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THE MAXIMUM NUMBER OF 2×2 ODD SUBMATRICES IN $(0,1)$-MATRICES*

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Abstract. Let A be an $m \times n$, $(0,1)$-matrix. A submatrix of A is odd if the sum of its entries is an odd integer and even otherwise. The maximum number of 2×2 odd submatrices in a $(0,1)$-matrix is related to the existence of Hadamard matrices and bounds on Turán numbers. Pinelis [On the minimal number of even submatrices of 0-1 matrices, *Designs, Codes and Cryptography*, 9:85–93, 1994] exhibits an asymptotic formula for the minimum possible number of $p \times q$ even submatrices of an $m \times n$ $(0,1)$-matrix. Assuming the Hadamard conjecture, specific techniques are provided on how to assign the 0's and 1's, in order to yield the maximum number of 2×2 odd submatrices in an $m \times n$ $(0,1)$-matrix. Moreover, formulas are determined that yield the exact maximum counts with one exception, in which case upper and lower bounds are given. These results extend and refine those of Pinelis.

Key words. $(0,1)$-matrices, Even and odd matrices, Hadamard matrices.

AMS subject classifications. 15A36, 15A37.

1. Introduction. Definition 1.1. *A $(0,1)$-matrix is* **odd** *if the sum of its entries is an odd integer. Otherwise, a $(0,1)$-matrix is* **even**.

Unless otherwise noted, all matrices in this paper are $(0,1)$-matrices. The matrix B below is odd while the matrix C is even:

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Modifying Pinelis' notation slightly [5], we will use the following notation throughout this paper.

Definition 1.2. *For an $m \times n$ $(0,1)$-matrix A, $e(A;\ p,\ q)$ denotes the number of even $p \times q$ submatrices. $E(m,\ n;\ p,\ q)$ denotes the minimum of $e(A;\ p,\ q)$ taken over all $m \times n$ $(0,1)$-matrices A. Similarly, $g(A;\ p,\ q)$ denotes the number of $p \times q$ odd submatrices of A while $G(m,\ n;\ p,\ q)$ denotes the maximum of $g(A;\ p,\ q)$ taken over all $m \times n$ $(0,1)$-matrices A.*

In this paper, we consider 2×2 submatrices. The number N of 2×2 submatrices of an $m \times n$ $(0,1)$-matrix A is given by

$$N = \binom{m}{2}\binom{n}{2} = \frac{mn(m-1)(n-1)}{4}.$$

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Thus $G(m,\ n;\ 2,\ 2) = N - E(m,\ n;\ 2,\ 2)$. In this paper, rather than determine $E(m,\ n;\ 2,\ 2)$, we determine $G(m,\ n;\ 2,\ 2)$ directly as a function of m and n.

Let $G = G(n,\ n;\ 2,\ 2)$. Assume that an $4k \times 4k$ Hadamard matrix exists and that $n > 2$. Column A gives Pinelis' bounds on G [5], column B gives the corresponding results in this paper. If $m \neq n$ we give exact values for $G(m,\ n;\ 2,\ 2)$.

n	A	B
$4k$	$G \leq 32k^4 - 8k^3$	$G = 32k^4 - 8k^3$
$4k+1$	$G \leq 32k^4 + 24k^3 + 4k^2$	$32k^4 + 24k^3 + 6k - 2 \leq\ G \leq 32k^4 + 24k^3 + 4k^2$
$4k-1$	$G \leq 32k^4 - 40k^3 + 16k^2 - 2k$	$G = 32k^4 - 40k^3 + 16k^2 - 2k$
$4k-2$	$G \leq 32k^4 - 72k^3 + 60k^2 - 22k + 3$	$G = 32k^4 - 72k^3 + 56k^2 - 16k + 1$

The quantities $E(m,\ n;\ 2,\ 2)$ and $G(m,\ n;\ 2,\ 2)$ are closely associated with two well-known problems, the problem of finding Turán numbers and the problem of constructing Hadamard matrices.

