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SOME NONLINEAR AND NONLOCAL STURM-LIOUVILLE PROBLEMS MOTIVATED BY THE PROBLEM OF FLUTTER

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

We study three internally connected Sturm-Liouville problems for nonlinear ordinary differential equations that are motivated by the problem of aeroelastic instability. Solutions are analyzed and asymptotic results are presented. A numerical study using a development of simple shooting reveals the spectrum and corresponding eigenfunctions.

Key words and phrases: Nonlinear boundary value problems, Sturm-Liouville theory, instability, Mach number.
AMS (MOS) Subject Classifications: 34B15, 34B24, 34C15, 58F10, 34L15, 34A45, 65L10

1 Introduction

The problem of instability of a wing in a supersonic flow, often known as flutter, has already attracted the attention of many researchers in both mathematics and engineering. Our project is motivated by the contributions of two distinguished authors in this area. Graef et al. [1, 2] studied stability (with respect to twisting) of an elastic wing in an airflow. This problem is well-known in engineering (see, e.g. [3]). The problem of stability is reduced in [1, 2] to a nonlinear Sturm-Liouville problem (SLP) for the angle of rotation of the wing about the point of its attachment to the airplane with Mach number M as the spectral parameter. The Liapunov function approach is used. A connection between the slope s of the wing at the point of attachment and the critical Mach number M is studied. Diverse numerical experiments were conducted for (M, s)-dependence. Comparisons between nonlinear and linear descriptions of stability were made. It is shown that the linear approximation is insufficient for some important ranges of the parameters.

Stability of an elastic wing (or panel) in an airflow may be lost by a mechanism other than the twisting studied by Graef et al. Librescu et al.[1] studied another sophisticated model [9, 8]. Specifics will be given below but for now we only mention that the problem in [9, 8] is described by a nonlinear and nonlocal partial differential equation (PDE) and that it is solved by Galerkin's method. Again the Mach number M appears as a parameter and the range of critical values of M represents one of the goals of this study.

In this paper we consider Sturm-Liouville problems for nonlinear ordinary differential equations (ODE) that represent a variation of both models above. Its structure resembles the structure of the problem considered by Graef et al. but also includes a non-local term of the type considered by Librescu et al. We call this model *fully nonlinear*. We also consider two simplified models, one of the type considered by Graef et al. (we call it *autonomous*) and one without the nonlinear term but retaining the nonlocal term of the type introduced by Librescu et al. (we call it *semilinear*). For all three models we study the connection between the Mach number (i.e. the spectral parameter) and the slope of the solution at the beginning of the interval as well as some properties of the eigenfunctions. We make some comparisons between the results for all three models. We view our SLP as a stability problem and hence we mostly are interested in the first eigenvalue and the corresponding mode. Our approach is partially analytic and partially numeric, in the style of the papers [1, 2] and [9, 8].

In Section 2 we briefly describe the models studied by Graef et al. and Librescu et al. We further formulate the three SLPs under consideration. In Section 3 we study the semilinear SLP. Then in Section 4 we develop results for the autonomous SLP, mostly (but not completely) following Graef et al. from [1, 2]. In Section 5 we study the fully nonlinear model and compare results for all three models in Section 6.

2 The Autonomous and Semilinear Models

In [1, 2], Graef et al. consider the stability of an elastic wing twisting in an airflow. The problem is reduced to finding nontrivial solutions of the SLP $L_A(y, \lambda) = 0$ where:

$$L_A(y, \lambda) = y'' + \lambda y + 2\lambda^3 y^3, y : (0, \ell) \to \mathbb{R} \tag{1}$$

such that

$$y(0) = y(\ell) = 0. \tag{2}$$

The alternative boundary conditions (BC)

$$y(0) = y'(0) = 0 \tag{3}$$

[1]Professor Liviu Librescu died after giving his students an opportunity to escape during the Virginia Tech shootings in Spring 2007. The first of the authors was honored to discuss the problem of aeroelastic stability with Professor Librescu during 2005.

are also considered. The function $y(x)$ is proportional to the angle of rotation of the wing and λ is proportional to the Mach number. For simplicity we will abuse terminology below and simply refer to λ as the Mach number.

It is shown that the function

$$\mathcal{L} = \frac{(y')^2}{2} + \lambda \frac{y^2}{2} + \lambda \frac{y^4}{2} \tag{4}$$

is the Liapunov function for the ODE (1) and hence by (2)

$$\mathcal{L}(y) = \frac{s_0^2}{2}, \tag{5}$$

where $s_0 = y'(0)$ is the slope of the eigenfunction at the left boundary. Solutions to (4)-(5) may be expressed in terms of elliptic integrals.

The second of the BC (2) results in some dependence between the critical Mach number λ and the slope s_0.

