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BOUNDARY VALUE PROBLEMS FOR THIRD ORDER DIFFERENTIAL EQUATIONS BY SOLUTION MATCHING

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

For the ordinary differential equation, $y''' = f(x, y, y', y'')$, solutions of 3-point boundary value problems on $[a, b]$ are matched with solutions of 3-point boundary value problems on $[b, c]$ to obtain solutions satisfying 5-point boundary conditions on $[a, c]$.

Key words and phrases: Boundary value problem, ordinary differential equation, solution matching.
AMS (MOS) Subject Classifications: 34B15, 34B10

1 Introduction

We are concerned with the existence and uniqueness of solutions of boundary value problems on an interval $[a, c]$ for the third order ordinary differential equation,

$$y''' = f(x, y, y', y''), \tag{1}$$

satisfying the 5-point boundary conditions,

$$y(a) - y(x_1) = y_1, \quad y(b) = y_2, \quad y'(x_2) - y'(c) = y_3, \tag{2}$$

where $a < x_1 < b < x_2 < c$ and $y_1, y_2, y_3 \in \mathbb{R}$.

It is assumed throughout that $f : [a, c] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and that solutions of initial value problems for (1) are unique and exist on all of $[a, c]$. Moreover, the points $a < x_1 < b < x_2 < c$ are fixed throughout.

Nonlocal boundary value problems, for which the number of boundary points is greater than the order of the ordinary differential equation, have received considerable interest. For a small sample of such works, we refer the reader to works by Bai and Fang [1], Gupta [8], Gupta and Trofimchuk [9], Infante [13], Ma [14, 15] and Webb [18].

Monotonicity conditions will be imposed on f and sufficient conditions will be given such that for certain solutions of 3-point boundary value problems, $y_1(x)$ on $[a,b]$ and $y_2(x)$ on $[b,c]$, then $y(x)$ defined by

$$y(x) = \begin{cases} y_1(x), & a \le x \le b, \\ y_2(x), & b \le x \le c, \end{cases}$$

will be a desired unique solution of (1), (2). In particular, a monotonicity condition is imposed on $f(x,y,z,w)$ insuring that 3-point boundary value problems for (1) satisfying any one of

$$y(a) - y(x_1) = y_1,\ y(b) = y_2,\ y'(b) = m, \quad m \in \mathbb{R}, \tag{3}$$

$$y(a) - y(x_1) = y_1,\ y(b) = y_2,\ y''(b) = m, \quad m \in \mathbb{R}, \tag{4}$$

$$y(b) = y_2,\ y'(b) = m,\ y'(x_2) - y'(c) = y_3, \quad m \in \mathbb{R}, \tag{5}$$

or

$$y(b) = y_2,\ y''(b) = m,\ y'(x_2) - y'(c) = y_3, \quad m \in \mathbb{R}, \tag{6}$$

has at most one solution.

With the added hypothesis that solutions exist to boundary value problems for (1) satisfying any of (3), (4), (5) or (6), a unique solution of (1), (2) is then constructed.

Solution matching techniques were first used by Bailey, Shampine, and Waltman [2] where they dealt with solutions of 2-point boundary value problems for the second order equation $y''(x) = f(x, y(x), y'(x))$ by matching solutions of initial value problems. Since then, there have been numerous papers in which solutions of 2-point boundary value problems on $[a,b]$ were matched with solutions of 2-point boundary value problems on $[b,c]$ to obtain solutions of 3-point boundary value problems on $[a,c]$; see, for example [3, 5, 10, 16, 17]. In 1973, Barr and Sherman [4] used solution matching techniques to obtain solutions of 3-point boundary value problems for third order differential equations from solutions of 2-point problems, and they also generalized their results to equations of arbitrary order obtaining solutions of nth order equations. More recently, Eggensperger *et al.* [6] and Henderson and Prasad [11] used matching methods for solutions of multipoint boundary value problems on time scales. Finally, Erke *et al.* [7] and Henderson and Tisdell [12] employed matching to obtain solutions of other 5-point problems for higher order differential equations. In this paper, we will adapt this matching method to obtain solutions of the 5-point boundary value problems (1), (2) on $[a,c]$.

