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POSITIVE SOLUTIONS OF SECOND ORDER NONLINEAR DIFFERENCE BOUNDARY VALUE PROBLEMS

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

We study a class of second order nonlinear difference boundary value problems with separated boundary conditions. A series of criteria are obtained for the existence of one, two, arbitrary number, and even an infinite number of positive solutions. A theorem for the nonexistence of positive solutions is also derived. Several examples are given to demonstrate the applications. Our results improve and supplement several results in the literature.

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AMS (MOS) Subject Classifications: 39A12, 39A70

1 Introduction

In this paper, we study the existence of positive solutions of the second order boundary value problem (BVP) consisting of the difference equation

$$-\Delta^2 u(k) = w(k)f(k, u(k+1)),\ k \in \mathbb{N}(0, K), \tag{1.1}$$

and the separated boundary condition (BC)

$$\begin{aligned} a_{11}u(0) - a_{12}\Delta u(0) = 0, \\ a_{21}u(K+1) + a_{22}\Delta u(K+1) = 0, \end{aligned} \tag{1.2}$$

where $\mathbb{N}(0, K) = \{0, 1, \ldots, K\}$.

Second order difference BVPs have drawn attention in research in recent years. Results have been obtained for the existence of positive solutions for different types of BVPs, see Ma and Raffoul [12] and Pang and Ge [13] for multi-point BVPs, Tian and Ge [14] for BVPs on the half-line, and Yu, Zhu, and Guo [15] for BVPs with parameters. As regard to BVP (1.1), (1.2), Aykut and Guseinov [3] derived conditions for the existence of one positive solution. However, to the best of the knowledge of the authors, very little is known on the existence of multiple positive solutions and on the nonexistence of positive solutions.

In this paper, we will establish a series of criteria for BVP (1.1), (1.2) to have one, two, any arbitrary number, and even a countably infinite number of positive solutions. Criteria for nonexistence of positive solutions are also derived. Several examples are given to demonstrate the applications. Our results improve the results in [3] even for the existence of one positive solution.

This paper is organized as follows: After this introduction, our main results are stated in Section 2. Several examples are given in Section 3. All the proofs are given in Section 4.

2 Main Results

Throughout this paper, we assume without further mention that

(H1) $a_{ij} \geq 0$ for $i, j = 1, 2$ such that $a_{22} > 0$, $a_{11} + a_{12} > 0$, and

$$b := a_{11}a_{21}(K+1) + a_{11}a_{22} + a_{12}a_{21} > 0; \tag{2.1}$$

(H2) $w(k) \geq 0$ on $\mathbb{N}(0, K)$ such that $\sum_{k=0}^{K} w(k) > 0$;

(H3) for fixed $k \in \mathbb{N}(0, K)$, $f(k, \cdot) : [0, \infty) \to [0, \infty)$ is continuous.

Define $G : \mathbb{N}(0, K+2) \times \mathbb{N}(0, K) \to \mathbb{R}$ as

$$G(k,l) = \frac{1}{b} \begin{cases} [a_{12} + a_{11}(l+1)][a_{22} + a_{21}(K+1-k)], & 0 \leq l \leq k-1, \\ (a_{12} + a_{11}k)[a_{22} + a_{21}(K-l)], & k \leq l \leq K, \end{cases} \tag{2.2}$$

where b is defined by (2.1). It is known that $G(k,l)$ is the Green's function of the BVP consisting of the equation

$$-\Delta^2 u(k) = 0, \ \ k \in \mathbb{N}(0, K), \tag{2.3}$$

and BC (1.2), see Agarwal [1] for the detail. Let

$$\alpha = \min\left\{ \frac{a_{11} + a_{12}}{a_{11}(K+1) + a_{12}}, \frac{a_{22}}{a_{11}K + a_{22}} \right\}, \tag{2.4}$$

$$\beta = \max_{k \in \mathbb{N}(1,K+1)} \left\{ \sum_{l=0}^{K} G(k,l)w(l) \right\}. \tag{2.5}$$

From Aykut and Guseinov [3], we know that $0 < \alpha < 1$ and

$$0 < \alpha G(l+1, l) \leq G(k,l) \leq G(l+1,l) \text{ on } \mathbb{N}(1, K+1) \times \mathbb{N}(0, K). \tag{2.6}$$

Definition 2.1 *A function* $u(k) : \mathbb{N}(0, K+2) \to \mathbb{R}$ *is said to be a positive solution of BVP* (1.1), (1.2) *if* $u(k)$ *satisfies BVP* (1.1), (1.2) *and* $u(k) > 0$ *on* $\mathbb{N}(1, K+1)$.

Define

$$\|u\| = \max_{k \in \mathbb{N}(1,K+1)} |u(k)|.$$

The first theorem is our basic result on the existence of positive solutions of BVP (1.1), (1.2).

