

EXISTENCE OF PERIODIC SOLUTIONS FOR A CLASS OF EVEN ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

Chengjun Guo¹, Donal O'Regan², Yuantong Xu³ and Ravi P. Agarwal⁴

¹Department of Applied Mathematics, Guangdong University of Technology
510006, P. R. China

²Department of Mathematics, National University of Ireland, Galway, Ireland
e-mail: donal.oregan@nuigalway.ie

³Department of Mathematics, Sun Yat-sen University
Guangzhou Guangdong 510275, P. R. China

⁴Department of Mathematical Sciences, Florida Institute of Technology
Melbourne, Florida 32901, USA
e-mail: agarwal@fit.edu

Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

Using Mawhin's continuation theorem we establish the existence of periodic solutions for a class of even order differential equations with deviating argument.

Key words and phrases: Even order differential equation, deviating argument, Mawhin's continuation theorem, Green's function.

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1 Introduction

In this paper, we discuss the even order differential equation with deviating argument of the form

$$x^{(2n)}(t) + \sum_{i=0}^{2n-2} a_i(t)x^{(i)}(t) + g(x(t - \tau(t))) = p(t), \quad (1)$$

where $\tau(t)$, $a_i(t)$ ($i = 0, 1, 2, \dots, n$), $p(t)$ are real continuous functions defined on \mathbf{R} with positive period T and $a_{2k-2}(t) > 0$ ($k = 1, 2, \dots, n$) for $t \in \mathbf{R}$, and $g(x)$ is a real continuous function defined on \mathbf{R} .

Periodic solutions for differential equations were studied in [2-12] and we note that most of the results in the literature concern lower order problems. There are only a few papers [1,13,14] which discuss higher order problems.

For the sake of completeness, we first state Mawhin's continuation theorem [3]. Let X and Y be two Banach space and $L : DomL \subset X \rightarrow Y$ is a linear mapping and

$N : X \longrightarrow Y$ is a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \text{Ker} L = \text{codim} \text{Im} L < +\infty$, and $\text{Im} L$ is closed in Y . If L is a Fredholm mapping of index zero, there exist continuous projectors $P : X \longrightarrow X$ and $Q : Y \longrightarrow Y$ such that $\text{Im} P = \text{Ker} L$ and $\text{Im} L = \text{Ker} Q = \text{Im}(I - Q)$. It follows that $L|_{\text{Dom} L \cap \text{Ker} P} : (I - P)X \longrightarrow \text{Im} L$ has an inverse which will be denoted by K_P . If Ω is an open and bounded subset of X , the mapping N will be called L -compact on Ω if $QN(\overline{\Omega})$ is bounded and $\overline{K_P(I - Q)N(\overline{\Omega})}$ is compact. Since $\text{Im} Q$ is isomorphic to $\text{Ker} L$, there exists an isomorphism $J : \text{Im} Q \longrightarrow \text{Ker} L$. The following theorem is called Mawhin's continuation theorem (see [3]).

Theorem 1.1 *Let L be a Fredholm mapping of index zero, and let N be L -compact on $\overline{\Omega}$. Suppose*

- (1) *for each $\lambda \in (0, 1)$ and $x \in \partial\Omega$, $Lx \neq \lambda Nx$, and*
- (2) *for each $x \in \partial\Omega \cap \text{Ker}(L)$, $QNx \neq 0$ and $\deg(QN, \Omega \cap \text{Ker}(L), 0) \neq 0$.*

Then the equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap D(L)$.

2 Main Result

Now we make the following assumptions on $a_i(t)$:

- (i) $M_{2k-2} = \max_{t \in [0, T]} a_{2k-2}(t) \geq a_{2k-2}(t) \geq m_{2k-2} = \min_{t \in [0, T]} a_{2k-2}(t) > 0$, ($k = 1, 2, \dots, n$) for each $t \in [0, T]$;
- (ii) $M_{2n-2} < (\frac{\pi}{T})^2$ and $\frac{M_{2n-2i}}{M_{2n-2i+2}} < (\frac{\pi}{T})^2$ ($i = 2, 3, \dots, n$);
- (iii) There exists a positive constant r with $m_0 > r$, such that with $A - \frac{2M_0 + m_0 + r}{2(m_0 - r)} B > 0$ and $1 - A^* > 0$, where $A = 1 - A^*$,