The monotonicity hypothesis on f which will play a fundamental role in uniqueness of solutions (and later existence as well), is given by:

(A) For all $w \in \mathbb{R}$,

$$f(x, v_1, v_2, w) > f(x, u_1, u_2, w),$$

(a) when $x \in (a,b], u_1 \ge v_1$ and $v_2 > u_2$, and

(b) when $x \in [b,c), u_1 \le v_1$ and $v_2 > u_2$.

2 Uniqueness of Solutions

In this section, we establish that under condition (A), solutions of the 3-point boundary value problems, as well as the 5-point problem, of this paper are unique, when they exist.

Theorem 2.1 *Let $y_1, y_2, y_3 \in \mathbb{R}$ be given and assume condition* (A) *is satisfied. Then, given $m \in \mathbb{R}$, each of the boundary value problems for* (1) *satisfying any of* (3), (4), (5) *or* (6) *has at most one solution.*

Proof. We will establish the results for (1) satisfying each of (3), (5), and (6). Arguments for (1), (4) are quite similar.

First, we establish the result for (1), (3). For the sake of contradiction, assume for some $m \in \mathbb{R}$, there are distinct solutions, p and q, of (1), (3), and set $w = p - q$. Then

$$w(a) - w(x_1) = w(b) = w'(b) = 0.$$

By uniqueness of solutions of initial value problems for (1), we may assume with no loss of generality that $w''(b) < 0$. It follows that there exists $a < r < b$ such that

$$w''(r) = 0 \text{ and } w''(x) < 0 \text{ on } (r, b].$$

This implies in turn that

$$w(x) < 0 \text{ and } w'(x) > 0 \text{ on } [r, b).$$

This leads to

$$w'''(r) = \lim_{x \to r^+} \frac{w''(x)}{x - r} \leq 0.$$

However, from condition (A),

$$\begin{aligned} w'''(r) &= p'''(r) - q'''(r) \\ &= f(r, p(r), p'(r), p''(r)) - f(r, q(r), q'(r), q''(r)) \\ &= f(r, p(r), p'(r), p''(r)) - f(r, q(r), q'(r), p''(r)) \\ &> 0, \end{aligned}$$

which is a contradiction. Thus, (1), (3) has at most one solution.

Now, we deal with uniqueness of solutions of (1), (5). Again, for contradiction purposes, assume for some $m \in \mathbb{R}$, there are distinct solutions, ρ and σ, of (1), (5), and set $z = \rho - \sigma$. Then,

$$z(b) = z'(b) = z'(x_2) - z'(c) = 0.$$

As before, by uniqueness of solutions of initial value problems for (1), we may assume with no loss of generality that $z''(b) < 0$. Then, there exists $b < r < c$ such that

$$z''(r) = 0 \text{ and } z''(x) < 0 \text{ on } [b, r).$$

It follows that

$$z(x) < 0 \text{ and } z'(x) < 0 \text{ on } (b, r].$$

This yields that

$$z'''(r) = \lim_{x \to r^-} \frac{z''(x)}{x - r} \geq 0.$$

But, by condition (A),

$$\begin{aligned} z'''(r) &= \rho'''(r) - \sigma'''(r) \\ &= f(r, \rho(r), \rho'(r), \rho''(r)) - f(r, \sigma(r), \sigma'(r), \sigma''(r)) \\ &= f(r, \rho(r), \rho'(r), \rho''(r)) - f(r, \sigma(r), \sigma'(r), \rho''(r)) \\ &< 0, \end{aligned}$$

which is again a contradiction. Thus, (1), (5) has at most one solution.

