CORRIGENDUM TO “THE MAXIMAL SPECTRAL RADIUS OF A DIGRAPH WITH $(M+1)^2 - S$ EDGES”

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Abstract. In [3] we claimed to have proved the following: given any fixed integer $s > 6$ it holds that for all sufficiently large m the maximal spectral radius of a digraph with $k = (m+1)^2 - s$ edges is obtained by the digraph whose adjacency matrix is the one that Friedland [1] called E_k. However, what we have in fact showed is the weaker result that the above digraph is optimal among digraphs with k edges **and $\mathbf{m+1}$ vertices**.

Key words. Spectral radius, digraphs, 0-1 matrices, Perron-Frobenius theorem, number of walks.

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In [3] the following notation and terminology was used: for positive integers m, s the set of digraphs on the $m+1$ vertices $\{1, 2, \ldots, m+1\}$ that have exactly $(m+1)^2 - s$ edges is denote by $\mathcal{DI}(m,s)$. We will always assume that m is large enough in comparison to s that $m^2 < (m+1)^2 - s$, i.e. that $2m+1 > s$. By $\mathcal{PDI}(m,s)$ we denote the subset of digraphs that have a adjacency matrix whose rows and columns are weakly decreasing. In other words, if $G \in \mathcal{PDI}(m,s)$ have adjacency matrix $A = (a_{ij})$, then there is some numerical partition λ of s such that

$$a_{ij} = \begin{cases} 0 & j + \lambda_{m+1-i} > m+1 \\ 1 & \text{otherwise} \end{cases}$$

Let us (in this note) denote this A by $B(m, \lambda)$. As an example,

$$B(4,(2,1)) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

We state again that $\mathcal{PDI}(m,s) = \{\, B(m,\lambda) \mid \lambda \vdash s \,\}$, in particular, $|\mathcal{PDI}(m,s)| = p(s)$ for all (sufficiently large) m, where $p(s)$ denotes the number of numerical partitions of s. Schwarz [2] proved that

$$\max\{\, \rho(A) | A \in \mathcal{DI}(m,s) \,\} = \max\{\, \rho(A) | A \in \mathcal{PDI}(m,s) \,\}, \tag{0.1}$$

where $\rho(A)$ is the spectral radius (i.e. largest modulus of an eigenvalue) of the adjacency matrix of A.

Now fix $s > 6$, and let $\tau = \tau(s)$ be the numerical partition of s such that $\tau = (\lfloor s/2 \rfloor, 1, \ldots, 1)$. In other words, τ is a “hook” such that the length of its arms

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differ by at most one. Friedland [1] defined the matrix E_k as

$$E_k = \begin{pmatrix} J_a & \alpha_p \\ \alpha_q^t & 0 \end{pmatrix}, \qquad k = a^2 + \ell, \quad p = \lfloor \ell/2 \rfloor, \quad q = \ell - p,$$

where J_a is the $a \times a$ matrix of all ones, and α_p is the row vector consisting of p ones. We have that $B(\tau, m) = E_{(m+1)^2-s}$.

Let σ be a different partition of s (if s is even, σ should be different from the conjugate partition of τ as well). We showed in [3] that there is an $w = w(\sigma)$ such that for all $m > w$ we have that $\rho(B(m,\tau)) > \rho(B(m,\sigma))$. Since there are only finitely many numerical partitions of s, we get that there is an $S = S(s)$ so that for all $m > S$,

$$\rho(B(m,\tau)) = \max\{\, \rho(A) | A \in \mathcal{PDI}(m,s) \,\} = \max\{\, \rho(A) | A \in \mathcal{DI}(m,s) \,\}.$$

In other words, for a fixed s, there is an S such that for all $m > S$ the digraph with adjacency matrix $B(m, \tau(s))$ has the largest spectral radius among all digraphs on the vertex set $\{1, 2, \ldots, m+1\}$ with $(m+1)^2 - s$ edges.

Friedland [1] conjectured that the maximal spectral radius of a digraph with $(m+1)^2 - s$ edges can be obtained by a digraph with $m+1$ vertices. Clearly, given this conjecture our result could be strengthened as follows: for a fixed $s > 6$, there is an S such that for all $m > S$ the digraph with adjacency matrix $B(m, \tau(s))$ has the largest spectral radius among all digraphs with $(m+1)^2 - s$ edges. This is what we claimed to have proved in [3].

In summary: the phrase "digraph with $(m+1)^2 - s$ edges" should be changed to "digraph with $(m+1)^2 - s$ edges **and $\mathbf{m+1}$ vertices**" in the title, in the abstract, and at the bottom of page 180 of [3].

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