$$B = M_1 \left(\frac{T}{2}\right)^{2n-2} + (M_2 - m_2) \left(\frac{T}{2}\right)^{2n-3} + M_3 \left(\frac{T}{2}\right)^{2n-4} + (M_4 - m_4) \left(\frac{T}{2}\right)^{2n-5} \\ + \dots + M_{2n-3} \left(\frac{T}{2}\right)^2 + (M_{2n-2} - m_{2n-2}) \frac{T}{2},$$

$$A^* = [M_{2n-2} \left(\frac{T}{2}\right)^2 + M_{2n-3} \left(\frac{T}{2}\right)^3 + M_{2n-4} \left(\frac{T}{2}\right)^4 + \dots + M_2 \left(\frac{T}{2}\right)^{2n-2} + M_1 \left(\frac{T}{2}\right)^{2n-1}]$$

and $M_{2k-1} = \max_{t \in [0, T]} |a_{2k-1}(t)|$ ($k = 1, 2, \dots, n - 1$).

Our main result is the following theorem.

Theorem 2.1 *Under the assumptions (i), (ii) and (iii), if*

$$\lim_{|x| \rightarrow \infty} \sup \left| \frac{g(x)}{x} \right| \leq r \tag{2}$$

and

$$\lim_{|x| \rightarrow \infty} \text{sgn}(x)g(x) = +\infty, \tag{3}$$

then Eq.(1) has at least one T -periodic solution.

In order to prove the main theorem we need some preliminaries. Set

$$X := \{x|x \in C^{2n-1}(\mathbf{R}, \mathbf{R}), x(t+T) = x(t), \forall t \in \mathbf{R}\}$$

and $x^{(0)}(t) = x(t)$, and define the norm on X by

$$\|x\| = \max_{0 \leq j \leq 2n-1} \max_{t \in [0, T]} |x^{(j)}(t)|,$$

and set

$$Y := \{y|y \in C(\mathbf{R}, \mathbf{R}), y(t+T) = y(t), \forall t \in \mathbf{R}\}.$$

We define the norm on Y by $\|y\|_0 = \max_{t \in [0, T]} |y(t)|$. Thus both $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_0)$ are Banach spaces.

Remark 2.1 *If $x \in X$, then it follows that $x^{(i)}(0) = x^{(i)}(T)$ ($i = 0, 1, 2, \dots, 2n - 1$).*

Define the operators $L : X \longrightarrow Y$ and $N : X \longrightarrow Y$, respectively, by

$$Lx(t) = x^{(2n)}(t), \quad t \in \mathbf{R}, \tag{4}$$

and

$$Nx(t) = p(t) - \sum_{i=0}^{2n-2} a_i(t)x^{(i)}(t) - g(x(t - \tau(t))), t \in \mathbf{R}. \tag{5}$$

Clearly,

$$KerL = \{x \in X : x(t) = c \in \mathbf{R}\} \tag{6}$$

and

$$ImL = \{y \in Y : \int_0^T y(t)dt = 0\} \tag{7}$$

is closed in Y . Thus L is a Fredholm mapping of index zero.

Let us define $P : X \rightarrow X$ and $Q : Y \rightarrow Y/Im(L)$, respectively, by

$$Px(t) = \frac{1}{T} \int_0^T x(t)dt = x(0), \quad t \in \mathbf{R}, \tag{8}$$

for $x = x(t) \in X$ and

$$Qy(t) = \frac{1}{T} \int_0^T y(t)dt, \quad t \in \mathbf{R} \tag{9}$$

for $y = y(t) \in Y$. It is easy to see that $ImP = KerL$ and $ImL = KerQ = Im(I - Q)$. It follows that $L|_{DomL \cap KerP} : (I - P)X \longrightarrow ImL$ has an inverse which will be denoted by K_P .

Furthermore for any $y = y(t) \in ImL$, if $n = 1$, it is well-known that

$$K_P y(t) = -\frac{t}{T} \int_0^T du \int_0^u y(s)ds + \int_0^t du \int_0^u y(s)ds. \tag{10}$$

If $n > 1$, let $x(t) \in \text{Dom}L \cap \text{Ker}P$ be such that $K_P y(t) = x(t)$. Then $x^{(2n)}(t) = y(t)$,

$$x^{(2n-1)}(t) = x^{(2n-1)}(0) + \int_0^t x^{(2n)}(s) ds \quad (11)$$

and

$$x^{(2n-2)}(t) = x^{(2n-2)}(0) + x^{(2n-1)}(0)t + \int_0^t du \int_0^u x^{(2n)}(s) ds. \quad (12)$$

