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Instability in Hamiltonian Systems

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Contents

Preface

From the time of the pioneer Poincaré's essay [20] up to the present days, chaos in conservative dynamics has been identified with the presence of heteroclinic motions in transversal cross sections to the flow. The existence of this unlimited dynamical richness leads, in an unmistakable way, to the instability of the studied system. V. I. Arnold discovered that, surprisingly, these situations often arise in a persistent way when an integrable Hamiltonian system is perturbed.

The global strategy designed by Arnold was based on the control of the so-called splitting of separatrices, which takes place when a parametric family of perturbations of the initial integrable system is considered. The method used by Arnold furnished orbits drifting along invariant objects and therefore giving rise to the presence of (nowadays called) Arnold diffusion. From the quantitative point of view, those events were observed for an open, but small, set of parameter values.

Besides proving the existence of Arnold diffusion for a new family of three degrees of freedom Hamiltonian systems, another goal of this book is not only to show how Arnold-like results can be extended to substantially larger sets of parameters, but also how to obtain effective estimates on the splitting of separatrices size when the *frequency* of the perturbation belongs to open real sets.

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Introduction

Preliminaries and historical remarks

This book discusses dynamical aspects concerning perturbed Hamiltonian systems. For the sake of clarity, let us start by giving a few basic definitions and properties related to conservative dynamics, biased toward the items needed in this book. For complete expositions of those subjects we cite [2] and [13].

Let us recall that a $2l$-dimensional vector field is said to be Hamiltonian if the associated flow satisfies an ordinary differential equation of the form:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \tag{0.0.1}$$

The variable $p \in \mathbb{R}^l$ is called the momentum and $q \in \mathbb{R}^l$ is called the position. The function $H : (q,p) \in G \subset \mathbb{R}^{2l} \to H(q,p) \in \mathbb{R}$, where G is some open set of $\mathbb{R}^{2l}$, is called the Hamiltonian of the system (0.0.1) and the equations are known as the Hamilton equations. Furthermore, the Hamiltonian H is said to have l degrees of freedom.

In order to introduce the notion of integrable Hamiltonian system, let us state three more definitions:

- A function $f : G \to \mathbb{R}$ is said to be a first integral of the Hamiltonian H, if it is constant along the orbits of the system (0.0.1).

- Two functions $f, g : G \to \mathbb{R}$, $f = f(q,p)$ and $g = g(q,p)$ are said to be in involution, if their Poisson bracket

  $$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}$$

  is identically zero.

- The functions $f_j : G \to \mathbb{R}$, $1 \leq j \leq n$, are said to be independent on G, if the vectors $\{\nabla f_j(x_0)\}_{1\leq j\leq n}$ are linearly independent for every point $x_0 \in G$. Furthermore, given a family of functions $F = \{f_j\}_{1\leq j\leq n}$, $f_j : G \to \mathbb{R}$, we denote by R_F the open subset of G where $\{f_j\}_{1\leq j\leq n}$ are independent.

Now, we can define the simplest Hamiltonian systems:

Definition 0.0.1 *Let $H = H(q, p)$ be a Hamiltonian system with l degrees of freedom. If there exists a family $F = \{f_1, \ldots, f_l\}$ of l first integrals of H which are two by two in involution, then we say that H is integrable in R_F.*

The intersections of the level manifolds of these l independent first integrals in involution are subsets of R_F which are invariant for the flow associated to the integrable Hamiltonian system. These invariant objects are completely described by the Arnold-Liouville Theorem, which ensures that they are diffeomorphic to either l-dimensional tori, $\mathbb{T}^l = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$, or l-dimensional cylinders, $C^l = \mathbb{T}^r \times \mathbb{R}^s$, with $r + s = l$ and $r \neq l$.

In this book we will study three degrees of freedom perturbed Hamiltonian systems, that is, the dynamical system associated to Hamiltonians of the type $H_\mu = H(q, p, \mu)$, where μ is a small parameter and H_0 is an integrable Hamiltonian. In particular, we will study two examples of such families H_μ and, in both cases, if $\mu = 0$ then the respective three first integrals in involution (and therefore the set R_F) are easily obtained.

Our particular examples of such parametric families (see (0.0.3) and (0.0.4)) belong to a large kind of Hamiltonians often arising in the gaps between the Kolmogorov-Arnold-Moser (KAM) tori. In order to introduce the notion of KAM tori and their relations with the Arnold-Liouville tori, let us make a general approach to such concepts in the l degrees of freedom context: It will be very useful to write an l degrees of freedom integrable Hamiltonian H in adequate coordinates (I, θ) in a way that it does not depend on θ. This can be done whenever we restrict ourselves to neighbourhoods of l-dimensional Arnold-Liouville tori of H (therefore, cylinders are not considered). The coordinates $I = (I_1, \ldots, I_l) \in \mathbb{R}^l$ are usually called actions while $\theta = (\theta_1, \ldots, \theta_l) \in \mathbb{R}^l$ are called angular coordinates. Hence, a complete expression for the solutions of the associated dynamical system

$$\dot{I}_j = 0, \qquad \dot{\theta}_j = \frac{\partial H}{\partial I_j}(I), \quad j = 1, \ldots, l$$

can be obtained and therefore every orbit of this integrable Hamiltonian system can be identified with an l-dimensional Arnold- Liouville torus and each one of these invariant tori can be labelled by its own frequency vector

$$\omega(I) = \left(\frac{\partial H}{\partial I_1}(I), \ldots, \frac{\partial H}{\partial I_l}(I) \right).$$

Kolmogorov's Theorem asserts that, for any non-degenerate close enough perturbation of H, most of those invariant tori do not vanish (they are only slightly deformed). Hence, there are invariant tori densely filled with orbits winding around them and these tori even form a majority in the sense that the measure of the complement of their union is small whenever the new Hamiltonian is sufficiently close to the original integrable one.

Those tori are called KAM tori and the presence of these invariant sets in the phase space of the perturbed system implies that any lower-dimensional object has to be confined to a very small phase space region.

In fact, one of the main purposes of this book consists in studying Lyapunov stability properties of two-dimensional tori living in a vicinity of (some of) the three-dimensional KAM tori associated to our two families of three degrees of freedom Hamiltonian systems. Namely, we are especially interested in proving the existence of instability phenomena in certain regions of the phase space and, more concretely, in finding what is usually called Arnold diffusion.

Let us comment that we have to work (at least) with more than two degrees of freedom Hamiltonian systems, because if we deal with two degrees of freedom Hamiltonians, then the associated KAM tori are two-dimensional while the phase space, once an energy level $H_\mu = ctant$ is fixed, is three-dimensional. Therefore, the KAM tori split the phase space acting as a barrier for the flow in such a way that the Lyapunov stability of any equilibrium object is fulfilled. This is no longer true in three (or more) degrees of freedom Hamiltonian systems as was proven by Arnold in his remarkable paper [1].

In order to describe Arnold's result, let us make a historical overview on one of the possible (see [3], [4] or [29] for a different way to obtain instability) fundamental pieces leading to instability: The splitting of separatrices.

In his famous essay on the stability of the solar system [20], Poincaré discovered the splitting of separatrices phenomenon by studying high-frequency perturbations of a pendulum. These high-frequency perturbations can be described by the following two-parameter family of Hamiltonians

$$\frac{y^2}{2} + \cos x + \mu \sin x \cos \frac{t}{\varepsilon}$$

or, using its equivalent autonomous form, by the following two-parameter family of two degrees of freedom Hamiltonian systems

$$H_{\varepsilon,\mu}(q,p) = \frac{I}{\varepsilon} + \frac{y^2}{2} + \cos x + \mu \sin x \cos \theta, \quad q = (\theta, x), \quad p = (I, y).$$

When the unique perturbing parameter μ vanishes we obtain integrable Hamiltonian systems. In fact, it suffices to consider the family $F = \{f_1, f_2\}$ of first integrals in involution defined by

$$f_1(\theta, x, I, y) = I, \quad f_2(\theta, x, I, y) = \frac{y^2}{2} + \cos x$$

in order to check that $H_{\varepsilon,0}$ are integrable Hamiltonian systems (see Definition 0.0.1) when taking R_F the complement, in the phase space, of the set $T = \{(\theta, x, I, y) : \ x = y = 0\}$. The set T can be written as

$$T = \bigcup_{\alpha \in \mathbb{R}} T_\alpha$$

where $T_\alpha = \{(\theta, x, I, y) : \ x = y = 0, I = \alpha\}$ are, for every $\alpha \in \mathbb{R}$, invariant sets for the unperturbed flow. Furthermore, each one of these invariant sets are connected to itself by a two-dimensional manifold filled by homoclinic motions to T_α. These two-dimensional manifolds are usually called separatrices and they coincide, for each value of α, with the invariant stable and unstable manifolds of T_α. Although the sets T_α survive the perturbation, it is no longer true that the perturbed systems present separatrices. In fact, the separatrices arising when $\mu = 0$ split when the perturbation is considered and the phase space associated to the perturbed Hamiltonians displays unlimited dynamical richness. This fact led Poincaré to conjecture that those situations explained the stochastic behaviour frequently appearing in Hamiltonian systems. Poincaré even related the measure of the associated stochastic behaviour with the length, in terms of the perturbing parameter μ, of the split of the separatrices of the unperturbed system. Nowadays (see [11], [28]) it is well-known that using first variational equations one may obtain an expression

$$D = \mu M + \mathrm{O}(\mu^2) \tag{0.0.2}$$

which gives the distance, D, at first order in the perturbing parameter μ, between the invariant perturbed manifolds of T_α. The method essentially works because although the function M (called Melnikov function) is exponentially small in ε, one may restrict the predictions for the case in which $|\mu|$ is exponentially small with respect to ε. This is the advantage to work with two-parameter families of perturbed Hamiltonians making one of the parameters exponentially small with respect to the other. We remark that, since we are going to work with more general situations (just several-parameter perturbations but without assuming any of them exponentially small with respect to one another) we have to make a more careful analysis to obtain the asymptotic behaviour of the splitting distance (see [6] where the predictions on the splitting size for the Hamiltonian introduced by Poincaré were already stated without assuming $|\mu|$ to be exponentially small with respect to ε).

Several decades after Poincaré's work, Arnold introduced in [1] the following two-parameter Hamiltonian family

$$H_{\varepsilon,\mu}(I_1, I_2, \theta_1, \theta_2, t) = \frac{1}{2}(I_1^2 + I_2^2) + \varepsilon(\cos\theta_1 - 1)\,(1 + \mu(\sin\theta_2 + \cos t))$$

with (I_1, I_2) in some subset G of $\mathbb{R}^2$, $(\theta_1, \theta_2) \in \mathbb{T}^2$, $\mathbb{T}^2$ denoting the bidimensional torus, and $t \in \mathbb{R}$. Of course, the family of non-autonomous Hamiltonian systems introduced by Arnold could be replaced by the respective equivalent family of autonomous three degrees of freedom Hamiltonian systems.

With this particular example, Arnold illustrated how the perturbations of integrable Hamiltonians may give rise to instability: There are orbits for which the action variables vary in a determined amount and this amount does not depend on the size of the perturbing parameter μ, even when μ goes to zero. This kind of instability is usually called diffusion. Diffusion and splitting of separatrices will be the dynamical aspects on perturbed Hamiltonian systems which we are going to work with.

We should not think that this instability (diffusion) depends on the particular Arnold's choice. Indeed, in [1] Arnold proposed his conjecture introducing diffusion as a generic phenomenon, in the sense that it should be present in almost all Hamiltonian systems.

As a first approach to give a positive answer to his conjecture, Arnold developed his ideas in the system introduced above; in Arnold's example a continuous family of two-dimensional invariant whiskered tori (i.e., tori having stable and unstable manifolds) arises when the perturbation is considered. Furthermore, a transversal intersection between the stable and unstable manifolds of each torus takes place. Hence, these splittings of separatrices form a rather tangled net of intersections connecting the invariant tori by means of what is usually called a transition chain. Once Arnold established the existence of these chains, he found orbits drifting along every tori taking part in the considered chain: In other words, he showed the existence of trajectories for which the values of (some of) the action variables (I_1, I_2) vary in a finite amount (not depending on μ) over a sufficiently large period of time: Nekhoroshev Theorem [18] states that, for initial conditions in an open set of the phase space, this period of time has to be exponentially large with respect to the parameter ε.

The global strategy designed by Arnold established a possible mechanism, strongly based on the control of the splitting size, for proving the existence of instability in Hamiltonian systems. Nevertheless, Arnold's scheme succeeded only by making the perturbing parameter μ exponentially small with respect to ε. This exponential smallness was strongly used by Arnold to obtain effective estimates on the splitting size.