Theorem 2.1 *Assume there exist* $0 < r_* < r^*$ *[respectively,* $0 < r^* < r_*$*] such that*

$$f(k, x) \leq \beta^{-1} r_* \quad \text{for all } (k, x) \in \mathbb{N}(0, K) \times [\alpha r_*, r_*] \tag{2.7}$$

and

$$f(k, x) \geq \beta^{-1} r^* \quad \text{for all } (k, x) \in \mathbb{N}(0, K) \times [\alpha r^*, r^*]. \tag{2.8}$$

Then BVP (1.1), (1.2) *has at least one positive solution* u *with* $r_* \leq \|u\| \leq r^*$ *[respectively,* $r^* \leq \|u\| \leq r_*$*].*

In the sequel, we will use the following notation:

$$f_0 = \liminf_{x \to 0} \min_{k \in \mathbb{N}(0,K)} f(k, x)/x, \quad f_\infty = \liminf_{x \to \infty} \min_{k \in \mathbb{N}(0,K)} f(k, x)/x;$$
$$f^0 = \limsup_{x \to 0} \max_{k \in \mathbb{N}(0,K)} f(k, x)/x, \quad f^\infty = \limsup_{x \to \infty} \max_{k \in \mathbb{N}(0,K)} f(k, x)/x.$$

The next three theorems are derived from Theorem 2.1 using f_0, f_∞, f^0, and f^∞.

Theorem 2.2 *BVP (1.1), (1.2) has at least one positive solution if either*

(a) $f^0 < \beta^{-1}$, *and* $f_\infty > (\alpha\beta)^{-1}$; *or*

(b) $f_0 > (\alpha\beta)^{-1}$ *and* $f^\infty < \beta^{-1}$.

Theorem 2.3 *Assume there exists* $r_* > 0$ *such that* (2.7) *holds.*

(a) If $f_0 > (\alpha\beta)^{-1}$, *then BVP (1.1), (1.2) has at least one positive solution* u *with* $\|u\| \leq r_*$;

(b) if $f_\infty > (\alpha\beta)^{-1}$, *then BVP (1.1), (1.2) has at least one positive solution* u *with* $\|u\| \geq r_*$.

Theorem 2.4 *Assume there exists* $r^* > 0$ *such that* (2.8) *holds.*

(a) If $f^0 < \beta^{-1}$, *then BVP (1.1), (1.2) has at least one positive solution* u *with* $\|u\| \leq r^*$;

(b) if $f^\infty < \beta^{-1}$, then BVP (1.1), (1.2) has at least one positive solution u with $\|u\| \geq r^*$.

Combining Theorems 2.3 and 2.4 we obtain results on the existence of at least two positive solutions.

Theorem 2.5 *Assume either*

(a) $f_0 > (\alpha\beta)^{-1}$ and $f_\infty > (\alpha\beta)^{-1}$, and there exists $r > 0$ such that

$$f(k,x) < \beta^{-1}r \ \ \textit{for all } (k,x) \in \mathbb{N}(0,K) \times [\alpha r, r]; \ \textit{or} \tag{2.9}$$

(b) $f^0 < \beta^{-1}$ and $f^\infty < \beta^{-1}$, and there exists $r > 0$ such that

$$f(k,x) > \beta^{-1}r \ \ \textit{for all } (k,x) \in \mathbb{N}(0,K) \times [\alpha r, r]. \tag{2.10}$$

Then BVP (1.1), (1.2) has at least two positive solutions u_1 and u_2 with $\|u_1\| < r < \|u_2\|$.

Note that in Theorem 2.5, the inequalities in (2.9) and (2.10) are strict and hence are different from those in (2.7) and (2.8) in Theorem 2.1. This is to guarantee that the two solutions u_1 and u_2 are different.

By applying Theorem 2.1 repeatedly, we can generalize the conclusions to obtain criteria for the existence of multiple positive solutions.

Theorem 2.6 *Let $\{r_i\}_{i=1}^N \subset \mathbb{R}$ such that $0 < r_1 < r_2 < r_3 < \cdots < r_N$. Assume either*

(a) f satisfies (2.9) with $r = r_i$ when i is odd, and satisfies (2.10) with $r = r_i$ when i is even; or

(b) f satisfies (2.9) with $r = r_i$ when i is even, and satisfies (2.10) with $r = r_i$ when i is odd.

Then BVP (1.1), (1.2) has at least $N-1$ positive solutions u_i with $r_i < \|u_i\| < r_{i+1}$, $i = 1, 2, \ldots, N-1$.

Theorem 2.7 *Let $\{r_i\}_{i=1}^\infty \subset \mathbb{R}$ such that $0 < r_1 < r_2 < r_3 < \cdots$. Assume either*

(a) f satisfies (2.7) with $r_* = r_i$ when i is odd, and satisfies (2.8) with $r^* = r_i$ when i is even; or

(b) f satisfies (2.7) with $r_* = r_i$ when i is even, and satisfies (2.8) with $r^* = r_i$ when i is odd.

Then BVP (1.1), (1.2) has an infinite number of positive solutions.

The following is an immediate consequence of Theorem 2.7.

