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EXISTENCE OF THREE SOLUTIONS FOR SYSTEMS OF MULTI–POINT BOUNDARY VALUE PROBLEMS

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

By using a variational approach, we obtain some sufficient conditions for the existence of three classical solutions of a boundary value problem consisting of a system of differential equations and some multi–point boundary conditions. Applications of our results are discussed. Our results extend some related work in the literature.

Key words and phrases: Solutions, boundary value problems, critical points.
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1 Introduction

In recent years, there are many papers published on the existence of solutions of boundary value problems (BVPs) with various multi-point boundary conditions (BCs). For a small sample of the recent work on this topic, we refer the reader to [6, 10, 11, 15, 16, 19, 22, 23] for second order problems and to [2, 7, 8, 9, 12, 13, 17, 18, 24] for higher order ones. In this paper, we study the BVP consisting of the system of differential equations

$$(\phi_{p_i}(u_i'))' + \lambda f_i(t, u_1, \ldots, u_n) = 0,\ t \in (0,1),\ i = 1, \ldots, n, \tag{1}$$

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and the multi–point BCs

$$u_i(0) = \sum_{j=1}^{m} a_j u_i(t_j),\ u_i(1) = \sum_{j=1}^{m} b_j u_i(t_j),\ i = 1, \ldots, n, \tag{2}$$

where λ is a real parameter, $m, n \geq 1$ are integers, $p_i > 1$, $\phi_{p_i}(x) = |x|^{p_i-2}x$, and $f_i \in C([0,1] \times \mathbb{R}^n)$ for $i = 1, \ldots, n$, $a_j, b_j \in \mathbb{R}$ for $j = 1, \ldots, m$, and $0 < t_1 \leq t_2 \leq \ldots \leq t_m < 1$. We will obtain sufficient conditions for the existence of an open interval $\Lambda \subseteq [0, \infty)$ such that, for each $\lambda \in \Lambda$, BVP (1), (2) has at least three classical solutions. Here, by a classical solution of BVP (1), (2), we mean a function $u = (u_1, \ldots, u_n)$ such that, for $i = 1, \ldots, n$, $u_i \in C^1[0,1]$, $\phi_{p_i}(u_i') \in C^1[0,1]$, and $u_i(t)$ satisfies (1), (2).

Our proof is based on a three critical point theorem of Ricceri [26]; see Lemma 1.2 below. For more applications of this theorem to various problems, we refer the reader to [3, 5, 6, 14, 21] for work on ordinary differential equations and [1, 4, 20] for work on partial differential equations. In particular, Bonannao [3] applied Ricceri's theorem to the BVP

$$u'' + \lambda f(u) = 0, \quad t \in (0,1), \tag{3}$$

$$u(0) = u(1) = 0, \tag{4}$$

and obtained the following interesting result.

Proposition 1.1 ([3, Theorem 2]) *Let $f \in C(\mathbb{R})$, $\bar{F}(x) = \int_0^x f(\xi)d\xi$, and the norm of the Sobolev space $W_0^{1,2}([0,1])$ be defined by $||u|| = \left(\int_0^1 |u'(s)|^2 ds\right)^{1/2}$. Assume that there exist four positive constants $\bar{c}, \bar{d}, \bar{a}, \bar{s}$ with $\bar{c} < \sqrt{2}\bar{d}$ and $\bar{\gamma} < 2$ such that*

- *(i) $f(x) \geq 0$ for all $x \in [-\bar{c}, \max\{\bar{c}, \bar{d}\}]$,*
- *(ii) $\bar{F}(\bar{c}) \leq (\bar{c}/2\bar{d})^2 \bar{F}(\bar{d})$,*
- *(iii) $\bar{F}(x) \leq \bar{a}(1 + |x|^{\bar{\gamma}})$ for all $x \in \mathbb{R}$.*

Then there exist an open interval $\Lambda \subseteq [0, \infty)$ and a positive real number δ such that, for each $\lambda \in \Lambda$, BVP (3), (4) has at least three solutions belonging to $C^2[0,1]$ whose norms in $W_0^{1,2}([0,1])$ are less than δ.

Candito [5] extended Proposition 1.1 to the nonautonomous case. He and Ge [14] further extended the result in [5] to the BVP consisting of the equation

$$(\phi_p(u'))' + \lambda f(t, u) = 0, \quad t \in (0,1), \tag{5}$$

and BC (4), where $p > 1$ and $f \in C([0,1] \times \mathbb{R})$. Recently, Du [6] extended the main results in [3, 5, 14] to the BVP consisting of Eq. (5) and the three–point BCs

$$u(0) = 0,\ u(1) = \alpha u(\eta), \tag{6}$$

where $\alpha \in \mathbb{R}$ and $\eta \in (0,1)$.

