Electronic Journal of Qualitative Theory of Differential Equations
Spec. Ed. I, 2009 No. **2**, 1–16; **http://www.math.u-szeged.hu/ejqtde/**

POPULATION DYNAMICS WITH SYMMETRIC AND ASYMMETRIC HARVESTING

Jaffar Ali[1], David Perry[2], Sarath Sasi[3], Jessica Schaefer[4]
Brian Schilling[3], R. Shivaji[3] and Matthew Williams[5]

[1] Department of Mathematics, Florida Gulf Coast University
e-mail: jahameed@fgcu.edu
[2] Department of Mathematics, New York University
e-mail: dwp234@nyu.edu
[3] Department of Mathematics, Center for Computational Sciences
Mississippi State University, Mississippi State, MS 39762-9715 USA
e-mail: ss885@msstate.edu, bls153@msstate.edu, shivaji@ra.msstate.edu
[4] Department of Mathematics, Northern Arizona University
e-mail: jrs244@dana.ucc.nau.edu
[5] Department of Mathematics, Clarkson State University
e-mail: williamr@clarkson.edu

Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

We study the positive solutions to steady state reaction diffusion equations with Dirichlet boundary conditions of the forms:

$$-u'' = \begin{cases} \lambda[au - bu^2 - c], & x \in (L, 1-L), \\ \lambda[au - bu^2], & x \in (0, L) \cup (1-L, 1), \end{cases} \qquad \text{(A)}$$
$$u(0) = 0 = u(1),$$

and

$$-u'' = \begin{cases} \lambda[au - bu^2 - c], & x \in (0, \frac{1}{2}), \\ \lambda[au - bu^2], & x \in (\frac{1}{2}, 1), \end{cases} \qquad \text{(B)}$$
$$u(0) = 0 = u(1).$$

Here λ, a, b, c and L are positive constants with $0 < L < \frac{1}{2}$. Such steady state equations arise in population dynamics with logistic type growth and constant yield harvesting. Here u is the population density, $\frac{1}{\lambda}$ is the diffusion coefficient and c is the harvesting effort. In particular, model A corresponds to a symmetric harvesting case and model B to an asymmetric harvesting case. Our objective is to study the existence of positive solutions and also discuss the effects of harvesting. We will develop appropriate quadrature methods via which we will establish our results.

Key words and phrases: Population dynamics, reaction diffusion, harvesting, symmetric, asymmetric.
AMS (MOS) Subject Classifications: 35K57, 34B15, 92D25

1 Introduction

In [9] the authors study the nonlinear boundary value problem

$$\begin{cases} -\Delta u = au - bu^2 - ch(x), & x \in \Omega, \\ \quad u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1}$$

arising in population dynamics. Here Ω is a smooth bounded region with $\partial\Omega \in C^2$, u is the population density, $au - bu^2$ represents logistic growth where $a > 0$, $b > 0$ are constants. Here h is assumed to be a smooth function representing the intrinsic properties of the region while $c \geq 0$ represents the harvesting effort. The assumptions in the development of this steady state model are: (i) the species disperses randomly in the bounded environment; (ii) the reproduction of the species follows logistic growth; (iii) the boundary is hostile to the species; and (iv) the environment is homogeneous (i.e., the diffusion coefficient is independent of spatial variable x).

Here we extend this study to cases where $h(x)$ is a non-smooth function. However we restrict our analysis to the one-dimensional case with $\Omega = (0, 1)$ and study the following two models:

Model A

$$\begin{aligned} -u'' &= \begin{cases} \lambda[au - bu^2 - c], & x \in (L, 1-L), \\ \lambda[au - bu^2], & x \in (0, L) \cup (1-L, 1), \end{cases} \\ u(0) &= 0 = u(1); \end{aligned} \tag{A}$$

Model B

$$\begin{aligned} -u'' &= \begin{cases} \lambda[au - bu^2 - c]; & x \in (0, \frac{1}{2}) \\ \lambda[au - bu^2]; & x \in (\frac{1}{2}, 1) \end{cases} \\ u(0) &= 0 = u(1). \end{aligned} \tag{B}$$

There has been a raging controversy over the decline of western Atlantic bluefin tuna since the 1970s (see [8]). The U.S. fishing industries demand that the bluefin tuna movements between the western and eastern Atlantic be considered for a better assessments of the effect of 'overfishing' of bluefin by fishermen in central and eastern Atlantic, including the Mediterranean Sea. The model B is a simplified representation of the bluefin tuna population in the Atlantic Ocean in which we focus on the diffusion effects (the convection effects are not considered). It tries to capture the scenario where fishing is restricted to one half of the ocean. The relevance of the study is to see if such regulations can improve the depleting stock of fish in the Atlantic (see also [3] and [10]).

The classes of such models in which the reaction term is negative at the origin are known as semipositone problems. It is well documented in the literature that the study of positive solutions to such classes of problems are mathematically challenging (see [1] and [7]). Also see [4], [5] and references therein.

We analyze these problems by modifying the quadrature method discussed in [6] to accommodate such models with discontinuous reaction terms. For further information on the quadrature method see [2] and [4].

In Section 2 the quadrature method that was developed in [6] is briefly outlined. In Section 3 and Section 4 we extend the quadrature method for model A and model B respectively, and study the existence of a positive solution. Though in both models (A) and (B), in one of the harvesting regions the harvesting effort is zero, our methods easily extend to models with unequal harvesting rates in different regions. In Section 5, we provide various computational results on the bifurcation diagrams of positive solutions.

