Electronic Journal of Qualitative Theory of Differential Equations
Spec. Ed. I, 2009 No. **3**, 1–12; **http://www.math.u-szeged.hu/ejqtde/**

DISCRETE FRACTIONAL CALCULUS WITH THE NABLA OPERATOR

Ferhan M. Atıcı[1] and Paul W. Eloe[2]

[1] Western Kentucky University
Department of Mathematics and Computer Science
Bowling Green Kentucky 42101-3576 USA
e-mail: ferhan.atici@wku.edu
[2] University of Dayton
Department of Mathematics
Dayton Ohio 45469-2316 USA
email: Paul.Eloe@notes.udayton.edu

Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

Properties of discrete fractional calculus in the sense of a backward difference are introduced and developed. Exponential laws and a product rule are developed and relations to the forward fractional calculus are explored. Properties of the Laplace transform for the nabla derivative on the time scale of integers are developed and a fractional finite difference equation is solved with a transform method. As a corollary, two new identities for the gamma function are exhibited.

Key words and phrases: Discrete fractional calculus.
AMS (MOS) Subject Classifications: 39A12, 44A25, 26A33

1 Introduction

In this article, we shall continue our recent work to study discrete fractional calculus. We shall continue to employ terminology from the theory of time scales calculus [5] and in this article, we shall employ the backward difference, or the nabla derivative.

There has been little work done in the study of discrete fractional calculus in the case of the forward difference. Miller and Ross [15] initiated the study and the authors [2, 3, 4] have more recently been developing discrete forward fractional calculus. There has been more work in the study of discrete fractional calculus using the backward difference [11, 8, 10, 12, 13]; applications arising in time series analysis have motivated the development in discrete backward fractional calculus. As is typical in fractional calculus, authors do not agree on basic definitions.

In Section 2, we shall present the definitions and fundamental identities that we shall employ. In Section 3, we present a discrete nabla analogue of the Laplace transform and develop some fundamental properties. We point out that this transform agrees with the transform for the alpha derivative on the time scale of integers when alpha is nabla. We close the article by applying the transform method to solve a fractional nabla difference equation and exhibit two new identities for the gamma function. Throughout this article, we shall compare and contrast the methods and results with the analogous methods and results for fractional forward difference calculus.

2 Preliminary Definitions and Properties

Define

$$t^{\overline{n}} = t(t+1)(t+2)...(t+n-1), \quad n \in \mathbb{N},$$

and $t^{\overline{0}} = 1$. $t^{\overline{n}}$ is well-known and has been called *t to the n rising* [6], *the rising factorial power* [9], or *the ascending factorial* [7]. Many authors employ the Pochhammer symbol [19] to denote the rising factorial function.

Let α be any real number. Then "t to the α rising" is defined to be

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)},$$

where $t \in \mathbb{R} \setminus \{..., -2, -1, 0\}$, and $0^{\overline{\alpha}} = 0$. Note that

$$\nabla(t^{\overline{\alpha}}) = \alpha t^{\overline{\alpha-1}}$$

where $\nabla y(t) = y(t) - y(t-1)$.

For $k = 2, 3, \ldots$, define ∇^k inductively by $\nabla^k = \nabla\nabla^{k-1}$. Let $f : \mathbb{N}_a \to \mathbb{R}$ be given and consider the following discrete $n-$th order initial value problem

$$\nabla^n y(t) = f(t),$$

$$\nabla^i y(a) = 0, \quad 0 \leq i \leq n-1,$$

where a is a real number and $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$.

The solution of this initial value problem [6, Theorem 3.99] is given by

$$y(t) = \sum_{s=a+1}^{t} \frac{(t-\rho(s))^{\overline{n-1}}}{(n-1)!} f(s)$$

where $t \equiv a+1 \pmod 1$, $\rho(s) = s-1$ and $\frac{(t-\rho(s))^{\overline{n-1}}}{(n-1)!}$ is the Cauchy function for $\nabla^n y(t) = 0$. To agree with Gray and Zhang [11], we shall define the $n-$th order sum of $f(t)$ by the formula

$$\nabla_a^{-n} f(t) = \sum_{s=a}^{t} \frac{(t-\rho(s))^{\overline{n-1}}}{\Gamma(n)} f(s). \tag{1}$$

Hence, the solution of the initial value problem is $\nabla_{a+1}^{-n} f(t)$. With this observation, define (as done in [11, Equation 2.3]) the $\nu-$th order fractional sum of f by

$$\nabla_a^{-\nu} f(t) = \sum_{s=a}^{t} \frac{(t-\rho(s))^{\overline{\nu-1}}}{\Gamma(\nu)} f(s) \tag{2}$$

where $\nu \in \mathbb{R}\backslash\{..., -2, -1, 0\}$.