Equation (1) describes the stability of a wing when third-order *piston theory* is used. Graef et al. also consider a fifth-order piston theory approximation, but that case is not considered in this work.

In [9, 8], Librescu et al. use third-order piston theory and the geometrically nonlinear elastic shell theory to model the elastic stability of a cylindrical panel (e.g. a wing) in an airflow. A (simplified) version of the governing PDE used in [9, 8] is as follows,

$$y_{xxxx} - 6 \left(\int_0^1 y_x^2 \, dx \right) y_{xx} + K \left\{ (\lambda y_x + \zeta y_t) + \epsilon \left(\lambda y_x + \zeta y_t \right)^3 \right\} + \tau_1^4 y_{tt} = 0. \tag{6}$$

The boundary is assumed to be simply supported,

$$y(0,t) = y_{xx}(0,t) = y(1,t) = y_{xx}(1,t) = 0. \tag{7}$$

Here $x \in [0,1]$, $t \in [0,\infty)$ are a dimensionless coordinate and time, $y(x,t) : [0,1] \times [0,\infty) \to \mathbb{R}$ is the dimensionless transverse deflection of the panel, and K, ζ, ϵ are (positive) constants. For the sake of simulation, the values of the parameters used in [9, 8] would produce a very small value for ϵ, of the order of 10^{-4} to 10^{-5}. If we consider the stationary problem, $\partial/\partial t = 0$, we come up with the SLP, $L_S(y,\lambda) = 0$,

$$L_S(y,\lambda) = y'''' - 6 \left(\int_0^1 (y')^2 \, dx \right) y'' + K \left(\lambda y' + \epsilon \lambda^3 (y')^3 \right), \tag{8}$$

$$y(0) = y''(0) = y(1) = y''(1) = 0, \tag{9}$$

which is similar to the SLP (1)-(2), yet differs by an essential nonlocal term.

The parameter λ is given in terms of the Mach number M, $\lambda = \frac{M^2}{\sqrt{M^2-1}} \approx M$ for $M \gg 1$. Below we will refer to λ simply as the Mach number. Librescu et al. study the problem (6)-(7) by Galerkin's method and find conditions for stability in terms of the Mach number λ.

Moreover, the physical parameter ϵ is taken as a fixed but small constant. It is not a small parameter as might be used in perturbation theory.

The stability problems considered in [1, 2, 3, 9, 8] require that the eigenfunction $y(x)$ be a positive solution on $x \in (0,1)$. For this reason, we assume that $y'(0) = s_0 > 0$ throughout this work. Further the aforementioned relationship between the spectral parameter λ and the Mach number M requires that only eigenfunctions for which λ is positive are considered. Though the model considered in [9, 8] was developed for large or moderate M, mathematically it is more convenient to remove this engineering restriction and study $\lambda \in (0, \infty)$.

Lemma 2.1 *The SLP (8) has no nontrivial solutions for any values of the parameters K, ϵ and any (positive) Mach number λ.*

Proof. Multiplying (8) by $y(x)$, integrating over $(0,1)$, and then integrating by parts yields

$$(y'''y - y''y')|_0^1 + \|y''\|^2 - 6\|y'\|^2 \left(y'y|_0^1 - \|y'\|^2 \right) + K \left(\lambda \left. \frac{y^2}{2} \right|_0^1 + \epsilon\lambda^3 \left. \frac{y^4}{4} \right|_0^1 \right) = 0.$$

Here and below $\|\cdot\|$ denotes the $L^2(0,1)$ norm. Using the boundary conditions (9) yields

$$\|y''\|^2 + 6\|y'\|^4 = 0.$$

This completes the proof.

The differential operators $L_A(y,\lambda)$ and $L_S(y,\lambda)$ are different yet share some similarities. This comparison moves the authors of this paper to study the following SLP, $L(y,\lambda) = 0$,

$$L(y,\lambda) = y'' - \left(\int_0^1 y^2\,dx \right) y' + \lambda y + \epsilon\lambda^3 y^3, \tag{10}$$

such that

$$y(0) = y(1) = 0 \tag{11}$$

and ϵ is a small number.

We also introduce two operators that represent some reductions of L. We first observe that if we neglect the integral term in L we get precisely the operator studied by Graef et al. in [1, 2],

$$L_A(y,\lambda) = y'' + \lambda y + \epsilon\lambda^3 y^3. \tag{12}$$

(Compare with (1) above.) And if we neglect the cubic term in L then we obtain a semilinear operator

$$L_{SL}(y,\lambda) = y'' - \left(\int y^2\,dx\right) y' + \lambda y. \tag{13}$$

(Compare with the analogous, but higher order, (8) above.)

Our goal is to study the (λ, s_0)-dependence, where $s_0 = y'(0)$, and to compare the properties of the three SLPs associated with the operators L, L_A, and L_{SL}.

