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BOUNDED NONOSCILLATORY SOLUTIONS OF NEUTRAL TYPE DIFFERENCE SYSTEMS

E. Thandapani [1], R.Karunakaran [2] and I.M.Arockiasamy [3]

[1] Ramanujan Institute for Advanced Study in Mathematics
University of Madras, Chepauk, Chennai-600 005, India
e-mail: ethandapani@yahoo.co.in
[2] Department of Mathematics, Periyar University
Salem-636 011, Tamil Nadu, India
[3] Department of Mathematics, St. Paul's Hr. Sec. School
Salem-636 007, Tamil Nadu, India

Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

This paper deals with the existence of a bounded nonoscillatory solution of nonlinear neutral type difference systems. Examples are provided to illustrate the main results.

Key words and phrases: Neutral difference system, bounded nonoscillatory solution, fixed point theorem.
AMS (MOS) Subject Classifications: 39A10, 39A12

1 Introduction

Consider nonlinear neutral type difference systems of the form

$$\begin{aligned}\Delta^2\left(x_1(n)-a_1(n)\,x_1(n-k_1)\right)&=p_1(n)\,f_1\left(x_2(n-\ell_2)\right)+e_1(n)\\ \Delta^2\left(x_2(n)-a_2(n)\,x_2(n-k_2)\right)&=p_2(n)\,f_2\left(x_1(n-\ell_1)\right)+e_2(n)\end{aligned}\tag{1}$$

where Δ is a forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$, $n \in \mathbb{N}(n_0) = \{n_0, n_0+1, \cdots\}$, $n_0 \geq 0$, k_i, ℓ_i are non negative integers $\{a_i(n)\}$, $\{p_i(n)\}, \{e_i(n)\}$ are real sequences and $f_i : \mathbb{R} \to \mathbb{R}$ is continuous function for $i = 1, 2$.

Let $\theta = \max\{k_1, k_2, \ell_1, \ell_2\}$. By a solution of the system (1) we mean a function $X = (\{x_1(n)\}, \{x_2(n)\})$, defined for all $n \geq n_0 - \theta$, and which satisfies the system (1) for all $n \geq n_0$.

In this paper, we obtain some sufficient conditions for the existence of a nonoscillatory bounded solutions of the system (1) via fixed point theorems and some new techniques. Here we allow the sequences $\{p_i(n)\}$ and $\{e_i(n)\}, i = 1, 2$, to be oscillatory. The existence of nonoscillatory solutions of neutral type difference equations has been treated in [1, 2, 6, 12] and in papers cited therein.

2 Main Results

In this section, we obtain sufficient conditions for the existence of bounded nonoscillatory solutions of the system (1). We consider the following cases:

(i) $0 < a_i(n) < 1$, (ii) $1 < a_i \equiv a_i(n) < \infty, i = 1, 2$, and their combination.

In this sequel we use the following fixed point theorem.

Lemma 2.1 *[Krasnoselskii's fixed point theorem] Let B be a Banach space. Let S be a bounded, closed, convex subset of B and let F, T be maps of S into B such that $Fx + Ty \in S$ for all $x, y \in S$. If F is contractive and T is completely continuous, then the equation*

$$Fx + Tx = x$$

has a solution in S.

The details of Lemma 2.1 can be found in [1] and [4] .

Theorem 2.1 *Assume that $0 < a_i(n) \leq a_i < 1$,*

$$\sum_{n=n_0}^{\infty} n\,|p_i(n)| < \infty \tag{2}$$

and

$$\left| \sum_{n=n_0}^{\infty} n e_i(n) \right| < \infty, i = 1, 2. \tag{3}$$

Then the system (1) has a bounded nonoscillatory solution.

Proof. In view of conditions (2) and (3), one can choose $N \in \mathbb{N}(n_0)$ sufficiently large that

$$\sum_{n=N}^{\infty} n\,|p_i(n)| \leq \frac{1-a_i}{5M_i}, \left| \sum_{n=N}^{\infty} n e_i(n) \right| \leq \frac{1-a_i}{10}, i = 1, 2,$$

where $M_i = \max\left\{ |f_i(y_{3-i}(n))| : \frac{2-2a_i}{5} \leq y_{3-i}(n) \leq 1 \right\}, i = 1, 2.$