Since $x^{(2n-2)}(T) = x^{(2n-2)}(0)$, we have

$$x^{(2n-1)}(0)T + \int_0^T du \int_0^u x^{(2n)}(s) ds = 0$$

or

$$x^{(2n-1)}(0) = -\frac{1}{T} \int_0^T du \int_0^u x^{(2n)}(s) ds.$$

From (12), we have

$$x^{(2n-2)}(t) = x^{(2n-2)}(0) - \frac{t}{T} \int_0^T du \int_0^u x^{(2n)}(s) ds + \int_0^t du \int_0^u x^{(2n)}(s) ds. \quad (13)$$

Now since $\int_0^T x^{(2n-2)}(s) ds = 0$, from (13) we have

$$x^{(2n-2)}(0)T - \frac{T}{2} \int_0^T du \int_0^u x^{(2n)}(s) ds + \int_0^T dw \int_0^w du \int_0^u x^{(2n)}(s) ds = 0,$$

or

$$x^{(2n-2)}(0) = \frac{1}{2} \int_0^T du \int_0^u x^{(2n)}(s) ds - \frac{1}{T} \int_0^T dw \int_0^w du \int_0^u x^{(2n)}(s) ds. \quad (14)$$

From (13) and (14), we have

$$\begin{aligned} x^{(2n-2)}(t) &= \frac{1}{2} \int_0^T du \int_0^u x^{(2n)}(s) ds - \frac{1}{T} \int_0^T dw \int_0^w du \int_0^u x^{(2n)}(s) ds \\ &\quad - \frac{t}{T} \int_0^T du \int_0^u x^{(2n)}(s) ds + \int_0^t du \int_0^u x^{(2n)}(s) ds \\ &= \left(\frac{1}{2} - \frac{t}{T}\right) \int_0^T du \int_0^u x^{(2n)}(s) ds + \int_0^t du \int_0^u x^{(2n)}(s) ds \\ &\quad - \frac{1}{T} \int_0^T dw \int_0^w du \int_0^u x^{(2n)}(s) ds. \end{aligned} \quad (15)$$

Let $y_0(t) = y(t)$ and $y_1(t) = x^{(2n-2)}(t)$. Since $y(t) = x^{(2n)}(t)$, we have from (15) that

$$\begin{aligned} x^{(2n-2)}(t) = y_1(t) &= \left(\frac{1}{2} - \frac{t}{T}\right) \int_0^T du \int_0^u y_0(s) ds \\ &\quad + \int_0^t du \int_0^u y_0(s) ds - \frac{1}{T} \int_0^T dw \int_0^w du \int_0^u y_0(s) ds. \end{aligned} \quad (16)$$

From (16), we obtain

$$x^{(2n-3)}(t) = x^{(2n-3)}(0) + \int_0^t y_1(s) ds$$

and

$$x^{(2n-4)}(t) = x^{(2n-4)}(0) + x^{(2n-3)}(0)t + \int_0^t du \int_0^u y_1(s) ds. \quad (17)$$

Since $x^{(2n-4)}(T) = x^{(2n-4)}(0)$, we have from (17) that

$$x^{(2n-3)}(0) = -\frac{1}{T} \int_0^T du \int_0^u y_1(s) ds. \tag{18}$$

Since $\int_0^T x^{(2n-4)}(s) ds = 0$, we have from (17) that

$$x^{(2n-4)}(0) = \frac{1}{2} \int_0^T du \int_0^u y_1(s) ds - \frac{1}{T} \int_0^T dw \int_0^w du \int_0^u y_1(s) ds. \tag{19}$$

Let $y_2(t) = x^{(2n-4)}(t)$ and we have from (17)-(19) that

$$\begin{aligned} x^{(2n-4)}(t) = y_2(t) &= \left(\frac{1}{2} - \frac{t}{T}\right) \int_0^T du \int_0^u y_1(s) ds \\ &+ \int_0^t du \int_0^u y_1(s) ds - \frac{1}{T} \int_0^T dw \int_0^w du \int_0^u y_1(s) ds. \end{aligned}$$

Let $y_i(t) = x^{(2n-2i)}(t)$ ($i = 1, 2, \dots, n - 1$) and as above it is easy to check that

$$\begin{aligned} x^{(2n-2i)}(t) = y_i(t) &= \left(\frac{1}{2} - \frac{t}{T}\right) \int_0^T du \int_0^u y_{i-1}(s) ds \\ &+ \int_0^t du \int_0^u y_{i-1}(s) ds - \frac{1}{T} \int_0^T dw \int_0^w du \int_0^u y_{i-1}(s) ds, \end{aligned}$$

for ($i = 1, 2, \dots, n - 1$), and

$$y_n(t) = y_n(0) - \frac{t}{T} \int_0^T du \int_0^u y_{n-1}(s) ds + \int_0^t du \int_0^u y_{n-1}(s) ds.$$