3 Semilinear SLP (Approximation $L \approx L_{SL}$)

We consider the semilinear SLP

$$L_{SL}(y,\lambda) = y'' - \|y\|^2 y' + \lambda y = 0, x \in (0,1), \tag{14}$$

$$y(0) = y(1) = 0. \tag{15}$$

If we temporarily assume $\|y\|$ to be known, then the first eigenfunction of the SLP (14)-(15) has the form

$$y(x) = Ce^{\frac{\|y\|^2}{2}x}\sin(\pi x) \tag{16}$$

with an arbitrary constant C while the first eigenvalue is

$$\lambda = \pi^2 + \frac{\|y\|^4}{4}. \tag{17}$$

We use s_0 as previously defined and introduce $s_1 = y'(1)$. Clearly $C = \frac{s_0}{\pi}$. Substituting $y(x)$ into $\|y\|^2$ yields

$$C^2 \int_0^1 e^{\|y\|^2 x}\sin^2(\pi x)\,dx = \|y\|^2$$

or

$$s_0^2 = \frac{\|y\|^4\left(\|y\|^4 + 4\pi^2\right)}{2\left(e^{\|y\|^2} - 1\right)}. \tag{18}$$

Equations (17)-(18) provide a parametric representation of the (λ, s_0)-dependence we seek. The (positive) parameter $\|y\|^2$ may be eliminated from (17)-(18) to produce an explicit representation $s_0 = S(\lambda)$.

Below we use the standard asymptotic notation from [4]. In particular if ζ is a small parameter, then $\eta = o(\zeta)$ means that $\eta/\zeta \to 0$ as $\zeta \to 0$ and $f(\zeta) \sim g(\zeta)$ means that $f(\zeta) - g(\zeta) = o(\zeta)$. The following result follows from such an asymptotic analysis. Additionally, we introduce the notation $(\hat{x}, \hat{y})$ to refer to the single maximum of the first mode.

Lemma 3.1

(i) If $\|y\| \to 0$*, then* $s_0 \to 0$*,* $\lambda \sim \pi^2 + \frac{s_0^4}{16\pi^4}$*,* $y(x) \sim \frac{s_0}{\pi}\sin(\pi x)$*,* $\hat{y} \sim \frac{s_0}{\pi}$*.*

(ii) If $\|y\| \to \infty$*, then* $s_0 \to 0$*,* $\lambda \to \infty$*,* $s_0 \sim \frac{4(\lambda-\pi^2)}{\sqrt{2}} e^{-\sqrt{\lambda-\pi^2}}$*,* $\hat{y} \sim \frac{4\sqrt{\lambda-\pi^2}}{\sqrt{2}e}$*.*

(iii) The function $s_0 = S(\lambda)$ *has a single maximum,* $s^* = S(\lambda^*)$ *in the neighborhood of which* $\lambda \sim \lambda^* \pm C_1\sqrt{s^* - s_0}$ *with some constant* C_1*.*

To complement, and extend, the results obtained by the analysis above, solutions of the SLP (14)-(15) were obtained numerically. The numerical results agree with the analytical results in the lemma above.

For a standard second order, linear boundary value problem, a standard simple shooting method [12] would suffice to obtain numerical solutions. However to account for the nonlocal first-order term in (14) we suggest a development of this approach. A standard simple shooting method rewrites the boundary value problem (BVP) as an initial value problem (IVP) and then defines a nonlinear function F whose zeros are solutions of the IVP which also satisfy the boundary conditions in the BVP. More precisely, if we denote by $y \equiv y(x; s_0)$ the solution of (14) with the initial conditions

$$y(0) = 0, y'(0) = s_0$$

then we define F by $F(s_0) \equiv y(1; s_0)$. Therefore F has a root when y satisfies (14) and the boundary conditions (15) also hold. Those roots can be found relatively easily with a standard nonlinear solver such as Newton's method with finite difference approximation of F' (as an explicit formula for F' is not typically available).

The key innovation here is to rewrite the problem slightly to accommodate the nonlocal nature of (14). We introduce an unknown scalar C and solve the related BVP

$$y'' - Cy' + \lambda y = 0, x \in (0, 1) \tag{19}$$

with the same boundary conditions (15). Then the nonlinear function becomes

$$F\left(\begin{bmatrix} s_0 \\ C \end{bmatrix}\right) = \begin{bmatrix} y(1; s_0) \\ C - \|y(x; s_0)\|^2 \end{bmatrix}. \tag{20}$$

Again Newton's method with finite difference Jacobians can find the roots of F and thereby numerical solutions of (14)-(15).