Let $B(n_0)$ be the set of all real valued sequences $X(n) = (\{x_1(n)\}, \{x_2(n)\})$ with the norm $\|x(n)\| = \sup\limits_{n \geq n_0} (|x_1(n)|, |x_2(n)|) < \infty$. Then $B(n_0)$ is a Banach space. We define a closed, bounded and convex subset S of $B(n_0)$ as follows:

$$S = \left\{ x(n) = (\{x_1(n)\}, \{x_2(n)\}) \in B(n_0) : \frac{2-2a_i}{5} \leq x_i(n) \leq 1, i = 1, 2, n \geq n_0 \right\}.$$

Let maps $F = (F_1, F_2)$ and $T = (T_1, T_2) : S \to B(n_0)$ be defined by

$$(F_i x)(n) = \begin{cases} a_i(n)\,x_i(n-k_i) + \dfrac{7-7a_i}{10}, n \geq N, \\ (F_i x)(N), n_0 \leq n \leq T, i = 1, 2, \end{cases}$$

and

$$(T_i x)(n) = \begin{cases} \sum\limits_{s=n}^{\infty} (s-n+1)\left(p_i(s) f_i\left(x_{3-i}\left(s-\ell_{3-i}\right)+e_i(s)\right)\right), n \geq N, \\ (T_i x)(N), \qquad n_0 \leq n \leq T, i=1,2. \end{cases}$$

First we show that if $x, y \in S$, then $Fx + Ty \in S$. Hence, for $x = \{x(n)\}$ and $y = \{y(n)\} \in S$ and $n \geq N$, we have

$$\begin{aligned}(F_i x)(n) + (T_i y)(n) &\leq a_i(n) x_i(n-k_i) + \frac{7-7a_i}{10} + \sum_{s=n}^{\infty}(s-n+1)\left|p_i(s)\right| \\ &\quad \left|f_i\left(y_{3-i}\left(s-\ell_{3-i}\right)\right)\right| + \left|\sum_{s=n}^{\infty}(s-n+1) e_i(s)\right| \\ &\leq a_i + \frac{7-7a_i}{10} + M_i \sum_{s=n}^{\infty} s\left|p_i(s)\right| + \left|\sum_{s=n}{}^{\infty} s e_i(s)\right| \\ &\leq a_i + \frac{7-7a_i}{10} + \frac{1-a_i}{5} + \frac{1-a_i}{10} = 1, i=1,2.\end{aligned}$$

Further we obtain

$$\begin{aligned}&(F_i x)(n) + (T_i y)(n) \\ &\geq \frac{7-7a_i}{10} - \left(\sum_{s=n}^{\infty}(s-n+1)\left|p_i(s)\right|\left|f_i\left(y_{3-i}\left(s-\ell_{3-i}\right)\right)\right| + \left|\sum_{s=n}^{\infty}(s-n+1) e_i(s)\right|\right) \\ &\geq \frac{7-7a_i}{10} - \frac{1-a_i}{5} - \frac{1-a_i}{10} = \frac{2-2a_i}{5}, i=1,2.\end{aligned}$$

Hence $Fx + Ty \in S$ for any $x, y \in S$, that is, $FS \cup TS \subset S$. Next we show that F is a contraction on S. In fact for $x, y \in S$ and $n \geq N$, we have

$$\begin{aligned}\left|(F_i x)(n) - (F_i y)(n)\right| &\leq a_i(n)\left|x_i(n-k_i) - y_i(n-k_i)\right| \\ &\leq \max(a_i)\left|x_i(n-k_i) - y_i(n-k_i)\right|, i=1,2.\end{aligned}$$

This implies that

$$\|Fx - Fy\| \leq |a|\,\|x-y\|.$$

Since $0 < |a| < 1, a = (a_1, a_2)$, we conclude that F is a contraction mapping on S. Next we show that T is completely continuous. For this, first we show that T is continuous. Let $x_k = (\{x_{1k}(n)\}, \{x_{2k}(n)\}) \in S$ and $x_{ik}(n) \to x_i(n)$ as $k \to \infty, i = 1, 2$. Since S is closed, $x = (x_1(n), x_2(n)) \in S$. For $n \geq N$, we obtain

$$\begin{aligned}&\left|(T_i x_k)(n) - (T_i x)(n)\right| \\ &\qquad \leq \sum_{s=n}^{\infty}(s-n+1)\left|p_i(s)\right|\left|f_i\left(x_{3-ik}\left(s-\ell_{3-i}\right)\right) - f_i\left(x_{3-i}\left(s-\ell_{3-i}\right)\right)\right| \\ &\qquad \leq \sum_{s=n}^{\infty} s\left|p_i(s)\right|\left|f_i\left(x_{3-ik}\left(s-\ell_{3-i}\right)\right) - f_i\left(x_{3-i}\left(s-\ell_{3-i}\right)\right)\right|, i=1,2.\end{aligned}$$

Since $|f_i(x_{3-ik}(s-\ell_{3-i})) - f_i(x_{3-i}(s-\ell_{3-i}))| \to 0$ as $k \to \infty$, for $i = 1, 2$, we have T is continuous.