Note that $y_n(t) = x(t) \in \text{Dom}L \cap \text{Ker}P$. Thus $y_n(0) = x(0) = 0$, and

$$K_P y(t) = -\frac{t}{T} \int_0^T du \int_0^u y_{n-1}(s) ds + \int_0^t du \int_0^u y_{n-1}(s) ds. \tag{20}$$

Let Ω be an open and bounded subset of X . In view of (5), (9) and (10) (or (20)), we can easily see that $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N(\overline{\Omega})$ is compact. Thus the mapping N is L -compact on $\overline{\Omega}$. That is, we have the following result.

Lemma 2.1 *Let L, N, P and Q be defined by (4), (5), (8) and (9) respectively. Then L is a Fredholm mapping of index zero and N is L -compact on $\overline{\Omega}$, where Ω is any open and bounded subset of X .*

In order to prove our main result, we need the following Lemmas [6, 7]. The first result follows from [6 and Remark 2.1] and the second from [7].

Lemma 2.2 *Let $x(t) \in C^{(n)}(\mathbf{R}, \mathbf{R}) \cap C_T$. Then*

$$\|x^{(i)}\|_0 \leq \frac{1}{2} \int_0^T |x^{(i+1)}(s)| ds, i = 1, 2, \dots, n - 1,$$

where $n \geq 2$ and $C_T := \{x | x \in C(\mathbf{R}, \mathbf{R}), x(t + T) = x(t), \forall t \in \mathbf{R}\}$.

Lemma 2.3 Suppose that M, λ are positive numbers and satisfy $0 < M < (\frac{\pi}{T})^2$ and $0 < \lambda < 1$, then for any function φ defined in $[0, T]$, the following problem

$$\begin{cases} x''(t) + \lambda Mx(t) = \lambda\varphi(t), \\ x(0) = x(T), x'(0) = x'(T), \end{cases}$$

has a unique solution

$$x(t) = \int_0^T G(t, s)\lambda\varphi(s)ds,$$

where $\alpha = \sqrt{\lambda M}$, and

$$G(t, s) = \begin{cases} w(t-s), & (k-1)T \leq s \leq t \leq kT, \\ w(T+t-s), & (k-1)T \leq t \leq s \leq kT, \quad (k \in \mathbf{N}), \end{cases}$$

with

$$w(t) = \frac{\cos \alpha(t - \frac{T}{2})}{2\alpha \sin \frac{\alpha T}{2}}.$$

Now, we consider the following auxiliary equation

$$x^{(2n)}(t) + \lambda \sum_{i=0}^{2n-2} a_i(t)x^{(i)}(t) + \lambda g(x(t - \tau(t))) = \lambda p(t), \quad (21)$$

where $0 < \lambda < 1$. We have

Lemma 2.4 Suppose the conditions of Theorem 2.1 are satisfied. If $x(t)$ is a T -periodic solution of Eq.(21), then there are positive constants D_i ($i = 0, 1, \dots, 2n-1$), which are independent of λ , such that

$$\|x^{(i)}\|_0 \leq D_i, \quad t \in [0, T] \quad \text{for } i = 0, 1, \dots, 2n-1. \quad (22)$$

Proof. Suppose that $x(t)$ is a T -periodic solution of (21). By (2) of Theorem 2.1 we know that there exists a $\overline{M}_1 > 0$, such that

$$|g(x(t))| \leq r|x(t)|, \quad |x(t)| > \overline{M}_1, \quad t \in \mathbf{R}. \quad (23)$$

Set

$$E_1 = \{t : |x(t)| > \overline{M}_1, \quad t \in [0, T]\}, \quad (24)$$

$$E_2 = [0, T] \setminus E_1 \quad (25)$$

and

$$\rho = \max_{|x| \leq \overline{M}_1} |g(x)|. \quad (26)$$

Let $\varepsilon = \frac{m_0-r}{2}$. By (21), (23), (24), (25), (26) and Lemma 2.2, we obtain