We shall define the fractional nabla difference in the analogous manner to the forward fractional difference as proposed by Miller and Ross [15]. Let $\mu > 0$ and assume that $m-1 < \mu < m$ where m denotes a positive integer. Set $-\nu = \mu - m$. Define

$$\nabla^{\mu} u(t) = \nabla^{m-\nu} u(t) = \nabla^{m}(\nabla^{-\nu} u(t)). \tag{3}$$

This definition for a fractional derivative does not agree with the definition employed by Gray and Zhang [11].

For the readers' benefit, we shall recall definitions of the fractional sum and Δ difference operators, see [2, 4, 15]. Define $t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$; so, $t^{(n)} = t(t-1)\ldots(t-n+1)$. Note that

$$t^{\overline{\alpha}} = (t+\alpha-1)^{(\alpha)}. \tag{4}$$

We shall employ (4) repeatedly. Define

$$\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-\sigma(s))^{(\nu-1)} f(s) \tag{5}$$

and

$$\Delta^{\mu} u(t) = \Delta^{m-\nu} u(t) = \Delta^{m}(\Delta^{-\nu} u(t)),$$

where $m-1 < \mu < m$, $-\nu = \mu - m$ and m is a positive integer.

It is already known and it can be easily shown that $\Delta^m y(t-m) = \nabla^m y(t)$ for any positive number m. Next we shall generalize this formula for any positive real number ν and also give a relation between the $\nabla-$fractional sum and the $\Delta-$fractional sum operators.

Lemma 2.1 *Let $0 \leq m-1 < \nu \leq m$ where m denotes an integer, let a be a positive integer, and let $y(t)$ be defined on $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$. Then the following statements are valid.*

(i) $\Delta_a^{\nu} y(t-\nu) = \nabla_a^{\nu} y(t)$ *for* $t \in \mathbb{N}_{m+a}$.

(ii) $\Delta_a^{-\nu} y(t+\nu) = \nabla_a^{-\nu} y(t)$ *for* $t \in \mathbb{N}_a$.

Proof. (i) Starting on the left side of the identity we have

$$\Delta_a^{\nu} y(t-\nu) = \Delta^m \Delta_a^{-(m-\nu)} y(t-\nu) = \Delta^m \sum_{s=a}^{t-m} \frac{(t-\nu-\sigma(s))^{(m-\nu-1)}}{\Gamma(m-\nu)} y(s)$$

$$= \nabla^m \sum_{s=a}^{t} \frac{(t+m-\nu-\sigma(s))^{(m-\nu-1)}}{\Gamma(m-\nu)} y(s)$$
$$= \nabla^m \sum_{s=a}^{t} \frac{(t-\rho(s))^{\overline{m-\nu-1}}}{\Gamma(m-\nu)} y(s) = \nabla^m \nabla_a^{-(m-\nu)} y(t) = \nabla_a^\nu y(t),$$

where we used the definitions of Δ and ∇ fractional operators (1)-(5).

The proof of (ii) is analogous to the proof of (i) and follows even more directly from (4).

□

Lemma 2.2 *Let $0 \le m-1 < \nu \le m$ where m denotes a positive integer and $y(t)$ be defined on $\mathbb{N}_{\nu-m} = \{\nu-m, \nu-m+1, \ldots\}$. Then the following statements are valid.*

(i) $\Delta_{\nu-m}^\nu y(t) = \nabla_{\nu-m}^\nu y(t+\nu)$ for $t \in \mathbb{N}_{-m}$.

(ii) $\Delta_{\nu-m}^{-(m-\nu)} y(t) = \nabla_{\nu-m}^{-(m-\nu)} y(t-m+\nu)$ for $t \in \mathbb{N}_0$.