Definition 1.3. *The* **Turán number** $T(n,\ l,\ k)$ *is the smallest possible number of k-subsets of an n-set such that every l-subset contains one of the chosen k-sets [1].*

It is known [1] that

$$T(2n, 5, 4) \leq 2\binom{n}{4} + E(n, n; 2, 2) = 2\binom{n}{4} + N - G(n, n; 2, 2).$$

The size of $G(m,\ n;\ 2,\ 2)$ is a function of how the 0's and 1's in the matrix A can be positioned so as to balance the number of times a pair of rows of A *agree* or *match* in a column (both 0 or both 1) with the number of times they *disagree* or *mismatch* in a column (one 0 and one 1). The current theory [6] of Hadamard matrices helps to answer questions about the size of $G(m,\ n;\ 2,\ 2)$. The relationship between Hadamard matrices and the size of $G(m,\ n;\ 2,\ 2)$ is presented in Section 4.

In Section 2 we present our approach to determining the value of $G(m,\ n;\ 2,\ 2)$. In Section 3, we review the Hadamard problem and its relationship to $G(m,\ n;\ 2,\ 2)$. Section 4 contains the formulas for computing $G(m,\ n;\ 2,\ 2)$. In Section 5 we discuss the asymptotic behavior of $G(m,\ n;\ 2,\ 2)$.

2. The Essence of the Problem. Consider the $m \times n$ matrix A containing all 1's. Each 2×2 submatrix of this matrix is even. This is an extreme case.

With N denoting the total number of 2×2 submatrices of A as in the previous section, we have

$$0 \leq G(m,\ n;\ 2,\ 2) \leq N.$$

If both m and n are at least two, and either m or n is three or greater, then an $m \times n$ matrix must have at least one even submatrix. To see this, consider two 2×2

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submatrices that share a column (or a row). If both are odd, then the 2×2 submatrix consisting of the columns (or rows) not shared is clearly even. Thus $G(m,\ n;\ 2,\ 2) = N$ only in the case $m = n = 2$. In that case either one of the following 2×2 matrices yields the largest number of 2×2 odd submatrices, namely one. That is, for A and B below, $g(A;\ 2,\ 2) \ = \ g(B;\ 2,\ 2) \ = \ 1$.

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Consider the order 3 matrices given below in which $g(A;\ 2,\ 2) = 4$, $e(A;\ 2,\ 2) = 5$, $g(B;\ 2,\ 2) = 6$, $e(B;\ 2,\ 2) = 3$. It turns out, by equation (2.4) below, that $G(3,\ 3;\ 2,\ 2) = 6$.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Definition 2.1. *Let s and t be rows of an $m \times n$ matrix A. Then $\mathbf{S(s,t)}$ denotes the number of columns where s and t agree (both zero or both one) and $\mathbf{D(s,t)}$ denotes the number of columns where s and t disagree (one zero and the other one).*

Then the number of 2×2 odd submatrices in rows s and t is given by

$$S(s,t)\cdot D(s,t). \tag{2.1}$$

This quantity is maximized when $S(s,t)$ and $D(s,t)$ can be made the same, or as nearly the same as possible given that they are integer values. Since $S(s,t)+D(s,t) = n$, this means the quantity in (2.1) above is maximum when n is even and each factor has the value $\frac{n}{2}$, or when n is odd and the two factors are $\frac{n+1}{2}$ and $\frac{n-1}{2}$. In other words, given an $m \times n$ matrix A, if $n = 2k$ the number of odd submatrices is maximized if the 0's and 1's can be assigned so every pair of rows of A agree and disagree in exactly $\frac{n}{2}$ positions, while if $n = 2k + 1$, the best we can hope for is that every pair of rows of A agree in $\lfloor \frac{n}{2} \rfloor$ positions and disagree in $\lceil \frac{n}{2} \rceil$ positions, or vice versa. Therefore,

$$G(m,n;2,2) \leq \binom{m}{2} \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \tag{2.2}$$

with equality if and only if there exists an $m \times n$ $(0,1)$-matrix in which every pair of rows s and t maximize $S(s,t)\cdot D(s,t)$.

Example 2.2. In the matrix C below, the 0's and 1's are assigned so as to yield the maximum number of 2×2 odd submatrices. Note that each pair, of the six possible pairs of rows, agree (and disagree) in exactly 2 column positions.

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

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Hence, from (2.2), G(4, 4; 2, 2) = 24.