$$\begin{aligned}
 \|x^{(2n-1)}\|_0 &\leq \frac{1}{2} \int_0^T |x^{(2n)}(s)| ds \\
 &\leq \frac{\lambda}{2} \int_0^T [|\sum_{i=0}^{2n-2} a_i(t)x^{(i)}(t)| + |g(x(t-\tau(t)))| + |p(t)|] dt \\
 &\leq \frac{\lambda T}{2} [M_{2n-2} \|x^{(2n-2)}\|_0 + M_{2n-3} \|x^{(2n-3)}\|_0 + \dots + M_2 \|x^{(2)}\|_0 + M_1 \|x^{(1)}\|_0 \\
 &\quad + M_0 \|x\|_0] + \frac{\lambda}{2} \int_0^T |g(x(t-\tau(t)))| dt + \frac{\lambda T}{2} \|p\|_0 \\
 &\leq \frac{T}{2} [M_{2n-2} \frac{T}{2} + M_{2n-3} (\frac{T}{2})^2 + \dots + M_2 (\frac{T}{2})^{2n-3} + M_1 (\frac{T}{2})^{2n-2}] \|x^{(2n-1)}\|_0 + \\
 &\quad + \frac{T}{2} M_0 \|x\|_0 + \frac{1}{2} [\int_{E_1} |g(x(t-\tau(t)))| dt + \int_{E_2} |g(x(t-\tau(t)))| dt] + \frac{T}{2} \|p\|_0 \\
 &\leq A^* \|x^{(2n-1)}\|_0 + \frac{T}{2} (M_0 + r + \varepsilon) \|x\|_0 + \frac{T}{2} C \\
 &= A^* \|x^{(2n-1)}\|_0 + \frac{T}{4} (2M_0 + r + m_0) \|x\|_0 + \frac{T}{2} C,
 \end{aligned} \tag{27}$$

where $C = (\rho + \|p\|_0)$ and

$$\begin{aligned}
 A^* &= [M_{2n-2} (\frac{T}{2})^2 + M_{2n-3} (\frac{T}{2})^3 + M_{2n-4} (\frac{T}{2})^4 \\
 &\quad + \dots + M_2 (\frac{T}{2})^{2n-2} + M_1 (\frac{T}{2})^{2n-1}].
 \end{aligned}$$

Now from (27), we have

$$\|x^{(2n-1)}\|_0 \leq (1 - A^*)^{-1} [\frac{T}{4} (2M_0 + r + m_0) \|x\|_0 + \frac{T}{2} C]. \tag{28}$$

On the other hand, from (21) and Lemma 2.3, we get

$$\begin{aligned}
 &x^{(2n-2)}(t) \\
 &= \int_0^T G_1(t, t_1) \lambda [(M_{2n-2} - a_{2n-2}(t_1)) x^{(2n-2)}(t_1) + p(t_1) \\
 &\quad - g(x(t-\tau(t_1)))] dt_1 - \lambda \int_0^T G_1(t, t_1) [\sum_{i=0}^{2n-3} a_i(t_1) x^{(i)}(t_1)] dt_1,
 \end{aligned} \tag{29}$$

where $\alpha_1 = \sqrt{\lambda M_{2n-2}}$, and

$$G_1(t, t_1) = \begin{cases} w_1(t - t_1), & (k - 1)T \leq t_1 \leq t \leq kT, \\ w_1(T + t - t_1), & (k - 1)T \leq t \leq t_1 \leq kT, \quad (k \in \mathbf{N}), \end{cases} \tag{30}$$

with

$$w_1(t) = \frac{\cos \alpha_1(t - \frac{T}{2})}{2\alpha_1 \sin \frac{\alpha_1 T}{2}} \tag{31}$$

and

$$\int_0^T G_1(t, t_1) dt_1 = \frac{1}{\lambda M_{2n-2}}. \tag{32}$$

From (29) and Lemma 2.3, we have

$$\begin{aligned}
& x^{(2n-4)}(t) \\
&= \lambda \int_0^T G_2(t, t_1) \int_0^T G_1(t_1, t_2) [p(t_2) - g(x(t - \tau(t_2)))] dt_2 dt_1 \\
&+ \lambda \int_0^T G_2(t, t_1) \int_0^T G_1(t_1, t_2) (M_{2n-2} - a_{2n-2}(t_2)) x^{(2n-2)}(t_2) dt_2 dt_1 \\
&+ \int_0^T G_2(t, t_1) \left[\frac{M_{2n-4}}{M_{2n-2}} x^{(2n-4)}(t_1) - \lambda \int_0^T G_1(t_1, t_2) a_{2n-4}(t_2) x^{(2n-4)}(t_2) dt_2 \right] dt_1 \\
&- \lambda \int_0^T G_2(t, t_1) \int_0^T G_1(t_1, t_2) \left[\sum_{i=0}^{2n-5} a_i(t_1) x^{(i)}(t_2) + a_{2n-3}(t_2) x^{(2n-3)}(t_2) \right] dt_2 dt_1,
\end{aligned} \tag{33}$$