Looking at this relation between parameters, an interesting research question arises: Obtain, in an analytic way, correct estimates on the splitting separatrix size for three degrees of freedom Hamiltonian systems, being these estimates valid in "large" regions in the parameter space, and prove that those Hamiltonian systems are unstable in the sense that they present Arnold's diffusion.

Let us point out again that, since the splitting size is going to be exponentially small in ε, the classical perturbation theory used by Poincaré and Arnold does not work to detect such splittings and we would not be able to detect the existence of Arnold's diffusion by using the classical perturbation theory designed by Poincaré. Hence, new analytical tools have to be implemented.

Splitting of separatrices

This book deals mostly with the phenomenon of the splitting of separatrices exhibited by two families of three degrees of freedom Hamiltonian systems. Moreover, for the second family (see (0.0.4)), the obtained estimates on the splitting size are used to construct a transition chain connecting two-dimensional invariant tori. Consequently, we prove the existence of diffusion, and therefore instability.

In Chapter 1 we consider our first family of Hamiltonian systems $H_{\varepsilon,\beta,\mu}$ depending on three parameters ε, β and μ. When the single perturbing parameter μ is zero, the Hamiltonian systems $H_{\varepsilon,\beta,0}$ are integrable (later, we will describe the respective subset R_F associated to the definition of integrability) and exhibit, for each value of β (called frequency) and every value of ε, a continuum of invariant bidimensional tori, each one of which having a three-dimensional homoclinic manifold (separatrix). Those tori survive the perturbation and some of their hyperbolic properties allow us to prove that, for every small enough value of μ, they are, in fact, whiskered tori in the sense described above: They exhibit stable and unstable manifolds. Nevertheless, it is no longer true that the invariant stable and unstable perturbed manifolds coincide but they intersect at homoclinic orbits of the respective whiskered tori.

Therefore, one of our main objectives consists in comparing the size of the perturbation with the angle (transversality) at which the perturbed manifolds intersect. In order to measure this transversality, or the splitting size, we will make use of the so-called splitting functions. The definition of these splitting functions involves the choice of suitable coordinates (called flow-box coordinates) and the whole process becomes one of the fundamental pieces leading to the proof of Theorem 0.0.2, which is the main result of Chapter 1. Let us remark that, although we will work with a three-parameter family of Hamiltonian systems, we will not need to assume any of the parameters to be exponentially small with respect to one another. In fact, it will be enough to assume that the perturbing parameter μ satisfies $\mu \in (0, \varepsilon^m)$, for some value of m which essentially depends on the properties of the considered perturbation.

Theorem 0.0.2 allows us to know the asymptotic values for the transversality along one intersection (homoclinic orbit) between the perturbed stable and unstable manifolds. To reach these estimates we will also use what we call renormalized Melnikov functions. Let us remark that, although the well-known Melnikov method has been used often from the time of the pioneering Poincaré's essay up to the present days, it is not enough for our purposes as we have already explained. To obtain computable formulae that yield correct estimates on how large the splittings of separatrices are, we need to follow a more elaborated strategy than the one used by Poincaré or Arnold. This is the reason why we are going to name our obtained computable formulae *renormalized* Melnikov functions and they are going to be, in fact, one of the most important tools used along this book.

During the last decade several kinds of strategies directed to control the splitting of separatrices asymptotic estimates have been performed. Among the papers dealing with these subjects, we cite [6], [7], [22] and [25] because they are the closest ones to the situations studied in this book. This closeness is understood by taking into account that, for our first Hamiltonian (0.0.3), the dynamics related to the pendulum $\frac{1}{2}y^2 + A(\cos x - 1)$ is extremely slower than the one related to the rotors $\varepsilon^{-1}(I_1 + \beta I_2)$. The same holds for our second case, see the Hamiltonian family (0.0.4) or its equivalent form given in (2.0.8). This is the reason why (0.0.3) and (0.0.4) are called two-time scales Hamiltonian systems as well as the Hamiltonians studied in the four references mentioned above. For three-time scales situations we cite [5] and [9] for a different way to get estimates on the splitting of separatrices in this scenario.

One of the main differences between the present work and the six references mentioned in the above paragraph is that we are going to deduce the splitting estimates without assuming that the frequency β is "irrational enough". In fact, we focus on establishing the necessary techniques for obtaining the correct asymptotic expressions for the splitting of separatrices whenever the frequency β belongs to certain open set of real numbers. We will put special emphasis in the relation between the predictions of this splitting size in open sets of frequencies with the Arnold's diffusion phenomenon.

Let us introduce our first family of Hamiltonian systems

$$H_{\varepsilon,\beta,\mu}(x,y,I_1,I_2,\theta_1,\theta_2) = \frac{I_1+\beta I_2}{\varepsilon} + \frac{y^2}{2} + A(\cos x - 1) + \mu y \sin x M_1(\theta_1,\theta_2) \quad (0.0.3)$$

where

$$M_1(\theta_1,\theta_2) = \sum_{(k_1,k_2)\in\Lambda} \frac{(a\varepsilon^p)^{|k_1|+|k_2|}}{f(k_1,k_2)} \sin(k_1\theta_1 + k_2\theta_2)$$

and μ denotes the unique perturbing parameter. The equations of motion associated to (0.0.3) are obtained by using (0.0.1) with $H = H_{\varepsilon,\beta,\mu}$, $q = (\theta_1,\theta_2,x)$ and $p = (I_1,I_2,y)$.

The assumptions imposed to the function f taking part in the definition of M_1 are given at the beginning of the first chapter where we also introduce the properties of the indexes subset Λ of $\mathbb{Z}^2$.

Let us observe that the Hamiltonian family introduced in (0.0.3) can be seen as a quasiperiodic perturbation of a planar pendulum

$$H_1(x,y) = \frac{y^2}{2} + A(\cos x - 1)$$

by considering the non-autonomous equivalent family

$$H_{\varepsilon,\beta,\mu}(x,y,t) = \frac{y^2}{2} + A(\cos x - 1) + \mu y \sin x M_1\left(\frac{t}{\varepsilon},\frac{\beta}{\varepsilon}\right).$$

According to Definition 0.0.1, the unperturbed systems ($\mu = 0$) are integrable in the complement of the set

$$T = \{(x,y,I_1,I_2,\theta_1,\theta_2):\ x = y = 0\} = \bigcup_{\alpha_1,\alpha_2} T_{\alpha_1,\alpha_2}$$

where

$$T_{\alpha_1,\alpha_2} = \{(x,y,I_1,I_2,\theta_1,\theta_2):\ x = y = 0,\ I_1 = \alpha_1,\ I_2 = \alpha_2\}$$

are two-dimensional invariant tori for the unperturbed flow. Each of these tori T_{α_1,α_2} has, for the unperturbed system, a three-dimensional separatrix (i.e., there exists a two-parameter family of three-dimensional manifolds, each of these manifolds filled by homoclinic orbits to the respective torus). The invariant tori T_{α_1,α_2} survives the perturbation, but their separatrices break when the perturbation is considered and the main purpose of the first chapter is to give the asymptotic estimates of these splittings.

Let us remark that we are choosing a perturbation which is an entire function on the angles, unlike in [6], [7] or [22] where the studied Hamiltonian families contain perturbations with poles. Surprisingly, as we will comment later on, the fact that the perturbation is entire makes the computing splitting size process more intricate.

In [25] Simó studied perturbed Hamiltonian systems in which the perturbation is an entire function. After one step of an averaging method, see [25], the Hamiltonian family considered by Simó is close to the one given in (0.0.3) by taking $f(k_1, k_2) = k_1^{k_1} k_2^{k_2}$ (observe that due to the condition (1.0.4) imposed at the beginning of the first chapter to our function f, the particular example studied by Simó is out of scope of this book). On the other hand, he obtained a lot of valid semi-numerical estimates for the splitting size for the case in which the frequency β coincides with the so-called golden mean number $\tilde{\beta} = \dfrac{\sqrt{5}+1}{2}$.

On the contrary, we are going to analytically prove that for our first family of Hamiltonian systems $H_{\varepsilon,\beta,\mu}$, see (0.0.3), and for any value of $(\varepsilon, \beta, \mu)$ in an open "large" set, there exist computable expressions, the already announced renormalized Melnikov functions, giving the correct estimates for the splitting size.

As a first approach to the computation of the transversality we would calculate, following Poincaré's ideas, the variation of the "unperturbed" energies H_1, I_1 and I_2 along the orbits of the perturbed systems. The difference between the values of these energies along the perturbed stable manifold and the perturbed unstable one define three functions (called splitting functions) which furnish a final formula for the splitting size. However, we are not able to compute these splitting functions, essentially because we do not know any explicit expression for the solutions of the perturbed ($\mu \neq 0$) Hamiltonian systems. This difficulty will be overcome by proving that the splitting functions are *close enough* to the following computable ones:

$$\mathcal{M}_i(\hat{\psi}) = \int_{\mathbb{R}} \{\mathcal{Q}_i, H_{\varepsilon,\beta,\mu}\}(\Delta^0(\hat{\psi}, t))dt, \quad i = 1, 2, 3$$

where $\mathcal{Q}_1 \equiv H_1$, $\mathcal{Q}_2 \equiv I_1$, $\mathcal{Q}_3 \equiv I_2$, $\{\cdot,\cdot\}$ denotes the Poisson bracket defined at the beginning of this chapter, Δ^0 is a convenient parameterization of the homoclinic separatrix of the unperturbed system and the variable $\hat{\psi} = (\psi_1, \psi_2)$ determines the different orbits on such homoclinic manifold.

The functions $\mathcal{M}_i$ play the same role as the function M in (0.0.2) does. Therefore, they could be called Melnikov functions. In fact, following Arnold's steps we can prove that the norm of the difference between Melnikov and splitting functions is of order μ^2. Nevertheless, this is not enough for our purposes since the Melnikov functions will turn out to be exponentially small in terms of the parameter ε (hence, to compare splitting and Melnikov functions, we would restrict our study to those μ which are exponentially small with respect to ε).

In order to get a method valid for larger sets of parameters, we need to use an analytical continuation of the unperturbed torus separatrix in some complex domain by following the ideas developed by Lazutkin [14], which were adapted to differential equations in [6] and [10]. This complex extension of the homoclinic separatrix leads to a complex extension of the above defined Melnikov functions giving rise to three complex functions (still denoted by $\mathcal{M}_i$, $i = 1, 2, 3$) which are going to be called *renormalized* Melnikov functions.

Now, we can describe briefly the contents of the different sections of the first chapter:

In Section 1.1 we use the weak hyperbolicity exhibited by our perturbed model to obtain analytic expressions for the local invariant perturbed manifolds of T_{α_1,α_2} (see Theorem 1.1.8 and Lemma 1.1.11). Moreover, as expected, the distance between local perturbed and unperturbed manifolds is of the same order as the perturbation. The techniques needed to obtain all the results in Section 1.1, especially Theorem 1.1.8, explain the differences between these results and those ones used in [7]: Since our results have to be valid in a continuum of frequencies (so, we can not use any Diophantine condition) we can not guarantee the existence of normal form coordinates (unlike in [7], where the results are only valid for the golden mean case) leading, in a final stage, to the existence of whiskered tori.

Once analytic expressions for the local invariant manifolds are found, we take advantage of the closeness between the local perturbed and unperturbed manifolds of each invariant torus to extend the local perturbed unstable manifold by using the Extension Theorem I, see Theorem 1.1.14. The Extension Theorem I is stated in Section 1.1, but its proof is given in Chapter 3.

In Section 1.2 we obtain expressions for the renormalized Melnikov functions in terms of Fourier series

$$\mathcal{M}_i(\hat{\psi}) = \sum_{(k_1,k_2)\in\Lambda} B_{\hat{k}}^{(i)} \mathcal{E}_{\hat{k}} \sin(k_1\psi_1 + k_2\psi_2),$$

where, for $\hat{k} = (k_1, k_2)$, $B_{\hat{k}}^{(i)}$ are explicitly computed coefficients and

$$\mathcal{E}_{\hat{k}} = \exp\left(-\frac{\pi\,|k_1 + \beta k_2|}{2\varepsilon\sqrt{A}}\right)(a\varepsilon^p)^{|k_1|+|k_2|}.$$

Unfortunately, these final Fourier series expressions of the renormalized Melnikov functions are too much complicated. But, for our purposes, it is enough to control the two dominant terms of the associated numerical series

$$\sum_{(k_1,k_2)\in\Lambda} \cos(k_1\overline{\psi}_1 + k_2\overline{\psi}_2) k_j B_{\hat{k}}^{(i)} \mathcal{E}_{\hat{k}}, \quad i = 2, 3, \quad j = 1, 2,$$

with $(\overline{\psi}_1, \overline{\psi}_2) \in \{0, \pi\} \times \{0, \pi\}$ certain initial phases given rise to homoclinic orbits for the perturbed system.

