21} & 1 & a_{23} & x_{24} \\ x_{31} & x_{32} & 1 & a_{34} \\ a_{41} & x_{42} & a_{43} & 1 \end{bmatrix}$ be a partial weakly sign symmetric P-matrix specifying the digraph $q = 7$, $n = 2$. Clearly the original minors, $1 - a_{12}a_{21}, 1 - a_{14}a_{41}, 1 - a_{34}a_{43}$ are greater than zero. We consider two cases: (1) $a_{12}a_{23}a_{34}a_{41} \leq 0$ and (2) $a_{12}a_{23}a_{34}a_{41} > 0$.

Case 1: $a_{12}a_{23}a_{34}a_{41} \leq 0$. Set $x_{13} = a_{14}a_{43}$, $x_{24} = a_{21}a_{14}$, and all other unspecified entries equal to zero. Each of the proper principal minors is equal to 1, or an original minor, or the product of two original minors. The determinant $\text{Det}A = (1 - a_{12}a_{21})(1 - a_{14}a_{41})(1 - a_{34}a_{43}) + a_{12}a_{21}a_{14}a_{41}a_{34}a_{43} - a_{12}a_{23}a_{34}a_{41}$.

Case 2: $a_{12}a_{23}a_{34}a_{41} > 0$. Set $x_{13} = a_{14}a_{43}, x_{24} = a_{21}a_{14} + a_{23}a_{34}$, and all other unspecified entries equal to zero. Each of the proper principal minors is equal to 1, or an original minor, or the product of two original minors. The determinant $\text{Det}A = (1 - a_{12}a_{21})(1 - a_{14}a_{41})(1 - a_{34}a_{43}) + a_{12}a_{21}a_{14}a_{41}a_{34}a_{43}$. Thus, this digraph has weakly sign symmetric P-completion.

Any partial weakly sign symmetric P-matrix specifying the digraph $q = 6$, $n = 4$ may be extended to a partial weakly sign symmetric P-matrix specifying the digraph $q = 7$, $n = 2$ by setting the unspecified (1,4)-entry equal to 0 (see Figure 4.3). The same reasoning applies to the digraphs $q = 4$, $n = 16$; $q = 5$, $n = 7$; and $q = 6$, $n = 7$. Thus, these digraphs also have weakly sign symmetric P-completion. Since all unspecified twins are assigned zero, all of these digraphs have sign symmetric P-completion by Lemma 2.2. □

LEMMA 4.3. *A 4×4 matrix specifying the patterns with digraphs $q = 4$, $n = 16$; $q = 5$, $n = 7$; $q = 6$, $n = 4, 7$; $q = 7$, $n = 2$; does not have weakly sign symmetric P_0-completion.*

Proof. A matrix satisfying the pattern with digraph $q = 4$, $n = 16$ does not have weakly sign symmetric P_0-completion by Example 9.8 in [6]. The same partial matrix with additional specified entries set to zero satisfying the pattern with one of the other four digraphs does not have completion. □

THEOREM 4.4. *(Classification of Patterns of 4 × 4 matrices). Let Q be a pattern for 4×4 matrices that includes all diagonal positions. The pattern Q has (weakly) sign symmetric P-completion if and only if its digraph is one of the following (numbered as in [5], q is the number of edges, n is the diagram number).*

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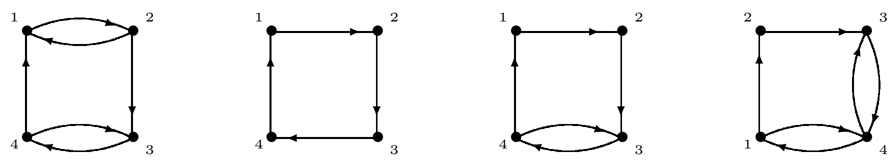

$q = 6$, $n = 4$ $q = 4$, $n = 16$ $q = 5$, $n = 7$ $q = 6$, $n = 7$

FIG. 4.3. *Digraphs associated with $q = 7$, $n = 2$*

$q = 0$;
$q = 1$;
$q = 2$ *n = 1-5;*
$q = 3$ *n = 1-11, 13;*
$q = 4$ *n = 1-12, 14-19, 21-23, 25-27;*
$q = 5$ *n = 1-5, 7-10, 14-17, 21-24, 26-29, 31, 33-34, 36-37;*
$q = 6$ *n = 1-8, 13, 15, 17, 19, 23, 26-27, 32, 35, 38-40, 43, 46;*
$q = 7$ *n = 2, 4-5, 9, 14, 24, 29, 34, 36;*
$q = 8$ *n = 1, 10, 12, 18;*
$q = 9$ *n = 8, 11;*
$q = 12$.