Corollary 2.1 *Let* $\{r_i\}_{i=1}^{\infty} \subset \mathbb{R}$ *such that* $0 < r_1 < r_2 < r_3 < \cdots$. *Let* $E_1 = \cup_{i=1}^{\infty}[\alpha r_{2i-1}, r_{2i-1}]$ *and* $E_2 = \cup_{i=1}^{\infty}[\alpha r_{2i}, r_{2i}]$. *Assume*

$$\limsup_{E_1 \ni x \to \infty} \max_{k \in \mathbb{N}(0,K)} \frac{f(k,x)}{x} < \beta^{-1} \quad \text{and} \quad \liminf_{E_2 \ni x \to \infty} \min_{k \in \mathbb{N}(0,K)} \frac{f(k,x)}{x} > (\alpha\beta)^{-1}.$$

Then BVP (1.1), (1.2) *has an infinite number of positive solutions.*

We observe that in the above theorems, if one of $f_0, f_\infty, f^0, f^\infty$ is involved and it is between β^{-1} and $(\alpha\beta)^{-1}$, then the corresponding conclusions fail. Motivated by the ideas in [6, 10], we employ the first eigenvalue of a Sturm-Liouville problem (SLP) associated with BVP (1.1), (1.2) and the topological homotopy invariance method to improve the criteria given in theorems 2.2-2.5.

Consider the SLP consisting of the equation

$$-\Delta^2 u(k) = \mu w(k) u(k+1), \ k \in \mathbb{N}(0,K), \tag{2.11}$$

and BC (1.2). Let μ_0 be its fist eigenvalue with an associated eigenfunction $v_0(k)$. It is known from assumption (H1) that $\mu_0 > 0$, $v_0(k) > 0$ on $\mathbb{N}(1, K+1)$, see [2, 8, 9, 11] for the detail. Now we have

Theorem 2.8 *BVP* (1.1), (1.2) *has at least one positive solution if* f *satisfies one of the following conditions:*

(a) $f^0 < \mu_0$ *and* $f_\infty > \mu_0$;

(b) $f_0 > \mu_0$ *and* $f^\infty < \mu_0$;

(c) $f_0 > \mu_0$ *and there exists* r_* *such that* (2.7) *holds;*

(d) $f_\infty > \mu_0$ *and there exists* r_* *such that* (2.7) *holds;*

(e) $f^0 < \mu_0$ *and there exists* r^* *such that* (2.8) *holds;*

(f) $f^\infty < \mu_0$ *and there exists* r^* *such that* (2.8) *holds.*

Theorem 2.9 *BVP* (1.1), (1.2) *has at least two positive solutions if either*

(a) $f_0 > \mu_0$, $f_\infty > \mu_0$, *and there exists* r *such that* (2.9) *holds; or*

(b) $f^0 < \mu_0$, $f^\infty < \mu_0$, *and there exists* r *such that* (2.10) *holds.*

Remark 2.1 We claim that

$$\beta^{-1} \le \mu_0 \le (\alpha\beta)^{-1}, \tag{2.12}$$

and hence Theorems 2.8 and 2.9 improve the results of Theorems 2.2-2.5. In fact, for $k \in \mathbb{N}(1, K+1)$ we have

$$v_0(k) = \mu_0 \sum_{l=0}^{K} G(k,l)w(l)v_0(l+1).$$

Let $k_1 \in \mathbb{N}(1, K+1)$ with $v_0(k_1) = \|v_0\|$. Then

$$\|v_0\| = \mu_0 \sum_{l=0}^{K} G(k_1,l)w(l)v_0(l+1) \le \mu_0\beta\|v_0\|. \tag{2.13}$$

On the other hand, we observe that $w(l)v_0(l+1) \not\equiv 0$ on $\mathbb{N}(0, K)$. For otherwise, every $\mu \in \mathbb{R}$ is an eigenvalue of SLP (2.11), (1.2). From (2.6) we have that for any $k \in \mathbb{N}(1, K+1)$

$$\begin{aligned} v_0(k) &\ge \mu_0 \sum_{l=0}^{K} \alpha G(l+1,l)w(l)v_0(l+1) \\ &\ge \alpha\mu_0 \sum_{l=0}^{K} G(k_1,l)w(l)v_0(l+1) = \alpha v_0(k_1) = \alpha\|v_0\|. \end{aligned}$$

Let $k_2 \in \mathbb{N}(1, K+1)$ satisfy

$$\sum_{l=0}^{K} G(k_2,l)w(l) = \beta. \tag{2.14}$$

Then

$$\|v_0\| \ge \mu_0 \sum_{l=0}^{K} G(k_2,l)w(l)v_0(l+1) \ge \mu_0\alpha\beta\|v_0\|. \tag{2.15}$$

The combination of (2.13) and (2.15) proves our claim.

Finally, we present a result on the nonexistence of positive solutions of BVP (1.1), (1.2).

Theorem 2.10 *BVP* (1.1), (1.2) *has no positive solutions if*

(a) $f(k,x)/x < \beta^{-1}$ *for all* $(k,x) \in \mathbb{N}(0,K) \times (0,\infty)$, *or*

(b) $f(k,x)/x > (\alpha\beta)^{-1}$ *for all* $(k,x) \in \mathbb{N}(0,K) \times (0,\infty)$.