The two dominant terms $\cos(k_1^{(t)}\overline{\psi}_1 + k_2^{(t)}\overline{\psi}_2) k_j^{(t)} B_{\hat{k}^{(t)}}^{(i)} \mathcal{E}_{\hat{k}^{(t)}}$, $t = n^0, n^1$, will be found by means of the Main Lemma I, see Lemma 1.3.10 (whose proof is given in Chapter 4) and they describe the leading order behaviour of the whole series. We will return to this point later on. Here, the main difference between two-time and three-time scales Hamiltonian systems arises. While in three-time scales systems the dominant terms of the above series are located in a direct way (they usually coincide with the first terms) this is no longer true in the two-time scales framework. Hence, we have to follow an elaborated strategy directed to search those dominant terms. Moreover, since the respective ones used in [7], [22] or [25] depend strongly on the Diophantine condition imposed to the considered frequencies, they are not completely satisfactory in our context.

In Section 1.3 we prove the Main Theorem I stated below. To this end, we consider intersections of the whiskers obtained in Section 1.1 with Poincaré sections located in some suitable domain $\mathcal{U}$.

The whiskers (stable and unstable) for the unperturbed problem coincide. This is no longer true in the perturbed case and the differences of the values of the unperturbed energies H_1, I_1 and I_2 on the mentioned sections can be used to measure how these manifolds break under the perturbation. The success of the method is based on the existence of a very specific change of variables (see Lemma 1.3.1), depending on μ, directed to construct an adequate frame of references useful to measure the transversality. The new coordinates are called flow-box and they are defined in some domain $\mathcal{U}$ visited (by use of the Extension Theorem I) by the stable and unstable tori invariant manifolds. These coordinates were already used in [7]; however, the existence of such coordinates in [7] was proven by using the existence of normal form coordinates defined in a small neighbourhood of the Diophantine torus. In our situation, we have to prove the existence of those flow-box coordinates by using different arguments.

The unperturbed energies H_1, I_1 and I_2 are respectively denoted by $\mathcal{K}^\mu$, $\mathcal{J}_1^\mu$ and $\mathcal{J}_2^\mu$ in the flow-box coordinates. By means of those functions the splitting vector is defined (in a natural way) as the difference of the values of those energies along the perturbed stable and unstable manifolds. The splitting vector will be denoted by $(\mathcal{K}_u^\mu, \mathcal{J}_{1,u}^\mu, \mathcal{J}_{2,u}^\mu)$, where each one of its components (called splitting functions) depend on $(s, \psi_1, \psi_2) \in \mathcal{C}_1' \times \mathbb{T}^2$, where $\mathcal{C}_1'$ is some complex subset (contained in the complex domain where the unperturbed torus separatrix was extended to) and $\mathbb{T}^2$ denotes the bidimensional torus. Once four homoclinic orbits are detected at $(\overline{\psi}_1, \overline{\psi}_2) \in \{0, \pi\} \times \{0, \pi\}$ for our original ($s = 0$) perturbed Hamiltonian system, in order to measure the transversality (or the size of the splitting), along one of these homoclinic solutions, we choose two components $\mathcal{J}_{1,u}^\mu$ and $\mathcal{J}_{2,u}^\mu$ of the splitting vector and define

$$\Upsilon = \Upsilon(\overline{\psi}_1, \overline{\psi}_2) = \det \begin{pmatrix} \dfrac{\partial \overline{\mathcal{J}}_{1,u}^\mu}{\partial \psi_1}(\overline{\psi}_1, \overline{\psi}_2) & \dfrac{\partial \overline{\mathcal{J}}_{1,u}^\mu}{\partial \psi_2}(\overline{\psi}_1, \overline{\psi}_2) \\ \dfrac{\partial \overline{\mathcal{J}}_{2,u}^\mu}{\partial \psi_1}(\overline{\psi}_1, \overline{\psi}_2) & \dfrac{\partial \overline{\mathcal{J}}_{2,u}^\mu}{\partial \psi_2}(\overline{\psi}_1, \overline{\psi}_2) \end{pmatrix},$$

where $\overline{\mathcal{J}}_{i,u}^\mu(\psi_1, \psi_2) = \mathcal{J}_{i,u}^\mu(0, \psi_1, \psi_2)$.

Now, we may state our first main result. Before that, let us point out that, for giving upper and lower bounds for the measure of the "good" set of the parameters ε (those ones for which our estimates for the transversality are valid) and due to the fact that we are dealing with entire perturbations, we find convenient to introduce the transformation

$$F : \varepsilon \in \mathbb{R}^+ \to F(\varepsilon) = \varepsilon \, |\ln \varepsilon| \, .$$

Finally, let us recall that the function M_1 appearing in the perturbing term of (0.0.3) depends on two parameters a and p, advice that by $\mathcal{L}$ we denote the Lebesgue measure on $\mathbb{R}$ and refer the reader to (1.0.4) where the constant N is introduced.

Theorem 0.0.2 (Main Theorem I) *Once the parameters a and p are fixed there exist $\varepsilon_0 \in (0,1)$ and a real open subset $\mathcal{U}_\varepsilon \subset (0,\varepsilon_0]$, $\mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon(a,p)$, with*

$$ctant\ \varepsilon_0^{8/3}\left|\ln\varepsilon_0\right|^{3/2} \leq \mathcal{L}(F(\mathcal{U}_\varepsilon)) \leq ctant\ \varepsilon_0^{8/3}\left|\ln\varepsilon_0\right|^{8/3}$$

satisfying the following property: For every $\varepsilon \in \mathcal{U}_\varepsilon$, there exists a neighbourhood $I_{\tilde\beta} = I_{\tilde\beta}(\varepsilon)$ of the golden mean $\tilde\beta = (\sqrt{5}+1)/2$ with

$$\frac{1}{100}\varepsilon^{5/3}\left|\ln\varepsilon\right|^{1/2} \leq length(I_{\tilde\beta}) \leq \frac{1}{2}\varepsilon^{5/3}\left|\ln\varepsilon\right|^{1/2}$$

and there exists $(\overline{\psi}_1,\overline{\psi}_2) \in \{0,\pi\}\times\{0,\pi\}$, $\overline{\psi}_i = \overline{\psi}_i(\varepsilon)$, $i=1,2$, in such a way that, for every $\beta \in I_{\tilde\beta}$ and any $\mu \in (0,\varepsilon^m)$, with $m > 3N+8$, it follows that

$$\mu^2\exp\left(-\frac{b_1\left|\ln\varepsilon\right|^{1/2}}{\varepsilon^{1/2}}\right) \leq \left|\Upsilon(\overline{\psi}_1,\overline{\psi}_2)\right| \leq \mu^2\exp\left(-\frac{b_2\left|\ln\varepsilon\right|^{1/2}}{\varepsilon^{1/2}}\right),$$

where the positive constants b_1 and b_2 do not depend neither on ε nor on μ.

Remark 0.0.3 *The set $\mathcal{U}_\varepsilon$ will be constructed in Chapter 1. We will see that $\mathcal{U}_\varepsilon$ can be written as:*

$$\mathcal{U}_\varepsilon = \bigcup_{n\geq n^*}\mathcal{U}_n$$

where $\{\mathcal{U}_n\}_{n\geq n^*}$ is a sequence of two by two disjoint open intervals (contained in $(0,\varepsilon_0]$) "converging" to 0. Therefore, the following statement is true: For any neighbourhood $\mathcal{U}$ of 0, the estimates on the transversality given by the Main Theorem I hold for values of ε belonging to an open subset of $\mathcal{U}$.

Hence, the real number ε_0 furnished by the Main Theorem I is only used to give estimates on the measure of the full set of values of ε for which we are able to prove the Main Theorem I.

One of the key tools used during the proof of the Main Theorem I is related to the fact that, once the Melnikov functions are extended to the complex framework mentioned above, one may write for $i = 1,2$,

$$\mathcal{J}^\mu_{i,u}(s,\psi_1,\psi_2) = \mathcal{M}_{i+1}(s,\psi_1,\psi_2) + E^\mu_{i+1}(s,\psi_1,\psi_2)$$

and later one may apply the Main Lemma I to get suitable bounds for the error functions E^μ_{i+1}.

The place in which the dominant terms of the renormalized Melnikov functions are located depends essentially on the chosen perturbation and, what is even more important, these places essentially determine the size of the transversality. Hence, since in our first example we are dealing with an entire perturbation, we finally obtain that the transversality is of order

$$\mu^2 \exp\left(-\frac{ctant\ \ |\ln\varepsilon|^{1/2}}{\varepsilon^{1/2}}\right).$$

On the contrary, if we were chosen a perturbation exhibiting poles, then the associated transversality would be of order (see our second example for similar estimates)

$$\mu^2 \exp\left(-\frac{ctant}{\varepsilon^{1/2}}\right).$$

Hence, since in the first case the splitting is much smaller than in the second one, the first case requires a much more careful analysis.

Fortunately, the value of the frequency β is not explicitly relevant in Chapter 1. However, the Main Lemma I depends strongly on the value of β. The proof of the Main Lemma I is developed in Chapter 4 where we look for the required dominant terms of certain numerical series by slightly (but, as much as we can) modifying the value of $\tilde{\beta} = (\sqrt{5}+1)/2$ in such a way that all the arguments work in the obtained open set of frequencies $I_{\tilde{\beta}} = I_{\tilde{\beta}}(\varepsilon)$.

Let us finish this section by pointing out that the techniques developed in the first chapter to prove the Main Theorem I can be applied to other Hamiltonian families. Namely, these ideas work in examples like fast quasiperiodic perturbations with poles (extending in this way the results obtained in [7] to open sets of frequencies) and also in the classical Arnold's example of diffusion with two equal parameters [21], [26] and [27]. Moreover, in Chapter 2 we apply those techniques to a family of weakly hyperbolic near-integrable Hamiltonian systems, see (0.0.4) (which is an example directly inspired by the one introduced in [22]), to analytically prove the existence of diffusion in the associated phase space: More concretely, we will obtain transition chains connecting a continuum of tori whose frequencies belong to a golden mean neighbourhood of length of order $\varepsilon^{5/6}$, being ε a small parameter but not the perturbing one. Nevertheless, we will describe these events in the following section.

Arnold's diffusion

In Chapter 1 we get the asymptotic estimates for the splitting size (transversality) associated to the Hamiltonian given in (0.0.3), for any value of the frequency β in certain neighbourhood of the golden mean. However, for the Hamiltonian family (0.0.3), no (real) diffusion can be achieved because the equations of motion related to the angular variables (θ_1, θ_2) are given by $\dot{\theta}_1 = \varepsilon^{-1}$, $\dot{\theta}_2 = \beta\varepsilon^{-1}$. Hence, once a value of the parameters

β and ε is chosen, the frequency vector $\omega(I) = (\varepsilon^{-1}, \beta\varepsilon^{-1}) = \tilde{\omega}$ remains constant (along the orbits) and therefore we can not expect the existence of orbits drifting along tori with different frequencies.

This will be not the case when considering the following family of Hamiltonian systems

$$H_{\varepsilon,\mu}(x, y, I_1, I_2, \theta_1, \theta_2) = I_1 + \frac{I_2^2}{2\sqrt{\varepsilon}} + \frac{y^2}{2} + \varepsilon(\cos x - 1)\,(1 + \mu m(\theta_1, \theta_2)) \tag{0.0.4}$$

where ε and μ are small parameters (μ the perturbing one) and m is a 2π-periodic function in θ_1 and θ_2 whose properties are stated below. The equations of motion associated to (0.0.4) are obtained from (0.0.1) by taking $H = H_{\varepsilon,\mu}$, $q = (\theta_1, \theta_2, x)$ and $p = (I_1, I_2, y)$. Let us remark that, just as in the first case, although we deal with a two-parameter family of Hamiltonian systems, we will not need to assume μ to be exponentially small with respect to ε. In fact, it will be enough to assume that the perturbing parameter μ satisfies $\mu \in (0, \varepsilon^w)$, for some value of w which essentially depends on the properties of the function m.

The difference between this model and the one considered in the first chapter becomes clear if one takes into account that, for the Hamiltonian family (0.0.4), the frequency vector is given by $\omega(I) = (1, I_2)$. Therefore, $\omega(I)$ varies along the orbits and we may expect to find real diffusion in the associated phase space.