where $\alpha_2 = \sqrt{\frac{M_{2n-4}}{M_{2n-2}}}$, and

$$G_2(t, t_2) = \begin{cases} w_2(t - t_2), & (k-1)T \leq t_2 \leq t \leq kT, \\ w_2(T + t - t_2), & (k-1)T \leq t \leq t_2 \leq kT, \quad (k \in \mathbf{N}), \end{cases} \tag{34}$$

with

$$w_2(t) = \frac{\cos \alpha_2(t - \frac{T}{2})}{2\alpha_2 \sin \frac{\alpha_2 T}{2}} \tag{35}$$

and

$$\int_0^T G_2(t, t_2) dt_2 = \frac{M_{2n-2}}{M_{2n-4}}. \tag{36}$$

By induction, we have

$$\begin{aligned}
& x(t) = \lambda \int_0^T G_n(t, t_1) \cdots \int_0^T G_1(t_{n-1}, t_n) [p(t_n) - g(x(t_n - \tau(t_n)))] dt_n \cdots dt_1 \\
&+ \lambda \int_0^T G_n(t, t_1) \cdots \int_0^T G_1(t_{n-1}, t_n) (M_{2n-2} - a_{2n-2}(t_n)) x^{(2n-2)}(t_n) dt_n \cdots dt_1 \\
&+ \int_0^T G_n(t, t_1) \cdots \int_0^T G_2(t_{n-2}, t_{n-1}) \left[\frac{M_{2n-4}}{M_{2n-2}} x^{(2n-4)}(t_{n-1}) - \right. \\
&\lambda \int_0^T G_1(t_{n-1}, t_n) a_{2n-4} x^{(2n-4)}(t_n) dt_n \left. \right] dt_{n-1} \cdots dt_1 \\
&+ \int_0^T G_n(t, t_1) \cdots \int_0^T G_3(t_{n-3}, t_{n-2}) \left[\frac{M_{2n-6}}{M_{2n-4}} x^{(2n-6)}(t_{n-2}) - \right. \\
&\lambda \int_0^T G_2(t_{n-2}, t_{n-1}) \int_0^T G_1(t_{n-1}, t_n) [a_{2n-6} x^{(2n-6)}(t_n) dt_n dt_{n-1}] dt_{n-2} \cdots dt_1 \\
&+ \cdots + \cdots \\
&+ \int_0^T G_n(t, t_1) \left[\frac{M_0}{M_2} x(t_1) - \lambda \int_0^T G_{n-1}(t_1, t_2) \int_0^T G_{n-2}(t_2, t_3) \right. \\
&\cdots \int_0^T G_1(t_{n-1}, t_n) a_0(t_n) x(t_n) dt_n \cdots dt_2 \left. \right] dt_1 \\
&- \lambda \int_0^T G_n(t, t_1) \cdots \int_0^T G_1(t_{n-1}, t_n) \left[\sum_{k=1}^{n-1} a_{2k-1}(t_n) x^{(2k-1)}(t_n) \right] dt_n \cdots dt_1
\end{aligned} \tag{37}$$

where $\alpha_i = \sqrt{\frac{M_{2n-2i}}{M_{2n-2i+2}}}$ ($2 \leq i \leq n$), and

$$G_i(t, t_i) = \begin{cases} w_i(t - t_i), & (k - 1)T \leq t_i \leq t \leq kT, \\ w_i(T + t - t_i), & (k - 1)T \leq t \leq t_i \leq kT, \quad (k \in \mathbf{N}), \end{cases} \quad (38)$$

with

$$w_i(t) = \frac{\cos \alpha_i(t - \frac{T}{2})}{2\alpha_i \sin \frac{\alpha_i T}{2}} \quad (39)$$

and

$$\int_0^T G_i(t, t_i) dt_i = \frac{M_{2n-2i+2}}{M_{2n-2i}} \quad (2 \leq i \leq n). \quad (40)$$