3 Examples

In this section, we give several examples to demonstrate the applications of the criteria obtained in Section 2. For simplicity we choose $w(k) \equiv 1$ and $f(k, x) \equiv f(x)$ in all the examples.

Example 3.1 Let $f(x) = x^p$.

If $p > 1$, then $\lim_{x\to 0+} f(x)/x = 0$ and $\lim_{x\to\infty} f(x)/x = \infty$. By Theorem 2.2 (a), BVP (1.1), (1.2) has at least one positive solution.

If $0 < p < 1$, then $\lim_{x\to 0+} f(x)/x = \infty$ and $\lim_{x\to\infty} f(x)/x = 0$. By Theorem 2.2 (b), BVP (1.1), (1.2) has at least one positive solution.

Example 3.2 Let $f(x) = h(x^{p_1} + x^{p_2})$, where $0 < p_1 < 1 < p_2 < \infty$. Let $r = \left(\frac{1-p_1}{p_2-1}\right)^{1/(p_2-p_1)}$. Then

(a) BVP (1.1), (1.2) has at least one positive solution when $h = r(r^{p_1} + r^{p_2})^{-1}\beta^{-1}$;

(b) BVP (1.1), (1.2) has at least two positive solutions u_1 and u_2 with $\|u_1\| < r < \|u_2\|$ when $0 < h < r(r^{p_1} + r^{p_2})^{-1}\beta^{-1}$;

(c) BVP (1.1), (1.2) has no positive solutions when $h > r(r^{p_1} + r^{p_2})^{-1}(\alpha\beta)^{-1}$.

In fact, it is clear that $\lim_{x\to 0+} f(x)/x = \lim_{x\to\infty} f(x)/x = \infty$, $f(x)$ is strictly increasing, and r is the minimum point of $f(x)/x$ on $(0, \infty)$.

When $h = r(r^{p_1} + r^{p_2})^{-1}\beta^{-1}$, we have $f(x) \le f(r) = \beta^{-1}r$ for all $x \in [\alpha r, r]$. Then from Theorem 2.3 (a), BVP (1.1), (1.2) has a positive solution u_1 with $\|u_1\| \le r$. Similarly from Theorem 2.3 (b), BVP (1.1), (1.2) has a positive solution u_2 with $\|u_2\| \ge r$. However, u_1 and u_2 may be the same for the case when $\|u_1\| = \|u_2\| = r$.

When $0 < h < r(r^{p_1} + r^{p_2})^{-1}\beta^{-1}$, by a similar argument and from Theorem 2.5 (a), we obtain the conclusion.

When $h > r(r^{p_1} + r^{p_2})^{-1}(\alpha\beta)^{-1}$, $f(x)/x \ge f(r)/r > (\alpha\beta)^{-1}$ on $(0, \infty)$. Then the conclusion follows from Theorem 2.10 (b).

Example 3.3 Let

$$f(x) = \begin{cases} h/(x^{-p_1} + x^{-p_2}), & x > 0, \\ 0, & x = 0, \end{cases}$$

where $h > 0$, $p_1 > 1$ and $0 \le p_2 < 1$. Let $r = \left(\frac{p_1-1}{1-p_2}\right)^{1/(p_1-p_2)}$. Then

(a) BVP (1.1), (1.2) has at least one positive solution when $h = (r^{1-p_1}+r^{1-p_2})(\alpha\beta)^{-1}$;

(b) BVP (1.1), (1.2) has at least two positive solutions u_1 and u_2 with $\|u_1\| < r < \|u_2\|$ when $h > (r^{1-p_1} + r^{1-p_2})(\alpha\beta)^{-1}$;

(c) BVP (1.1), (1.2) has no positive solutions when $0 < h < (r^{1-p_1} + r^{1-p_2})\beta^{-1}$.

In fact, it is clear that $\lim_{x\to 0+} f(x)/x = \lim_{x\to\infty} f(x)/x = 0$, $f(x)$ is strictly increasing and r is the maximum point of $f(x)/x$ on $(0,\infty)$.

When $h = (r^{1-p_1} + r^{1-p_2})(\alpha\beta)^{-1}$, $f(r) = (\alpha\beta)^{-1}r$. Then $f(x) \ge f(r) = (\alpha\beta)^{-1}r$ on $[r, \alpha^{-1}r]$, i.e., $f(x) \ge \beta^{-1}r^*$ on $[\alpha r^*, r^*]$, where $r^* = \alpha^{-1}r$. By Theorem 2.4 (a) or (b), There exists at least one positive solution.

When $h > (r^{1-p_1} + r^{1-p_2})(\alpha\beta)^{-1}$, by a similar argument and from Theorem 2.5 (b), we obtain the conclusion.

When $0 < h < (r^{1-p_1} + r^{1-p_2})\beta^{-1}$, $f(x)/x \le f(r)/r < \beta^{-1}$ on $(0,\infty)$. Then the conclusion follows from Theorem 2.10 (a).