Let us assume that the function m taking part in our second family (0.0.4) is given by

$$m(\theta_1, \theta_2) = \sum_{(k_1,k_2)\in\Lambda} m_{k_1,k_2} \cos(k_1\theta_1 + k_2\theta_2)$$

where, for some positive constants r_1 and r_2, the function m turns out to be analytic on the strip

$$\left\{(\theta_1, \theta_2) \in \mathbb{C}^2 : |\mathrm{Im}\ \theta_1| < r_1, |\mathrm{Im}\ \theta_2| < r_2\right\}.$$

At the beginning of Chapter 2 we will introduce a second assumption (see (2.0.7)) on the analytic function m, which will imply that m can not be analytically prolonged onto a larger strip. See also [7] where the authors also impose the same kind of hypotheses on the considered perturbing term.

We must point out that the case in which m was an entire function is closer to the one chosen in Chapter 1 than the case considered above: The method developed for the Hamiltonian (0.0.3) applies in a more direct way here if we assume that m is an entire function.

The Hamiltonian family given in (0.0.4) is directly related with the one studied in [22]. The objective of Chapter 2 is to offer a new procedure for obtaining estimates on the splitting of separatrices valid for open sets of frequencies. This new procedure leads us to our second main result stating the presence of transition chains (whose length depends on ε) in the phase space. We refer to this phenomenon by saying that the Hamiltonian (0.0.4) exhibits *micro-diffusion*.

Since most of the results proved in Chapter 1 have to be adjusted to the Hamiltonian (0.0.4), we are going to pay special attention to those modifications needed in order the global strategy used in the first chapter remains fruitful for proving the Main Theorem II, see Theorem 0.0.4.

We start by proving the existence of two-dimensional whiskered tori

$$T_{\beta_1,\beta_2} = \{(x, y, I_1, I_2, \theta_1, \theta_2) : \ x = y = 0, \ I_i = \beta_i, \ i = 1, 2\}$$

for the perturbed systems ($\mu \neq 0$) associated to the Hamiltonian (0.0.4). In this context, and denoting by $W^-(T_{\beta_1,\beta_2})$ the unstable manifold of the invariant tori T_{β_1,β_2}, we may present the following result (see (2.0.5) where the constant N is introduced):

Theorem 0.0.4 (Main Theorem II) *There exist $\varepsilon_0 \in (0, 1)$ and a real open subset $\mathcal{U}^*_\varepsilon \subset (0, \varepsilon_0]$, $\mathcal{U}^*_\varepsilon = \mathcal{U}^*_\varepsilon(r_1, r_2)$ with*

$$ctant\ \varepsilon_0^{11/6} \leq \mathcal{L}(\mathcal{U}^*_\varepsilon) \leq ctant\ \varepsilon_0^{11/6}$$

satisfying the following property: For every $\varepsilon \in \mathcal{U}^*_\varepsilon$ there exists a neighbourhood $I^*_{\tilde{\beta}} = I^*_{\tilde{\beta}}(\varepsilon)$ of the golden mean with

$$\frac{1}{100}\varepsilon^{5/6} \leq length(I^*_{\tilde{\beta}}) \leq \frac{1}{2}\varepsilon^{5/6}$$

such that, for every $\mu \in (0, \varepsilon^w)$, with $w > \dfrac{3N}{2} + 7$, and for every positive constants β_1^0 and β_2^0, with $\beta_2^0 \in I^*_{\tilde{\beta}}$, there exist β_1^n and β_2^n with $|\beta_2^n - \beta_2^0| > ctant\ \varepsilon^{5/6}$ such that

$$W^-(T_{\beta_1^0,\beta_2^0}) \subset \overline{W^-(T_{\beta_1^n,\beta_2^n})}.$$

By means of the Main Theorem II, we construct a transition chain connecting "distant" tori from which, in the same way as in [1], one may conclude the existence of trajectories drifting along each one of the tori taking part in the considered chain.

This fact obviously implies the instability of the respective system $H_{\varepsilon,\mu}$ for every μ not null but sufficiently small.

Remark 0.0.5 *We point out that the length of the transition chain given by the Main Theorem II only depends on ε. Moreover, since $|\beta_2^n - \beta_2^0| > ctant\ \varepsilon^{5/6}$, for ε small enough, one has*

$$|\beta_2^n - \beta_2^0| >> \varepsilon >> \varepsilon^{\frac{3N}{2}+7} > \varepsilon^w > \mu$$

thus that the length of the transition chain is much bigger than the size of the perturbing parameter.

As we said before, most of the results introduced in Chapter 1 have to be adapted for proving the Main Theorem II. More concretely, once local invariant manifolds of

T_{β_1,β_2} are located, one must extend the local unstable one until it reaches again some specific neighbourhood of the respective torus. We guarantee the existence (with suitable properties) of these "extended" manifolds by demonstrating the Extension Theorem II, whose proof, as in the case of the Extension Theorem I, is given in Chapter 3.

Once again, renormalized Melnikov functions will play a crucial role in the whole strategy directed to prove Theorem 0.0.4. In this new scenario, they are defined by

$$\mathcal{M}_i(\hat{\psi}) = \sum_{\hat{k}\in\Lambda} B_{\hat{k}}^{(i)} \mathcal{E}_{\hat{k}}^* \sin(k_1\psi_1 + k_2\psi_2), \quad i = 1, 2, 3,$$

where

$$\mathcal{E}_{\hat{k}}^* = \exp\left(-\frac{\pi\left|\hat{k}\omega\right|}{2\sqrt{\varepsilon}}\right)\exp\left(-(|k_1|\,r_1 + |k_2|\,r_2)\right).$$

Once homoclinic orbits for the perturbed system are localized at initial phases $(\overline{\psi}_1, \overline{\psi}_2) \in \{0, \pi\} \times \{0, \pi\}$, we also select the two dominant terms of the associated series

$$\sum_{\hat{k}\in\Lambda} \cos(k_1\overline{\psi}_1 + k_2\overline{\psi}_2) k_j B_{\hat{k}}^{(i)} \mathcal{E}_{\hat{k}}^*,$$

for $i = 2, 3$, $j = 1, 2$, by using the Main Lemma II, see Lemma 2.3.8, whose proof is postponed to Chapter 4. Finally, we construct the suitable coordinates (i.e., flow-box coordinates are furnished by Lemma 2.3.1) in which the splitting functions are defined. Then, keeping in mind the notation used for the previous Hamiltonian case, we will prove the following result giving estimates for the transversality along the intersection between the perturbed manifolds:

Theorem 0.0.6 *There exist $\varepsilon_0 \in (0, 1)$ and a real open subset $\mathcal{U}_\varepsilon^* \subset (0, \varepsilon_0]$, $\mathcal{U}_\varepsilon^* = \mathcal{U}_\varepsilon^*(r_1, r_2)$ with*

$$ctant\ \varepsilon_0^{11/6} \leq \mathcal{L}(\mathcal{U}_\varepsilon^*) \leq ctant\ \varepsilon_0^{11/6}$$

satisfying the following property: For every $\varepsilon \in \mathcal{U}_\varepsilon^*$ there exists a neighbourhood $I_{\tilde{\beta}}^* = I_{\tilde{\beta}}^*(\varepsilon)$ of the golden mean with

$$\frac{1}{100}\varepsilon^{5/6} \leq length(I_{\tilde{\beta}}^*) \leq \frac{1}{2}\varepsilon^{5/6}$$

and there exists $(\overline{\psi}_1, \overline{\psi}_2) \in \{0, \pi\} \times \{0, \pi\}$, $\overline{\psi}_i = \overline{\psi}_i(\varepsilon)$, $i = 1, 2$, such that for every $\beta_2 \in I_{\tilde{\beta}}^*$ and every $\mu \in (0, \varepsilon^w)$, with $w > \dfrac{3N}{2} + 7$, it holds that

$$\mu^2 \exp\left(-\frac{b_1'}{\varepsilon^{1/4}}\right) \leq \left|\Upsilon(\overline{\psi}_1, \overline{\psi}_2)\right| \leq \mu^2 \exp\left(-\frac{b_2'}{\varepsilon^{1/4}}\right)$$

where the positive constants b_1' and b_2' do not depend neither on ε nor on μ.

Remark 0.0.7 *The set $\mathcal{U}_\varepsilon$ introduced in the statements of Theorem 0.0.4 and Theorem 0.0.6 displays the same topological properties that of the set $\mathcal{U}_\varepsilon$, see Remark 0.0.3. Therefore, we may also claim that, given any neighbourhood $\mathcal{U}^*$ of 0, the conclusions of Theorem 0.0.4 and Theorem 0.0.6 hold for any value of ε in an open subset of $\mathcal{U}^*$.*

Theorem 0.0.6 implies that the unstable manifold of the invariant torus $T_{\beta_1^0,\beta_2^0}$ transversally intersect (at the respective energy level) the stable manifold of every invariant tori $T_{\beta_1',\beta_2'}$ whenever

$$\left|\beta_i^0 - \beta_i'\right| = \mathrm{O}\left(\mu^2 \exp\left(-\frac{ctant}{\varepsilon^{1/4}}\right)\right)$$

for $i = 1, 2$.

This fact, together with a nice extension of the classical Inclination Lemma (or lambda-lemma) to a not completely hyperbolic scenario (see [8], [16]) will imply the Main Theorem II.

Summary

The main objective of this book is to provide the asymptotic estimates on the splitting of separatrices for two families of perturbed Hamiltonian systems.

As we have already commented, a similar kind of estimates were obtained in [7], [25] or [22] among others. In particular, in [25] a family of perturbed Hamiltonian systems similar to our first one (see (0.0.3)) is studied and, moreover, the guidelines leading to the proof of the Main Theorem in [7] (see Theorem 4 in [7]) are basically those ones used in this book. Nevertheless, most of the proofs in the above references do not work when one try to extend those asymptotic estimates for open sets of frequencies and, in particular, the Diophantine conditions, which are essentially useful to achieve the results in the above mentioned papers, are completely useless in our context. At this point, it seems crucial to note, once again, that if one try to give some partial positive answer to Arnold's conjecture by using techniques dealing with splitting of separatrices estimates, then it is necessary to get those estimates to be valid when the value of the respective frequency belongs to open sets (as large as possible). The length of the sets of frequencies for which our results are valid (see the statements of the Main Theorem I and the Main Theorem II) depends on the value of the parameter ε (which is already present in the definition of our two families (0.0.3) and (0.0.4), and, in fact, the length of those sets of frequencies goes to zero as ε does. Of course, it remains a hardest work in order to extend the asymptotic estimates on the respective splitting of separatrices for sets of frequencies whose lengths do not depend on the parameter ε, see Remark 1.3.14 and Remark 2.3.11 for related comments.

Since we have to deal with resonances (rational values of the frequencies) we can not use a result similar to Theorem 5 (Normal Form Theorem) in [7]. This powerful result provides, in [7], a canonical change of variables in which the respective perturbed

Hamiltonian systems become integrable. Many results in [7] are direct consequences of the Normal Form Theorem. Among them, we emphasize the two following ones:

1.- In [7] the local invariant manifolds of the perturbed tori are easily obtained by using the variables furnished by the Normal Form Theorem. Let us recall that, as we said along this section, it is extremely important to have available parameterizations of those invariant manifolds in order to give asymptotic estimates for the splitting size (in fact, these manifolds are those which coincide when the perturbing parameter μ is zero but split when the perturbation is considered). Hence, our respective results giving parameterizations of those invariant manifolds, see Lemma 1.1.11 and Lemma 2.1.3, which are extensively used along the book, would be a direct consequence of a like-Normal Form Theorem.

2.- From the Normal Form Theorem it easily follows in [7] the existence of new coordinates (called flow-box coordinates) displaying the same properties as the flow-box coordinates constructed in this book (see Lemma 1.3.1 and Lemma 2.3.1). These flow-box coordinates play a crucial role for proving the Main Theorem in [7], as well as to prove our two main results. Here is where the advantages of the Normal Form Theorem arise: Essentially, we must prove the existence of certain domains in the phase space in which our perturbed Hamiltonian systems become integrable. The main difference between our results (Lemma 1.3.1 and Lemma 2.3.1) and the Normal Form Theorem in [7] is that the domains in which we are able to transform our perturbed systems into integrable ones do not contain the invariant perturbed tori (which can be though as the singularities of the vector field) and this is enough for our purposes. However, since the flow-box coordinates are needed to be holomorphic, the standard proofs of the existence of flow-box coordinates are completely unfruitful in our context. We have to develop a more elaborated strategy in order to obtain such coordinates and this process becomes one of the hardest points of this book. In fact, see especially Remark 1.3.2, we have to consider a previously introduced complex parameter $\mathbf{s}$ as a new variable of motion in order to get a complex holomorphic time-function (see (1.3.62)) which essentially plays the same role as the (real) time-functions which are frequently used to contruct conjugations between vector fields without singularities. Of course, all these arguments could be avoided if some like-Normal Form Theorem could be applied independently of the arithmetics of the frequency.