Proof. (i) Starting on the right side of the identity we have

$$\nabla_{\nu-m}^\nu y(t+\nu) = \nabla^m \nabla_{\nu-m}^{-(m-\nu)} y(t+\nu) = \nabla^m \sum_{s=\nu-m}^{t+\nu} \frac{(t+\nu-\rho(s))^{\overline{m-\nu-1}}}{\Gamma(m-\nu)} y(s)$$

$$= \Delta^m \sum_{s=\nu-m}^{t-m+\nu} \frac{(t-\sigma(s))^{(m-\nu-1)}}{\Gamma(m-\nu)} y(s)$$
$$= \Delta^m \Delta_{\nu-m}^{-(m-\nu)} y(t) = \Delta_{\nu-m}^\nu y(t),$$

where we used the definitions of Δ and ∇ fractional operators (1)-(5).

The proof of (ii) is analogous.

□

Next we prove the power rule which plays an important role for proving some basic properties of the $\nabla-$fractional operators and for solving some fractional difference equations.

Lemma 2.3

$$\nabla_1^{-\nu} t^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} t^{\overline{\mu+\nu}}.$$

Proof. $\nabla_1^{-\nu} t^{\overline{\mu}} = \dfrac{1}{\Gamma(\nu)} \displaystyle\sum_{s=1}^{t} (t-\rho(s))^{\overline{\nu-1}} s^{\overline{\mu}}$

$$= \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-1} \frac{\Gamma(t-s+\nu-1)}{\Gamma(t-s)} \frac{\Gamma(s+\mu+1)}{\Gamma(s+1)}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-1}\binom{t-1}{s}\frac{\Gamma(t-s+\nu-1)\Gamma(s+\mu+1)}{\Gamma(t)} \\
&= \frac{\Gamma(\mu+1)}{\Gamma(t)}\sum_{s=0}^{t-1}\binom{t-1}{s}(\mu+1)^{\overline{s}}\nu^{\overline{t-s-1}} \\
&= \frac{\Gamma(\mu+1)}{\Gamma(t)}(\nu+\mu+1)^{\overline{t-1}} \\
&= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}t^{\overline{\mu+\nu}}.
\end{aligned}
$$

A proof of the identity

$$\sum_{s=0}^{t-1}\binom{t-1}{s}(\mu+1)^{\overline{s}}\nu^{\overline{t-s-1}} = (\nu+\mu+1)^{\overline{t-1}}$$

can be found in [1, Remark 2.2.1]. □

Remark 2.1 *The authors [2] have proved a power rule*

$$\Delta_{\mu}^{-\nu}t^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}t^{(\mu+\nu)}.$$

(4) and Lemma 2.1 (ii) can be employed to give an alternate verification of Lemma 2.3.

$$\nabla_1^{-\nu}t^{\overline{\mu}} = \Delta_1^{-\nu}(t+\nu)^{\overline{\mu}} = \frac{1}{\Gamma(\nu)}\sum_{s=1}^{t}(t+\nu-\sigma(s))^{(\nu-1)}s^{\overline{\mu}}$$

$$= \frac{1}{\Gamma(\nu)}\sum_{s=1}^{t}(t+\nu-\sigma(s))^{(\nu-1)}(s+\mu-1)^{(\mu)} = \frac{1}{\Gamma(\nu)}\sum_{s=\mu}^{t+\mu-1}(t+\mu-1+\nu-\sigma(s))^{(\nu-1)}s^{(\mu)}$$

$$= \Delta_{\mu}^{-\nu}(t+\nu+\mu-1)^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t+\nu+\mu-1)^{(\mu+\nu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}t^{\overline{\mu+\nu}}.$$

Next we shall state and prove the law of exponents for the $\nabla-$fractional sums. Gray and Zhang [11] also consider laws of exponents; our definitions for fractional derivatives are not in total agreement.

Theorem 2.1 *Let f be a real valued function, and let $\mu, \nu > 0$. Then*

$$\nabla_a^{-\nu}[\nabla_a^{-\mu}f(t)] = \nabla_a^{-(\mu+\nu)}f(t) = \nabla_a^{-\mu}[\nabla_a^{-\nu}f(t)].$$

Proof. The proof follows from the definition of the $\nabla-$ fractional sum operator and the power rule.