Finally, we address uniqueness for solutions of (1), (6). As in the pattern, assume there exists an $m \in \mathbb{R}$ for which there are distinct solutions, h and k, of (1), (6), and set $l = h - k$. Then,

$$l(b) = l''(b) = l'(x_2) - l'(c) = 0.$$

By uniqueness of solutions of initial value problems for (1), we may assume that $l'(b) < 0$; in particular, we have

$$h(b) = k(b), \; h''(b) = k''(b), \text{ and } h'(b) < k'(b).$$

Now, for each $\delta > 0$, let $k_\delta(x)$ be the solution of (1) satisfying the initial conditions

$$k_\delta(b) = k(b), k_\delta'(b) = k'(b), \text{ and } k_\delta''(b) = k''(b) + \delta.$$

By continuous dependence of solutions of (1) on initial conditions, it follows that $\lim_{\delta \to 0} k_\delta^{(i)}(x) = k^{(i)}(x)$ uniformly on $[b, c]$, for each $i = 0, 1, 2$. Next, we set $l_\delta = h - k_\delta$. Then,

$$l_\delta(b) = 0, \; l_\delta'(b) < 0, \text{ and } l_\delta''(b) = -\delta < 0,$$

and $\lim_{\delta \to 0} l_\delta^{(i)}(x) = l^{(i)}(x)$ uniformly on $[b, c]$, for each $i = 0, 1, 2$. It follows that, for δ sufficiently small, there exists $b < r < c$ such that,

$$l_\delta''(r) = 0 \text{ and } l_\delta''(x) < 0 \text{ on } [b, r).$$

In turn, then

$$l_\delta(x) < 0 \text{ and } l_\delta'(x) < 0 \text{ on } (b, r].$$

This yields that

$$l_\delta'''(r) = \lim_{x \to r^-} \frac{l_\delta''(x)}{x - r} \geq 0,$$

whereas, from condition (A),

$$\begin{aligned} l_\delta'''(r) &= h'''(r) - k_\delta'''(r) \\ &= f(r, h(r), h'(r), h''(r)) - f(r, k_\delta(r), k_\delta'(r), k_\delta''(r)) \\ &= f(r, h(r), h'(r), h''(r)) - f(r, k_\delta(r), k_\delta'(r), h''(r)) \\ &< 0, \end{aligned}$$

again a contradiction. Thus, (1), (6) also has at most one solution.

The proof is complete.

Theorem 2.2 *Let $y_1, y_2, y_3 \in \mathbb{R}$ be given and assume condition* (A) *is satisfied. Then, the boundary value problem* (1), (2) *has at most one solution.*

Proof. Again, we argue by contradiction. Assume for some values $y_1, y_2, y_3 \in \mathbb{R}$, there are distinct solutions, p and q, of (1), (2), and let $w = p - q$. Then

$$w(a) - w(x_1) = w(b) = w'(x_2) - w'(c) = 0.$$

By Theorem 2.1, $w'(b) \neq 0$, and $w''(b) \neq 0$. We assume with no loss of generality that $w'(b) > 0$. Then, there are points $a < r_1 < b < r_2 \leq c$ so that

$$w'(r_1) = 0 \text{ and } w'(x) > 0 \text{ on } (r_1, r_2).$$

There are two cases to analyze; that is, $w''(b) > 0$ or $w''(b) < 0$.

We will first treat the case $w''(b) > 0$. In view of the fact that $w'(b) > 0$, there exists $b < r < c$ so that

$$w''(r) = 0 \text{ and } w''(x) > 0 \text{ on } [b, r).$$

Then

$$w(x) > 0 \text{ and } w'(x) > 0 \text{ on } (b, r].$$

This leads to

$$w'''(r) = \lim_{x \to r^-} \frac{w''(x)}{x - r} \leq 0.$$

However, from condition (A) again,

$$\begin{aligned} w'''(r) &= p'''(r) - q'''(r) \\ &= f(r, p(r), p'(r), p''(r)) - f(r, q(r), q'(r), q''(r)) \\ &= f(r, p(r), p'(r), p''(r)) - f(r, q(r), q'(r), p''(r)) \\ &> 0, \end{aligned}$$

which is a contradiction.