Unfortunately, we do not known how to prove a Normal Form Theorem (similar to Theorem 5 in [7]) without imposing Diophantine conditions on the value of the frequency. It seems that the price we must pay in order to get asymptotic estimates valid for open sets of frequencies is to give alternative proofs to all the results in [7] directly depending on the Normal Form Theorem. Nevertheless, this is not enough.

This shortage essentially arises due to the fact that, in this book, the dominant terms of the renormalized Melnikov functions have to be obtained by using more results than the ones properly coming from the Continued Fraction Theory. We must get the leading behaviour of the renormalized Melnikov functions not only for the case in which the frequency coincides with a sufficiently enough irrational number (say the golden mean $\tilde{\beta} = (\sqrt{5}+1)/2$) but also when this frequency ranges in a neighbourhood of $I_{\tilde{\beta}}$, see the statements of our two Main Lemmas (Lemma 1.3.10 and Lemma 2.3.8) and especially Chapter 4 where these two crucial results are proved.

Finally, we also want to point out that the global strategy designed along this book is applied to two different kind of families of Hamiltonian systems (the first one is a quasiperiodic family and the second a non-quasiperiodic one) in order to show how the arguments needed to get splitting estimates in the quasiperiodic case can be adapted to get transition chains, or Arnold s difussion, in the non-quasiperiodic one.

Chapter 1

Splitting in open sets of frequencies

In this chapter we will give the asymptotic value of the splitting of separatrices exhibited by the following family of Hamiltonian systems (see also (0.0.3))

$$H_{\varepsilon,\beta,\mu}(x,y,I_1,I_2,\theta_1,\theta_2)=\frac{I_1+\beta I_2}{\varepsilon}+\frac{y^2}{2}+A(\cos x-1)+\mu y\sin x M_1(\theta_1,\theta_2). \quad (1.0.1)$$

When $\mu=0$, there exists a two-parameter family of two-dimensional tori

$$T_{\alpha_1,\alpha_2}=\{(x,y,I_1,I_2,\theta_1,\theta_2):\ x=y=0,\ I_1=\alpha_1,\ I_2=\alpha_2\},$$

which are invariant by the unperturbed flow.

We will prove that each one of these invariant tori has a three-dimensional invariant manifold foliated by homoclinic solutions of the unperturbed system. We choose the perturbing term

$$\mu y\sin x M_1(\theta_1,\theta_2)$$

in such a way that T_{α_1,α_2} are still invariant tori for the complete (perturbed) system but, however, we can not expect that the *homoclinic* manifold survives the perturbation. Nevertheless, certain symmetry properties (see Remark 1.1.15) of our Hamiltonian $H_{\varepsilon,\beta,\mu}$ allow us to check the existence of homoclinic orbits for the perturbed systems (at least when μ is small enough) in such a way that the unstable and stable manifolds of T_{α_1,α_2} intersect along these orbits. The splitting is identified with the intersection angle between those manifolds (which is null if $\mu=0$) and, in order to analytically obtain lower and upper bounds for this splitting, both of the same order, we have to impose some assumptions on the function M_1 as well as on the parameters appearing in the definition of the Hamiltonians $H_{\varepsilon,\beta,\mu}$.

We assume that M_1 is given by

$$M_1(\theta_1,\theta_2)=\sum_{(k_1,k_2)\in\Lambda}\frac{(a\varepsilon^p)^{|k_1|+|k_2|}}{f(k_1,k_2)}\sin(k_1\theta_1+k_2\theta_2). \quad (1.0.2)$$

The set of indexes Λ can be any subset of $\mathbb{Z}^2$ which could be written as

$$\Lambda=\Lambda(\mathcal{N}_1,\mathcal{N}_2,\mathcal{N}_3)=\bigcup_{i=1}^{3}\Lambda_i(\mathcal{N}_i) \quad (1.0.3)$$

where, for some positive constants $\mathcal{N}_i$,

$$\begin{aligned}\Lambda_i(\mathcal{N}_i) &= \{(k_1,k_2)\in\mathbb{Z}^2 : |k_i|\geq\mathcal{N}_i,\ k_{3-i}=0\}, \quad i=1,2\\ \Lambda_3(\mathcal{N}_3) &= \{(k_1,k_2)\in\mathbb{Z}^2 : |k_1|+|k_2|\geq\mathcal{N}_3,\ k_1k_2\neq 0\}.\end{aligned}$$

Moreover, for some positive constants L_1, L_2 and N, we assume that

$$L_1(|k_1|+|k_2|)^{-N}\leq|f(k_1,k_2)|\leq L_2(|k_1|+|k_2|)^N, \tag{1.0.4}$$

for every $(k_1,k_2)\in\Lambda$.

Furthermore, for every $(k_1,k_2)\in\Lambda$, we assume that

$$f(-k_1,-k_2)=-f(k_1,k_2). \tag{1.0.5}$$

This last condition will be only used to write the renormalized Melnikov functions obtained in Section 1.2 in a convenient way (see, in particular, how (1.3.111) is obtained).

The parameters A,a,p,ε and μ are positive and we assume that $a<1$ and the perturbing parameter μ satisfies $\mu\in(0,\varepsilon^m)$ with $m=m(b,N)$ a sufficiently large constant, where N is given in (1.0.4) and b is a positive constant introduced in (1.1.7).

Hence, we observe that we are dealing with entire perturbations of integrable Hamiltonian systems. In this context, in [25] the asymptotic expressions of the splitting size were numerically predicted in a case close to the above one by taking $f(k_1,k_2)=k_1^{k_1}k_2^{k_2}$ and β equal to the golden mean number.

The main goal of this chapter is to prove the Main Theorem I, see Theorem 0.0.2, which gives the splitting size estimates for our Hamiltonian family, whenever ε belongs to an open real subset and β to an open neighbourhood of the golden mean.

Before going into details, let us summarize the different steps in which the proof of the Main Theorem I can be divided:

1. We will use (local) adequate coordinates $(q,p,I_1,I_2,\theta_1,\theta_2)$ (see Lemma 1.1.4) to obtain explicit expressions for the local perturbed stable and unstable manifolds of the invariant tori T_{α_1,α_2} (see Theorem 1.1.8). Since it will be very useful to get those parameterizations of perturbed invariant manifolds in a convenient complex scenario we will start by extending our Hamiltonian family (1.0.1) by letting the angular variables (θ_1,θ_2) belong to certain complex strip

$$\mathcal{B}_1'=\left\{(\theta_1,\theta_2)\in\mathbb{C}^2:\ |\mathrm{Im}\ \theta_i|\leq r_i,\ i=1,2\right\}.$$

Furthermore, as we have already pointed out in the Introduction, the three dimensional unperturbed separatrix of the invariant tori T_{α_1,α_2} has also to be extended to certain complex domain (the final bounds for the splitting size depend on the width of this complex domain as well as on the constants r_1 and r_2 used to define $\mathcal{B}_1'$). Once Lemma 1.1.11 will be proved, we will deduce, in particular, that the local unperturbed manifold of T_{α_1,α_2} is close to the unperturbed separatrix. This closeness is enough to ensure that the Extension Theorem I (see Theorem 1.1.14) guarantees that the unstable manifold comes back, by the action of the flow up to finite time, to the domain where the coordinates $(q, p, I_1, I_2, \theta_1, \theta_2)$ were defined. Let us remark that the Extension Theorem I can be easily proved (by using Gronwall's estimates) for the real Hamiltonian case. Nevertheless, this is no longer true when the complex Hamiltonian is considered. As a consequence of the Extension Theorem I, one easily obtain a small domain $\mathcal{U}$ (containing certain pieces of perturbed stable and unstable manifolds and contained in the domain of definition of the coordinates $(q, p, I_1, I_2, \theta_1, \theta_2)$) which will be used as the scenario for measuring the splitting of separatrices.

2. The distance between perturbed manifolds can be measured by computing the difference of the three unperturbed energies (first integrals for the unperturbed system) when they are evaluated at points in the perturbed stable manifold and at points in the perturbed unstable one. Unfortunately, we are not able to obtain explicit expressions for these differences. Nevertheless, in Section 1.2 we compute the three renormalized Melnikov functions associated to the Hamiltonian family (1.0.1) and prove that they essentially (up to small error) coincide with the above mentioned differences of unperturbed energies (see Lemma 1.2.1). The obtained errors are too long in order to state that the distances between perturbed manifolds can be replaced by the Melnikov functions. The reason why this replacement can not be done is that, while the Melnikov functions are exponentially small in ε, the errors are potentially small in μ (as we have said in the Introduction, Lemma 1.2.1 would directly give asymptotic estimates for the splitting size in the case in which μ was exponentially small with respect to ε).

3. To get more accurate bounds for the error made when replacing the difference of unperturbed energies by renormalized Melnikov functions we consider, in Section 1.3, flow-box coordinates (see Lemma 1.3.1) as canonical coordinates near a convenient piece of the unperturbed separatrix. We take advantage from the fact that, by using flow-box coordinates, the differences of unperturbed energies (these differences are now called splitting functions and are defined at (1.3.73)) coincide with the renormalized Melnikov functions up to certain error functions whose norms are, actually, exponentially small with respect to ε (just as the norm of the renormalized Melnikov functions).

Finally, in order to show that, in fact, the error functions are smaller than the renormalized Melnikov functions, we must restrict the range of (ε, β) (ε and β those parameters appearing in the Hamiltonian family (1.0.1)) to certain open real subset of $\mathbb{R}^2$. Proceeding in this way, we are able to use the Main Lemma I (see Lemma 1.3.10) in order to replace Melnikov and error functions (which, up to this moment, were given in terms of Fourier series) by simpler expressions. These useful asymptotic expressions for Melnikov and error functions allow us to get an asymptotic formula for the splitting size (see (1.3.115)) leading to the proof of the Main Theorem I.

1.1 Invariant local manifolds

Let us begin this section by recalling that the unperturbed system ($\mu = 0$) associated to the Hamiltonian family (1.0.1) presents invariant tori located at $x = y = 0$. Those invariant tori survive the considered perturbation and one of the objectives of this section is not only to prove that, in fact, for every μ small enough, these tori are whiskered in the sense that they have (local) stable and unstable manifolds, but also to obtain analytic expressions for these invariant manifolds. This is what Theorem 1.1.8 furnishes.

Furthermore, it will be necessary to extend the definition of our Hamiltonian system to a complex one and, moreover, to obtain fine properties of the global unstable manifold of the above mentioned invariant tori in this complex extension. This will be done by means of the main result of this section, see Theorem 1.1.14, whose proof is postponed to Chapter 3.

Let us start by considering the equations of motion associated to the Hamiltonian (1.0.1), namely:

$$\begin{aligned} \dot{x} &= y + \mu \sin x M_1(\hat{\theta}) \\ \dot{y} &= A \sin x - \mu y \cos x M_1(\hat{\theta}) \\ \dot{I}_j &= -\mu y \sin x \frac{\partial M_1}{\partial \theta_j}(\hat{\theta}), \quad j = 1, 2 \\ \dot{\theta}_1 &= \frac{1}{\varepsilon}, \quad \dot{\theta}_2 = \frac{\beta}{\varepsilon} \end{aligned} \tag{1.1.6}$$

where $\hat{\theta} = (\theta_1, \theta_2)$ and M_1 is the function given in (1.0.2). Let us denote by $\psi_i = \theta_i(0)$ the initial phases of θ_i, $i = 1, 2$, respectively.

The validity of the whole argument to be developed along this chapter goes through studying the dynamics of the system (1.1.6) not only for real values of ψ_i, but also when $\hat{\psi} = (\psi_1, \psi_2)$ belongs to some complex strip

$$\mathcal{B}_1' = \left\{ (\psi_1, \psi_2) \in \mathbb{C}^2 : |\mathrm{Im}\ \psi_1| \leq r_1,\ |\mathrm{Im}\ \psi_2| \leq r_2 \right\}.$$

The way in which the Main Theorem I is proven strongly depends on the value of the width of the complex strip $\mathcal{B}_1'$. In a few words, it is well-known that one can obtain small bounds for the norm of those real analytic periodic functions admitting an analytic complex extension onto a large strip. This fact will be used along the proof of Theorem 0.0.2 and, of course, it will be greatly useful to work with large values of r_i. However, at the same time, we can not consider arbitrarily large strips because we need, at least, to control the behaviour of the perturbation (and, more concretely, of the function M_1) on the whole complex domain. In this first example, and due to the fact that M_1 is an entire function, we must choose r_i depending on the parameter ε and, in fact, it will be necessary to let r_i go to infinity as ε tends to zero. More concretely, we take

$$r_i = r_i(\varepsilon) = -\ln(a\varepsilon^p) - \varepsilon^b, \qquad i = 1, 2. \tag{1.1.7}$$

The parameter b is positive and it will be fixed when the final arguments of this chapter (see, in particular, (1.3.107)) will be given.