Motivated by these works, in this paper, we establish some criteria for the existence of three classical solutions of BVP (1), (2) (see Theorem 2.1). As applications, we present some new existence results (see Corollaries 2.1 and 2.2) for the scalar BVP consisting of Eq. (5) and the multi–point BCs

$$u(0) = \sum_{j=1}^{m} a_j u(t_j),\ u(1) = \sum_{j=1}^{m} b_j u(t_j). \tag{7}$$

Observe that BCs (7) include BCs (4) and (6) as special cases. We also give an application to BVP (5), (4) (see Corollary 2.3). Our results extend the main results in [3, 5, 6, 14] to more general problems (see Remark 2.1).

For the reader's convenience, we now recall the following two results that are fundamental tools in our discussion.

Lemma 1.1 ([25, Proposition 3.1]) *Let X be a separable and reflexive real Banach space, and Φ, J two real functions on X. Assume that there exist $r > 0$ and $u_0, u_1 \in X$ such that*

$$\Phi(u_0) = J(u_0) = 0, \quad \Phi(u_1) > r,$$

$$\sup_{u\in\Phi^{-1}((-\infty,r])} J(u) < r\frac{J(u_1)}{\Phi(u_1)}.$$

Then, for each η satisfying

$$\sup_{u\in\Phi^{-1}((-\infty,r])} J(u) < \eta < r\frac{J(u_1)}{\Phi(u_1)},$$

one has

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda(\eta - J(u))) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda(\eta - J(u))).$$

Lemma 1.2 ([26, Theorem 1]) *Let X be a separable and reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

$$\lim_{||u||\to\infty}(\Phi(u) + \lambda\Psi(u)) = \infty$$

for all $\lambda \in [0,\infty)$, and that there exists a continuous concave function $h : [0,\infty) \to \mathbb{R}$ such that

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda\Psi(u) + h(\lambda)) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda\Psi(u) + h(\lambda)).$$

Then there exist an open interval $\Lambda \subseteq [0,\infty)$ and a positive real number δ such that, for each $\lambda \in \Lambda$, the equation

$$\Phi'(u) + \lambda \Psi'(u) = 0$$

has at least three solutions in X whose norms are less than δ.

The rest of this paper is organized as follows. After this introduction, in Section 2, we state the main results and give one simple example for illustrative purposes. The proofs of the main results, together with some technical lemmas, are presented in Section 3.

2 Main Results

In the sequel, for $i = 1, \dots, n$, let

$$X_i = \left\{ y \in W^{1,p_i}([0,1]) \ : \ y(0) = \sum_{j=1}^{m} a_j y(t_j), \ y(1) = \sum_{j=1}^{m} b_j y(t_j) \right\}$$

and let X be defined by

$$X = X_1 \times X_2 \times \ldots \times X_n$$

with the norm

$$||u|| = ||(u_1, \dots, u_n)|| = \sum_{i=1}^{n} ||u_i'||_{p_i},$$

where

$$||u_i'||_{p_i} = \left(\int_0^1 |u'(s)|^{p_i} ds \right)^{1/p_i}.$$

Then, X is a separable and reflexive real Banach space.

We first make the following assumption.

(H1) $\sum_{j=1}^{m} a_j \neq 1$ and $\sum_{j=1}^{m} b_j \neq 1$.

We now introduce some notations. For any nonempty set S, let

$$S^n = \underbrace{S \times S \times \ldots \times S}_{n},$$

and for $x > 0$, define

$$S_{1,n}(x) = \begin{cases} [0, t_1/2] \times [x \sum_{j=1}^{m} a_j, x]^n & \text{if} \quad \sum_{j=1}^{m} a_j < 1, \\ [0, t_1/2] \times [x, x \sum_{j=1}^{m} a_j]^n & \text{if} \quad \sum_{j=1}^{m} a_j > 1, \end{cases} \tag{8}$$

$$S_{2,n}(x) = \begin{cases} [(1+t_m)/2, 1] \times [x\sum_{j=1}^m b_j, x]^n & \text{if} \quad \sum_{j=1}^m b_j < 1, \\ [(1+t_m)/2, 1] \times [x, x\sum_{j=1}^m b_j]^n & \text{if} \quad \sum_{j=1}^m b_j > 1. \end{cases} \tag{9}$$

Let the positive constants κ_i, $i = 1, \dots, n$, and ρ be defined by

$$\kappa_i = 2^{p_i-1}\left(t_1^{1-p_i}\left|1 - \sum_{j=1}^m a_j\right|^{p_i} + (1-t_m)^{1-p_i}\left|1 - \sum_{j=1}^m b_j\right|^{p_i}\right) \tag{10}$$

and

$$\rho = \frac{1}{2}\left(1 + \frac{\sum_{j=1}^m |a_j|}{|1 - \sum_{j=1}^m a_j|} + \frac{\sum_{j=1}^m |b_j|}{|1 - \sum_{j=1}^m b_j|}\right). \tag{11}$$

Throughout this paper, the following assumptions are also needed.