2 Preliminaries

Here we describe the quadrature method that has been used to prove our results. Consider the boundary value problem

$$\begin{cases} -u'' = \lambda f(u), \quad x \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{2}$$

where $f : [0, \infty) \to (0, \infty)$ is a C^1 function and λ is a positive parameter. Suppose u is a solution of (2) in $(0,1)$ such that $u'(x_0) = 0$ for $x_0 \in (0,1)$. Then $u(x_0+x) = u(x_0-x)$ for all $x \in [0, c)$ where $c = \min\{x_0, 1 - x_0\}$. This follows by noting the fact that both $v(x) := u(x_0 + x)$ and $w(x) := u(x_0 - x)$ satisfy the initial value problem

$$\begin{aligned} -z''(x) &= \lambda f(z(x)), \\ z(0) &= u(x_0), \\ z'(0) &= 0. \end{aligned}$$

Hence if u is a positive solution of (2) then u must be symmetric about $x = \frac{1}{2}$, increasing on $(0, \frac{1}{2})$ and decreasing on $(\frac{1}{2}, 1)$. Let $\rho = ||u||_\infty$ (note that $u(\frac{1}{2}) = \rho$). Now multiplying (2) by $u'(x)$ we obtain

$$-\Big(\frac{u'(x)^2}{2}\Big)' = \lambda (F(u(x)))' \tag{3}$$

where $F(s) := \int_0^s f(t)dt$, and integrating from 0 to $\frac{1}{2}$ we get

$$u'(x) = \sqrt{2\lambda[F(\rho) - F(u(x))]} \qquad 0 < x < \frac{1}{2}. \tag{4}$$

Here we have used $u(\frac{1}{2}) = \rho; u'(\frac{1}{2}) = 0$.
We rewrite (4) as

$$\frac{u'(x)}{\sqrt{[F(\rho) - F(u(x))]}} = \sqrt{2\lambda} \tag{5}$$

and integrate from $u(0) = 0$ to $u(x)$ after making the substitution $z = u(x)$. This gives

$$\int_0^{u(x)} \frac{dz}{\sqrt{[F(\rho) - F(z)]}} = \sqrt{2\lambda}x \qquad 0 < x < \frac{1}{2}. \tag{6}$$

Further since $u(\frac{1}{2}) = \rho$ we obtain

$$\sqrt{\lambda} = \sqrt{2}\int_0^{\rho} \frac{dz}{\sqrt{[F(\rho) - F(z)]}} := G(\rho). \tag{7}$$

Thus, if there exists a positive solution u of (2) with $u(\frac{1}{2}) = \rho$, then ρ must be such that $G(\rho)$ exists and satisfy $\sqrt{\lambda} = G(\rho)$. Since $f(\rho) > 0$ and $F(\rho) > F(z)$ for all $0 \leq z \leq \rho$, it follows that $G(\rho)$ exists for all $\rho > 0$. (Also $G(\rho)$ is a continuous function with $G(0) = 0$.)

Now suppose there exists $\rho > 0$ such that $\sqrt{\lambda} = G(\rho)$. Define $u(x)$ for $x \in [0, \frac{1}{2}]$ using (6). It can be shown that u is a C^2 function that satisfies (2). Thus (2) has a positive solution u with $u(\frac{1}{2}) = \rho > 0$ iff $\sqrt{\lambda} = G(\rho)$. Hence by analyzing $G(\rho)$ for $\rho \in (0, \infty)$ we can precisely discuss the existence, non-existence and multiplicity of positive solutions as λ varies. Further, when the solution exists, (6) defines the solution.

3 Symmetric Constant Yield Harvesting

In this section we consider the model A with harvesting allowed only in the interior. Quadrature methods are developed for the regions with and without harvesting separately to prove the existence of a positive solution. We call such classes of positive symmetric solutions as 'Type I' solutions. In this section we consider Type I solutions. Let $f(u) := au - bu^2$ and $\tilde{f}(u) := au - bu^2 - c$.

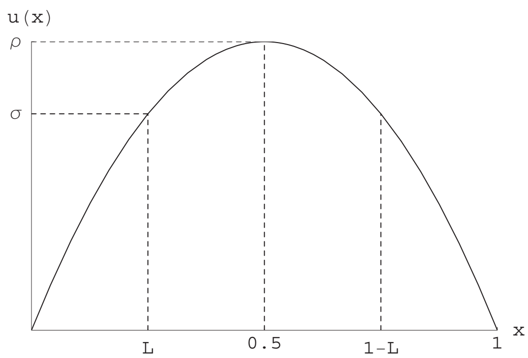

Figure 1: Typical Type I solution

We first consider the interval $(L, \frac{1}{2})$. On the interval $(L, \frac{1}{2})$ we have

$$-u'' = \lambda \tilde{f}(u).$$

Using similar calculations as in Section 2, we have

$$u' = \sqrt{2\lambda[\tilde{F}(\rho) - \tilde{F}(u)]}, \qquad L < x \leq \frac{1}{2},$$

where $\tilde{F}(s) := \int_0^s \tilde{f}(t)dt$ and $\rho = u(\frac{1}{2})$. Hence on $(L, \frac{1}{2})$, solutions of the model A will satisfy

$$\int_{u(x)}^{\rho} \frac{1}{\sqrt{\tilde{F}(\rho) - \tilde{F}(v)}} dv = \sqrt{2\lambda}\Big[\frac{1}{2} - x\Big]. \tag{8}$$