The pattern Q has weakly sign symmetric P_0-completion if and only if in its digraph the induced subdigraph of any 3-cycle or 4-cycle is a clique. Equivalently, Q's digraph is one of those with (weakly) sign symmetric P-completion and it is not one of the following: $q = 4$, $n = 16$; $q = 5$, $n = 7$; $q = 6$, $n = 4, 7$; $q = 7$, $n = 2$.

Proof. We first consider (weakly) sign symmetric P-matrices.

Part 1. Digraphs that have (weakly) sign symmetric P-completion.

The patterns of the digraphs listed below have (weakly) sign symmetric P-completion because every strongly connected nonseparable induced subgraph has (weakly) sign symmetric P-completion [6]: $q = 0$; $q = 1$; $q = 2$, $n =$ 1-5; $q = 3$, $n =$ 1-11, 13; $q = 4$, $n =$ 1-12, 14-15, 17-19, 21-23, 25-27; $q = 5$, $n =$ 1-5, 8-10, 14-17, 21-24, 26-29, 31, 33-34, 36-37; $q = 6$, $n =$ 1-3, 5-6, 8, 13, 15, 17, 19, 23, 26-27, 32, 35, 38-40, 43, 46; $q = 7$, $n =$ 4, 5, 9, 14, 24, 29, 34, 36; $q = 8$, $n =$ 1, 10, 12, 18; $q = 9$, $n =$ 8, 11; $q = 12$.

The patterns of the digraphs $q = 4$, $n = 16$; $q = 5$, $n = 7$; $q = 6$, $n = 4$, 7; and $q = 7$, $n = 2$ have (weakly) sign symmetric P-completion by Lemma 4.2.

Part 2. Digraphs that do not have (weakly) sign symmetric P completion.

The following digraphs do not have (weakly) sign symmetric P-completion because each of these digraphs contains one of the order three digraphs in Figure 4.1 as an induced subdigraph: $q = 3$, $n = 12$; $q = 4$, $n =$ 13, 20, 24; $q = 5$, $n =$ 6, 11-13, 18-20, 25, 30, 32, 35, 38; $q = 6$, $n =$ 9-12, 14, 16, 18, 20-22. 24-25, 28-31, 33-34, 36-37, 41-42, 44-45, 47-48; $q = 7$, $n =$ 1, 3, 6, 7, 8, 10-13, 15-23, 25-28, 30-33, 35, 37-38; $q = 8$, $n =$ 3-9, 11, 13-17, 19-27; $q = 9$, $n =$ 1-7, 9-10, 12-13; $q = 10$, $n =$ 2-5; $q = 11$.

The digraph $q = 10$, $n = 1$ (i.e., the double triangle) does not have (weakly) sign symmetric P-completion by the example given in Lemma 2.3 in [3]. The digraph

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$q =$ 8, $n =$ 2 (i.e., the symmetric cycle) does not have (weakly) sign symmetric P-completion by Example 3.3 in [3].

Now consider weakly sign symmetric P_0-matrices. The digraphs $q = 4$, $n = 16$; $q = 5$, $n = 7$; $q = 6$, $n =$ 4,7; $q = 7$, $n = 2$ do not have weakly sign symmetric P_0-completion by Lemma 4.3. Each of the other patterns can be classified by one of the following facts: By Theorem 2.1, if a pattern does not have weakly sign symmetric P-completion, it also does not have weakly sign symmetric P_0-completion. Also, a pattern has weakly sign symmetric P_0-completion if each of the strongly connected components of its digraph has weakly sign symmetric P_0-completion [6]. □

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