(H2) there exists a function $F : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ such that $F(t, x_1, \dots, x_n)$ is continuous in t and differentiable in x_i, $i = 1, \dots, n$. Moreover, $\partial F/\partial x_i = f_i$ for $i = 1, \dots, n$;

(H3) there exist two positive constants c and d with $c < d$ such that

$$F(t, x_1, \dots, x_n) \geq 0 \quad \text{for } (t, x_1, \dots, x_n) \in S_{1,n}(d) \cup S_{2,n}(d)$$

and

$$\sum_{i=1}^n \frac{\kappa_i d^{p_i}}{p_i} \max_{(t,x_1,\dots,x_n)\in K} F(t, x_1, \dots, x_n) \leq \sum_{i=1}^n \frac{\kappa_i c^{p_i}}{p_i} \int_{t_1/2}^{(1+t_m)/2} F(s, d, \dots, d)ds,$$

where

$$K = \left\{(t, x_1, \dots, x_n) \ : \ t \in [0,1], \quad \sum_{i=1}^n \frac{|x_i|^{p_i}}{p_i} < \rho^{p_i} \sum_{i=1}^n \frac{\kappa_i c^{p_i}}{p_i}\right\};$$

(H4) there exist $\theta \in L^1(0,1)$ and n positive constants γ_i with $\gamma_i < p_i$, $i = 1, \dots, n$, such that

$$F(t, x_1, \dots, x_n) \leq \theta(t)\left(1 + \sum_{i=1}^n |x_i|^{\gamma_i}\right)$$

for $t \in [0,1]$ and $x_i \in \mathbb{R}$, $i = 1, \dots, n$;

(H5) $F(t, 0, \dots, 0) = 0$ for $t \in [0,1]$.

We say that a function $u = (u_1, \dots, u_n) \in X$ is a weak solution of BVP (1), (2) if

$$\int_0^1 \sum_{i=1}^n \phi_{p_i}(u_i'(s))v_i'(s)ds - \lambda \int_0^1 \sum_{i=1}^n f_i(s, u_1(s), \dots, u_n(s))v_i(s)ds = 0$$

for any $v = (v_1, \ldots, v_n) \in W_0^{1,p_1}([0,1]) \times W_0^{1,p_2}([0,1]) \times \ldots \times W_0^{1,p_n}([0,1])$. Under the assumptions (H1)–(H5), Theorem 2.1 below shows that BVP (1), (2) has at least three classical solutions. To prove the theorem, we will first apply Lemmas 1.1 and 1.2 to obtain the existence of three weak solutions of BVP (1), (2), then we show that the three weak solutions are indeed the classical solutions. In the process of the proof, three functionals Φ, Ψ, and J are constructed in such a way that all the conditions of Lemmas 1.1 and 1.2 are satisfied.

We now make some brief comments about the assumptions (H1)–(H5). (H1) is needed to obtain some useful bounds for functions in X (see Lemma 3.1). The function F introduced in (H2) is used in the construction of the functional Ψ. (H3) is required in the proof of the existence of a function $w \in X$ with some nice properties (see Lemma 3.2). Both Lemmas 3.1 and 3.2 are crucial to prove our existence results. (H4), together with Lemma 3.1, is used to show the functional $\Phi(u) + \lambda\Psi(u)$ is weakly coercive for $\lambda \in [0,\infty)$, i.e.,

$$\lim_{||u||\to\infty} (\Phi(u) + \lambda\Psi(u)) = \infty, \ \lambda \in [0,\infty).$$

Finally, (H5) is needed to show that $J(0,\ldots,0) = 0$, which is necessary in order to apply Lemma 1.1.

Now, we state our main results. The first one is concerned with BVP (1), (2).

Theorem 2.1 *Assume (H1)–(H5) hold. Then there exist an open interval $\Lambda \subseteq [0,\infty)$ and a positive real number δ such that, for each $\lambda \in \Lambda$, BVP (1), (2) has at least three classical solutions whose norms in X are less than δ.*

For $n = 1$, let

$$\tilde{F}(t,x) = \int_0^x f(t,\xi)d\xi. \tag{12}$$

The following corollaries are direct consequences of Theorem 2.1.

Corollary 2.1 *Assume (H1) and the following conditions hold:*

(A1) there exist two positive constants c and d with $c < d$ such that

$$\tilde{F}(t,x) \geq 0 \quad \text{for } (t,x) \in S_{1,1}(d) \cup S_{2,1}(d)$$

and

$$\max_{t\in[0,1],|x|<\rho c(\kappa)^{1/p}} \tilde{F}(t,x) \leq \left(\frac{c}{d}\right)^p \int_{t_1/2}^{(1+t_m)/2} \tilde{F}(s,d)ds,$$

where $S_{1,1}$ and $S_{2,1}$ are defined by (8) and (9), respectively, with $n = 1$, ρ is defined by (11), and κ is defined by (10) with $p_i = p$, i.e.,

$$\kappa = 2^{p-1}\left(t_1^{1-p}\Big|1 - \sum_{j=1}^m a_j\Big|^p + (1-t_m)^{1-p}\Big|1 - \sum_{j=1}^m b_j\Big|^p\right); \tag{13}$$

(A2) there exist $\theta \in L^1(0,1)$ *and* $\gamma > 0$ *with* $\gamma < p$ *such that*

$$\tilde{F}(t,x) \le \theta(t)\,(1 + |x|^{\gamma})$$

for $t \in [0,1]$ *and* $x \in \mathbb{R}$.