The lemma above gives some indication of the complex interplay between s_0 and λ (which are also connected with $\|y\|$), and the numerical resolution of (14)-(15) provides a convenient means to visualize that relationship. The graph on the left in Figure 1 shows a curve in the (λ, s_0)-plane for which solutions of (14)-(15) exist. The value of $\|y\|$ is monotonic increasing along this curve moving from left to right. Notice that for a given value of $y'(0) = s_0$ there may exist a pair of values of λ for which the SLP (14)-(15) has a solution. A pair of such values has been highlighted in the left graph

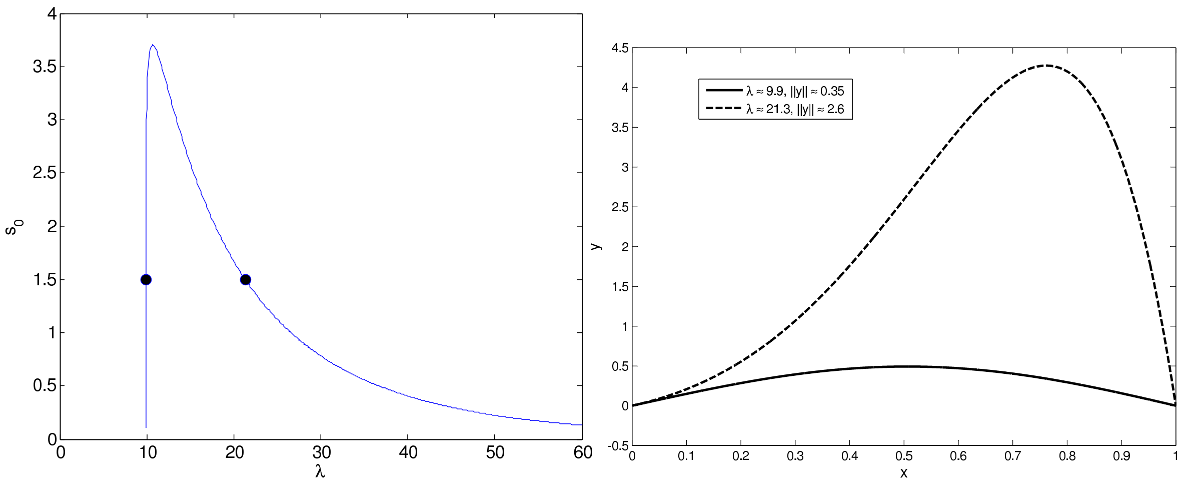

Figure 1: (left) A curve in the (λ, s_0)-plane for which (14)-(15) has a positive solution. (right) The solutions corresponding to the two points highlighted in the graph on the left. For both solutions $y'(0) = s_0 \approx 1.5$, so they share the same slope at the left boundary. However the value of $\|y\|$ is obviously different for each, and in fact they also correspond to different eigenvalues λ.

of Figure 1 and the corresponding solutions of (14)-(15) for those values of s_0 and λ are shown in the right graph.

The spectrum shown on the left in Figure 1 corresponds to the first mode. The solutions y corresponding to the points along the curve in the (λ, s_0)-plane are positive solutions of the given semilinear SLP. With further analysis we can describe similar curves in the (λ, s_0)-plane corresponding to higher modes, none of which are positive solutions. We note in passing that the (λ, s_0)-curve for one mode intersects the (λ, s_0)-curve for the next highest mode.

Finally, the solutions in the right graph of Figure 1 fit well with the asymptotic results of Lemma 3.1. In particular, the maximum of each eigenfunction is easily observed to be in line with the estimates given above.

4 Autonomous SLP (Approximation $L \approx L_A$)

We consider the autonomous SLP

$$L_A(y, \lambda) = y'' + \lambda y + \epsilon \lambda^3 y^3 = 0, x \in (0, 1), \tag{21}$$

$$y(0) = y(1) = 0 \tag{22}$$

following mostly the approach used by Graef et al. in [1, 2], where almost the same SLP is considered. We introduce the Liapunov function for this model

$$\mathcal{L} = \frac{(y')^2}{2} + \lambda \frac{y^2}{2} + \epsilon \lambda^3 \frac{y^4}{4}. \tag{23}$$

Then using the first of the boundary conditions in (22) and $y'(0) = s_0(> 0)$, we arrive at

$$(y')^2 + \lambda y^2 + \frac{\epsilon\lambda^3}{2} y^4 = s_0^2. \tag{24}$$

The following results follow from a similar analysis in [1, 2].

Lemma 4.1

(i) The first eigenfunction of SLP (21)-(22) has a single maximum at $x = \frac{1}{2}, y\left(\frac{1}{2}\right) = \hat{y}$, *such that*

$$s_0^2 = \lambda \hat{y}^2 + \frac{\epsilon\lambda^3 \hat{y}^4}{2}. \tag{25}$$

(ii) $|y'(1)| = s_0$.