From (32), (37), (40) and Lemma 2.2, we obtain

$$\begin{aligned} & \|x\|_0 \\ & \leq \max_{t \in [0, T]} \lambda \int_{E_1} |G_n(t, t_1)| \cdots \int_0^T |G_1(t_{n-1}, t_n)| |p(t_n) - g(x(t_n - \tau(t_n)))| dt_n \cdots dt_1 \\ & + \max_{t \in [0, T]} \lambda \int_{E_2} |G_n(t, t_1)| \cdots \int_0^T |G_1(t_{n-1}, t_n)| |p(t_n) - g(x(t_n - \tau(t_n)))| dt_n \cdots dt_1 + \\ & \max_{t \in [0, T]} \lambda \int_0^T |G_n(t, t_1)| \cdots \int_0^T |G_1(t_{n-1}, t_n)| (M_{n-1} - a_{n-1}(t_n)) |x^{(2n-2)}(t_n)| dt_n \cdots dt_1 \\ & + \max_{t \in [0, T]} \int_0^T |G_n(t, t_1)| \cdots \int_0^T |G_2(t_{n-2}, t_{n-1})| \left| \frac{M_{n-2}}{M_{n-1}} x^{(2n-4)}(t_{n-1}) - \right. \\ & \left. \lambda \int_0^T G_1(t_{n-1}, t_n) a_{n-2} x^{(2n-4)}(t_n) dt_n \right| dt_{n-1} \cdots dt_1 \\ & + \max_{t \in [0, T]} \int_0^T |G_n(t, t_1)| \cdots \int_0^T |G_3(t_{n-3}, t_{n-2})| \left| \frac{M_{n-3}}{M_{n-2}} x^{(2n-6)}(t_{n-2}) - \right. \\ & \left. \lambda \int_0^T G_2(t_{n-2}, t_{n-1}) \int_0^T G_1(t_{n-1}, t_n) [a_{n-3} x^{(2n-6)}(t_n) dt_n dt_{n-1}] dt_{n-2} \right| dt_{n-1} \cdots dt_1 \\ & + \cdots + \cdots \\ & + \max_{t \in [0, T]} \int_0^T |G_n(t, t_1)| \left| \frac{M_0}{M_1} x(t_1) - \lambda \int_0^T G_{n-1}(t_1, t_2) \int_0^T G_{n-2}(t_2, t_3) \right. \\ & \left. \cdots \int_0^T G_1(t_{n-1}, t_n) a_0(t_n) x(t_n) dt_n \cdots dt_2 \right| dt_1 \\ & + \max_{t \in [0, T]} \lambda \int_0^T |G_n(t, t_1)| \cdots \int_0^T |G_1(t_{n-1}, t_n)| \left[\sum_{k=1}^{n-1} a_{2k-1}(t_n) x^{(2k-1)}(t_n) \right] dt_n \cdots dt_1 \\ & \leq \frac{1}{M_0} [\|p\|_0 + \rho + (r + \varepsilon) \|x\|_0] + \frac{M_0 - m_0}{M_0} \|x\|_0 + \frac{1}{M_0} [(M_{2n-2} - m_{2n-2}) \|x^{(2n-2)}\|_0 \\ & + (M_{2n-4} - m_{2n-4}) \|x^{(2n-4)}\|_0 + \cdots + (M_2 - m_2) \|x^{(2)}\|_0] \\ & + \frac{1}{M_0} [M_1 \|x^{(1)}\|_0 + M_3 \|x^{(3)}\|_0 + \cdots + M_{2n-3} \|x^{(2n-3)}\|_0] \\ & \leq \frac{1}{M_0} [C + (M_0 - m_0 + r + \varepsilon) \|x\|_0] + \frac{1}{M_0} [M_1 \left(\frac{T}{2}\right)^{2n-2} + (M_2 - m_2) \left(\frac{T}{2}\right)^{2n-3} \\ & + M_3 \left(\frac{T}{2}\right)^{2n-4} + (M_4 - m_4) \left(\frac{T}{2}\right)^{2n-5} + \cdots + M_{2n-3} \left(\frac{T}{2}\right)^2 \\ & + (M_{2n-2} - m_{2n-2}) \frac{T}{2}] \|x^{(2n-1)}\|_0. \end{aligned} \quad (41)$$

Now (41) and $\varepsilon = \frac{m_0-r}{2}$ give

$$\|x\|_0 \leq \frac{2(B\|x^{(2n-1)}\|_0+C)}{(m_0-r)}, \tag{42}$$

where

$$B = M_1\left(\frac{T}{2}\right)^{2n-2} + (M_2 - m_2)\left(\frac{T}{2}\right)^{2n-3} + M_3\left(\frac{T}{2}\right)^{2n-4} + (M_4 - m_4)\left(\frac{T}{2}\right)^{2n-5} \\ + \dots + \dots + M_{2n-3}\left(\frac{T}{2}\right)^2 + (M_{2n-2} - m_{2n-2})\frac{T}{2}$$

and $M_{2k-1} = \max_{t \in [0, T]} |a_{2k-1}(t)| \quad (k = 1, 2, \dots, n - 1)$.