Then there exist an open interval $\Lambda \subseteq [0,\infty)$ *and a positive real number* δ *such that, for each* $\lambda \in \Lambda$, *BVP* (5), (7) *has at least three classical solutions whose norms in* X *are less than* δ.

Corollary 2.2 *Assume (H1) and the following conditions hold:*

(B1) $f(t,x) = g(t)h(x)$ *with* $g(t)$ *and* $H(x) = \int_0^x h(\xi)d\xi$ *being nonnegative;*

(B2) there exist two positive constants c *and* d *with* $c < d$ *such that*

$$\max_{t\in[0,1]} g(t) \max_{|x|<\rho c(\kappa)^{1/p}} H(x) \le \left(\frac{c}{d}\right)^p H(d)(G((1+t_m)/2) - G(t_1/2)),$$

where ρ *is defined by* (11) *and* κ *is defined by* (13), *and* $G(t) = \int_0^t g(s)ds$.

(B3) there exists $\sigma > 0$ *and* $\gamma > 0$ *with* $\gamma < p$ *such that*

$$H(x) \le \sigma(1 + |x|^{\gamma}) \quad \text{for } x \in \mathbb{R}.$$

Then there exist an open interval $\Lambda \subseteq [0,\infty)$ *and a positive real number* δ *such that, for each* $\lambda \in \Lambda$, *BVP* (5), (7) *has at least three classical solutions whose norms in* X *are less than* δ.

Corollary 2.3 *Assume (A2) and the following condition hold:*

(C1) there exist two positive constants c *and* d *with* $c < d$ *such that*

$$\tilde{F}(t,x) \ge 0 \quad \text{for } (t,x) \in ([0,1/4] \cup [3/4,1]) \times [0,d]$$

and

$$\max_{t\in[0,1],|x|<2^{1-1/p}c} \tilde{F}(t,x) \le \left(\frac{c}{d}\right)^p \int_{1/4}^{3/4} \tilde{F}(s,d)ds.$$

Then there exist an open interval $\Lambda \subseteq [0,\infty)$ *and a positive real number* δ *such that, for each* $\lambda \in \Lambda$, *BVP* (5), (4) *has at least three classical solutions whose norms in* X *are less than* δ.

Remark 2.1 *Corollaries 2.1–2.3 improve and extend the main results in [3, 5, 6, 14]. In particular, for the case when* $p = 2$, *by taking* $\bar{c} = \sqrt{2}c$ *and* $\bar{d} = d$, *it is easy to see that Proposition 1.1 is a special case of Corollary 2.3.*

We conclude this section with the following simple example.

Example Consider the BVP consisting of the equation

$$(|u'|u')' + \lambda t(e^u + 2u) = 0, \quad t \in (0,1), \tag{14}$$

and the three–point BCs

$$u(0) = u(1) = \frac{1}{2} u\left(\frac{1}{2}\right), \tag{15}$$

where λ is a real parameter.

We claim that there exist an open interval $\Lambda \subseteq [0,\infty)$ and a positive real number δ such that, for each $\lambda \in \Lambda$, BVP (14), (15) has at least three classical solutions whose norms in X are less than δ.

In fact, with $p = 3$, $m = 1$, $a_1 = b_1 = t_1 = 1/2$, and $f(t,x) = t(e^x + 2x)$, it is easy to see that BVP (14), (15) is of the form BVP (5), (7). Let $g(t) = t$ and $h(x) = e^x + 2x$. Then

$$\max_{t\in[0,1]} g(t) = 1, \ G(t) = \frac{t^2}{2}, \ \text{and } H(x) = e^x + x^2.$$

Clearly, (B1) and (B3) with $\sigma = 1$ and $\gamma = 2$ hold.

Let $c = 1$ and $d = 12$. For ρ and κ defined by (11) and (13), by a simple calculation, we have that $\rho = 3/2$, $\kappa = 4$,

$$\left(\frac{c}{d}\right)^p H(d)(G(1+t_1)/2 - G(t_1/2)) \approx 23.57,$$

and

$$\max_{t\in[0,1]} g(t) \max_{|x|<\rho c(\kappa)^{1/p}} H(x) \approx 16.49.$$

Then

$$\max_{t\in[0,1]} g(t) \max_{|x|<\rho c(\kappa)^{1/p}} H(x) \le \left(\frac{c}{d}\right)^p H(d)(G(1+t_1)/2 - G(t_1/2)),$$

i.e., (B2) holds. The conclusion now readily follows from Corollary 2.2.