$$\nabla_a^{-\nu}[\nabla_a^{-\mu}f(t)] = \frac{1}{\Gamma(\nu)}\sum_{s=a}^{t}(t-\rho(s))^{\overline{\nu-1}}\nabla_a^{-\mu}f(s)$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\nu)}\sum_{s=a}^{t}(t-\rho(s))^{\overline{\nu-1}}\frac{1}{\Gamma(\mu)}\sum_{\tau=a}^{s}(s-\rho(\tau))^{\overline{\mu-1}}f(\tau)\\
&= \sum_{\tau=a}^{t}\sum_{s=\tau}^{t}\frac{(t-\rho(s))^{\overline{\nu-1}}(s-\rho(\tau))^{\overline{\mu-1}}}{\Gamma(\nu)\Gamma(\mu)}f(\tau)\\
&= \frac{1}{\Gamma(\mu)}\sum_{\tau=a}^{t}\nabla_1^{-\nu}(t-\rho(\tau))^{\overline{\mu-1}}f(\tau)\\
&= \frac{1}{\Gamma(\mu+\nu)}\sum_{\tau=a}^{t}(t-\rho(\tau))^{\overline{\nu+\mu-1}}f(\tau)\\
&= \nabla_a^{-(\mu+\nu)}f(t).
\end{aligned}
$$

□

3 An Application of a Discrete Transform

In this section we employ a discrete Laplace transform which is the Laplace transform for the alpha derivative on the time scale of integers [6] applied to $\alpha = \nabla$.

Define the discrete transform ($\mathcal{N}$-transform) by

$$
\mathcal{N}_{t_0}(f(t))(s) = \sum_{t=t_0}^{\infty}(1-s)^{t-1}f(t). \tag{6}
$$

If the domain of the function f is $\mathbb{N}_1$, then we use the notation $\mathcal{N}$ or $\mathcal{N}_1$.

Lemma 3.1 *For any $\nu \in \mathbb{R}\setminus\{...,-2,-1,0\}$,*

(i) $\mathcal{N}(t^{\overline{\nu-1}})(s) = \dfrac{\Gamma(\nu)}{s^{\nu}}$, $|1-s|<1$, *and*

(ii) $\mathcal{N}(t^{\overline{\nu-1}}\alpha^{-t})(s) = \dfrac{\alpha^{\nu-1}\Gamma(\nu)}{(s+\alpha-1)^{\nu}}$, $|1-s|<\alpha$.

Proof. First we assume $0<\nu\le 1$. Then

$$
\begin{aligned}
\mathcal{N}(t^{\overline{\nu-1}})(s) &= \sum_{t=1}^{\infty}(1-s)^{t-1}t^{\overline{\nu-1}} = \sum_{t=1}^{\infty}(1-s)^{t-1}\frac{\Gamma(t+\nu-1)}{\Gamma(t)}\\
&= \sum_{t=0}^{\infty}(1-s)^{t}\frac{\Gamma(t+\nu)}{\Gamma(t+1)} = \Gamma(\nu)\,{}_2F_1(1,\nu;1;1-s)\\
&= \frac{1}{\Gamma(1-\nu)}\int_0^1\frac{u^{\nu-1}(1-u)^{1-\nu-1}}{(1-u(1-s))}du = \frac{\Gamma(\nu)}{s^{\nu}}.
\end{aligned}
$$

For identities with respect to the hypergeometric functions we refer the reader to [1]; we have employed the following identities,

$$
{}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1\frac{u^{b-1}(1-u)^{c-b-1}}{(1-uz)^{a}}du
$$

and
$$\int_0^1 \frac{u^{x-1}(1-u)^{y-1}}{(au+b(1-u))^{x+y}}du = \frac{\Gamma(x)\Gamma(y)}{a^x b^y \Gamma(x+y)}.$$
The radius of convergence in (i) is given by the radius of convergence for the series expansion for ${}_2F_1(1,\nu;1;1-s)$.

Note that, in general,
$$\mathcal{N}(t^{\overline{\nu}})(s) = \frac{\nu}{s}\mathcal{N}(t^{\overline{\nu-1}}) \tag{7}$$
where $\nu \in \mathbb{R}\backslash\{...,-2,-1,0\}$. To see this, note that summation by parts implies
$$\sum_{t=1}^{\infty}(1-s)^{t-1}t^{\overline{\nu-1}} = \frac{1}{\nu}\sum_{t=1}^{\infty}(1-s)^{t-1}\nabla(t^{\overline{\nu}}) =$$
$$\frac{1}{\nu}\sum_{t=1}^{\infty}(\nabla((1-s)^t t^{\overline{\nu}}) - (\nabla(1-s)^t)(t^{\overline{\nu}})) = -\frac{1}{\nu}0^{\overline{\nu}} + \frac{s}{\nu}\sum_{t=1}^{\infty}(1-s)^{t-1}t^{\overline{\nu}}.$$
Since the gamma function has a pole at zero, $0^{\overline{\nu}} = 0$ and the proof of (i) is complete.