Definition 2.3. *Suppose s and t denote two rows of an $m \times n$ $(0,1)$-matrix. The pair (s,t) is* **perfect** *if $|S(s,t) - D(s,t)| = 0$ or, equivalently, $S(s,t) = D(s,t) = \frac{n}{2}$. The pair (s,t) is* **near-perfect** *if $|S(s,t) - D(s,t)| = 1$. A (0,1)-matrix is* **perfect** *if each pair of its rows is perfect. A (0,1)-matrix is* **near-perfect** *if each pair of its rows is near-perfect.*

Note, an $m \times n$ $(0,1)$-matrix for which n is odd cannot be perfect. If n is even, the matrix may or not be perfect, but it certainly cannot be near-perfect. Clearly if a $(0,1)$-matrix of order $m \times n$ is perfect, then we have equality in (2.2).

When can the 0's and 1's of a $(0,1)$-matrix be assigned so that it is perfect? Or, if the 0's and 1's cannot be assigned so that it is perfect, how close to perfect can we get? This will lead us to the subject of Hadamard matrices. However, first we will consider some basic facts about $(0,1)$-matrices and the count of their 2×2 odd submatrices.

Theorem 2.4. *Suppose A is a $(0,1)$-matrix and B is a matrix obtained from A by any one, or any combination of the following operations:*
a) *Exchange any pair of rows or any pair of columns.*
b) *In any row or column, replace the 0's by 1's and the 1's by 0's.*
c) *Take the transpose.*
Then $g(A;\ 2,\ 2) = g(B;\ 2,\ 2)$.

Proof. The indicated operations do not change the parity of any 2×2 $(0,1)$-submatrix. □

Definition 2.5. *A $(0,1)$-matrix A is* **normalized** *if its first row and first column contain only zeroes.*

Theorem 2.6. *If A is any $m \times n$ $(0,1)$-matrix, then there exists a normalized $m \times n$ $(0,1)$-matrix B, such that $g(A;\ 2,\ 2) = g(B;\ 2,\ 2)$.*

Proof. Apply Theorem 2.4. □

Theorem 2.7. *Suppose A is an $m \times n$ $(0,1)$-matrix. If A is perfect or near-perfect, then $g(A;\ 2,\ 2)$ is maximum.*

Proof. If A is perfect or near-perfect, then equality holds in (2.2) and $g(A;\ 2,\ 2) = G(m,\ n;\ 2,\ 2)$. If $n = 2t$, then

$$G(m,\ 2t;\ 2,\ 2) = \frac{m(m-1)}{8} \cdot n^2 = \frac{m(m-1)}{2} \cdot t^2. \tag{2.3}$$

On the other hand, if $n = 2t+1$, then

$$G(m,\ 2t+1;\ 2,\ 2) = \binom{m}{2} \cdot \frac{n^2-1}{4} = \frac{m(m-1)}{2} \cdot t(t+1). \tag{2.4}$$ □

Theorem 2.8. *Suppose A is an $m \times n$ $(0,1)$-matrix. If A contains three or more rows which are pairwise perfect, then n is divisible by 4.*

Proof. (We adapt the standard argument for showing that the order of any Hadamard matrix of order greater than 2 is a multiple of 4.)

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By Theorem 2.6, we may assume A is normalized. Because two rows of A are pairwise perfect, n is even, say $n = 2t$. Moreover, we may assume the first three rows are pairwise perfect. Since the first row is all zeroes, rows two and three must be half zeroes and half ones. Since we can rearrange columns, we may as well assume that the second row has zeroes in the first half and ones in the second half. Since the third row has an equal number of zeroes and ones, the number of zeroes in its first half must equal the number of ones in its second half. Call this number w. There are therefore $2w$ positions where rows two and three agree, and so necessarily $2w$ positions where they disagree. Therefore $n = 4w$. □

In summary, the question of determining $G(m,\ n;\ 2,\ 2)$ becomes one of assessing how 0's and 1's may be assigned in a matrix of order $m \times n$ in such a way so as to simultaneously maximize the number $S(s,t) \cdot D(s,t)$ for all pairs of rows. We now turn to Hadamard matrices to find an answer to that question in the case $n = 4w$.