Letting $\sigma := u(L)$ and evaluating (8) at $x = L$, we have

$$\int_{\sigma}^{\rho} \frac{1}{\sqrt{\tilde{F}(\rho) - \tilde{F}(v)}} dv = \sqrt{2\lambda}\Big[\frac{1}{2} - L\Big]. \tag{9}$$

Simplifying (9) we get

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}(\frac{1}{2} - L)} \int_{\sigma}^{\rho} \frac{1}{\sqrt{\tilde{F}(\rho) - \tilde{F}(v)}} dv =: G_1(\sigma, \rho). \tag{10}$$

We next consider the interval $(0, L)$. On the interval $(0, L)$ we have

$$-u'' = \lambda[f(u)].$$

As proceeding in the previous case we observe that the solution u of model A on $(0, L)$ satisfies

$$\int_0^{u(x)} \frac{1}{\sqrt{\frac{m^2}{2} - \lambda F(v)}} dv = \sqrt{2}x \tag{11}$$

where $F(s) := \int_0^s f(t)dt$ and $m = u'(0)$. Letting $\sigma = u(L)$, and evaluating (11) at $x = L$ we get

$$\int_0^{\sigma} \frac{1}{\sqrt{\frac{m^2}{2} - \lambda F(v)}} dv = \sqrt{2}L. \tag{12}$$

In order for u to be a C^1 solution of model A, it is necessary that

$$u'(L^+) = u'(L^-).$$

Hence

$$\frac{m^2}{2} = \lambda[\tilde{F}(\rho) - \tilde{F}(\sigma) + F(\sigma)].$$

Substituting in (12) and simplifying we get

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}L} \int_0^{\sigma} \frac{1}{\sqrt{F(\rho) - F(v) - c(\rho - \sigma)}} dv =: G_2(\sigma, \rho). \tag{13}$$

Lemma 3.1. *Let* $S := \{\rho : \hat{\alpha} < \rho < \hat{\beta}\}$ *where* $\hat{\alpha} = \frac{3}{4b}\left(a - \sqrt{\frac{3a^2-16bc}{3}}\right)$ *and* $\hat{\beta} = \frac{a+\sqrt{a^2-4bc}}{2b}$. *If* $\rho \in S$ *then* G_1, G_2 *are well-defined.*

Proof. We note here that $\hat{\alpha}$ is the first non-zero root of $\tilde{F}(u) = \frac{au^2}{2} - \frac{bu^3}{3} - cu$ and $\hat{\beta}$ is the first largest positive root of $\tilde{F}'(u) = \tilde{f}(u)$.

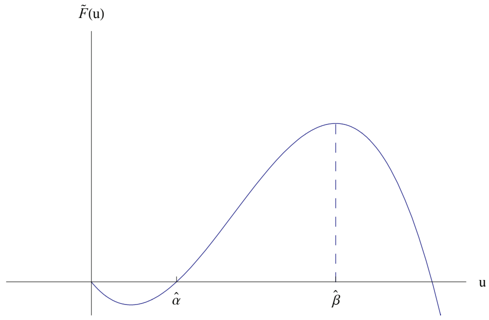

Figure 2: Graph of $\tilde{F}(u)$

It is clear from the graph of $\tilde{F}$ that $\tilde{F}(\rho) - \tilde{F}(v)$ is positive for all $v \in (0, \rho)$ only if $\rho \in (\hat{\alpha}, \hat{\beta})$. Further, for $\rho \in S$, $\tilde{f}(\rho) > 0$. Hence G_1 is well-defined for $\rho \in S$. Now we observe that for $v \leq \sigma$,

$$\begin{aligned} F(\rho) - F(v) - c(\rho - \sigma) &\geq F(\rho) - F(v) - c(\rho - v) \\ &= \tilde{F}(\rho) - \tilde{F}(v). \end{aligned}$$

Hence $G_2(\sigma, \rho)$ is also well-defined for all $\rho \in S$. □

Remark 3.1. *Note that* S *is non-empty iff* $c \leq \frac{3a^2}{16b}$.

Lemma 3.2. *Let* $\rho \in S$ *be fixed, then there exists a unique* $\sigma^*(\rho)$ *such that* $G_1(\sigma^*(\rho), \rho) = G_2(\sigma^*(\rho), \rho)$. *Moreover, if* u *is a positive solution of (A) then* $u(L) = \sigma^*(\rho)$.

Proof. We observe from (10) that G_1 is non-negative, decreasing and $G_1(\rho, \rho) = 0$. Similarly from (13), we have G_2 is non-negative, increasing and $G_2(0, \rho) = 0$. We also have that $G_1(0, \rho) > 0$ and $G_2(\rho, \rho) > 0$. Thus $G_1 - G_2$ a continuous function with $(G_1 - G_2)(0, \rho) > 0$ and $(G_1 - G_2)(\rho, \rho) < 0$. Hence by Intermediate Value Theorem there exists a $\sigma^*(\rho)$ such that $(G_1 - G_2)(\sigma^*(\rho), \rho) = 0$. i.e. $G_1(\sigma^*(\rho), \rho) = G_2(\sigma^*(\rho), \rho)$. As $G_1 - G_2$ is strictly decreasing such a $\sigma^*(\rho)$ is unique.