Remark 2.2 *To the best of our knowledge, no known criteria in the literature can be applied to BVP* (14), (15) *to obtain the same conclusion as what we get here.*

3 Proofs of the Main Results

Lemma 3.1 *Assume (H1) holds. If $u = (u_1, \ldots, u_n) \in X$, then*

$$\max_{t\in[0,1]} |u_i(t)| \le \rho ||u_i'||_{p_i}, \quad i = 1, \ldots, n,$$

where ρ is defined by (11).

Proof. For $i = 1, \dots, n$ and $t \in [0, 1]$, from

$$u_i(t) = \int_0^t u_i'(s)ds + C_1,$$

it follows that

$$u_i(0) = C_1 \quad \text{and} \quad \sum_{j=1}^m a_j u_i(t_j) = \sum_{j=1}^m a_j \int_0^{t_j} u_i'(s)ds + C_1 \sum_{j=1}^m a_j.$$

Since $u_i(0) = \sum_{j=1}^m a_j u_i(t_j)$, we have

$$C_1 = \sum_{j=1}^m a_j \int_0^{t_j} u_i'(s)ds + C_1 \sum_{j=1}^m a_j.$$

Then,

$$C_1 = \frac{1}{1 - \sum_{j=1}^m a_j} \sum_{j=1}^m a_j \int_0^{t_j} u_i'(s)ds.$$

Thus,

$$u_i(t) = \int_0^t u_i'(s)ds + \frac{1}{1 - \sum_{j=1}^m a_j} \sum_{j=1}^m a_j \int_0^{t_j} u_i'(s)ds. \tag{16}$$

Similarly, from

$$u_i(t) = \int_1^t u_i'(s)ds + C_2 \quad \text{and} \quad u_i(1) = \sum_{j=1}^m b_j u_i(t_j),$$

we have that

$$u_i(t) = \int_1^t u_i'(s)ds + \frac{1}{1 - \sum_{j=1}^m b_j} \sum_{j=1}^m b_j \int_1^{t_j} u_i'(s)ds. \tag{17}$$

Now, (16) and (17) imply that

$$|u_i(t)| \le \int_0^t |u_i'(s)|ds + \frac{1}{|1 - \sum_{j=1}^m a_j|} \sum_{j=1}^m |a_j| \int_0^{t_j} |u_i'(s)|ds$$

and

$$|u_i(t)| \le \int_t^1 |u_i'(s)|ds + \frac{1}{|1 - \sum_{j=1}^m b_j|} \sum_{j=1}^m |b_j| \int_{t_j}^1 |u_i'(s)|ds.$$

Hence,

$$\begin{aligned} 2|u_i(t)| &\le \int_0^1 |u_i'(s)|ds + \frac{1}{|1-\sum_{j=1}^m a_j|}\sum_{j=1}^m |a_j|\int_0^{t_j}|u_i'(s)|ds \\ &\quad + \frac{1}{|1-\sum_{j=1}^m b_j|}\sum_{j=1}^m |b_j|\int_{t_j}^1 |u_i'(s)|ds \\ &\le \left(1+\frac{\sum_{j=1}^m |a_j|}{|1-\sum_{j=1}^m a_j|}+\frac{\sum_{j=1}^m |b_j|}{|1-\sum_{j=1}^m b_j|}\right)\int_0^1 |u_i'(s)|ds \\ &= 2\rho\int_0^1 |u_i'(s)|ds. \end{aligned}$$

Then, from Hölder's inequality,

$$|u_i(t)| \le \rho\int_0^1 |u_i'(s)|ds \le \rho\left(\int_0^1 |u'(s)|^{p_i}\right)^{1/p_i} = \rho||u_i'||_{p_i} \quad \text{on } [0,1].$$

This completes the proof of the lemma.

Lemma 3.2 *Assume (H1)–(H3) hold. Then there exists* $w = (w_1, \ldots, w_n) \in X$ *such that*

$$\sum_{i=1}^n \frac{||w_i'||_{p_i}^{p_i}}{p_i} > r$$

and

$$\sum_{i=1}^n \frac{||w_i'||_{p_i}^{p_i}}{p_i} \max_{(t,x_1,\ldots,x_n)\in K} F(t,x_1,\ldots,x_n) < \sum_{i=1}^n \frac{\kappa_i c^{p_i}}{p_i}\int_0^1 F(s,w_1(s),\ldots,w_n(s))ds,$$

where κ_i *is defined by* (10) *and*

$$r = \sum_{i=1}^n \frac{\kappa_i c^{p_i}}{p_i}.$$

Proof. For $i = 1, \ldots, n$, let

$$w_i(t) = \begin{cases} d\left(\sum_{j=1}^m a_j + \frac{2\left(1-\sum_{j=1}^m a_j\right)}{t_1}t\right), & t\in[0,t_1/2], \\ d, & t\in[t_1/2,(1+t_m)/2], \\ d\left(\frac{2-\sum_{j=1}^m b_j-\left(\sum_{j=1}^m b_j\right)t_m}{1-t_m} - \frac{2\left(1-\sum_{j=1}^m b_j\right)}{1-t_m}t\right), & t\in[(1+t_m)/2,1]. \end{cases}$$