3. Hadamard Matrices. The history and research about Hadamard matrices is rich (see [3], [4], [6], [7]). Here we present only the basic information on Hadamard matrices needed in order to determine $G(m,\ n;\ 2,\ 2)$. While Hadamard was not the first person to consider such matrices, the use of his name to describe them is universal. In 1893, Hadamard [3] shows that if $A = (a_{ij})$ is a square matrix of order n that satisfies $|a_{ij}| \leq 1$ for all i and j, then $|det(A)| \leq n^{\frac{n}{2}}$. Furthermore, Hadamard shows that equality is achieved only when A is a special type of matrix now known as a *Hadamard matrix.*

DEFINITION 3.1. *A square $(1,-1)$-matrix of order n is called a* **Hadamard matrix** *if the rows (hence, columns) are pairwise orthogonal.*

DEFINITION 3.2. *Suppose A is a square $(0,1)$-matrix and B is a square $(1,-1)$-matrix. Let* $\mathbf{J}(\mathbf{B})$ *denote the $(0,1)$-matrix obtained from B by replacing the 1's by 0's and -1's by 1's. Let* $\mathbf{K}(\mathbf{A})$ *be the $(1,-1)$-matrix obtained from A by replacing the 0's by 1's and the 1's by -1's.*

THEOREM 3.3. *Suppose H is a $(1,-1)$-matrix of order n. The following are equivalent:*

a) *H is a Hadamard matrix.*
b) *$HH^T = nI_n$.*
c) *$J(H)$ is perfect.*

Proof. The proof is straightforward. □

COROLLARY 3.4. *$J(B)$ is perfect if and only if B is Hadamard. $K(A)$ is Hadamard if and only if A is perfect.*

DEFINITION 3.5. *We call a $(1,-1)$-matrix H* **perfect** *if J(H) is perfect and* **normalized** *if J(H) is normalized. Thus a Hadamard matrix is normalized if and only if its first row and column contain only 1's.*

By Corollary 3.4, the existence of perfect $(0,1)$-matrices is equivalent to the existence of Hadamard matrices.

The Hadamard Conjecture: [3] If $n = 1, 2$ or if $n = 4k$, then a Hadamard matrix of order n exists. The converse is known to be true [4] and also follows easily from Theorems 2.8 and 3.3.

There are only five values of $4k < 1000$ for which Hadamard matrices are not

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known [4], and as k grows larger the number of distinct Hadamard matrices of order $4k$ seems to increase rapidly. In this paper, we use the assumption that Hadamard matrices of order $4k$ exist to determine the values of $G(m,\ n;\ 2,\ 2)$ for all possible m, n, except when $m = n = 4k + 1$. In this last case we obtain upper and lower bounds.

4. Formulas for G(m, n; 2, 2). Our results in this section depend on the Hadamard conjecture.

Since $G(m,\ n;\ 2,\ 2) = G(n,\ m;\ 2,\ 2)$, we may assume that in all of the matrices considered in this section the number of rows is less than or equal to the number of columns. As we have already observed, $\mathrm{G}(2, \mathrm{n}; 2, 2) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$. To establish formulas for all $m, n > 2$, we consider the four cases $n = 4k,\ 4k-1,\ 4k-2$, and $4k+1$. In each proof given below, we assume $A = J(H_{4k})$ is a perfect normalized square $(0,1)$-matrix of order $4k$.

THEOREM 4.1. *If a Hadamard matrix of order $4k$ exists, then for any $m \leq 4k$,*

$$G(m,\ 4k;\ 2,\ 2) = 2m(m-1)k^2. \tag{4.1}$$

Proof. Suppose A is a perfect normalized $(0,1)$-matrix of order $4k$. Deleting all but m rows of A yields a perfect $m \times n$ matrix. Therefore, from (2.3), $G(m,\ 4k;\ 2,\ 2) = 2m(m-1)k^2$. □

THEOREM 4.2. *If a Hadamard matrix of order $4k$ exists, then for any $m \leq 4k-1$,*

$$G(m,\ 4k-1;\ 2,\ 2) = m(m-1)k(2k-1). \tag{4.2}$$

Proof. Suppose A is a perfect normalized $(0,1)$-matrix of order $4k$. Delete the first row and first column of A to obtain the square matrix W of order $4k-1$. The matrix W is near-perfect. If we delete any number of rows from W we get another near-perfect matrix. Therefore, from (2.4), for any $m \leq 4k-1$, $G(m,\ 4k-1;\ 2,\ 2) = m(m-1)k(2k-1)$. □

(The case where $m = n = 4k-1$ is covered in [1].)