We may directly combine the results above into a single relation for y' as follows

$$y' = \pm\sqrt{\lambda}\sqrt{\hat{y}^2 - y^2}\sqrt{1 + \frac{\epsilon\lambda^2}{2}\left(\hat{y}^2 + y^2\right)}, \tag{26}$$

where the "+" is taken for $x \le \frac{1}{2}$ and the "−" is taken for $x \ge \frac{1}{2}$. If we utilize the right boundary condition in (22) and make the substitution $y = \hat{y}\sin(\theta)$, then we may integrate (26) to obtain

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 + \frac{\epsilon\lambda^2\hat{y}^2}{2}\left(1 + \sin^2(\theta)\right)}} = \frac{\sqrt{\lambda}}{2}. \tag{27}$$

Equations (25) and (27) give a parametric representation of the (λ, s_0)-dependence that we seek. The parameter $\hat{y}^2$ may be eliminated to produce a single implicit relation between λ and s_0. For the lemma below, recall that the positive solutions we seek require $s_0 > 0$.

Lemma 4.2 *(i) For a given* $\hat{y} > 0$ *there exists a unique couple* (λ, s_0) *that satisfies (25) and (27).*

(ii) $\lambda < \pi^2$; $\pi^2 - \lambda \sim s_0^2$ *as* $s_0 \to 0$ *and* $\lambda \sim s_0^{-2/3}$ *as* $s_0 \to \infty$.

(iii) $\hat{y} \sim \frac{s_0}{\pi}$ *as* $s_0 \to 0$.

Proof. For (i) see [1, 2].

To prove (ii) we first estimate the denominator of the left hand side of (27). The inequality $0 \le \sin^2(\theta) \le 1$ implies the following inclusion

$$\frac{\sqrt{\lambda}}{2} \in \left[\frac{\frac{\pi}{2}}{\sqrt{1 + \epsilon\lambda^2\hat{y}^2}}, \frac{\frac{\pi}{2}}{\sqrt{1 + \frac{1}{2}\epsilon\lambda^2\hat{y}^2}}\right]$$

or after simplification

$$\frac{\pi^2 - \lambda}{\epsilon \lambda^3 \hat{y}^2} \in \left[\frac{1}{2}, 1 \right], \tag{28}$$

which immediately implies that $\lambda < \pi^2$. Equation (25) implies that

$$\epsilon \lambda^2 \hat{y}^2 = -1 + \sqrt{1 + 2\epsilon \lambda s_0^2}. \tag{29}$$

Combining (28) and (29) yields the inclusion

$$\frac{\pi^2 - \lambda}{\lambda \left(-1 + \sqrt{1 + 2\epsilon \lambda s_0^2} \right)} \in \left[\frac{1}{2}, 1 \right]. \tag{30}$$

Let $s_0 \to 0$. Since λ is bounded the quantity $\frac{\pi^2-\lambda}{\lambda^2 s_0^2}$ is bounded above which implies $\lambda \sim \pi^2$ and hence $\pi^2 - \lambda \sim s_0^2$. Now let $s_0 \to \infty$. Since λ is bounded the quantity $\frac{\pi^2-\lambda}{\lambda\sqrt{\lambda} s_0}$ is bounded below which implies that $\lambda \sim s_0^{-2/3}$. This establishes all of (ii).

Finally letting $s_0 \to 0$ in (25) and using (ii) immediately implies (iii). This completes the proof.

We note the similarity between the estimates for the maximum of the first mode, $\hat{y}$, given for the semilinear problem in Section 3 and here for the autonomous problem.

As with the semilinear problem, a numerical treatment fills in the part of the curve in the (λ, s_0)-plane between the limiting cases described by the asymptotic results of Lemma 4.2. Using the fixed value $\epsilon = 10^{-4}$ and an approach similar to the one described in Section 3, numerical solutions to the autonomous SLP (21)-(22) can be found. The corresponding (λ, s_0)-curve is shown in Figure 2. We note that for each $s_0 > 0$ there exists a unique λ on the spectrum for the autonomous SLP (21)-(22). This is not the case for the other two models – semilinear and fully nonlinear – that we consider in this work.

5 Fully Nonlinear SLP (Operator L)

We consider the SLP

$$L(y, \lambda) = y'' - \|y\|^2 y' + \lambda y + \epsilon \lambda^3 y^3 = 0, x \in (0,1), \tag{31}$$

$$y(0) = y(1) = 0. \tag{32}$$

The Liapunov function for the SLP above is

$$\mathcal{L} = \frac{(y')^2}{2} - \|y\|^2 \int_0^x (y')^2 \, d\eta + \lambda \frac{y^2}{2} + \epsilon \lambda^3 \frac{y^4}{4} \tag{33}$$