Let $w(t) = (w_1(t), \ldots, w_n(t))$. Then, $w \in X$, and by a simple calculation, we obtain that

$$\begin{aligned} ||w_i'||_{p_i}^{p_i} &= \int_0^1 |w_i'(s)|^{p_i} ds \\ &= \left(\int_0^{t_1/2} \left(\frac{2|1 - \sum_{j=1}^m a_j|}{t_1} \right)^{p_i} ds + \int_{(1+t_m)/2}^1 \left(\frac{2|1 - \sum_{j=1}^m b_j|}{1 - t_m} \right)^{p_i} ds \right) d^{p_i} \\ &= 2^{p_i - 1} \left(t_1^{1-p_i} \left| 1 - \sum_{j=1}^m a_j \right|^{p_i} + (1 - t_m)^{1-p_i} \left| 1 - \sum_{j=1}^m b_j \right|^{p_i} \right) d^{p_i} = \kappa_i d^{p_i}. \end{aligned} \tag{18}$$

Thus, in view of the fact that $c < d$, we have

$$\sum_{j=1}^m \frac{||w_i||_{p_i}^{p_i}}{p_i} = \sum_{j=1}^m \frac{\kappa_i d^{p_i}}{p_i} > \sum_{j=1}^m \frac{\kappa_i c^{p_i}}{p_i} = r.$$

Note that for $i = 1, \ldots, n$,

$$d \sum_{j=1}^m a_j \le w_i(t) \le d \quad \text{on } [0, t_1, /2] \quad \text{if } \sum_{j=1}^m a_j < 1,$$

$$d \le w_i(t) \le d \sum_{j=1}^m a_j \quad \text{on } [0, t_1, /2] \quad \text{if } \sum_{j=1}^m a_j > 1,$$

$$d \sum_{j=1}^m b_j \le w_i(t) \le d \quad \text{on } [(1 + t_m)/2, 1] \quad \text{if } \sum_{j=1}^m b_j < 1,$$

and

$$d \le w_i(t) \le d \sum_{j=1}^m b_j \quad \text{on } [(1 + t_m)/2, 1] \quad \text{if } \sum_{j=1}^m b_j > 1.$$

Then, from (H3) and (18), it follows that

$$\begin{aligned} & \sum_{i=1}^n \frac{\kappa_i c^{p_i}}{p_i} \int_0^1 F(s, w_1(s), \ldots, w_n(s)) ds \\ \ge & \sum_{i=1}^n \frac{\kappa_i c^{p_i}}{p_i} \int_{t_1/2}^{(1+t_m)/2} F(s, w_1(s), \ldots, w_n(s)) ds \\ \ge & \sum_{i=1}^n \frac{\kappa_i d^{p_i}}{p_i} \max_{(t, x_1, \ldots, x_n) \in K} F(t, x_1, \ldots, x_n) \\ = & \sum_{i=1}^n \frac{||w_i'||_{p_i}^{p_i}}{p_i} \max_{(t, x_1, \ldots, x_n) \in K} F(t, x_1, \ldots, x_n). \end{aligned}$$

This completes the proof of the lemma.

In what follows, for $i = 1, \ldots, n$, let $\phi_{p_i}^{-1}$ denote the inverse of ϕ_{p_i}. Then, $\phi_{p_i}^{-1}(x) = \phi_{q_i}(x)$, where $1/p_i + 1/q_i = 1$. Clearly, ϕ_{p_i} is increasing on $\mathbb{R}$, and

$$\lim_{x\to-\infty} \phi_{p_i}(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} \phi_{p_i}(x) = \infty. \tag{19}$$

Lemma 3.3 *For any fixed $\lambda \in \mathbb{R}$, $u = (u_1, \ldots, u_n) \in (C[0,1])^n$, and $i = 1, \ldots, n$, define $\alpha_i(\cdot; u) : \mathbb{R} \to \mathbb{R}$ by*

$$\begin{aligned} \alpha_i(x; u) &= \int_0^1 \phi_{p_i}^{-1}\left(x - \lambda \int_0^\tau f_i(s, u_1(s), \ldots, u_n(s))ds\right) d\tau \\ &\quad + \sum_{j=1}^m a_j u_i(t_j) - \sum_{j=1}^m b_j u_i(t_j). \end{aligned}$$

Then the equation

$$\alpha_i(x; u) = 0 \tag{20}$$

has a unique solution $x_{u,i}$.