Clearly, if u is a positive solution of (A) satisfying (10) and (13), then $u(L) = \sigma^*(\rho)$. □

Theorem 3.1. *Let* $\rho \in S$, $\lambda > 0$ *and let* r_1 *and* r_2 *be the zeros of* $\tilde{f}(u)$. *(A) has a positive solution iff* $\sqrt{\lambda} \in Range(H)$, *where* $H(\rho) = G_2(\sigma^*(\rho), \rho)$ *[or* $H(\rho) = G_1(\sigma^*(\rho), \rho)$*]. Moreover, if* $\rho = r_1$ *(or* $\rho = r_2$*) and satisfies* $F(r_1) = \frac{m^2}{2\lambda}$ *(or* $F(r_2) = \frac{m^2}{2\lambda}$*) then* $u \equiv r_1$ *(or* r_2*) on* $(L, \frac{1}{2})$ *and thus* $\sigma^*(\rho) = r_1$ *(or* r_2*).*

Proof. Let u be a positive solution of (A). By Lemma 3.2 $\sqrt{\lambda} \in Range(H)$. Conversely let $\sqrt{\lambda} \in Range(H)$. Let $u(x)$ be symmetric at $x = \frac{1}{2}$ and be defined as

$$u(x) = \begin{cases} u_1(x), & x \in (0, L), \\ u_2(x), & x \in (L, \frac{1}{2}], \end{cases} \tag{14}$$

where u_1, u_2 satisfies (11) and (8), respectively. Since $\sqrt{\lambda} \in Range(H)$, there exists $\rho \in S$ such that $\sqrt{\lambda} = H(\rho)$, that is $G_2(\sigma^*(\rho)) = \sqrt{\lambda} = G_1(\sigma^*(\rho))$, thus u clearly satisfies (A). Hence u is positive solution of (A).

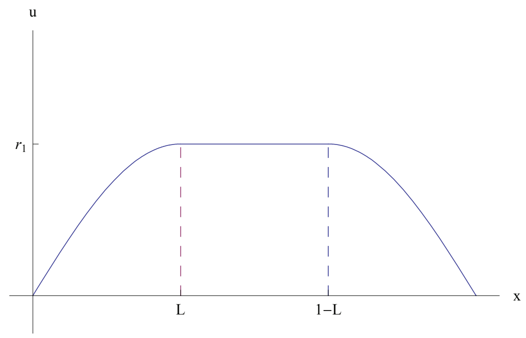

Figure 3: Typical Type I solution when $\rho = r_1$

We note that $\tilde{f}(r_1) = 0$ and thus $u''(L) = 0$. Also since $F(r_1) = \frac{m^2}{2\lambda}$, we have $u'(L^-) = 0$, and by continuity on u', $u'(L) = 0$. It is easy to see that $u \equiv r_1$ on $(L, 1-L)$ since $u(L)-r_1 = 0 = u(1-L)-r_1$, $(u(L)-r_1)' = u'(L) = 0 = u'(1-L) = (u(1-L)-r_1)'$ and $(u(L) - r_1)'' = u''(L) = 0 = u''(1 - L) = (u(1 - L) - r_1)''$. Hence the claim. □

4 Asymmetric Harvesting

In this section we analyze existence of solutions to model B using the quadrature method. We call such classes of positive asymmetric solutions as ‘Type II’ solutions. In this section we consider Type II solutions. Let $\tilde{f}(u) := \lambda[au - bu^2 - c], f(u) := \lambda[au - bu^2]$ and L be the point at which u is maximum.We consider the following two cases:

Case 1: Maximum value is achieved at a point $L > \frac{1}{2}$

Case 2: Maximum value is achieved at a point $L < \frac{1}{2}$

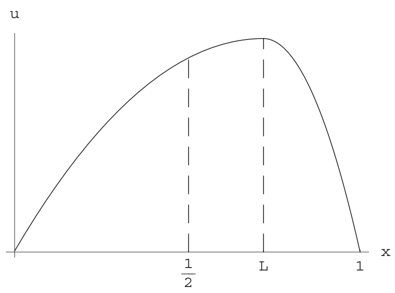

Figure 4: Typical Type II solution for case 1

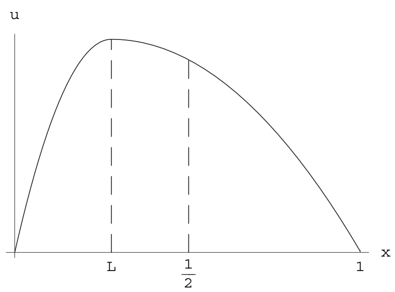

Figure 5: Typical Type II solution for case 2

4.1 Case 1 : $L > \frac{1}{2}$

We develop the quadrature method for the interval $0 \le x \le \frac{1}{2}$ first; the intervals $\frac{1}{2} \le x \le L$ and $L \le x \le 1$ are considered separately. As explained in Section 2 in the interval $0 \le x \le \frac{1}{2}$, we have

$$\int_0^\sigma \frac{dv}{\sqrt{\frac{m^2}{2} - \lambda\tilde{F}(v)}} = \tfrac{1}{2}\sqrt{2}$$

where $F(s) := \int_0^s f(t)dt$ and $m = u'(0)$. To solve for $\frac{m^2}{2}$ we set $u'(\frac{1}{2}^+) = u'(\frac{1}{2}^-)$, which on substitution gives

$$\sqrt{2\lambda}\sqrt{F(\rho) - F(\sigma)} = \sqrt{2}\sqrt{\tfrac{m^2}{2} - \lambda\tilde{F}(\sigma)}$$
$$\lambda(F(\rho) - F(\sigma)) = (\tfrac{m^2}{2} - \lambda\tilde{F}(\sigma))$$
$$\frac{m^2}{2} = \lambda(F(\rho) - F(\sigma) + \tilde{F}(\sigma)).$$