Example 3.4 Let

$$f(x) = \begin{cases} (\alpha^{-1}+1)\beta^{-1}x(\sin(h\ln x)+1)/2, & x > 0, \\ 0, & x = 0, \end{cases}$$

where $0 < h < (\pi - 2\sin^{-1}\delta)/\ln(\alpha^{-1})$ with $\delta = (\alpha^{-1}-1)/(\alpha^{-1}+1)$. We claim that BVP (1.1), (1.2) has an infinite number of positive solutions.

To show this, for $j = 2i+1$, $i \in \mathbb{N}$, let

$$\xi_j = \exp(h^{-1}(\sin^{-1}\delta + (j-1)\pi)),\ \eta_j = \exp(h^{-1}(j\pi - \sin^{-1}\delta)).$$

Then

$$\eta_j/\xi_j = \exp(h^{-1}(\pi - 2\sin^{-1}\delta)) > \exp(\ln(\alpha^{-1})) = \alpha^{-1},$$

hence $\xi_j < \alpha\eta_j$. Note that for $x \in [\alpha\eta_j, \eta_j] \subset [\xi_j, \eta_j]$, $\sin(h\ln x) \ge \sin(\sin^{-1}\delta) = \delta$. Therefore for $x \in [\alpha\eta_j, \eta_j]$,

$$f(x) \ge (\alpha^{-1}+1)\beta^{-1}\alpha\eta_j(\delta+1)/2 = \beta^{-1}\eta_j,$$

i.e., (2.8) holds with $r_* = \eta_j$.

For $j = 2i$, $i \in \mathbb{N}$, let

$$\xi_j = \exp(h^{-1}((j-1)\pi - \sin^{-1}\delta)),\ \eta_j = \exp(h^{-1}(j\pi + \sin^{-1}\delta)).$$

Then

$$\eta_j/\xi_j = \exp(h^{-1}(\pi + 2\sin^{-1}\delta)) > \exp(\ln(\alpha^{-1})) = \alpha^{-1},$$

hence $\xi_j < \alpha\eta_j$. Note that for $x \in [\alpha\eta_j, \eta_j] \subset [\xi_j, \eta_j]$, $\sin(h\ln x) \le -\delta$. Therefore for $x \in [\alpha\eta_j, \eta_j]$,

$$f(x) \le (\alpha^{-1}+1)\beta^{-1}\eta_j(-\delta+1)/2 = \beta^{-1}\eta_j,$$

i.e., (2.7) holds with $r^* = \eta_j$.

Therefore by Theorem 2.7, BVP (1.1), (1.2) has an infinite number of positive solutions.

4 Proofs

Let X be a Banach space, $\Omega \subset X$ a cone in X, and $\Gamma : X \to X$ a completely continuous operator. For $r > 0$, define

$$\Omega_r = \{u \in \Omega | \|u\| < r\} \quad \text{and} \quad \partial\Omega_r = \{u \in \Omega | \|u\| = r\}.$$

Let $\mathfrak{i}(\Gamma, \Omega_r, \Omega)$ be the fixed point index of Γ on Ω_r with respect to Ω. We will use the following well-known lemmas on fixed-point indices to prove our main results. For the detail of the fixed point index theory, see [5, 7].

Lemma 4.1 *Assume $\Gamma u \neq u$ for $u \in \partial\Omega_r$. Then*
(a) If $\|\Gamma u\| \geq \|u\|$ for $u \in \partial\Omega_r$, then $\mathfrak{i}(\Gamma, \Omega_r, \Omega) = 0$.
(b) If $\|\Gamma u\| \leq \|u\|$ for $u \in \partial\Omega_r$, then $\mathfrak{i}(\Gamma, \Omega_r, \Omega) = 1$.

Lemma 4.2 *Let $0 < r_1 < r_2$ satisfy*

$$\mathfrak{i}(\Gamma, \Omega_{r_1}, \Omega) = 0 \quad \textit{and}\ \mathfrak{i}(\Gamma, \Omega_{r_2}, \Omega) = 1;$$

or

$$\mathfrak{i}(\Gamma, \Omega_{r_1}, \Omega) = 1 \quad \textit{and}\ \mathfrak{i}(\Gamma, \Omega_{r_2}, \Omega) = 0.$$

Then Γ has a fixed point in $\bar{\Omega}_{r_2} \setminus \Omega_{r_1}$.

Define $D = \{u : \mathbb{N}(1, K+1) \to \mathbb{R}\}$, and let $\|u\| = \max_{k\in\mathbb{N}(1,K+1)} |u(k)|$ for $u \in D$. It is easy to see $(D, \|\cdot\|)$ is a Banach space. Let α be defined by (2.4). Define a cone Ω in D by

$$\Omega = \{u \in D \mid u(k) \geq 0,\ k \in \mathbb{N}(1, K+1) \text{ and } \min_{k\in\mathbb{N}(1,K+1)} u(k) \geq \alpha\|u\|\} \tag{4.1}$$

and an operator $\Gamma :\ D \to D$ by

$$\Gamma u = \sum_{l=0}^{K} G(k,l)w(l)f(l, u(l+1)), \quad k \in \mathbb{N}(1, K+1). \tag{4.2}$$