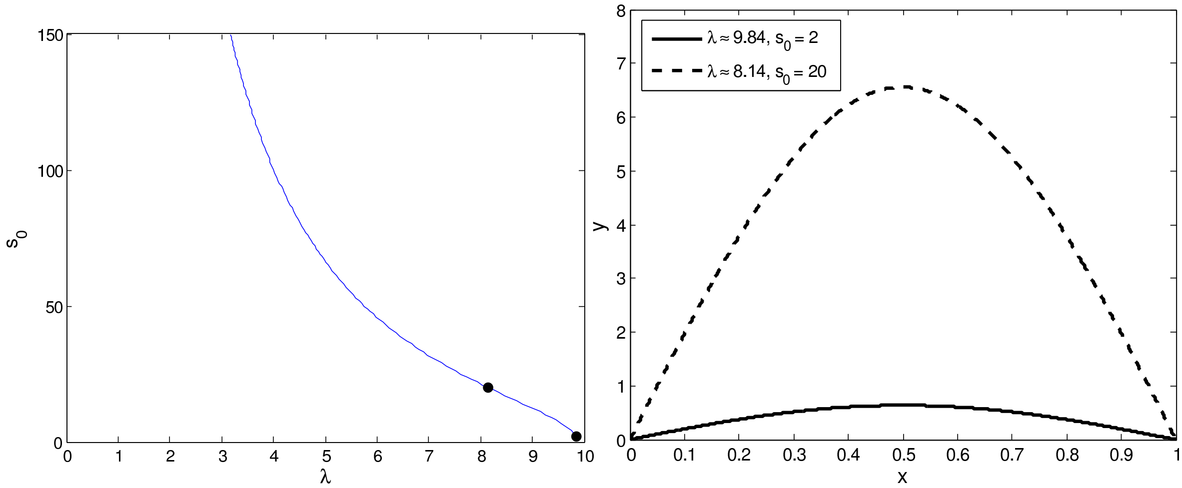

Figure 2: (left) The curve in the (λ, s_0)-plane for which (21)-(22) has a positive solution. (right) The solutions corresponding to the two points shown in the graph on the left.

and along with the slope $s_0 = y'(0)$ we obtain

$$(y')^2 - 2\|y\|^2 \int_0^x (y')^2 \, d\eta + \lambda y^2 + \frac{\epsilon\lambda^3}{2} y^4 = s_0^2. \tag{34}$$

This last identity, evaluated at $x = 1$ and using $s_1 = y'(1)$, yields

$$s_1^2 - 2\|y\|^2\|y'\|^2 = s_0^2. \tag{35}$$

For the point where the first eigenfunction has a maximum, $(\hat{x}, \hat{y})$, identity (34) allows us to obtain

$$\lambda\hat{y}^2 + \frac{\epsilon\lambda^3}{2}\hat{y}^4 = s_0^2 + 2\|y\|^2\|y'\|^2. \tag{36}$$

We also introduce the following SLP

$$\overline{L}(\overline{y}, \lambda) = \overline{y}'' + \|\overline{y}\|^2\overline{y}' + \lambda\overline{y} + \epsilon\lambda^3\overline{y}^3 = 0, x \in (0,1), \tag{37}$$

$$\overline{y}(0) = \overline{y}(1) = 0 \tag{38}$$

which is similar to (31)-(32), differing only in the sign of the first-order term.

Some properties of the first eigenfunctions for (31)-(32), as well as (37)-(38), are given by the following Lemma.

Lemma 5.1

(i)

$$s_0 < |s_1| < s_0 e^{\|y\|^2}. \tag{39}$$

In particular the eigenfunction $y(x)$ is asymmetric.

(ii)

$$|y'(x)| \le s_1, \forall x \in [0,1]. \tag{40}$$

(iii)

$$s_0^2 < \lambda \hat{y}^2 + \frac{\epsilon\lambda^3}{2}\hat{y}^4 = s_1^2. \tag{41}$$

(iv)

$$y(x) = \overline{y}(1-x). \tag{42}$$

Proof. The first of the inequalities, i.e. $s_0 < |s_1|$, follows immediately from (35). Denoting $z(x) = (y')^2(x)$ and $f(x) = s_0^2 - \lambda y^2(x) - \frac{\epsilon\lambda^3}{2}y^4(x)$ in (34) yields the following integral equation

$$z(x) - 2\|y\|^2 \int_0^x z(\eta)\,d\eta = f(x), \tag{43}$$

which may be reduced to the Cauchy problem

$$z'(x) - 2\|y\|^2 z(x) = f'(x), z(0) = s_0^2 (= f(0)). \tag{44}$$

The solution of this Cauchy problem is found in the standard way. Integrating by parts and utilizing the definition of $f(x)$ in the expression for the solution, we obtain

$$\begin{aligned} z(x) &= s_0^2 e^{2\|y\|^2 x} + \int_0^x e^{2\|y\|^2(x-\eta)} f'(\eta)\,d\eta \\ &= s_0^2 - \lambda y^2(x) - \frac{\epsilon\lambda^3}{2}y^4(x) + 2\|y\|^2 \int_0^x e^{2\|y\|^2(x-\eta)} \left(s_0^2 - \lambda y^2(\eta) - \frac{\epsilon\lambda^3}{2}y^4(\eta)\right) d\eta. \end{aligned}$$