Hence on $0 \le x \le \frac{1}{2}$,

$$G_1(\sigma, \rho) = \sqrt{2}\int_0^\sigma \frac{dv}{\sqrt{F(\rho) - F(\sigma) + \tilde{F}(\sigma) - \tilde{F}(v)}} = \sqrt{\lambda}. \tag{15}$$

When $\frac{1}{2} \leq x \leq L$ we have $\frac{-u'(x)^2}{2} = \lambda(F(u) - F(\sigma)) - \frac{\hat{m}^2}{2}$ where $\hat{m} = u'(\frac{1}{2})$. Hence

$$\int_{\sigma}^{u(x)} \frac{dv}{\sqrt{\frac{\hat{m}^2}{2} - \lambda(F(v) - F(\sigma))}} = \sqrt{2}(x - \frac{1}{2}),$$

and solving for $\frac{\hat{m}^2}{2}$ and then substituting $x = L$ we get

$$\int_{\sigma}^{\rho} \frac{dv}{\sqrt{F(\rho) - F(v)}} = \sqrt{2\lambda}(L - \frac{1}{2}). \tag{16}$$

Similarly for $L \leq x \leq 1$ we get

$$\int_{0}^{\rho} \frac{dv}{\sqrt{F(\rho) - F(v)}} = \sqrt{2\lambda}(1 - L). \tag{17}$$

Note here that u is decreasing and thus u' is negative on $L \leq x \leq 1$, i.e., $u'(x) = -\sqrt{2\lambda(F(\rho) - F(v))}$. Hence for $\frac{1}{2} \leq x \leq 1$ we have

$$G_2(\sigma, \rho) = \sqrt{2}\int_{0}^{\rho} \frac{dv}{\sqrt{F(\rho) - F(v)}} + \sqrt{2}\int_{\sigma}^{\rho} \frac{dv}{\sqrt{F(\rho) - F(v)}} = \sqrt{\lambda}. \tag{18}$$

Lemma 4.1. *Let $T := \{\rho : \hat{\alpha} < \rho < \beta\}$ where $\hat{\alpha}$ is as defined in Lemma 3.2 and $\beta = \frac{a}{b}$. If $\rho \in T$ then G_1, G_2 are well-defined.*

Proof. We have

$$F(\rho) - F(\sigma) + \tilde{F}(\sigma) - \tilde{F}(v) = \tilde{F}(\rho) - \tilde{F}(v) + c(\rho - \sigma),$$

and $\hat{\alpha} \leq \beta \leq \hat{\beta}$. Hence $\tilde{F}(\rho) - \tilde{F}(v) > 0$, for all $\rho \in T$. Further since $\tilde{f}(\rho) > 0$ for all

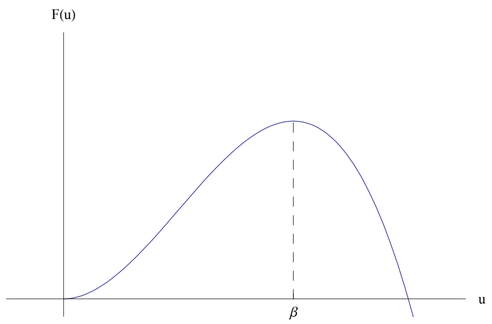

Figure 6: Graph of $F(u)$

$\rho \in T$, G_1 is well-defined for $\rho \in T$. It is clear from the graph of F that $F(\rho) - F(v)$ is positive for all $v \in (0, \rho)$ and $f(\rho) > 0$, if $\rho \in (0, \beta)$. So G_2 is also well-defined for all $\rho \in T$. □

Lemma 4.2. *Let $\rho \in T$ be fixed, then there exists a unique $\sigma^*(\rho)$ such that $G_1(\sigma^*(\rho),\rho) = G_2(\sigma^*(\rho),\rho)$. Moreover, if u is a positive solution of (B) of Type II with $L > \frac{1}{2}$, then $u(\frac{1}{2}) = \sigma^*(\rho)$.*

Proof. It is easy to see that $G_1 - G_2$ is continuous and $(G_1-G_2)(0,\rho) < 0$. We observe here that if $\sigma = \rho$,

$$F(\rho) - F(\sigma) + \tilde{F}(\sigma) - \tilde{F}(v) = F(\rho) - F(v) - c(\sigma - v) < F(\rho) - F(v).$$

Thus we have

$$\int_0^\rho \frac{dv}{\sqrt{F(\rho) - F(\sigma) + \tilde{F}(\sigma) - \tilde{F}(v)}} > \int_0^\rho \frac{dv}{\sqrt{F(\rho) - F(v)}}$$

and hence $(G_1 - G_2)(\rho,\rho) > 0$. So by the Intermediate Value Theorem there exists a $\sigma^*(\rho)$ such that $(G_1-G_2)(\sigma^*(\rho),\rho) = 0$, i.e., $G_1(\sigma^*(\rho),\rho) = G_2(\sigma^*(\rho),\rho)$. We also have

$$G_1'(\sigma,\rho) = \sqrt{2}\Big(\int_0^\sigma \frac{c \quad dv}{2[F(\rho) - F(v) - c(\sigma - v)]^{3/2}} + \frac{1}{\sqrt{F(\rho) - F(\sigma)}}\Big) > 0$$

and

$$G_2'(\sigma,\rho) = \frac{-\sqrt{2}}{\sqrt{F(\rho) - F(\sigma)}} < 0,$$

i.e., $G_1 - G_2$ is strictly increasing and hence such a $\sigma^*(\rho)$ is unique. The rest follows similarly as in the Lemma 3.2. □