Proof. From (19), we see that

$$\lim_{x\to-\infty} \alpha(x; u) = -\infty \quad \text{and} \quad \lim_{x\to\infty} \alpha(x; u) = \infty.$$

Hence, the existence and uniqueness of a solution of Eq. (20) follow from the fact that $\alpha(\cdot; u)$ is continuous and increasing on $\mathbb{R}$.

Lemma 3.4 *The function $u(t) = (u_1(t), \ldots, u_n(t))$ is a solution of BVP (1), (2) if and only if $u_i(t)$, $i = 1, \ldots, n$, is a solution of the system of the integral equations*

$$u_i(t) = \sum_{j=1}^m a_j u_i(t_j) + \int_0^t \phi_{p_i}^{-1}\left(x_{u,i} - \lambda \int_0^\tau f_i(s, u_1(s), \ldots, u_n(s))ds\right) d\tau, \; i = 1, \ldots, n,$$

where $x_{u,i}$ is the unique solution of Eq. (20).

Proof. This can be verified by direct computations.

Proof of Theorem 2.1. For $u = (u_1, \ldots, u_n) \in X$, let

$$\Phi(u) = \sum_{i=1}^n \frac{||u_i'||_{p_i}^{p_i}}{p_i}$$

and

$$\Psi(u) = -\int_0^1 F(s, u_1(s), \ldots, u_n(s))ds.$$

Then, Φ is a continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and Ψ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. In particular, for $u = (u_1, \ldots, u_n) \in X$ and $v = (v_1, \ldots, v_n) \in X$, we have

$$\Phi'(u)(v) = \int_0^1 \sum_{i=1}^n \phi_{p_i}(u_i'(s))v_i'(s)ds$$

and

$$\Psi'(u)(v) = -\int_0^1 \sum_{i=1}^n f_i(s, u_1(s), \ldots, u_n(s))v_i(s)ds.$$

Hence, the weak solutions of BVP (1), (2) are exactly the solutions of the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0. \tag{21}$$

Let $\lambda \in [0, \infty)$. From (H4) and Lemma 3.1, we see that

$$\begin{aligned}\Phi(u) + \lambda\Psi(u) &\geq \sum_{i=1}^n \frac{||u_i'||_{p_i}^{p_i}}{p_i} - \lambda\int_0^1 \theta(s)\left(1 + \sum_{i=1}^n |u_i(s)|^{\gamma_i}\right)ds \\ &\geq \sum_{i=1}^n \frac{||u_i'||_{p_i}^{p_i}}{p_i} - \lambda\left(1 + \rho\sum_{i=1}^n ||u_i'||_{p_i}^{\gamma_i}\right)\int_0^1 \theta(s)ds.\end{aligned}$$

Since $\gamma_i < p_i$ for $i = 1, \ldots, n$, we have that

$$\lim_{||u||\to\infty}(\Phi(u) + \lambda\Psi(u)) = \infty.$$

Lemma 3.1 implies that

$$\sup_{t\in[0,1]} \sum_{i=1}^n \frac{|u_i(t)|^{p_i}}{p_i} \leq \rho^{p_i}\sum_{i=1}^n \frac{||u_i'||_{p_i}^{p_i}}{p_i}. \tag{22}$$

Let w and r be as introduced in Lemma 3.2. Then,

$$\Phi(w) = \sum_{i=1}^n \frac{||w_i'||_{p_i}^{p_i}}{p_i} > r = \sum_{i=1}^n \frac{\kappa_i c^{p_i}}{p_i}.$$

Let $u_0 = (0, \ldots, 0)$, $u_1 = w$, and $J = -\Psi$. Then, in view of (H5), it is clear that

$$\Phi(u_0) = J(u_0) = 0, \quad \Phi(u_1) > r.$$

Moreover, from Lemma 3.2 and (22), it follows that

$$\begin{aligned}
\sup_{u\in\Phi((-\infty,r])}(J(u)) &= \sup_{u\in\Phi((-\infty,r])}(-\Psi(u))\\
&= \sum_{\sum_{i=1}^{n}\frac{\|u_i'\|_{p_i}^{p_i}}{p_i}\le r}\int_0^1 F(s,u_1(s),\ldots,u_n(s))ds\\
&\le \max_{(t,x_1,\ldots,x_n)\in K} F(t,x_1,\ldots,x_n)\\
&< \frac{\sum_{i=1}^{n}\frac{\kappa_i c^{p_i}}{p_i}}{\sum_{i=1}^{n}\frac{\|w_i'\|_{p_i}^{p_i}}{p_i}}\int_0^1 F(s,w_1(s),\ldots,w_n(s))ds\\
&< r\frac{-\Psi(w)}{\Phi(w)} = r\frac{J(u_1)}{\Phi(u_1)}.
\end{aligned}$$

Fix η such that

$$\sup_{u\in\Phi((-\infty,r])}(J(u)) < \eta < r\frac{J(u_1)}{\Phi(u_1)}$$

and define $h(\lambda)=\lambda\eta$ for $\lambda\ge 0$. Then, from Lemma 1.1, we have

$$\sup_{\lambda\ge 0}\inf_{u\in X}(\Phi(u)+\lambda\Psi(u)+h(\lambda)) < \inf_{u\in X}\sup_{\lambda\ge 0}(\Phi(x)+\lambda\Psi(x)+h(\lambda)).$$