Next we show that TS is relatively compact. Using the result [[3], Theorem 3.3], we need only to show that TS is uniformly Cauchy. Let $x = (\{x_1(n)\}, \{x_2(n)\})$ be in S. From (2) and (3) it follows that for $\epsilon > 0$, there exists $N^* > N$ such that for $n \geq N^*$,

$$\sum_{s=n}^{\infty}(s-n+1)|p_i(s)||f_i(x_{3-i}(s-\ell_{3-i}))| + \left|\sum_{s=n}^{\infty}(s-n+1)e_i(s)\right| < \frac{\varepsilon}{2}, i = 1, 2.$$

Then for $n_2 > n_2 \geq N^*$, we have

$$|(T_i x)(n_2) - (T_i x)(n_1)| < \varepsilon.$$

Thus TS is uniformly Cauchy. Hence it is relatively compact.

Thus by Lemma 2.1, there is a $x_0(n) = (\{x_{01}(n)\}, \{x_{02}(n)\}) \in S$ such that $Fx_0 + Tx_o = x_0$. We see that $\{x_0(n)\}$ is a bounded nonoscillatory solution of the system (1). The proof is now complete.

Theorem 2.2 *Suppose that $1 < a_i \equiv a_i(n) < \infty$ and conditions (2) and (3) hold. Then the system (1) has a bounded nonoscillatory solution.*

Proof. In view of conditions (2) and (3), we can choose $N > n_0$ sufficiently large that

$$\sum_{s=N+k_i}^{\infty} s|p_i(s)| \leq \frac{a_i - 1}{3D_i}, \left|\sum_{s=N+k_i}^{\infty} se_i(s)\right| \leq \frac{a_i - 1}{6}$$

where $D_i = \max\limits_{a_i-1 \leq x_{3-i}^{(n)} \leq 2a_i} \left\{\left|f_i\left(x_{3-i}^{(n)}\right)\right|\right\}, i = 1, 2.$

Let $B(n_0)$ be the Banach space defined in the proof of Theorem 2.1. We define a closed,bounded and convex subset S of $B(n_0)$ as follows:

$$S = \{x = (\{x_1(n)\}, \{x_2(n)\}) \in B(n_0) : a_i - 1 \leq x_i(n) \leq 2a_i, i = 1, 2, n \geq n_0\}.$$

Let maps $F = (F_1, F_2)$ and $T = (T_1, T_2) : S \to B(n_0)$ be defined by

$$(F_i x)(n) = \begin{cases} \dfrac{3a_i - 3}{2} + \dfrac{1}{a_i(n)} x_i(n+k_i), n \geq N, \\ (F_i x)(N), n_0 \leq n \leq N, i = 1, 2. \end{cases}$$

$$(T_i x)(n) = \begin{cases} -\dfrac{1}{a_i(n)} \displaystyle\sum_{s=n+k_i}^{\infty}(s-n-k_i)(p_i(s) f_i(x_{3-i}(s-\ell_{3-i})) + e_i(s)), n \geq N, \\ (T_i x)(N), \qquad n_0 \leq n \leq N, i = 1, 2. \end{cases}$$

Now we show that F is a contractive mapping on S. For any $x, y \in S$ and $n \geq N$, we obtain

$$\begin{aligned}\left|\left(F_i x\right)(n)-\left(F_i y\right)(n)\right| &\leq \frac{1}{a_i(n)}\left|x_i\left(n+k_i\right)-y_i\left(n+k_i\right)\right| \\ &\leq \frac{1}{a_i}\|x-y\| \leq \max\left\{\frac{1}{a_i}\right\}\|x-y\|, i=1,2.\end{aligned}$$

This implies that

$$\|Fx-Fy\| \leq \max\left\{\frac{1}{a_i}, i=1,2\right\}\|x-y\|.$$

Since $0<\max\left\{\frac{1}{a_i}, i=1,2\right\}<1$ we conclude that F is a contraction mapping on S.