Thus combining (27) and (42), we see that

$$\left(A - \frac{2M_0+m_0+r}{2(m_0-r)}B\right)\|x^{(2n-1)}\|_0 \leq \frac{TC}{2}\left(\frac{2M_0+m_0+r}{(m_0-r)} + 1\right) \\ = \frac{(M_0+m_0)TC}{(m_0-r)}, \tag{43}$$

where $A = 1 - A^*$.

From (42) and (43), we have

$$\|x^{(2n-1)}\|_0 \leq \frac{(M_0+m_0)TC}{(m_0-r)}\left(A - \frac{2M_0+m_0+r}{2(m_0-r)}B\right)^{-1} \\ = D_{2n-1} \tag{44}$$

and

$$\|x\|_0 \leq \frac{2(BD_{2n-1} + C)}{(m_0 - r)} = D_0. \tag{45}$$

Finally from (44), (45) and Lemma 2.2, we get

$$\|x^{(i)}\|_0 \leq D_i \quad (1 \leq i \leq 2n - 2). \tag{46}$$

The proof of Lemma 2.4 is complete.

Proof of Theorem 2.1. Suppose that $x(t)$ is a T-periodic solution of Eq.(21). By Lemma 2.4, there exist positive constants $D_i \ (i = 0, 1, \dots, 2n - 1)$ which are independent of λ such that (22) is true. By (3), we know that there exists a $M_2 > 0$, such that

$$sgn(x)g(x(t)) > 0, \quad |x(t)| > M_2, \quad t \in \mathbf{R}.$$

Consider any positive constant $\bar{D} > \max_{0 \leq i \leq 2n-1} \{D_i\} + M_2$.

Set

$$\Omega := \{x \in X : \|x\| < \bar{D}\}.$$

We know that L is a Fredholm mapping of index zero and N is L -compact on $\overline{\Omega}$ (see [3]).

Recall

$$Ker(L) = \{x \in X : x(t) = c \in \mathbf{R}\}$$

and the norm on X is

$$\|x\| = \max_{0 \leq j \leq 2n-1} \max_{t \in [0, T]} |x^{(j)}(t)|.$$

Then we have

$$x = \overline{D} \quad \text{or} \quad x = -\overline{D} \quad \text{for} \quad x \in \partial\Omega \cap Ker(L). \tag{47}$$

From (3) and (47), we have (if \overline{D} is chosen large enough)

$$a_0(t)\overline{D} + g(\overline{D}) - \|p\|_0 > 0 \quad \text{for} \quad t \in [0, T] \tag{48}$$

and

$$x^{(i)}(t) = 0, \quad \forall x \in \partial\Omega \cap Ker(L) (i = 1, 2, \dots, 2n - 1). \tag{49}$$

Finally from (5), (9) and (47)-(49), we have

$$\begin{aligned} (QNx) &= \frac{1}{T} \int_0^T [- \sum_{i=0}^{2n-2} a_i(t)x^{(i)}(t) - g(x(t - \tau(t))) + p(t)]dt \\ &= -\frac{1}{T} \int_0^T [a_0(t)x(t) + g(x(t - \tau(t))) - p(t)]dt \\ &\neq 0, \quad \forall x \in \partial\Omega \cap Ker(L). \end{aligned}$$

Then, for any $x \in KerL \cap \partial\Omega$ and $\eta \in [0, 1]$, we have

$$\begin{aligned} xH(x, \eta) &= -\eta x^2 - \frac{x}{T}(1 - \eta) \int_0^T [\sum_{i=0}^{2n-2} a_i(t)x^{(i)}(t) + g(x(t - \tau(t))) - p(t)]dt \\ &\neq 0. \end{aligned}$$

Thus

$$\begin{aligned} deg\{QN, \Omega \cap Ker(L), 0\} &= deg\{-\frac{1}{T} \int_0^T [\sum_{i=0}^{2n-2} a_i(t)x^{(i)}(t) \\ &\quad + g(x(t - \tau(t))) - p(t)]dt, \Omega \cap Ker(L), 0\} \\ &= deg\{-x, \Omega \cap Ker(L), 0\} \\ &\neq 0. \end{aligned}$$

From Lemma 2.4 for any $x \in \partial\Omega \cap Dom(L)$ and $\lambda \in (0, 1)$ we have $Lx \neq \lambda Nx$. By Theorem 1.1, the equation $Lx = Nx$ has at least a solution in $Dom(L) \cap \overline{\Omega}$, so there exists a T -periodic solution of Eq.(1). The proof is complete.

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