Therefore, by Lemma 2, there exist an open interval $\Lambda\subseteq[0,\infty)$ and a positive real number δ such that, for each $\lambda\in\Lambda$, Eq. (21) has at least three solutions

$$u^k=(u_{k1},\ldots,u_{kn}),\quad k=1,2,3,$$

in X that are three weak solutions of BVP (1), (2) and whose norms are less than δ. Then, for $k=1,2,3$, and $i=1,\ldots,n$, we have

$$\int_0^1\sum_{i=1}^{n}\phi_{p_i}(u_{ki}'(s))v_i'(s)ds-\lambda\int_0^1\sum_{i=1}^{n}f_i(s,u_{k1}(s),\ldots,u_{kn}(s))v_i(s)ds=0 \tag{23}$$

for any $v=(v_1,\ldots,v_n)\in W_0^{1,p_1}([0,1])\times W_0^{1,p_2}([0,1])\times\ldots\times W_0^{1,p_n}([0,1])$. Recall that, in one dimension, any weakly differentiable function is absolutely continuous, so that its classical derivative exists almost everywhere, and that the classical derivative coincides with the weak derivative. Now, applying integration by parts to (23) yields that

$$\sum_{i=1}^{n}\int_0^1((\phi_{p_i}(u_{ki}'(s)))'+\lambda f_i(s,u_{k1}(s),\ldots,u_{kn})(s))v_i(s)ds=0,$$

from which it follows that

$$(\phi_{p_i}(u_{ki}'))'+\lambda f_i(t,u_{k1},\ldots,u_{kn})=0\quad\text{for a.e. } t\in(0,1). \tag{24}$$

Then, by Lemmas 3.3 and 3.4, it is easy to see that

$$u_{ki}(t) = \sum_{j=1}^{m} a_j u_{ki}(t_j) + \int_0^t \phi_{p_i}^{-1}\left(x_{u^k,i} - \lambda \int_0^\tau f_i(s, u_{k1}(s), \ldots, u_{kn}(s))ds \right) d\tau,$$

where $x_{u^k,i}$ is the unique solution of Eq. (20) with $u = u^k$. Consequently, $u_{ki} \in C^1[0,1]$ and $\phi(u'_{ki}) \in C^1[0,1]$, i.e., u^k, $k = 1, 2, 3$, are classical solutions of BVP (1), (2). This completes the proof of the theorem.

Proof of Corollary 2.1. The conclusion follows directly from Theorem 2.1.

Proof of Corollary 2.2. Let $\tilde{F}$ be defined by (12). Then, from (B1) and (B2), we see that

$$\tilde{F}(t,x) = \int_0^x g(t)h(\xi)d\xi = g(t)H(x) \geq 0 \quad \text{for } (t,x) \in [0,1] \times \mathbb{R}$$

and

$$\begin{aligned}
\max_{t\in[0,1],|x|<(\rho\kappa)^{1/p}c} \tilde{F}(t,x) &= \max_{t\in[0,1],|x|<\rho c(\kappa)^{1/p}} \int_0^x g(t)h(\xi)d\xi \\
&= \max_{t\in[0,1]} g(t) \max_{|x|<\rho c(\kappa)^{1/p}} H(x) \\
&\leq \left(\frac{c}{d}\right)^p H(d)(G((1+t_m)/2) - G(t_1/2)) \\
&= \left(\frac{c}{d}\right)^p \int_0^d h(\xi)d\xi \int_{t_1/2}^{(1+t_m)/2} g(s)ds \\
&= \left(\frac{c}{d}\right)^p \int_{t_1/2}^{(1+t_m)/2} \int_0^d g(s)h(\xi)d\xi ds \\
&= \left(\frac{c}{d}\right)^p \int_{t_1/2}^{(1+t_m)/2} \int_0^d f(s,\xi)d\xi ds \\
&= \left(\frac{c}{d}\right)^p \int_{t_1/2}^{(1+t_m)/2} \tilde{F}(s,d)ds.
\end{aligned}$$

Thus, (A1) holds. By (B1) and (B3), we have

$$\tilde{F}(t,x) = \int_0^x g(t)h(\xi)d\xi = g(t)H(x) \leq \sigma g(t)(1 + |x|^\gamma)$$

for $(t,x) \in [0,1] \times \mathbb{R}$. Then (A2) holds with $\theta(t) = \sigma g(t)$. The conclusion then readily follows from Corollary 2.1.

Proof of Corollary 2.3. By taking $m = 1$, $a_1 = b_1 = 0$, and $t_1 = 1/2$, it is easy to see that the assumption (A1) reduces to (C1). The conclusion then readily follows from Corollary 2.1.

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