Next we show that for any $x, y \in S, Fx+Ty \in S$. For every $x, y \in S$ and $n \geq N$, we have

$$\begin{aligned}\left(F_i x\right)(n)+\left(T_i y\right)(n) \leq & \frac{3a_i-3}{2}+\frac{1}{a_i(n)} x_i\left(n+k_i\right) \\ & +\frac{1}{a_i(n)} \sum_{s=n+k_i}^{\infty}\left(s-k_i-n+1\right)\left|p_i(s)\right|\left|f_i\left(y_{3-i}\left(s-\ell_{3-i}\right)\right)\right| \\ & +\left|\sum_{s=n+k_i}^{\infty}\left(s-k_i-n+1\right) e_i(s)\right| \\ \leq & \frac{3a_i-3}{2}+2+\left(D_i \sum_{s=N+k_i}^{\infty} s\left|p_i(s)\right|+\left|\sum_{s=N+k_i}^{\infty} s e_i(s)\right|\right) \\ \leq & \frac{3a_i-3}{2}+2+\frac{a_i-1}{3}+\frac{a_i-1}{6}=2a_i, i=1,2.\end{aligned}$$

Further, we obtain

$$\begin{aligned}\left(F_i x\right)(n)-\left(T_i y\right)(n) \geq & \frac{3a_i-3}{2}+\frac{1}{a_i(n)} x_i\left(n+k_i\right) \\ & -\frac{1}{a_i(n)}\left[\sum_{s=n+k_i}^{\infty}\left(s-k_i-n+1\right)\left|p_i(s)\right|\left|f_i\left(y_{3-i}\left(s-\ell_{3-i}\right)\right)\right|\right. \\ & \left.+\left|\sum_{s=n+k_i}^{\infty}\left(s-k_i-n+1\right) e_i(s)\right|\right] \\ \geq & \frac{3a_i-3}{2}-\left(D_i \sum_{s=N+k_i}^{\infty} s\left|p_i(s)\right|+\left|\sum_{s=N+k_i}^{\infty} s e_i(s)\right|\right) \\ \geq & \frac{3a_i-3}{2}-\frac{a_i-1}{3}-\frac{a_i-1}{6}=a_i-1, i=1,2.\end{aligned}$$

Hence

$$a_i - 1 \le (F_i x)(n) + (T_i y)(n) \le 2a_i, i = 1, 2 \text{ for } n \ge n_0.$$

Thus we proved that $Fx + Ty \in S$ for any $x, y \in S$.

Proceeding similarly as in the proof of Theorem 2.1, we obtain that the mapping T is completely continuous. By Lemma 2.1, there is a $x_0 \in S$ such that $Fx_0 + Tx_0 = x_0$. We see that $\{x_0(n)\}$ is a non oscillatory bounded solution of the system (1). This completes the proof of Theorem 2.2.

Theorem 2.3 *Suppose that $0 < a_1(n) \le a_1 < 1, 1 < a_2 \equiv a_2(n) < \infty$ and conditions (2) and (3) hold. Then the difference system (1) has a bounded nonoscillatory solution.*

Proof. In view of conditions (2) and (3), we can choose a $N > n_0$ sufficiently large that

$$\sum_{n=N}^{\infty} n|p_1(n)| \le \frac{1-a_1}{5M_1}, \left|\sum_{n=N}^{\infty} ne_1(n)\right| \le \frac{1-a_1}{10}$$

and

$$\sum_{n=N+k_2}^{\infty} n|p_2(n)| \le \frac{a_2-1}{3D_2}, \left|\sum_{n=N+k_2}^{\infty} ne_2(n)\right| \le \frac{a_2-1}{6}.$$

Let $B(n_0)$ be the Banach space defined as in Theorem 2.1. We define a closed, bounded and convex subset S of $B(n_0)$ as follows:

$$\begin{aligned} S &= \{x = (\{x_1(n)\}, \{x_2(n)\}) \in B(n_0) : \frac{2-2a_1}{5} \le x_1(n) \le 1, \\ &\qquad a_2 - 1 \le x_2(n) \le 2a_2, n \ge n_0\}. \end{aligned}$$