Lemma 4.3 $\Gamma(\Omega) \subset \Omega$ *and* Γ *is completely continuous.*

Proof: For any $u \in \Omega$, $\Gamma u \geq 0$ on $\mathbb{N}(1, K+1)$. By (2.6),

$$\begin{aligned} \min_{k\in\mathbb{N}(1,K+1)} (\Gamma u)(k) &= \min_{k\in\mathbb{N}(1,K+1)} \sum_{l=0}^{K} G(k,l)w(l)f(l, u(l+1)) \\ &\geq \alpha \sum_{l=0}^{K} G(l+1,l)w(l)f(l, u(l+1)) \\ &\geq \alpha \max_{k\in\mathbb{N}(1,K+1)} \sum_{l=0}^{K} G(k,l)w(l)f(l, u(l+1)) = \alpha\|\Gamma u\|. \end{aligned}$$

Therefore, $\Gamma(\Omega) \subset \Omega$. The complete continuity of Γ can be shown by a standard argument using the Arzela-Arscoli Theorem. We omit the details. ∎

Proof of Theorem 2.1. We observe that BVP (1.1), (1.2) has a positive solution $u(k)$, $k \in \mathbb{N}(0, K+2)$, if and only if the operator Γ defined by (4.2) has a positive fixed point $u(k)$, $k \in \mathbb{N}(1, K+1)$. In fact, the fixed point of Γ can be extended to $\mathbb{N}(0, K+2)$ so that BC (1.2) is satisfied. Therefore, it is enough to show that Γ has a positive fixed point.

For any $u \in \partial\Omega_{r_*}$, $\|u\| = r_*$ and $\alpha r_* \le u(k) \le r_*$ on $\mathbb{N}(1, K+1)$. Without loss of generality, we assume $\Gamma u \ne u$. For otherwise, $\Gamma u = u$ implies u is a positive fixed point. From (2.7), $f(k, u(k+1)) \le \beta^{-1} r_*$ on $\mathbb{N}(0, K)$. For any $k \in \mathbb{N}(1, K+1)$

$$\begin{aligned} (\Gamma u)(k) &= \sum_{l=0}^{K} G(k,l) w(l) f(l, u(l+1)) \\ \le\ & \beta^{-1} r_* \sum_{l=0}^{K} G(k,l) w(l) \le \beta^{-1} r_* \beta = r_* = \|u\|. \end{aligned}$$

Thus $\|\Gamma u\| \le \|u\|$. By Lemma 4.1 (b), $\mathfrak{i}(\Gamma, \Omega_{r_*}, \Omega) = 1$.

For any $u \in \partial\Omega_{r^*}$, $\|u\| = r^*$ and $\alpha r^* \le u(k) \le r^*$ on $\mathbb{N}(1, K+1)$. From (2.8), $f(k, u(k+1)) \ge \beta^{-1} r^*$ on $\mathbb{N}(0, K)$. Let $k_2 \in \mathbb{N}(1, K+1)$ be defined by (2.14). Then

$$\begin{aligned} (\Gamma u)(k_2) &= \sum_{l=0}^{K} G(k_2,l) w(l) f(l, u(l+1)) \\ \ge\ & \beta^{-1} r^* \sum_{l=0}^{K} G(k_2,l) w(l) = \beta\beta^{-1} r^* = \|u\|. \end{aligned}$$

Thus $\|\Gamma u\| \ge \|u\|$. By Lemma 4.1 (a), $\mathfrak{i}(\Gamma, \Omega_{r^*}, \Omega) = 0$.

If $r_* < r^*$, then by Lemma 4.2, Γ has a fixed point $u \in \bar{\Omega}_{r^*} \setminus \Omega_{r_*}$. Similarly, if $r_* > r^*$, then Γ has a fixed point $u \in \bar{\Omega}_{r_*} \setminus \Omega_{r^*}$. In each case, u is a positive function with $\min\{r_*, r^*\} \le \|u\| \le \max\{r_*, r^*\}$. ∎

Proof of Theorem 2.2. (a) If $f^0 < \beta^{-1}$, there exists $r_* > 0$ such that

$$f(k,x) < \beta^{-1} x \le \beta^{-1} r_*, \quad (k,x) \in \mathbb{N}(0,K) \times [0, r_*].$$

If $f_\infty > (\alpha\beta)^{-1}$, there exists $\hat{r} > r_*$ such that

$$f(k,x) > (\alpha\beta)^{-1} x, \quad (k,x) \in \mathbb{N}(0,K) \times [\hat{r}, \infty).$$

Then for any r^* with $\alpha r^* \ge \hat{r}$

$$f(k,x) > (\alpha\beta)^{-1} x \ge \beta^{-1} r^* \ \text{ for all } (k,x) \in \mathbb{N}(0,K) \times [\alpha r^*, r^*].$$

Then the conclusion follows from Theorem 2.1.

(b) The proof is similar to Part (a) and hence is omitted. ∎

The proofs of Theorems 2.3 and 2.4 are in the same way and hence are omitted.