Theorem 4.1. *Let $\rho \in T$, $L \geq \frac{1}{2}$ and $\lambda > 0$. (B) has a positive solution iff $\sqrt{\lambda} \in$ Range(H), where $H(\rho) = G_2(\sigma^*(\rho),\rho)$ [or $H(\rho) = G_1(\sigma^*(\rho),\rho)$].*

Proof. Proof follows similarly as in the Theorem 3.1. □

4.2 Case 2 : $L < \frac{1}{2}$

Theorem 4.2. (B) *has no positive solution of Type II with $L < \frac{1}{2}$.*

Proof. We derive formulas for the solution using the quadrature method; again the regions $0 \leq x \leq \frac{1}{2}$ and $\frac{1}{2} \leq x \leq 1$ are considered separately. We further subdivide the region $0 \leq x \leq \frac{1}{2}$ into $0 \leq x \leq L$ and $L \leq x \leq \frac{1}{2}$.
As explained in Section 2 we have in the interval $0 \leq x \leq L$

$$\int_0^{u(x)} \frac{dv}{\sqrt{\tilde{F}(\rho) - \tilde{F}(v)}} = \sqrt{2\lambda}x,$$

and substituting $x = L$ we have

$$\int_0^\rho \frac{dv}{\sqrt{\tilde{F}(\rho) - \tilde{F}(v)}} = \sqrt{2\lambda} L.$$

Similarly for $L \leq x \leq \frac{1}{2}$ we get

$$\int_\sigma^\rho \frac{dv}{\sqrt{\tilde{F}(\rho) - \tilde{F}(v)}} = \sqrt{2\lambda}(\tfrac{1}{2} - L).$$

Note here that u is decreasing and thus u' is negative on $L \leq x \leq 1$, i.e., $u'(x) = -\sqrt{2\lambda(\tilde{F}(\rho) - \tilde{F}(v))}$. Thus, for $0 \leq x \leq \frac{1}{2}$

$$G_1(\sigma, \rho) = \sqrt{2} \int_0^\rho \frac{dv}{\sqrt{\tilde{F}(\rho) - \tilde{F}(v)}} + \sqrt{2} \int_\sigma^\rho \frac{dv}{\sqrt{\tilde{F}(\rho) - \tilde{F}(v)}} = \sqrt{\lambda}. \tag{19}$$

Now consider the interval $\frac{1}{2} \leq x \leq 1$. Since $u' < 0$ in this interval we have,

$$\int_0^\sigma \frac{dv}{\sqrt{\frac{m^2}{2} - \lambda F(v)}} = \tfrac{1}{2}\sqrt{2},$$

where $F(s) := \int_0^s f(t)dt$ and $m = u'(1)$. To solve for $\frac{m^2}{2}$ we set $u'(\frac{1}{2}^-) = u'(\frac{1}{2}^+)$, which on substitution gives

$$-\sqrt{2\lambda}\sqrt{\tilde{F}(\rho) - \tilde{F}(\sigma)} = -\sqrt{2}\sqrt{\tfrac{m^2}{2} - \lambda F(\sigma)}$$

$$\lambda(\tilde{F}(\rho) - \tilde{F}(\sigma)) = (\tfrac{m^2}{2} - \lambda F(\sigma))$$

$$\frac{m^2}{2} = \lambda(\tilde{F}(\rho) - \tilde{F}(\sigma) + F(\sigma)).$$

Thus on $\frac{1}{2} \leq x \leq 1$ we have

$$G_2(\sigma, \rho) = \sqrt{2} \int_0^\sigma \frac{dv}{\sqrt{\tilde{F}(\rho) - \tilde{F}(\sigma) + F(\sigma) - F(v)}} = \sqrt{\lambda}. \tag{20}$$

If a positive solution exists, then $G_1(\sigma^*(\rho), \rho) = G_2(\sigma^*(\rho), \rho)$ for some $\sigma^* \in (0, \rho]$. We observe here that for $0 \leq v \leq \sigma$,

$$\tilde{F}(\rho) - \tilde{F}(\sigma) + F(\sigma) - F(v) = \tilde{F}(\rho) - \tilde{F}(v) + c(\sigma - v) \geq \tilde{F}(\rho) - \tilde{F}(v).$$

Therefore

$$\begin{aligned} G_2(\sigma,\rho) &= \sqrt{2}\int_0^{\sigma} \frac{dv}{\sqrt{\tilde{F}(\rho)-\tilde{F}(\sigma)+F(\sigma)-F(v)}} \\ &\le \sqrt{2}\int_0^{\sigma} \frac{dv}{\sqrt{\tilde{F}(\rho)-\tilde{F}(v)}} \\ &< \sqrt{2}\int_0^{\rho} \frac{dv}{\sqrt{\tilde{F}(\rho)-\tilde{F}(v)}} \le G_1(\sigma,\rho). \end{aligned}$$

Hence no positive solution exists for Case 2. □

5 Computational Results

In this section we discuss the results obtained using *Mathematica* computation of bifurcation curves of positive solutions developed in Sections 3 and 4 for models A & B. Figure 7 is the bifurcation diagram of the symmetric harvesting problem (A) with parameter values $a = 10$, $b = \frac{1}{4}$, $c = 74$ and $L = \frac{1}{3}$. Based on our discussion in Section 3, ρ is restricted to the interval $(26.536, 30.198)$ for the set S in Theorem 3.1 to be non-empty.