THEOREM 4.3. *If a Hadamard matrix of order $4k$ exists, then for any m such that $2 < m \leq 4k-2$,*

$$G(m,\ 4k-2;\ 2,\ 2) = \binom{\frac{m^2}{4}}{}(2k-1)^2 + 2\binom{\frac{m}{2}}{2}(2k)(2k-2), \qquad (m\ \textit{even}) \tag{4.3}$$

$$G(m,\ 4k-2;\ 2,\ 2) = \left(\frac{m^2-1}{4}\right)(2k-1)^2 + \frac{(m-1)^2}{2}(k)(2k-2). \qquad (m\ \textit{odd}) \tag{4.4}$$

Proof. Let C be a $(0,1)$-matrix of dimension $m \times (4k-2)$, with $m > 2$. By Theorem 2.8, the graph whose vertices are the rows of C, and whose edges join pairs of

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perfect rows, is a triangle-free graph on m vertices. Hence, by Turán's Theorem [2] , C has at most $\lfloor \frac{m}{2} \rfloor \lceil \frac{m}{2} \rceil$ pairs of perfect rows. Thus the number of 2×2 odd submatrices of C is at most $\left(\frac{m^2}{4}\right)(2k-1)^2 + 2\binom{m/2}{2}(2k)(2k-2)$ if m is even, and at most $\left(\frac{m^2-1}{4}\right)(2k-1)^2 + \frac{(m-1)^2}{2}(k)(2k-2)$ if m is odd. This gives upper bounds on $G(m,\ 4k-2;\ 2,\ 2)$.

We now construct matrices which achieve these upper bounds.

Let A be a perfect normalized $(0,1)$-matrix of order $4k$. Separate the rows of A into two sets, T and B, according to whether their second entry is a zero or a one. Rearrange the rows of A so that the rows in T are on the top and the rows in B are on the bottom. The two sets T and B are equal in size because the transpose of a perfect square matrix is a perfect matrix, so the second column of A has an equal number of 0's and 1's.

Now, form a new square matrix W, of order $4k-2$, by deleting the first two columns of A, and then deleting the top row of T (a zero row) and the top row of B. Let T' be the set consisting of the top $2k-1$ rows of W, and let B' be the set consisting of the bottom $2k-1$ rows of W. Compare a row in T' with a row in B'. When we created W, we deleted columns 1 and 2 of A. In column 1 of A a row in the top half and a row in the bottom half agreed, in column 2 they disagreed. Therefore any row from T' and any row from B' form a perfect pair. By our graph theory observation above, this is the maximum possible number of perfect pairs in any square matrix of order $4k-2$.

Now consider two rows from the same set, that is either two rows of T' or two rows of B'. These rows were formed by deleting two matching pairs of entries in the corresponding rows of A. So they will have $2k-2$ matches and $2k$ mismatches. Thus any pair of rows of W that do not form a perfect pair are as close to being perfect as possible without actually being perfect.

Delete all but m rows from W, alternately deleting a row from T' and then a row from B'. The resulting matrix has the maximum possible number of 2×2 odd submatrices for an $m\times(4k-2)$ matrix.

The given formulas reveal the correct counts. We will explain the count for m even and leave the count for m odd to the reader.

Let m be even. Let T'' and B'' be the sets formed above by deleting all but $m/2$ rows from each of T' and B'. Each of the $(\frac{m}{2})^2$ choices of one row from T'' and one row from B'' gives a perfect pair of rows which contains $(2k-1)^2$ $2\ \times\ 2$ odd submatrices. Each of the $2\binom{m/2}{2}$ choices of a pair of rows either both from T'' or both from B'' gives a pair with $2k-2$ matches and $2k$ mismatches which contains $2k(2k-2)$ $2\ \times\ 2$ odd submatrices.

This completes the count of the $2\ \times\ 2$ odd submatrices for m even. □

The case where $n\ =\ 4k+1$ is divided into two parts, first where $m\ <\ n$ and then where $m\ =\ n$.