Now evaluating the above expression at $x = 1$, recalling the notation $s_1 = y'(1)$, evaluating the first term in the integral, and estimating two other terms by zero yields

$$\begin{aligned} s_1^2 &= s_0^2 e^{2\|y\|^2} - 2\|y\|^2 \int_0^1 e^{2\|y\|^2(1-\eta)} \left(s_0^2 - \lambda y^2(\eta) - \frac{\epsilon\lambda^3}{2}y^4(\eta)\right) d\eta \\ &< s_0^2 e^{2\|y\|^2}. \end{aligned}$$

This establishes the right half of (i).

To prove inequality (ii) above, we subtract (35) from (34) to find

$$(y')^2 = s_1^2 - \left[\left(\lambda y^2 + \frac{\epsilon\lambda^3}{2}y^4\right) + 2\|y\|^2 \int_x^1 (y')^2\,d\eta\right] < s_1^2.$$

The left-hand side of (iii) immediately follows from (36) and the very same identity along with (35) further shows that

$$\lambda\hat{y}^2 + \frac{\epsilon\lambda^3}{2}\hat{y}^4 = s_0^2 + 2\|y\|^2\|y'\|^2 = s_1^2,$$

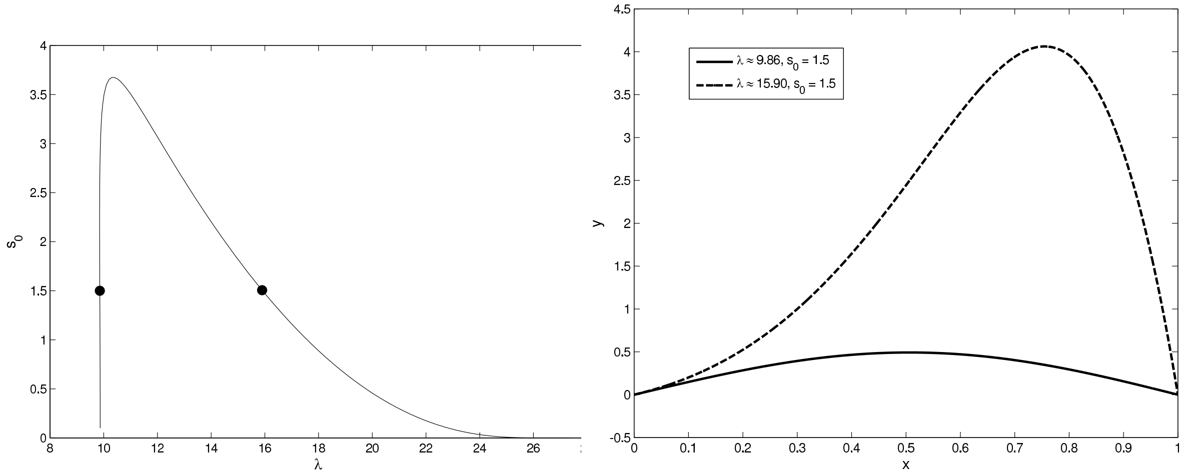

Figure 3: (left) A curve in the (λ, s_0)-plane for which (31)-(32) has a positive solution. (right) The solutions corresponding to the two points shown in the graph on the left.

completing the right-hand side of (iii).

We finally observe that the substitution of $\bar{x} = 1 - x$ transforms (31)-(32) into (37)-(38). This completes the proof.

We note that for the autonomous SLP (21)-(22) the inequalities in (i) and (iii) of the lemma above become equalities. (See Section 4.)

For the semilinear SLP (14)-(15) all statements of Lemma 5.1 remain the same if we let $\epsilon = 0$. Yet for the semilinear SLP we may sharpen (39) into

$$|s_1| = s_0 e^{\frac{1}{2}\|y\|^2}.$$

As with the semilinear model of Section 3 and the autonomous model of Section 4, we obtain numerical solutions to the fully nonlinear SLP (31)-(32) using a similar numerical technique. The first mode of the spectrum of the fully nonlinear SLP and two particular solutions are shown in Figure 3.