(ii) will follow from (i) once we write
$$\sum_{t=1}^{\infty}(1-s)^{t-1}t^{\overline{\nu-1}}\alpha^{-t} = \frac{1}{\alpha}\sum_{t=1}^{\infty}(1-\frac{s+\alpha-1}{\alpha})^{t-1}t^{\overline{\nu-1}}.$$
□

Lemma 3.2 *Let f be defined on $\mathbb{N}_{a+1}$. Then the following is valid:*
$$\mathcal{N}_a f(t+1) = (1-s)^{-1}\mathcal{N}_{a+1}f(t).$$

Proof.
$$\begin{aligned}\mathcal{N}_a f(t+1) &= \sum_{t=a}^{\infty}(1-s)^{t-1}f(t+1)\\ &= (1-s)^{-1}\sum_{t=a+1}^{\infty}(1-s)^{t-1}f(t)\\ &= (1-s)^{-1}\mathcal{N}_{a+1}f(t).\end{aligned}$$
□

Lemma 3.3 *For any positive real number ν,*
$$\mathcal{N}_a(\nabla_a^{-\nu}f(t)) = s^{-\nu}\mathcal{N}_a(f(t))(s),$$
where f is defined on N_a.

Proof.

$$\begin{aligned}
\mathcal{N}_a(\nabla_a^{-\nu} f(t)) &= \sum_{t=a}^{\infty}(1-s)^{t-1}\nabla_a^{-\nu} f(t)\\
&= \sum_{t=a}^{\infty}(1-s)^{t-1}\frac{1}{\Gamma(\nu)}\sum_{\tau=a}^{t}(t-\rho(\tau))^{\overline{\nu-1}}f(\tau)\\
&= \sum_{\tau=a}^{\infty}\sum_{t=\tau}^{\infty}(1-s)^{t-1}\frac{1}{\Gamma(\nu)}(t-\rho(\tau))^{\overline{\nu-1}}f(\tau)\\
&= \sum_{\tau=a}^{\infty}\sum_{r=1}^{\infty}(1-s)^{r-1+\tau-1}\frac{1}{\Gamma(\nu)}r^{\overline{\nu-1}}f(\tau)\\
&= \frac{1}{\Gamma(\nu)}\sum_{\tau=a}^{\infty}(1-s)^{\tau-1}f(\tau)\sum_{r=1}^{\infty}(1-s)^{r-1}r^{\overline{\nu-1}}\\
&= \frac{1}{\Gamma(\nu)}\mathcal{N}_a(f)(s)\mathcal{N}(t^{\overline{\nu-1}}) = s^{-\nu}\mathcal{N}_a(f(t))(s).
\end{aligned}$$

□

Lemma 3.4 *For $0<\nu\le 1$,*

$$\mathcal{N}_{a+1}(\nabla_a^{\nu} f(t))(s) = s^{\nu}\mathcal{N}_a(f(t))(s) - (1-s)^{a-1}f(a),$$

where f is defined on $\mathbb{N}_a$.

Proof. First note that

$$\mathcal{N}_{a+1}(\nabla f(t))(s) = s\mathcal{N}_a(f(t))(s) - (1-s)^{a-1}f(a).$$

In fact,

$$\begin{aligned}
\mathcal{N}_{a+1}(\nabla f(t))(s) &= \sum_{t=a+1}^{\infty}(1-s)^{t-1}\nabla f(t) = \sum_{t=a}^{\infty}(1-s)^{t-1}[f(t)-f(t-1)]\\
&= \sum_{t=a+1}^{\infty}(1-s)^{t-1}f(t) - \sum_{t=a+1}^{\infty}(1-s)^{t-1}f(t-1)\\
&= \sum_{t=a}^{\infty}(1-s)^{t-1}f(t) - (1-s)^{a-1}f(a) - \sum_{t=a}^{\infty}(1-s)^{t}f(t)\\
&= s\sum_{t=a}^{\infty}(1-s)^{t-1}f(t) - (1-s)^{a-1}f(a).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\mathcal{N}_{a+1}(\nabla_a^{\nu} f(t))(s) &= \mathcal{N}_{a+1}(\nabla\nabla_a^{-(1-\nu)}f(t))(s)\\
&= s\mathcal{N}_a(\nabla_a^{-(1-\nu)}f(t))(s) - (1-s)^{a-1}\nabla_a^{-(1-\nu)}f(t)|_{t=a}\\
&= s^{\nu}\mathcal{N}_a(f(t))(s) - (1-s)^{a-1}f(a),
\end{aligned}$$