Now, we deal with the case $w''(b) < 0$. It follows that there exists $r_1 < r < b$ such that

$$w''(r) = 0 \text{ and } w''(x) < 0 \text{ on } (r, b].$$

In turn, then

$$w(x) < 0 \text{ and } w'(x) > 0 \text{ on } (r, b].$$

But, then we have both

$$w'''(r) = \lim_{x \to r^+} \frac{w''(x)}{x - r} \leq 0,$$

and

$$\begin{aligned} w'''(r) &= p'''(r) - q'''(r) \\ &= f(r, p(r), p'(r), p''(r)) - f(r, q(r), q'(r), q''(r)) \\ &= f(r, p(r), p'(r), p''(r)) - f(r, q(r), q'(r), p''(r)) \\ &> 0, \end{aligned}$$

giving the usual contradiction.

Thus, (1), (2) has at most one solution, and the proof is complete.

3 Existence of Solutions

In this section, we establish monotonicity of higher order derivatives, as functions of m, of solutions of (1) satisfying each of (3), (4), (5) and (6). We use these monotonicity properties then to obtain solutions of (1), (2).

For notation purposes, given $m \in \mathbb{R}$, let $\alpha(x, m), u(x, m), \beta(x, m)$ and $v(x, m)$ denote the solutions, when they exist, of the boundary value problems for (1) satisfying, respectively, (3), (4), (5) and (6).

Theorem 3.1 *Suppose* (A) *is satisfied and that, for each* $m \in \mathbb{R}$, *there exist solutions of* (1) *satisfying each of* (3), (4), (5) *and* (6). *Then,* $\alpha''(b, m)$ *and* $\beta''(b, m)$ *are, respectively, strictly increasing and decreasing functions of* m *with ranges all of* $\mathbb{R}$.

Proof: The "strictness" of the conclusion arises from Theorem 2.1. Let $m_1 > m_2$ and let

$$w(x) = \beta(x, m_1) - \beta(x, m_2).$$

Then,

$$w(b) = 0,\ w'(b) = m_1 - m_2 > 0,\ w'(x_2) - w'(c) = 0, \text{ and } w''(b) \neq 0.$$

Contrary to the conclusion, assume $w''(b) > 0$. It follows that there exists $b < r < c$ such that

$$w''(r) = 0 \text{ and } w''(x) > 0 \text{ on } [b, r).$$

We also have,

$$w(x) > 0 \text{ and } w'(x) > 0 \text{ on } (b, r].$$

As in the other proofs above, we arrive at the same contradiction, $w'''(r) \leq 0$ and $w'''(r) > 0$. Thus, $w''(b) < 0$ and as a consequence, $\beta''(b, m)$ is a strictly decreasing function of m.

We next argue that $\{\beta''(b, m) \mid m \in \mathbb{R}\} = \mathbb{R}$. So, let $k \in \mathbb{R}$, and consider the solution, $v(x, k)$ of (1), (6), with v as defined above. Consider also the solution $\beta(x, v'(b, k))$ of (1), (5). Then $\beta(x, v'(b, k))$ and $v(x, k)$ are solutions of the same type boundary value problem (1), (5), and hence by Theorem 2.1, the functions are identical. Therefore,

$$\beta''(b, v'(b, k)) = v''(b, k) = k,$$

and the range of $\beta''(b, m)$, as a function of m, is the set of real numbers.

The argument for $\alpha''(b, m)$ is somewhat similar. This completes the proof.

In a similar way, we also have a monotonicity result on first order derivatives of $u(x, m)$ and $v(x, m)$.

Theorem 3.2 *Assume the hypotheses of Theorem 3.1. Then, $u'(b, m)$ and $v'(b, m)$ are, respectively, strictly increasing and decreasing functions of m with ranges all of $\mathbb{R}$.*

We now provide our existence result.

Theorem 3.3 *Assume the hypotheses of Theorem 3.1. Then* (1), (2) *has a unique solution.*

Proof. The existence is immediate from either Theorem 3.1 or Theorem 3.2. Making use of Theorem 3.1, there exists a unique $m_0 \in \mathbb{R}$ such that $\alpha''(b, m_0) = \beta''(b, m_0)$. Then

$$y(x) = \begin{cases} \alpha(x, m_0), & a \leq x \leq b, \\ \beta(x, m_0), & b \leq x \leq c, \end{cases}$$

is a solution of (1), (2), and by Theorem 2.2, $y(x)$ is the unique solution. The proof is complete.

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