THEOREM 4.4. *If a Hadamard matrix of order* $4k$ *exists, then for any* $m \le 4k$,

$$G(m,\ 4k+1;\ 2,\ 2)\ =\ m(m-1)k(2k+1). \tag{4.5}$$

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Proof. Suppose A is a perfect normalized $(0,1)$-matrix of order $4k$. No pair of rows in a matrix with an odd number of columns can be perfect, so we can find $G(m,\ 4k+1;\ 2,\ 2)$ using a matrix which is near-perfect. Such a matrix is easily obtained by augmenting A with any column of 0's and 1's, creating a near-perfect $4k \times (4k+1)$ matrix W. As before, we may delete any number of rows from W to get another near-perfect matrix. Hence from (2.4), for any $m \le 4k$, $G(m,\ 4k+1;\ 2,\ 2) = m(m-1)k(2k+1)$. ☐

THEOREM 4.5. *If a Hadamard matrix of order* $4k$ *exists, then*

$$G(4k+1,\ 4k+1;\ 2,\ 2) \ \ge\ 32k^4 + 24k^3 + 6k - 2. \tag{4.6}$$

Proof. We begin by constructing a square matrix W of order $4k+1$ as follows: Let w denote the last row of A. Define the row vector a of length $4k$ by setting $a(1) = 1$ and $a(i) = w(i)$ for $2 \le i \le 4k$. Now define a column vector c of length $4k$ all of whose entries are ones except the last entry, which is zero. Consider the augmented matrix below

$$W = \begin{pmatrix} A & c \\ a & 1 \end{pmatrix}.$$

What properties does W have? First, all pairs of rows from the top $4k$ rows are near-perfect, because A is perfect (the number of matches for any pair being either $2k$ or $2k+1$). Second, the last row, $(a\ 1)$ is near-perfect when paired with any row except the next to last row, $(w\ 0)$. In this last case, $(a\ 1)$ matches $(w\ 0)$ in every position, except the first and the last. Hence, there are just two mismatches while there are $4k-1$ matches. Therefore the number of 2×2 odd submatrices in W is given by

$$\begin{aligned} g(W;\ 2,\ 2) &= \left(\binom{4k+1}{2} - 1\right)(2k)(2k+1) + 2(4k-1) \\ &= 32k^4 + 24k^3 + 6k - 2. \end{aligned} \tag{4.7}$$

Clearly this is a lower bound on $G(4k+1,\ 4k+1;\ 2,\ 2)$. ☐

The possibility remains that there may exist square matrices of order $n = 4k+1$ with a larger number of 2×2 odd submatrices. There may even be a near-perfect square matrix of this order, whereas in the matrix constructed in the proof of Theorem 4.5 one pair of rows fails to be near-perfect. If we could find a near-perfect matrix of order $n = 4k+1$, then $G(4k+1,\ 4k+1;\ 2,\ 2)$ would be equal to $32k^4+24k^3+4k^2$. This is, in any case, an upper bound on $G(4k+1,\ 4k+1;\ 2,\ 2)$. The difference between the bounds is $4k^2-6k+2$, which is insignificant as a fraction of G for large n. To summarize,

$$32k^4 + 24k^3 + 6k - 2 \le\ G(4k+1,\ 4k+1;\ 2,\ 2) \le\ 32k^4 + 24k^3 + 4k^2.$$

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5. Asymptotic Behavior. Equations (4.1) through (4.7) imply that for any given pair of integers m and n with $m, n \geq 2$, when we consider the ratio of the maximum count of 2×2 odd submatrices to the count of all 2×2 submatrices, the limit approaches $\frac{1}{2}$ from above as either n or m go to infinity. That is,

$$\lim_{m\to\infty} \frac{G(m,n;2,2)}{\binom{m}{2}\binom{n}{2}} = \lim_{n\to\infty} \frac{G(m,n;2,2)}{\binom{m}{2}\binom{n}{2}} = \frac{1}{2}$$

and $G(m, n; 2, 2) > E(m, n; 2, 2)$ for all $m, n \geq 2$. This agrees with Pinelis' results in [5]. Though for any m and n there exist $(0, 1)$-matrices with all 2×2 submatrices even, and though every $m \times n$ $(0, 1)$-matrix with $m, n > 2$ has some 2×2 submatrices even, still we can always find an $m \times n$ $(0, 1)$-matrix with more 2×2 odd submatrices than even, as the examples in Section 4 show. Furthermore, as either m or n go to infinity, for $m \times n$ matrices, the ratio of the maximum number of 2×2 odd submarines to the minimum number of even 2×2 submatrix approaches one.

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