since

$$\nabla_a^{-(1-\nu)}f(t)|_{t=a} = \frac{1}{\Gamma(1-\nu)}\sum_{s=a}^{t}(t-\rho(s))^{\overline{-\nu}}f(s)|_{t=a} = f(a).$$

□

Example 3.1 *Consider the following initial value problem*

$$\nabla_0^{1/2} y(t) = 5 \text{ for } t = 1, 2, ...$$
$$\nabla_0^{-1/2} y(t)|_{t=0} = y(0) = a.$$

Applying the $\mathcal{N}_1$-transform to each side of the equation, we have

$$\mathcal{N}_1(\nabla_0^{1/2} y(t)) = \mathcal{N}_1(5).$$

Using Lemma 3.1 and Lemma 3.4, we obtain

$$s^{1/2}\mathcal{N}_0(y(t))(s) - (1-s)^{-1}y(0) = \frac{5}{s}.$$

It follows that

$$\mathcal{N}_0(y(t))(s) = \frac{5}{s^{3/2}} + \frac{y(0)}{(1-s)s^{1/2}}.$$

Using Lemma 3.2, we have

$$\mathcal{N}_0(y(t))(s) = \frac{5}{\Gamma(3/2)}\mathcal{N}_0(t^{\overline{1/2}}) + \frac{y(0)}{\Gamma(1/2)}\mathcal{N}_0((t+1)^{\overline{-1/2}}).$$

Applying the inverse $\mathcal{N}_0$-transform to each side of the last equation, we obtain the solution of the initial value problem

$$y(t) = \frac{5}{\Gamma(3/2)}t^{\overline{1/2}} + \frac{y(0)}{\Gamma(1/2)}(t+1)^{\overline{-1/2}},$$

where $t = 0, 1, 2, ...$

In the next example we compare the fractional $\nabla-$difference equation and the fractional $\Delta-$difference equation and their solutions.

Example 3.2 *Let's consider the following initial value problem*

$$\Delta^{1/2} y(t) = 5 \text{ for } t = 0, 1, ...$$
$$\Delta^{-1/2} y(t)|_{t=0} = y(-1/2) = a.$$

The solution of this initial value problem is

$$y(t) = \frac{5}{\Gamma(3/2)}t^{(1/2)} + \frac{y(-1/2)}{\Gamma(1/2)}t^{(-1/2)}. \tag{8}$$

For the details of the calculation of the solution we refer the paper [2] by the authors.

Next we convert the above fractional $\Delta-$ difference equation to a fractional $\nabla-$ difference equation using Lemma 2.2. Hence we have

$$\Delta\Delta_{-1/2}^{-1/2} y(t) = \Delta\nabla_{-1/2}^{-1/2} y(t-1+1/2) = \nabla\nabla_{-1/2}^{-1/2} y(t+1/2) = 5$$

for $t = 0, 1, 2, ...$ In order to apply the $\mathcal{N}_1$-transform, we also prove the following equality

$$\nabla_{-1/2}^{-1/2} y(t+1/2) = \nabla_0^{-1/2} z(t+1), \text{ for } t = 0, 1, 2, ...$$

where $z(t) = y(t - 1/2)$. Then

$$\begin{aligned}\nabla_{-1/2}^{-1/2} y(t+1/2) &= \sum_{s=-1/2}^{t+1/2} (t + 1/2 - \rho(s))^{\overline{-1/2}} y(s) \\ &= \sum_{u=0}^{t+1} (t + 1 - \rho(u))^{\overline{-1/2}} y(u - 1/2) \\ &= \nabla_0^{-1/2} z(t+1)\end{aligned}$$

where $z(t) = y(t - 1/2)$. As a result of this equality, the corresponding $\nabla-$difference equation becomes

$$\begin{aligned}&\nabla_0^{1/2} z(t) = 5 \text{ for } t = 1, 2, ... \\ &\nabla_0^{-1/2} z(t)|_{t=0} = z(0) = a.\end{aligned}$$