Before stating our first result we recall that the constant N was introduced in (1.0.4) and point out that, along this book, we denote by *ctant* several different constants not depending on the parameters ε and μ.

Lemma 1.1.1 *Under the above choice of $r_i = r_i(\varepsilon)$ (see (1.1.7)) we have*

$$\|M_1\|_{\mathcal{B}'_1} = \sup_{(\theta_1,\theta_2)\in\mathcal{B}'_1} |M_1(\theta_1,\theta_2)| \le ctant\ \varepsilon^{-b(N+2)}.$$

Proof

From the expression of M_1 given in (1.0.2), and using (1.0.4), we obtain

$$\|M_1\|_{\mathcal{B}'_1} \le L_1^{-1} \sum_{(k_1,k_2)\in\Lambda} (a\varepsilon^p)^{|\hat{k}|} \left|\hat{k}\right|^N e^{|k_1|r_1+|k_2|r_2},$$

where $\left|\hat{k}\right| = |k_1| + |k_2|$ for $\hat{k} = (k_1, k_2)$.

Now, using the definition of $r_i = r_i(\varepsilon)$, it follows that

$$\|M_1\|_{\mathcal{B}'_1} \le L_1^{-1} \sum_{(k_1,k_2)\in\Lambda} \left|\hat{k}\right|^N e^{-|\hat{k}|\varepsilon^b}.$$

Hence, using that for every natural numbers n_0, N_0 and any $\alpha \in (0,1)$, there exists a constant $C^* = C^*(n_0, N_0, \alpha)$ such that

$$\sum_{n\in\mathbb{N},\ n\ge n_0} n^{N_0} e^{-\alpha n} = \left| \frac{d^{N_0}}{d\alpha^{N_0}} \left(\frac{e^{-\alpha n_0}}{1-e^{-\alpha}} \right) \right| \le C^* \alpha^{-(N_0+1)},$$

the lemma follows by taking into account that

$$\sum_{(k_1,k_2)\in\Lambda_3} \left|\hat{k}\right|^N e^{-|\hat{k}|\alpha} \le 4 \sum_{t>\mathcal{N}_3} t^{N+1} e^{-\alpha t}.$$

□

Let us introduce the following standard lemma on the Cauchy estimates on a holomorphic function which will be used several times along this book:

Lemma 1.1.2 *Let $D \subset \mathbb{C}^m$ be a complex domain with ∂D smooth and $g : D \to \mathbb{C}$ an analytic function. Then, for any subdomain $D' \subset D$ with $\delta = dist(D', \partial D)$ and any $(n_1, \ldots, n_m) \in \mathbb{N}^m$, one has:*

$$\left\|\partial^M g\right\|_{D'} = \sup_{z\in D'} \left| \frac{\partial^M g(z)}{\partial_{z_1}^{n_1} \ldots \partial_{z_m}^{n_m}} \right| \le M! \delta^{-M} \|g\|_D\,,$$

where $M = n_1 + \ldots + n_m$.

From now on, we will restrict the variation of the initial phases ψ_i, $i = 1, 2$, to the set

$$\mathcal{B}_1'' = \left\{ (\psi_1, \psi_2) \in \mathbb{C}^2 : |\text{Im } \psi_i| \leq -\ln(a\varepsilon^p) - 2\varepsilon^b, \ i = 1, 2 \right\}.$$

Therefore, Lemma 1.1.1 and Lemma 1.1.2 imply

$$\left\| \frac{\partial M_1}{\partial \theta_j} \right\|_{\mathcal{B}_1''} \leq ctant \ \varepsilon^{-b(N+3)}, \tag{1.1.8}$$

for $j = 1, 2$.

Remark 1.1.3 *In order to guarantee that the system (1.1.6) is a small perturbation of an integrable one whenever $(x, y, \hat{I}, \hat{\theta}) \in \mathbb{C}^4 \times \mathcal{B}_1''$, $\hat{I} = (I_1, I_2)$, $\hat{\theta} = (\theta_1, \theta_2)$, we must choose, according to Lemma 1.1.1 and Lemma 1.1.2*

$$\mu \in (0, \varepsilon^m) \quad \textit{with} \quad m > b(N+3).$$

In fact, a stronger condition will be necessary along this chapter, thus that, from now on, we will take

$$m \geq \frac{5}{4}\left(\frac{3N}{2} + b(2N+11) + 1\right). \tag{1.1.9}$$

We will make a change of variables in order to express in a useful way the integrable part of the system (1.1.6). This change only accounts for the (x, y)-variables, so let us skip for the moment the equations of motion in the coordinates $(\hat{I}, \hat{\theta})$. Let

$$X = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{A}}y - x\right), \quad Y = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{A}}y + x\right). \tag{1.1.10}$$

Then, we obtain that the initial system can be written in the new variables in the following way

$$\begin{aligned} \dot{X} &= -\sqrt{A}X + \frac{\sqrt{A}}{\sqrt{2}}f'(x) - \frac{\mu}{\sqrt{2}}\sin x M_1(\hat{\theta}) - \frac{\mu}{\sqrt{2A}}y\cos x M_1(\hat{\theta}) \\ \dot{Y} &= \sqrt{A}Y + \frac{\sqrt{A}}{\sqrt{2}}f'(x) + \frac{\mu}{\sqrt{2}}\sin x M_1(\hat{\theta}) - \frac{\mu}{\sqrt{2A}}y\cos x M_1(\hat{\theta}) \end{aligned}$$

where $f(x) = -\cos x + 1 - \frac{1}{2}x^2$ and $x = x(X, Y)$, $y = y(X, Y)$.

Observe that the (unperturbed) equations

$$\dot{X} = -\sqrt{A}X + \frac{\sqrt{A}}{\sqrt{2}}f'(x), \qquad \dot{Y} = \sqrt{A}Y + \frac{\sqrt{A}}{\sqrt{2}}f'(x)$$

are associated to the Hamiltonian $H(X, Y) = -\sqrt{A}XY + \sqrt{A}f(x(X, Y))$. In this context we may apply the following well-known standard result (see, for instance, [17]):

Lemma 1.1.4 *There exist* $\sigma > 0$ *small enough and an analytic change of variables* $X = X(q,p)$, $Y = Y(q,p)$ *defined on* $B^2_\sigma = \{(q,p) \in \mathbb{C}^2 : |p| < \sigma, |q| < \sigma\}$ *which transforms the Hamiltonian* $H(X,Y) = -\sqrt{A}XY + \sqrt{A}f(x(X,Y))$ *into its normal form*

$$H(X(q,p),Y(q,p)) = \tilde{H}(q,p) = -\sqrt{A}\,(pq + F(pq))\,.$$

Moreover, $q = X + \mathrm{O}\left((|X|+|Y|)^2\right)$, $p = Y + \mathrm{O}\left((|X|+|Y|)^2\right)$ *and, denoting* $J = pq$, *then* $F(J) = \mathrm{O}(J^2)$.

Therefore, using the normal form coordinates (q,p), the vector fields given in (1.1.6) can be written in the following way

$$\begin{aligned}
\dot{q} &= -\sqrt{A}q(1+F_J) - (q^X - q^Y)\frac{\mu}{\sqrt{2}}\sin\tilde{x}M_1(\hat{\theta}) - (q^X + q^Y)\frac{\mu}{\sqrt{2A}}\tilde{y}\cos\tilde{x}M_1(\hat{\theta}) \\
\dot{p} &= \sqrt{A}p(1+F_J) + (p^Y - p^X)\frac{\mu}{\sqrt{2}}\sin\tilde{x}M_1(\hat{\theta}) - (p^X + p^Y)\frac{\mu}{\sqrt{2A}}\tilde{y}\cos\tilde{x}M_1(\hat{\theta}) \\
\dot{I}_j &= -\mu\tilde{y}\sin\tilde{x}\frac{\partial M_1}{\partial\theta_j}(\hat{\theta}), \quad j = 1,2 \\
\dot{\theta}_1 &= \frac{1}{\varepsilon}, \quad \dot{\theta}_2 = \frac{\beta}{\varepsilon}
\end{aligned} \tag{1.1.11}$$

where we take the notation $q^X = q^X(q,p) = \dfrac{\partial q}{\partial X}(X(q,p),Y(q,p))$ (the equivalent for q^Y, p^X and p^Y), $\tilde{x} = \tilde{x}(q,p) = x(X(q,p),Y(q,p))$ (the equivalent for $\tilde{y}$) and

$$F_J = F'(J).$$

Remark 1.1.5 *We point out that the complete (perturbed) vector field given in (1.1.11) is not Hamiltonian. This will not be a handicap for our purposes because we will only use equations (1.1.11) for giving (topological) properties of the invariant tori* T_{α_1,α_2} *for the Hamiltonian systems (1.0.1).*

Let us consider

$$\tilde{M} = B^2_\sigma \times \mathbb{C}^2 \times \mathcal{B}''_1 = \left\{(q,p,\hat{I},\hat{\theta}) \in \mathbb{C}^4 \times \mathcal{B}''_1 :\ |q| < \sigma, |p| < \sigma\right\}$$

and denote by ϕ_t the flow associated to the system (1.1.11).

Let us remark that the subsets

$$T_{\alpha_1,\alpha_2} = \left\{(q,p,\hat{I},\hat{\theta}) \in \tilde{M} : p = q = 0, \hat{I} = (\alpha_1,\alpha_2)\right\}$$

are (complex) invariant tori by ϕ_t.

Let $r(t) = (q(t),p(t),I_1(t),I_2(t),\theta_1(t),\theta_2(t))$ be the solution of the whole system with $r(0) = (q^0,p^0,I^0_1,I^0_2,\theta^0_1,\theta^0_2) \in \tilde{M}$. We say that $r(0)$ belongs to the stable manifold of

T_{α_1,α_2}, $W^+(T_{\alpha_1,\alpha_2})$, if $\lim_{t\to\infty} q(t) = \lim_{t\to\infty} p(t) = 0$ and $\lim_{t\to\infty} I_i(t) = \alpha_i$, $i = 1, 2$. We also define the unstable manifold, $W^-(T_{\alpha_1,\alpha_2})$, by taking limits as t tends to $-\infty$.

As we said at the beginning, in the present section we pursue two objectives: First, we are going to prove that $W^+(T_{\alpha_1,\alpha_2})$ and $W^-(T_{\alpha_1,\alpha_2})$ are non-trivial subsets (non-trivial means that none of these manifolds reduces to T_{α_1,α_2}) and second, we are going to obtain explicit expressions for these manifolds in a neighbourhood of T_{α_1,α_2}.

To this end, we are going to decompose the sets $W^+(T_{\alpha_1,\alpha_2})$ and $W^-(T_{\alpha_1,\alpha_2})$ by means of Poincaré sections located at $\theta_1 = \theta_1^0$, θ_1^0 fixed.

Let $r(0) \in \tilde{M}$ and let us compute the value of the respective solution $r(t)$ at time $2\pi\varepsilon$ to get

$$
\begin{aligned}
&q(2\pi\varepsilon) = q^0 \exp(-2\pi\varepsilon\sqrt{A}(1+F_J)) - \frac{\mu}{\sqrt{2}} \int_0^{2\pi\varepsilon} (q^X - q^Y)\sin\tilde{x} M_1(\hat{\theta})ds - \\
&\quad - \frac{\mu}{\sqrt{2A}} \int_0^{2\pi\varepsilon} (q^X + q^Y)\tilde{y}\cos\tilde{x} M_1(\hat{\theta})ds - \sqrt{A}\int_0^{2\pi\varepsilon} (1+F_J)(q(s) - \tilde{q}(s))ds, \\
&p(2\pi\varepsilon) = p^0 \exp(2\pi\varepsilon\sqrt{A}(1+F_J)) + \frac{\mu}{\sqrt{2}} \int_0^{2\pi\varepsilon} (p^Y - p^X)\sin\tilde{x} M_1(\hat{\theta})ds - \\
&\quad - \frac{\mu}{\sqrt{2A}} \int_0^{2\pi\varepsilon} (p^X + p^Y)\tilde{y}\cos\tilde{x} M_1(\hat{\theta})ds + \sqrt{A}\int_0^{2\pi\varepsilon} (1+F_J)(p(s) - \tilde{p}(s))ds, \\
&I_j(2\pi\varepsilon) = I_j^0 - \mu \int_0^{2\pi\varepsilon} \tilde{y}\sin\tilde{x} \frac{\partial M_1}{\partial\theta_j}(\hat{\theta})ds, \; j = 1, 2, \\
&\theta_1(2\pi\varepsilon) = \theta_1^0 + 2\pi, \quad \theta_2(2\pi\varepsilon) = \theta_2^0 + 2\pi\beta,
\end{aligned}
\tag{1.1.12}
$$

where $(\tilde{q}, \tilde{p})$ is the solution of the system

$$
\dot{q} = -\sqrt{A}q(1+F_J), \qquad \dot{p} = \sqrt{A}p(1+F_J)
$$

satisfying the initial condition $(\tilde{q}(0), \tilde{p}(0)) = (q^0, p^0)$.