Proof of Theorem 2.5. (a) If there exists $r > 0$ such that (2.9) holds, then there exist r_1 and r_2 such that $r_1 < r < r_2$ and $f(k,x) < \beta^{-1}x$ for all $(k,x) \in \mathbb{N}(0,K) \times [\alpha r_i, r_i]$, $i = 1,2$. By Theorem 2.3 (a) and (b), BVP (1.1), (1.2) has two positive solutions u_1 and u_2 satisfying $\|u_1\| \le r_1$ and $\|u_2\| \ge r_2$.

Similarly, case (b) follows from Theorem 2.4. ∎

The proofs of Theorems 2.6 and 2.7 are in the same way and are hence omitted.

Proof of Corollary 2.1. From the assumption we see that for sufficiently large i

$$\frac{f(k,x)}{x} < \beta^{-1} \text{ for all } (k,x) \in \mathbb{N}(0,K) \times [\alpha r_{2i-1}, r_{2i-1}]$$

and

$$\frac{f(k,x)}{x} > (\alpha\beta)^{-1} \text{ for all } (k,x) \in \mathbb{N}(0,K) \times [\alpha r_{2i}, r_{2i}].$$

This shows that for sufficiently large i,

$$f(k,x) < \beta^{-1}x \le \beta^{-1}r_{2i-1}$$

for all $(k,x) \in \mathbb{N}(0,K) \times [\alpha r_{2i-1}, r_{2i-1}]$ and

$$f(k,x) > (\alpha\beta)^{-1}x \ge (\alpha\beta)^{-1}\alpha r_{2i} = \beta^{-1}r_{2i}$$

for all $(k,x) \in \mathbb{N}(0,K) \times [\alpha r_{2i}, r_{2i}]$.

Therefore, the conclusion follows from Theorem 2.7. ∎

To prove Theorems 2.8 and 2.9, we need the following lemma.

Lemma 4.4 *(a) Assume $f_0 > \mu_0$. Then $\mathfrak{i}(\Gamma, \Omega_r, \Omega) = 0$ for all sufficiently small $r > 0$.*

(b) Assume $f_\infty > \mu_0$. Then $\mathfrak{i}(\Gamma, \Omega_r, \Omega) = 0$ for all sufficiently large $r > 0$.

(c) Assume $f^0 < \mu_0$. Then $\mathfrak{i}(\Gamma, \Omega_r, \Omega) = 1$ for all sufficiently small $r > 0$.

(d) Assume $f^\infty < \mu_0$. Then $\mathfrak{i}(\Gamma, \Omega_r, \Omega) = 1$ for all sufficiently large $r > 0$.

Proof. In this proof, we will use the following integration by parts formulas given in [1, 4]

$$\sum_{k=a}^{b} f(k+1)\Delta g(k) = f(k)g(k)|_a^{b+1} - \sum_{k=a}^{b} \Delta f(k)g(k) \tag{4.3}$$

and

$$\sum_{k=a}^{b} f(k)\Delta g(k) = f(k)g(k)|_a^{b+1} - \sum_{k=a}^{b} \Delta f(k)g(k+1). \tag{4.4}$$

(a) Let $0 < p < 1$ be fixed and define $\Gamma_1 : D \to D$ by

$$(\Gamma_1 u)(k) = \sum_{l=0}^{K} G(k,l)w(l)u^p(l+1).$$

Similar to the proof of Lemma 4.3, we can show that Γ_1 is compact and $\Gamma_1\Omega \subset \Omega$. Define $r_1 = (\alpha^{p+1}\sum_{l=0}^{K} G(l+1,l)w(l))^{1/(1-p)}$. Then for $0 < r \le r_1$ and $u \in \partial\Omega_r$, $u^p(l) \ge (\alpha r)^p$ on $\mathbb{N}(1, K+1)$ and hence

$$\begin{aligned} \|\Gamma_1 u\| = & \max_{k\in\mathbb{N}(1,K+1)} (\Gamma_1 u)(k) = \max_{k\in\mathbb{N}(1,K+1)} \sum_{l=0}^{K} G(k,l)w(l)u^p(l+1) \\ \ge & \ \alpha \sum_{l=0}^{K} G(l+1,l)w(l)(\alpha r)^p = \alpha(\alpha r)^p \sum_{l=0}^{K} G(l+1,l)w(l) \\ = & \ \alpha(\alpha r_1)^p (r/r_1)^p \sum_{l=0}^{K} G(l+1,l)w(l) \ge r_1 r/r_1 = r = \|u\|. \end{aligned} \tag{4.5}$$

By Lemma 4.1 (a), $\mathfrak{i}(\Gamma_1, \Omega_r, \Omega) = 0$.