It follows from the Example 3.1 that the solution of the above fractional $\nabla-$difference equation is

$$z(t) = \frac{5}{\Gamma(3/2)} t^{\overline{1/2}} + \frac{z(0)}{\Gamma(1/2)} (t+1)^{\overline{-1/2}}, \tag{9}$$

where $t \in \mathbb{N}_0$. A straight forward calculation shows that the solutions in (8) and (9) are same.

We shall close this paper with two identities for the gamma function. In [2], the authors employed the $\alpha-$Laplace transform associated with the forward finite difference, $\alpha = \Delta$. In particular, the authors employed

$$\mathcal{R}_a(f(t))(s) = \sum_{t=a}^{\infty} (\frac{1}{s+1})^{t+1} f(t)$$

and showed

$$\mathcal{R}_{\nu-1}(t^{(\nu-1)})(s) = \frac{\Gamma(\nu)}{s^{\nu}}.$$

Evaluate this identity at $s = 1$, to obtain

$$\Gamma(\nu) = \sum_{t=\nu-1}^{\infty} (\frac{1}{2})^{t+1} t^{(\nu-1)}, \quad \nu \in \mathbb{R} \setminus \{..., -2, -1, 0\}. \tag{10}$$

With the introduction of the nabla Laplace transform in this article, evaluate the identity given in Lemma 3.1 (i) at $s = \frac{1}{2}$ to obtain

$$\Gamma(\nu) = \sum_{t=1}^{\infty} (\frac{1}{2})^{t-\nu-1} t^{\overline{\nu-1}}, \quad \nu \in \mathbb{R} \setminus \{..., -2, -1, 0\}. \tag{11}$$

We believe that identities (10) and (11) are new. If one employs (4) and Lemma 3.1 (i), then the derivations to equate the right hand sides of (10) and (11) are straightforward.

References

[1] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.

[2] F. M. Atıcı and P. W. Eloe, A transform method in discrete fractional calculus, *International Journal Difference Equations* **2** (2) (2007), 165-176.

[3] F. M. Atıcı and P. W. Eloe, Fractional q-calculus on a time scale, *J. Nonlinear Mathematical Physics* **14** (3) (2007), 333-344.

[4] F. M. Atıcı and P. W. Eloe, Initial value problems in discrete fractional calculus, *Proc. Amer. Math. Soc.* **137** (2009), 981-989.

[5] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2001.

[6] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[7] G. Boros and V. Moll, *Irresistible Integrals: Symbols, Analysis and Experiments in the Evaluation of Integrals*, Cambridge University Press, Cambridge, 2004.

[8] J. B. Diaz and T. J. Osler, Differences of fractional order, *Math. Comp.* **28** (1974), 185-201.

[9] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, 2nd ed., Addison-Wesley, Reading, MA, 1994.

[10] C. W. J. Granger and R. Joyeux, An introduction to long-memory time series models and fractional differences, *J. Time Ser. Anal.* **1** (1980), 15-29.

[11] H. L. Gray and N. F. Zhang, On a new definition of the fractional difference, *Mathematics of Computation* **50** (182) (1988), 513-529.

[12] J. R. M. Hosking, Fractional differencing, *Biometrika* **68** (1981), 165-176.

[13] G. L. Isaacs, Exponential laws for fractional differences, *Math. Comp.* **35** (1980), 933-936.

[14] W. Kelley and A. Peterson, *Difference Equations: An Introduction with Applications*, Academic Press, London, 1991.

[15] K. S. Miller and B. Ross, Fractional difference calculus, *Proceedings of the International Symposium on Univalent Functions, Fractional Calculus and Their Applications*, Nihon University, Koriyama, Japan, May (1988), 139-152; *Ellis Horwood Ser. Math. Appl.*, Horwood, Chichester, 1989.

[16] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, Inc., New York, 1993.

[17] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[18] G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[19] J. Spanier and K. B. Oldham, The Pochhammer Polynomials $(x)_n$, *An Atlas of Functions*, pp. 149-165, Hemisphere, Washington, DC, 1987.