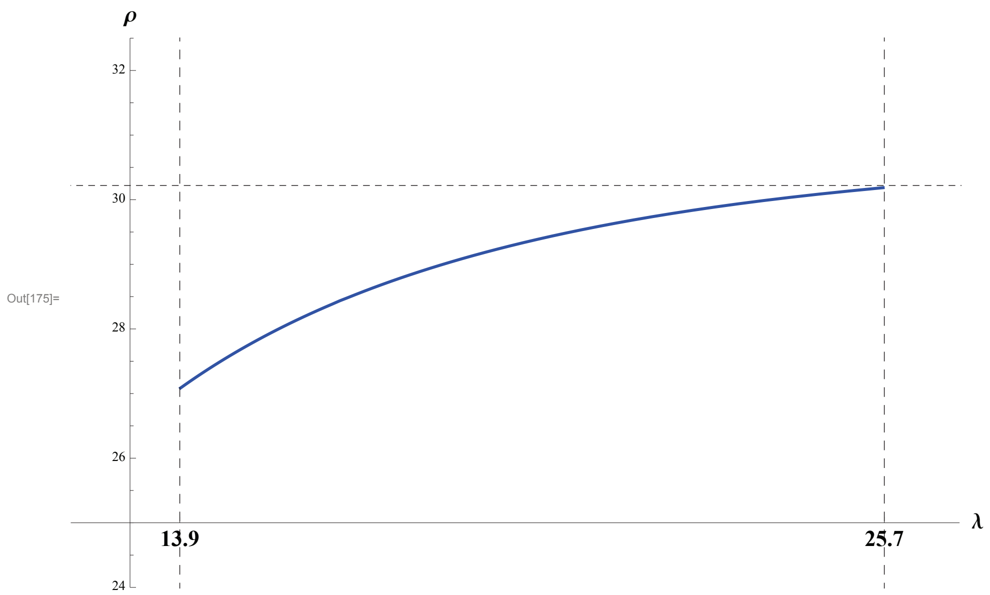

Figure 7: Symmetric harvesting: λ vs ρ when $a = 10$,$b = \frac{1}{4}$ and $c = 74$

Figures 8 and 9 give the bifurcation diagram of the symmetric harvesting problem (A) with parameter values $a = 100$, $b = \frac{1}{4}$, $c = 74$ and $L = \frac{1}{3}$.

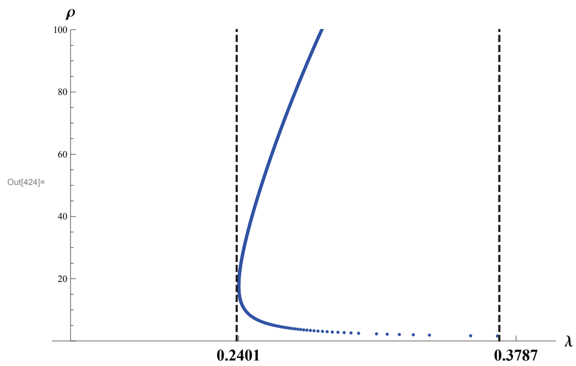

Figure 8: Symmetric harvesting: λ vs ρ when $a = 100$,$b = \frac{1}{4}$ and $c = 74$

Here again the the set S is non-empty only for $\rho \in (1.484, 399.259)$. In Figure 8, we notice that for higher birth rates (i.e. large a such as $a = 100$) there exists two solutions for $\lambda \in (0.2401, 0.3787)$.

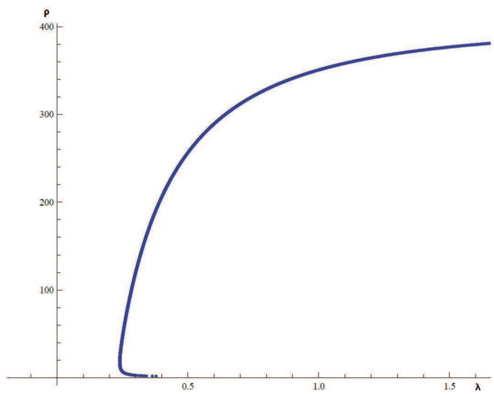

Figure 9: λ vs ρ when $a = 100$,$b = \frac{1}{4}$ and $c = 74$ - (2 solutions)

Figure 10 is the bifurcation diagram of the asymmetric harvesting problem (B) with parameter values $a = 10$, $b = \frac{1}{4}$ and $c = 74$. Here ρ has to lie in the interval $(26.536, 40)$ for the set T in Theorem 4.2 to be non-empty.

Figures 11 and 12 are the bifurcation diagrams of the asymmetric harvesting problem (B) with parameter values $a = 100$, $b = \frac{1}{4}$ and $c = 74$. As in the symmetric harvesting problem we get two solutions for higher birth rates when $\lambda \in (0.1053, 0.142)$. The T in Theorem 4.2 is non-empty if $\rho \in (1.48367, 400)$.