Define a homotopy operator $H : [0,1] \times \Omega \to \Omega$ by

$$H(s,u) = (1-s)\Gamma u + s\Gamma_1 u.$$

Then $H(s,\cdot)$ is compact for $0 \le s \le 1$. Since $f_0 > \mu_0$, we can choose $\varepsilon > 0$ and $0 < r_2 \le r_1$ such that for $(k,x) \in \mathbb{N}(0,K) \times [0, r_2]$

$$f(k,x) \ \ge \ (\mu_0 + \varepsilon)x \ \text{ and } \ x^p \ge (\mu_0 + \varepsilon)x.$$

Let $0 < r \le r_2$. We now show that $H(s,u) \ne u$ for all $0 \le s \le 1$ and $u \in K \cap \partial\Omega_r$. Assume the contrary, i.e., there exists $s_1 \in [0,1]$ and $u_1 \in \partial\Omega_r$ with $H(s_1, u_1) = u_1$. Then u_1 satisfies

$$-\Delta^2 u_1(k) = (1-s_1)w(k)f(k, u_1(k+1)) + s_1 w(k) u_1^p(w+1)$$

and BC (1.2). Hence

$$\begin{aligned}
&\sum_{l=0}^{K}\left[-\Delta^2 u_1(l)v_0(l+1)\right]\\
=\ &\sum_{l=0}^{K} w(l)\left[(1-s_1)f(l,u_1(l+1))v_0(l+1)+s_1u_1^p(l+1)v_0(l+1)\right], \qquad (4.6)
\end{aligned}$$

where v_0 is the eigenfunction of SLP (2.11), (1.2) associated to the eigenvalue μ_0. Using the integration by parts formulas (4.3) and (4.4) and BC (1.2), we have

$$\sum_{l=0}^{K}[-\Delta^2 u_1(l)v_0(l+1)] = \mu_0\sum_{l=0}^{K} w(l)u_1(l+1)v_0(l+1). \qquad (4.7)$$

Combining (4.6) and (4.7) we obtain

$$\begin{aligned}
&\mu_0\sum_{l=0}^{K} w(l)u_1(l+1)v_0(l+1)\\
=\ &\sum_{l=0}^{K} w(l)[(1-s_1)f(l,u_1(l+1))v_0(l+1)+s_1u_1^p(l+1)v_0(l+1)]\\
\geq\ &(\mu_0+\varepsilon)\sum_{l=0}^{K} w(l)[(1-s_1)u_1(l+1)v_0(l+1)+s_1u_1(l+1)v_0(l+1)]\\
=\ &(\mu_0+\varepsilon)\sum_{l=0}^{K} w(l)u_1(l+1)v_0(l+1),
\end{aligned}$$

which is a contradiction since u_1 and v_0 are positive on $\mathbb{N}(1,K+1)$ and $u_1(0)\geq 0$ which is implied by (H1). Hence by (4.5)

$$\mathfrak{i}(\Gamma,\Omega_r,\Omega)=\mathfrak{i}(H(0,\cdot),\Omega_r,\Omega)=\mathfrak{i}(H(1,\cdot),\Omega_r,\Omega)=\mathfrak{i}(\Gamma_1,\Omega_r,\Omega)=0.$$

The proofs of Parts (b), (c) and (d) are similar to Part (a) and hence are omitted. ∎

Proof of Theorem 2.8. We only prove case (a). The rest can be proved similarly.

By Lemma 4.4, $f^0<\mu_0$ implies $\mathfrak{i}(\Gamma,\Omega_r,\Omega)=1$ when $r>0$ small enough. $f_\infty>\mu_0$ implies $\mathfrak{i}(\Gamma,\Omega_R,\Omega)=0$ when $R>r$ large enough. Therefore, by Lemma 4.2, Γ has a positive fixed point and hence BVP (1.1), (1.2) has a positive solution. ∎

The proof of Theorem 2.9 is similar and hence is omitted.

Proof of Theorem 2.10. (a) Assume BVP (1.1), (1.2) has a positive solution u with $\|u\|=r$ for some $r>0$. Then u is a fixed point of the operator Γ defined by (4.2).

From the assumption, $f(k,u(k+1)) < \beta^{-1}u(k) \le \beta^{-1}r$ on $\mathbb{N}(0,K)$. Thus for any $k \in \mathbb{N}(1,K+1)$

$$\begin{aligned} u(k) &= (\Gamma u)(k) = \sum_{l=0}^{K} G(k,l)w(l)f(l,u(l+1)) \\ &< \beta^{-1}r\sum_{l=0}^{K} G(k,l)w(l) \le r \end{aligned}$$

which contradicts that $\|u\| = r$. Therefore, BVP (1.1), (1.2) has no positive solutions.

(b) Assume BVP (1.1), (1.2) has a positive solution u with $\|u\| = r$. Similar to the proof of Lemma 4.3 we can show that $\Gamma u \in \Omega$ and hence $\alpha r \le u(k) \le r$ on $\mathbb{N}(1,K+1)$. From the assumption, $f(k,u(k+1)) > (\alpha\beta)^{-1}u(k) \ge \beta^{-1}r$ on $\mathbb{N}(0,K)$. Let $k_2 \in \mathbb{N}(1,K+1)$ be defined as in (2.14). Then

$$\begin{aligned} u(k_2) &= \Gamma u(k_2) = \sum_{l=0}^{K} G(k_2,l)w(l)f(l,u(l+1)) \\ &> \beta^{-1}r\sum_{l=0}^{K} G(k_2,l)w(l) = r \end{aligned}$$

which contradicts that $\|u\| = r$. Therefore, BVP (1.1), (1.2) has no positive solutions. ∎

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