One key observation we make about the three models presented here – semilinear, autonomous, and fully nonlinear – is that the first mode for each model is distinct. As seen in Figure 4, the first modes all agree near the point $(\pi^2, 0)$, but as $\|y\|$ increases the spectra diverge. This shows that while all of these models share some characteristics, notably near the point on the spectrum corresponding to the trivial solution of the respective SLP, the terms unique to each model are essential in determining the dynamics of the spectrum and, therefore, the solutions. We note that the fundamental difference between the (λ, s_0)-curves for the autonomous model and its linear counterpart was recognized by Graef et al. in [1, 2].

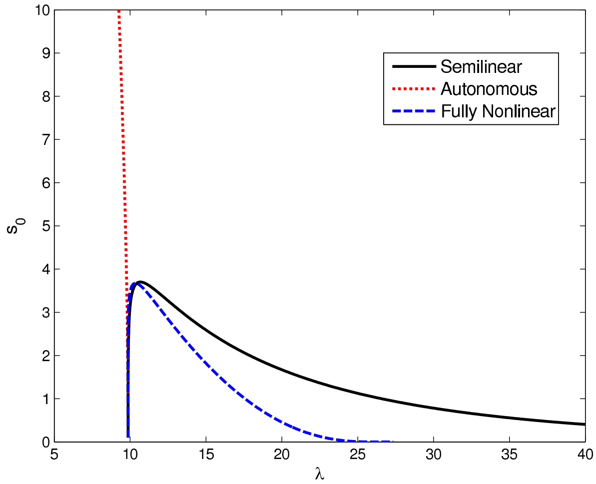

Figure 4: Each of the three models – semilinear, autonomous, and fully nonlinear – has a distinct first mode. All agree near $(\pi^2, 0)$.

6 Unsolved Problems

There are two unsolved problems that the authors would like to briefly mention in connection with the fully nonlinear SLP, (31)-(32).

Problem 6.1 – *The* (λ, s_0)*-dependence for the fully nonlinear SLP (31)-(32) has been studied numerically. How can the same problem be tackled analytically?*

Moreover, if we temporarily assume $\|y\|$ to be known, we may invert the linear (actually semilinear) operator L and reduce the SLP (31)-(32) to the following integral equation,

$$y(x) = \epsilon\lambda^3 \int_0^1 K(x, t, \|y\|, \lambda) y^3(t)\, dt \tag{45}$$

where the kernel K has the following representation

$$\begin{aligned} K(x, t, \|y\|, \lambda) &= \frac{K_1(x, \|y\|, \lambda)}{K_1(1, \|y\|, \lambda)} K_1(1-t, \|y\|, \lambda) - K_1(x-t, \|y\|, \lambda)\, \chi_{(0,x)}(t), \\ K_1(x, \|y\|, \lambda) &= \frac{1}{2\pi i} \int_C e^{px} \frac{1}{p^2 - \|y\|^2 p + \lambda}\, dp, \end{aligned}$$

with $\chi_{(0,x)}(t)$ the characteristic function and C the standard contour in the complex plane that appears in the Laplace transform. So more specifically, Problem 6.1 has the the form

Problem 6.1 (revised) – *Is it possible to embed the nonlinear integral equation (45) in the well-developed theory of nonlinear integral equations based on Krasnosel'skii's fixed* *point theorem [6, 5, 7]?*

The SLP for the nonlinear differential equation

$$y'' + q(x)y + \lambda\left[a(x) - f(x, y, y')\right]y = 0, x \in (0, 1) \tag{46}$$

subject to standard BC is studied in a series of papers [10, 11]. It is assumed there that

$$f(x, \xi, \eta) \geq 0 \tag{47}$$

for $x \in [0, 1]$, $0 < |\xi|, |\eta| < \rho$ for some constant $\rho > 0$. The dependence (λ, j), where j is the number of zeros of the corresponding eigenfunction, is studied extensively.

Scaling $y(x)$ in the autonomous SLP (21)-(22) (see [1, 2]) yields the ODE

$$y'' + \lambda\left[1 + 2y^2\right]y = 0 \tag{48}$$

so that $f(x, \xi, \eta) = -2\xi^2$ which obviously fails to satisfy the requirement (47).

Problem 6.2 – *Is it possible to develop the results of the current paper in terms of the (λ, j)-dependence?*

7 Conclusion

We have studied analytically and numerically three Sturm-Liouville problems. The main SLP is fully nonlinear and nonlocal and has a structure similar to the SLP which describes the stability of a wing (or panel) in an airflow. In that case, the spectral parameter λ is proportional to the Mach number. Two other Sturm-Liouville problems represent simplifications of the main model, in particular when we neglect either the nonlocal term or the higher order nonlinear term. Some information about the first eigenfunction $y(x)$ and the corresponding curve (λ, s_0) is presented (where s_0 is the slope of $y(x)$ at one endpoint). It is established to what extent the two approximations of the main model produce similar (λ, s_0)-dependencies. A robust numerical simulation is used to illuminate in detail the parts of the spectrum between the limiting cases given in the asymptotic analyses.

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