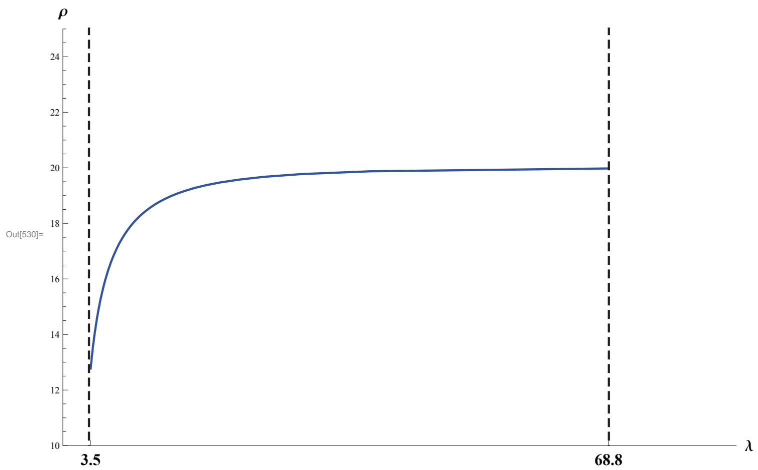

Figure 10: Asymmetric harvesting: λ vs ρ when $a = 10$, $b = \frac{1}{4}$ and $c = 74$

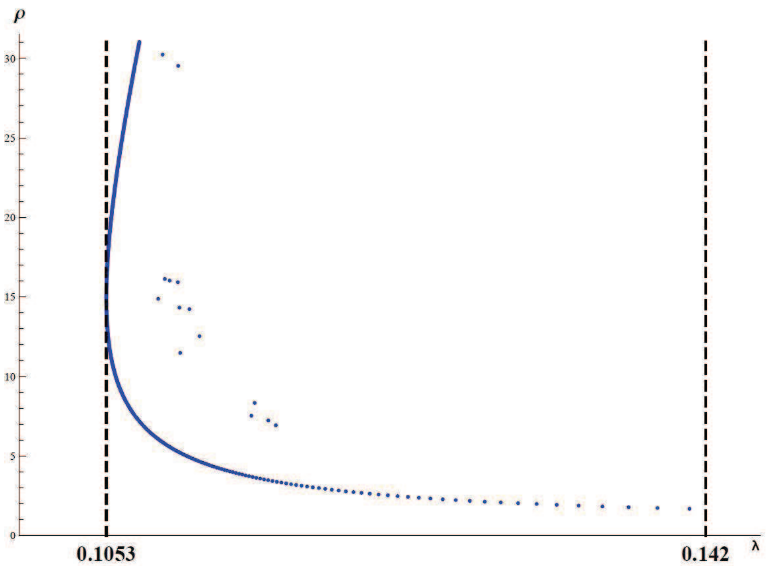

Figure 11: Asymmetric harvesting: λ vs ρ when $a = 100$, $b = \frac{1}{4}$ and $c = 74$

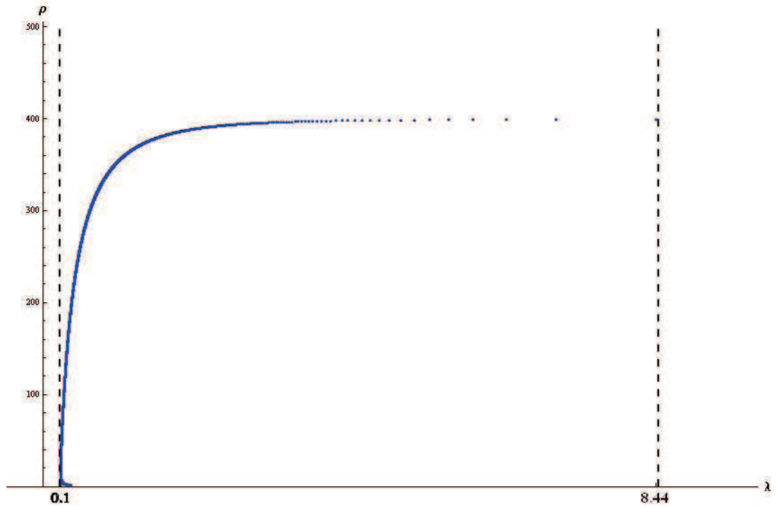

Figure 12: Asymmetric harvesting: λ vs ρ when $a = 100$, $b = \frac{1}{4}$ and $c = 74$

References

[1] H. Berestycki, L. A. Caffarelli and L. Nirenberg, Inequalities for second order elliptic equations with applications to unbounded domains I. A celebration of John F. Nash, Jr., *Duke Math. J.* **81** (1996), 467-494.

[2] K. J. Brown, M. M. A. Ibrahim and R. Shivaji, S-shaped bifurcation curves, *Nonlin. Anal.* **5** (5) (1981), 475-486.

[3] B. A. Block, *et al.*, Migratory movements, depth preferences, and thermal biology of Atlantic bluefin tuna, *Science* **293** (2001), 1310-1314 [http://www.sciencemag.org].

[4] A. Castro and R. Shivaji, Nonnegative solutions for a class of nonpositone problems, *Proc. Roy. Soc. Edin. Sect. A* **108** (1988), 291-302.

[5] A. Castro, C. Maya and R. Shivaji, Nonlinear eigenvalue problems with semipositone structure, *Electron. J. Diff. Eqns.* **5** (2000), 33-49.

[6] T. Laetsch, The number of solutions of a nonlinear two point boundary value problem, *Indiana Univ. Math. J.* **20** (1) (1970), 1-13.

[7] P. L. Lions, On the existence of positive solutions of semiliniear elliptic equations, *Siam Rev.* **24** (1982), 441-467.

[8] J. J. Magnuson, C. Safina and M. P. Sissenwine, Whose fish are they anyway?, *Science* **293** (2001), 1310-1314 [http://www.sciencemag.org].

[9] S. Oruganti, J. Shi and R. Shivaji, Diffusive equations with constant yield harvesting, I: steady states, *Trans. Amer. Math. Soc.* **354** (2002), 3601-3619.

[10] Dan Vergano. No place for fish to hide, *USA Today*, May 15, 2003.