Let maps $F = (F_1, F_2)$ and $T = (T_1, T_2) : S \to B(n_0)$ be defined by

$$(F_1x)(n) = \begin{cases} a_1(n)x_1(n-k_1) + \dfrac{7-7a_1}{10}, & n \ge N, \\ (F_1x)(N), & n_0 \le n \le N, \end{cases}$$

$$(T_1x)(n) = \begin{cases} \sum\limits_{s=n}^{\infty} (s-n+1)(p_i(s) f_1(x_2(s-k_2)) + e_1(s)), & n \ge N, \\ (T_1x)(N), & n_0 \le n \le N, \end{cases}$$

and

$$(F_2x)(n) = \begin{cases} \dfrac{1}{a_2(n)} x_2(n+k_2) + \dfrac{3a_2-3}{2}, & n \ge N, \\ (F_2x)(N), & n_0 \le n \le N, \end{cases}$$

$$(T_2x)(n) = \begin{cases} -\sum\limits_{s=n+k_2}^{\infty} (s-n-k_2+1)(p_2(s) f_2(x_1(s-k_1)) + e_2(s)), & n \ge N, \\ (T_2x)(N), & n_0 \le n \le N. \end{cases}$$

As in the proof of Theorems 2.1 and 2.2 one can show that F_1, F_2 are contractive mappings on S. It is easy to show that for any $x, y \in S.$ It is easy to show that for any $x, y \in S, F_1x + T_1y \in S$ and also $F_2x + T_2y \in S$. Proceeding as in the proof of Theorem 2.1, we obtain that the mappings T_1, T_2 are completely continuous. By Lemma 2.1, there are $x_{01}, x_{02} \in S$ such that $F_1x_{01} + T_1x_{01} = x_{01}, F_2x_{02} + T_2x_{02} = x_{02}$. We see that $x_0(n) = (\{x_{01}(n)\}, \{x_{02}(n)\})$ is a nonoscillatory bounded solution of the difference system (1). The proof is now complete.

In the following we provide some examples to illustrate the results.

Example 2.1 *Consider the difference system*

$$\begin{aligned} \Delta^2\left(x_1(n) - \frac{1}{2}x_1(n-1)\right) &= \frac{1}{3^n}x_2^3(n-1) - \frac{2^7}{3^{4n}} \\ \Delta^2\left(x_2(n) - \frac{1}{2}x_2(n-1)\right) &= \frac{1}{2^n}x_1^3(n-1) - \frac{8}{2^{4n}}, n \geq 1. \end{aligned} \tag{4}$$

It is easy to see that all conditions of Theorem 2.1 are satisfied, and hence the system (4) has a bounded nonoscillatory solution. In fact $x(n) = \left(\left\{\frac{1}{2^n}\right\}, \left\{\frac{1}{3^n}\right\}\right)$ *is one such solution of the system (4).*

Example 2.2 *Consider the difference system*

$$\begin{aligned} \Delta^2(x_1(n) - 2x_1(n-1)) &= \tfrac{-12}{n(n+2)(n+3)}x_2(n-1) - \tfrac{2}{(n+1)(n+2)(n+3)} \\ \Delta^2(x_2(n) - 2x_2(n-2)) &= \tfrac{-(26n+48)}{(n+1)(n+2)(n+3)(n+4)}x_1(n-1) - \tfrac{2n}{(n+1)(n+2)(n+3)(n+4)}, n \geq 1. \end{aligned} \tag{5}$$

By Theorem 2.2, the system (5) has a bounded nonoscillatory solution. In fact $x(n) = \left(\left\{\frac{1}{n+1}\right\}, \left\{\frac{1}{n+2}\right\}\right)$ *is one such solution of the system (5).*

Example 2.3 *Consider the difference system*

$$\begin{aligned} \Delta^2\left(x_1(n) - \frac{1}{2}x_1(n-1)\right) &= -\frac{1}{n^3}x_2^3(n-1) + \frac{1}{n^6} \\ \Delta^2(x_2(n) - 2x_2(n-1)) &= \frac{1}{2^{n+1}}x_1(n-1) - \frac{2(n+6)}{n(n+1)(n+2)(n+3)} - \frac{1}{4^n}. \end{aligned} \tag{6}$$

It is easy to see that all conditions of Theorem 2.3 are satisfied, and hence the system (6) has a bounded nonoscillatory solution. In fact $x(n) = \left(\left\{\frac{1}{2^n}\right\}, \left\{\frac{1}{n+1}\right\}\right)$ *is one such solution of the system.*

Remark 2.1 *It is easy to see that the difference system (1) includes different types of (ordinary, delay, neutral) fourth order difference equations, and hence the results obtained in this paper generalize many of the existing results for the fourth order difference equations, see for example [5, 7, 8, 9, 10, 11] and the references cited therein.*

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