Then, we are defining a map $P = \phi_{2\pi\varepsilon}$ given by $P(q, p, \hat{I}, \hat{\theta}) = (q', p', \hat{I}', \hat{\theta}')$ which we are going to write in the following way:

$$
\begin{aligned}
q' &= qL^{-1} + \frac{\mu\varepsilon^{-b(N+3)+1}}{\sqrt{2}} f_1(q, p, \theta_1, \theta_2), \\
p' &= pL - \frac{\mu\varepsilon^{-b(N+3)+1}}{\sqrt{2}} f_2(q, p, \theta_1, \theta_2), \\
I_j' &= I_j - \mu\varepsilon^{-b(N+3)+1} f_{2+j}(q, p, \theta_1, \theta_2), \quad j = 1, 2, \\
\theta_1' &= \theta_1 + 2\pi, \\
\theta_2' &= \theta_2 + 2\pi\beta,
\end{aligned}
\tag{1.1.13}
$$

where we have introduced the notation

$$L = \exp(2\pi\varepsilon\sqrt{A}(1+F_J)).$$

Lemma 1.1.6 *For every* $(q,p,\theta_1,\theta_2) \in B^2_\sigma \times \mathcal{B}''_1$, *it follows that*

$$|f_i(q,p,\theta_1,\theta_2)| \le ctant\ (|p|+|q|),$$

for $i = 1,2,3,4$.

Proof

From (1.1.10) and Lemma 1.1.4 we have

$$|\tilde{x}(q,p)| \le ctant\ (|p|+|q|) \quad \text{and} \quad |\tilde{y}(q,p)| \le ctant\ (|p|+|q|).$$

Then, since σ is a small constant, it also follows that

$$|\sin(\tilde{x}(q,p))| \le ctant\ (|p|+|q|) \quad \text{and} \quad |\tilde{y}(q,p)\cos(\tilde{x}(q,p))| \le ctant\ (|p|+|q|). \quad (1.1.14)$$

Therefore, using Gronwall's estimates, the expression of the complete system (1.1.11) and the bounds for the functions M_1 and $\partial M_1/\partial\theta_j$ respectively obtained in Lemma 1.1.1 and (1.1.8), we deduce

$$|q(s) - \tilde{q}(s)| \le ctant\ \mu\varepsilon^{-b(N+3)}(|p|+|q|)K^{-1}(\exp(2\pi K\varepsilon)-1),$$

whenever μ is small enough and K is some constant satisfying $K \le 4\sqrt{A}$ and the constant *ctant* also involves upper bounds for the norm of q^X, q^Y, p^X and p^Y. Hence,

$$\frac{1}{\mu\varepsilon^{-b(N+3)+1}}\left|\int_0^{2\pi\varepsilon}(1+F_J)(q(s)-\tilde{q}(s))ds\right| \le ctant\ (|p|+|q|).$$

Looking at (1.1.12), it is now easy to see that Lemma 1.1.1 leads to

$$|f_1(q,p,\theta_1,\theta_2)| \le ctant\ (|p|+|q|).$$

With the same arguments we conclude the required estimate for f_2, f_3 and f_4. □

Notice that, once θ_1^0 is fixed, with $\left|\mathrm{Im}\ \theta_1^0\right| \le -\ln(a\varepsilon^p) - 2\varepsilon^b$, the set

$$\mathcal{T}_{\alpha_1,\alpha_2}(\theta_1^0) = \left\{(q,p,\hat{I},\hat{\theta}) \in T_{\alpha_1,\alpha_2} : \theta_1 = \theta_1^0\right\}$$

is invariant by P. Let us consider the stable manifold of $\mathcal{T}_{\alpha_1,\alpha_2}(\theta_1^0)$ (with respect to the transformation P given in (1.1.13)) which is defined by

$$W^+(\mathcal{T}_{\alpha_1,\alpha_2}(\theta_1^0)) = \left\{r \in \tilde{M} : \lim_{n\to\infty} \mathrm{dist}\left(P^n(r), \mathcal{T}_{\alpha_1,\alpha_2}(\theta_1^0)\right) = 0\right\}$$

and the unstable one

$$W^-(\mathcal{T}_{\alpha_1,\alpha_2}(\theta_1^0)) = \left\{r \in \tilde{M} : \lim_{n\to\infty} \mathrm{dist}\left(P^{-n}(r), \mathcal{T}_{\alpha_1,\alpha_2}(\theta_1^0)\right) = 0\right\}.$$

Remark 1.1.7 *Let us point out that, if θ_1^0 and $\tilde{\theta}_1^0$ satisfy* $\mathrm{Im}\,\theta_1^0 = \mathrm{Im}\,\tilde{\theta}_1^0$, *then the associated Poincaré transformations P and $\tilde{P}$ are conjugated via the respective τ-flow diffeomorphism ϕ_τ. Therefore, we may write*

$$W^+(T_{\alpha_1,\alpha_2}) = \bigcup_{\theta_1^0} W^+(\mathcal{T}_{\alpha_1,\alpha_2}(\theta_1^0)), \quad W^-(T_{\alpha_1,\alpha_2}) = \bigcup_{\theta_1^0} W^-(\mathcal{T}_{\alpha_1,\alpha_2}(\theta_1^0))$$

and, since (1.1.6) is an analytic vector field, all the properties related to the smoothness of $W^+(\mathcal{T}_{\alpha_1,\alpha_2}(\theta_1^0))$ can be applied to $W^+(T_{\alpha_1,\alpha_2})$. In particular, once Theorem 1.1.8 will be proved, one may claim that $W^+_{loc}(T_{\alpha_1,\alpha_2})$ is an analytic manifold. The same holds for $W^-_{loc}(T_{\alpha_1,\alpha_2})$.

Now, it is enough to prove the existence of $W^+_{loc}(\mathcal{T}_{\alpha_1,\alpha_2}(\theta_1^0))$ and $W^-_{loc}(\mathcal{T}_{\alpha_1,\alpha_2}(\theta_1^0))$. The existence of these local invariant manifolds is stated in Theorem 1.1.8. The key property we are going to use is related to the fact that, roughly speaking, the origin $(0,0)$ is a (weak) hyperbolic fixed point for the system formed by the two first equations of (1.1.13). Therefore, for proving Theorem 1.1.8, we basically follow the arguments used to prove the existence of invariant manifolds in a hyperbolic system (see [12]). Nevertheless, since we also need explicit expressions for the local invariant manifolds of $\mathcal{T}_{\alpha_1,\alpha_2}(\theta_1^0)$, we include the proof of Theorem 1.1.8 (see also [8] where this kind of results are extensively studied).

To begin with, let us fix $\theta_1 = \theta_1^0$ and write $\theta = \theta_2$. Let us take σ the positive constant given by Lemma 1.1.4 and let $r_2 = r_2(\varepsilon)$ be the function given in (1.1.7). Let us consider, for every positive constant

$$r \le r_2 - \varepsilon^b = -\ln(a\varepsilon^p) - 2\varepsilon^b,$$

the space of holomorphic functions $\mathcal{H}(\Delta(\sigma, r))$ consisting of the analytic functions defined on $(-\sigma,\sigma) \times [0, 2\pi]$ that admit an holomorphic extension to the complex domain

$$\Delta(\sigma, r) = \{(q, \theta) \in \mathbb{C}^2 : \ |q| < \sigma, |\mathrm{Im}\ \theta| < r\}$$

and are continuous on the closure of $\Delta(\sigma, r)$.

The space $\mathcal{H}(\Delta(\sigma, r))$ endowed with the norm

$$\|g\|_{\sigma,r} = \sup_{(q,\theta)\in\overline{\Delta}(\sigma,r)} |g(q, \theta)|$$

is a Banach space.

Let us take a subset $\overline{\mathcal{H}}$ of $\mathcal{H}(\Delta(\sigma, r))$ defined by

$$\overline{\mathcal{H}} = \{g \in \mathcal{H}(\Delta(\sigma, r)) : \|g\|_{\sigma,r} \le A_0\},$$

where A_0 is a constant chosen in the following way: Let f_i, $i = 1, \dots, 4$, be the analytic functions defined on $B^2_\sigma \times \mathcal{B}''_1$ implicitly given in (1.1.13). Let us set, for every $C \in \overline{\mathcal{H}}$, the functions

$$C_i : (q, \theta) \in \Delta(\sigma, r) \to C_i(q, \theta) = f_i\left(q, \mu\varepsilon^{-b(N+3)}qC(q, \theta), \theta\right), \tag{1.1.15}$$

for $i = 1, \ldots, 4$. Since the value of θ_1 is fixed, we are skipping the dependence of the functions f_i with respect to θ_1. Let us observe that, from Lemma 1.1.6, there exists a positive constant $\tilde{\mathcal{F}}$ such that

$$|C_i(q,\theta)| \leq \tilde{\mathcal{F}}(1 + \mu\varepsilon^{-b(N+3)}A_0)\,|q| = \mathcal{F}\,|q|\,. \tag{1.1.16}$$

Then, we take A_0 large enough (and μ sufficiently small) so that

$$A_0 \geq \frac{\mathcal{F}}{\sqrt{2}\pi(1 + \mathrm{Re}\ F_J^*)\sqrt{A}}, \tag{1.1.17}$$

where $F_J^* = F_J^*(q,\theta) = F_J(q, \mu\varepsilon^{-b(N+3)}qC(q,\theta))$ satisfies, according to Lemma 1.1.4, $|\mathrm{Re}\ F_J^*| \leq ctant\ \mu\varepsilon^{-b(N+3)}A_0$, for every $C \in \overline{\mathcal{H}}$.

We will prove the existence of the local stable manifold of $\mathcal{T}_{\alpha_1,\alpha_2}(\theta_1^0)$ by means of a contractive operator defined on $\overline{\mathcal{H}}^3$.

Theorem 1.1.8 *For any $(\alpha_1,\alpha_2) \in \mathbb{C}^2$, every $\gamma \in (0,1)$ and any positive parameters β, ε and μ with $\mu \in (0, \varepsilon^m)$, $m = m(b,N)$ sufficiently large, there exists a unique $(A^+_{\beta,\varepsilon,\mu}, B^{+,1}_{\beta,\varepsilon,\mu}, B^{+,2}_{\beta,\varepsilon,\mu}) \in \overline{\mathcal{H}}^3$ such that*

$$\begin{aligned} W^+_{loc}(\mathcal{T}_{\alpha_1,\alpha_2}(\theta_1^0)) \;&=\; \big\{\big(q, \mu\varepsilon^{-b(N+3)}qA^+_{\beta,\varepsilon,\mu}(q,\theta), \alpha_1 + \mu\varepsilon^{-b(N+3)}qB^{+,1}_{\beta,\varepsilon,\mu}(q,\theta), \\ &\quad \alpha_2 + \mu\varepsilon^{-b(N+3)}qB^{+,2}_{\beta,\varepsilon,\mu}(q,\theta), \theta\big) : \ |\mathrm{Im}\ \theta| < r,\ |q| < (1-\gamma)\sigma\big\}\,. \end{aligned}$$

Proof

Since $(A^+_{\beta,\varepsilon,\mu}, B^{+,1}_{\beta,\varepsilon,\mu}, B^{+,2}_{\beta,\varepsilon,\mu})$ will be found as the fixed point of a contractive map defined from $\overline{\mathcal{H}}^3$ into itself, the uniqueness in $\overline{\mathcal{H}}^3$ follows immediately.

Let us define, for every $C \in \overline{\mathcal{H}}$, the function

$$C_1^*(q,\theta) = \frac{1}{L} + \frac{\mu\varepsilon^{-b(N+3)+1}}{q\sqrt{2}}C_1(q,\theta), \quad i = 1,2,3,4,$$

where the function C_1 was defined at (1.1.15) and

$$L = \exp(2\pi\varepsilon\sqrt{A}(1 + F_J^*)), \quad F_J^* = F_J(q, \mu\varepsilon^{-b(N+3)}qC(q,\theta)).$$

Furthermore, for every $(C,D,E) \in \overline{\mathcal{H}}^3$ let

$$\begin{aligned} M_1(C,D,E)(q,\theta) \;&=\; \frac{\varepsilon C_2(q,\theta)}{Lq\sqrt{2}} + \frac{1}{L}C_1^*(q,\theta)C(qC_1^*(q,\theta), \theta + 2\pi\beta) \\ M_2(C,D,E)(q,\theta) \;&=\; \frac{\varepsilon C_3(q,\theta)}{q} + C_1^*(q,\theta)D(qC_1^*(q,\theta), \theta + 2\pi\beta) \end{aligned}$$

and

$$M_3(C,D,E)(q,\theta) = \frac{\varepsilon C_4(q,\theta)}{q} + C_1^*(q,\theta)E(qC_1^*(q,\theta), \theta + 2\pi\beta)$$

for $(q,\theta)\in\Delta(\sigma,r)$.

Let us remark that the functions C_i, $i=1,\ldots,4$ are analytic and, from Lemma 1.1.6 (see also (1.1.15)), we may write

$$C_i(q,\theta)=C_{i,1}(\theta)q+C_{i,2}(\theta)q^2+\ldots$$

for $i=1,\ldots,4$ and every $(q,\theta)\in\Delta(\sigma,r)$. Hence, C_1^* and $M_j(C,D,E)$ are analytic on $\Delta(\sigma,r)$, for $j=1,2,3$.

On the other hand, from (1.1.13) we conclude that, if there exist three functions $A^+_{\beta,\varepsilon,\mu}$, $B^{+,1}_{\beta,\varepsilon,\mu}$ and $B^{+,2}_{\beta,\varepsilon,\mu}$ with

$$\begin{aligned}\big(M_1(A^+_{\beta,\varepsilon,\mu},B^{+,1}_{\beta,\varepsilon,\mu},B^{+,2}_{\beta,\varepsilon,\mu}),M_2(A^+_{\beta,\varepsilon,\mu},B^{+,1}_{\beta,\varepsilon,\mu},B^{+,2}_{\beta,\varepsilon,\mu}),M_3(A^+_{\beta,\varepsilon,\mu},B^{+,1}_{\beta,\varepsilon,\mu},B^{+,2}_{\beta,\varepsilon,\mu})\big)=\\=\big(A^+_{\beta,\varepsilon,\mu},B^{+,1}_{\beta,\varepsilon,\mu},B^{+,2}_{\beta,\varepsilon,\mu}\big)\,,\end{aligned}$$

then the set

$$\begin{aligned}\mathcal{R}(q,\theta)=\\=\big\{\big(q,\mu\varepsilon^{-b(N+3)}qA^+_{\beta,\varepsilon,\mu}(q,\theta),\alpha_1+\mu\varepsilon^{-b(N+3)}qB^{+,1}_{\beta,\varepsilon,\mu}(q,\theta),\alpha_2+\mu\varepsilon^{-b(N+3)}qB^{+,2}_{\beta,\varepsilon,\mu}(q,\theta),\theta\big)\big\}\end{aligned}$$

is P-invariant.

Moreover, if we expand $|L^{-1}|=\exp(-2\pi\varepsilon\sqrt{A}(1+\mathrm{Re}\ F_J^*))$ to obtain

$$\left|L^{-1}\right|=1-2\pi\varepsilon\sqrt{A}(1+\mathrm{Re}\ F_J^*)+\mathrm{O}(\varepsilon^2) \tag{1.1.18}$$

we conclude, by taking μ small enough, that

$$\frac{1}{|L|}+\frac{\mu\varepsilon^{-b(N+3)+1}}{\sqrt{2}}\mathcal{F}<1$$

where $\mathcal{F}$ is the constant given in (1.1.16). Hence, the dynamics of P over $\mathcal{R}(q,\theta)$ tends to $p=q=0$, $I_1=\alpha_1$, $I_2=\alpha_2$. Therefore, we conclude that $W^+_{loc}(\mathcal{T}_{\alpha_1,\alpha_2})=\mathcal{R}(p,\theta)$ according to the following remark:

Remark 1.1.9 *Observe that the inclusion $\mathcal{R}(p,\theta)\subset W^+_{loc}(\mathcal{T}_{\alpha_1,\alpha_2})$ easily follows from the above arguments. On the other hand, the inclusion $W^+_{loc}(\mathcal{T}_{\alpha_1,\alpha_2})\subset\mathcal{R}(p,\theta)$ follows from the fact that $\mathcal{R}(p,\theta)$ contains the set $\mathcal{T}_{\alpha_1,\alpha_2}$ (by putting $q=0$ in the above expression of $\mathcal{R}(p,\theta)$) and that dim $\mathcal{R}(p,\theta)=$ dim $W^+_{loc}(\mathcal{T}_{\alpha_1,\alpha_2})$.*

Let us show the existence of a solution of the equation $M(C,D,E)=(C,D,E)$, where

$$M(C,D,E)=(M_1(C,D,E),M_2(C,D,E),M_3(C,D,E))\,.$$

First, we will prove that the operator M is well defined: Taking μ small enough, (1.1.18) implies that

$$\frac{1}{|L|}+\frac{\mu\varepsilon^{-b(N+3)+1}}{\sqrt{2}}\mathcal{F}+\sqrt{2A}\pi\varepsilon(1+\mathrm{Re}\ F_J^*)\leq 1.$$

Therefore, from (1.1.16), (1.1.17) and the fact that $\|C\|_{\sigma,r} \le A_0$, we have that

$$\|M_1(C,D,E)\|_{\sigma,r} \le \frac{1}{|L|}\left(\frac{\varepsilon}{\sqrt{2}}\mathcal{F} + \frac{A_0}{|L|} + \frac{\mu\varepsilon^{-b(N+3)+1}\mathcal{F}A_0}{\sqrt{2}}\right) \le \frac{A_0}{|L|} < A_0,$$

where we have also used that, for every $(q,\theta) \in \Delta(\sigma, r)$, $|qC_1^*(q,\theta)| < \sigma$ and therefore it makes sense to evaluate C in $(qC_1^*(q,\theta), \theta + 2\pi\beta)$.

Moreover, for $i = 2, 3$, it is easy to see that

$$\|M_i(C,D,E)\|_{\sigma,r} \le \varepsilon\mathcal{F} + \left(\frac{1}{|L|} + \frac{\mu\varepsilon^{-b(N+3)+1}}{\sqrt{2}}\mathcal{F}\right) A_0 \le A_0.$$

Now, let us prove that M is a contractive operator: To this end, let us take $\gamma \in (0,1)$ and restrict the domain of definition of M_j, $j = 1,2,3$, to $\Delta(\sigma', r)$, $\sigma' = (1-\gamma)\sigma$, for proving the existence of a constant $c \in (0,1)$ such that

$$\|M(C,D,E) - M(C',D',E')\|_{\sigma',r} \le c\,\|(C,D,E) - (C',D',E')\|_{\sigma',r}\,,$$

for every (C,D,E), $(C',D',E') \in \overline{\mathcal{H}}^3$. We will make use of the following facts: According to Lemma 1.1.2, for every $F \in \overline{\mathcal{H}}$ we have

$$\left\|\frac{\partial F}{\partial q}\right\|_{\sigma',r} \le \frac{1}{\gamma\sigma}\|F\|_{\sigma,r} \le \frac{1}{\gamma\sigma}A_0.$$

Moreover, from Lemma 1.1.2 and Lemma 1.1.6, we get, for $i = 1,\ldots,4$, that

$$\left|\frac{\partial f_i}{\partial p}(q,p,\theta_1,\theta_2)\right| \le ctant,$$

for every $(q,p,\theta_1,\theta_2) \in B_\sigma^2 \times \mathcal{B}_1''$ also satisfying $|p| < \frac{1}{2}\sigma$.

Therefore, taking $C_i'(q,\theta) = f_i(q, \mu\varepsilon^{-b(N+3)}qC'(q,\theta),\theta)$ and bearing in mind that we may take μ small enough so that $\mu\varepsilon^{-b(N+3)}\|C'\|_{\sigma,r} < 1/2$, we deduce that

$$|C_i(q,\theta) - C_i'(q,\theta)| \le ctant\ \ |q|\,\mu\varepsilon^{-b(N+3)}\|C - C'\|_{\sigma',r}\,,$$

for every $(q,\theta) \in \Delta(\sigma', r)$ and also

$$\begin{aligned}&|F(qC_1^*(q,\theta),\theta+2\pi\beta) - F'(q(C_1')^*(q,\theta),\theta+2\pi\beta)| \le \\ &\quad\le \left(1 + \frac{ctant}{\gamma\sigma}A_0\,|q|\,\mu^2\varepsilon^{-2b(N+3)+1}\right)\|(C,D,E) - (C',D',E')\|_{\sigma',r}\,, \qquad (1.1.19)\end{aligned}$$

where (F, F') stands for (C, C'), (D, D') or (E, E').

In the same way, we obtain

$$\begin{aligned}&|C_1(q,\theta)F(qC_1^*(q,\theta),\theta+2\pi\beta) - C_1'(q,\theta)F'(q(C_1')^*(q,\theta),\theta+2\pi\beta)| \le \\ &\quad\le ctant\ \ |q|\left(1 + \mu\varepsilon^{-b(N+3)} + \frac{|q|}{\gamma\sigma}\mu^2\varepsilon^{-2b(N+3)+1}\right)\|(C,D,E) - (C',D',E')\|_{\sigma',r}\,.\end{aligned}$$

Then, denoting

$$c = \frac{1}{|L|^2} + ctant\ \mu\varepsilon^{-b(N+3)+1}\left(1 + \gamma^{-1}\mu\varepsilon^{-b(N+3)}\left(1 + \mu\varepsilon^{-b(N+3)+1}\right)\right),$$

it finally follows that

$$\|M(C,D,E) - M(C',D',E')\|_{\sigma',r} \leq c\,\|(C,D,E) - (C',D',E')\|_{\sigma',r},$$

where, for μ sufficiently small, the positive constant c is less than one. Therefore, the theorem is proved. □

An analogous result to Theorem 1.1.8 can be formulated to find the local unstable manifold of each set $\mathcal{T}_{\alpha_1,\alpha_2}(\theta_1^0)$ for the transformation P given in (1.1.13).

We finish this section, devoted to the study of the invariant tori manifolds of the system given in (1.1.6), by introducing some tools, which will be crucial to obtain information about the behaviour of the global unstable manifold. This behaviour is described by the Extension Theorem I.

Let us start by considering our unperturbed system (write $\mu = 0$ in (1.1.6))

$$\dot{x} = y, \quad \dot{y} = A\sin x, \quad \dot{I}_1 = \dot{I}_2 = 0, \quad \dot{\theta}_1 = \frac{1}{\varepsilon}, \quad \dot{\theta}_2 = \frac{\beta}{\varepsilon} \tag{1.1.20}$$

and by observing that, since θ_1 and θ_2 do not appear in the right-hand sides of the equations, we may restrict the range of variation of the (x, y, I_1, I_2)-variables to $\mathbb{R}^4$, although θ_i are complex variables. This means that, in a first approach, we may parameterize the separatrix of the invariant torus T_{α_1,α_2} in the following way

$$\begin{aligned}(\hat{\psi}, t) &= (\psi_1, \psi_2, t) \in \mathcal{B}_1'' \times \mathbb{R} \to (x^0(t), y^0(t), \hat{I}^0(t), \hat{\theta}^0(\hat{\psi}, t)) = \\ &= \left(4\arctan(e^{\sqrt{A}t}), \frac{2\sqrt{A}}{\cosh(\sqrt{A}t)}, \alpha_1, \alpha_2, \psi_1 + \frac{t}{\varepsilon}, \psi_2 + \frac{\beta t}{\varepsilon}\right).\end{aligned} \tag{1.1.21}$$

Nevertheless, the proof of the Main Theorem I depends strongly on a good control of the global behaviour of the perturbed unstable manifold of each invariant torus T_{α_1,α_2} and, in particular, we will often use that during a sufficiently large period of time (see Theorem 1.1.14) this global perturbed unstable manifold remains close enough to the respective unperturbed one. This forces us to extend the above parameterizations of the unperturbed separatrix to a complex domain.

To this end, let us define the complex subset

$$\mathcal{C}_1 = \left\{ s \in \mathbb{C} : \ |\mathrm{Im}\ s| < \frac{\pi}{2\sqrt{A}} \right\}$$

and consider the following complex extension of the unperturbed separatrix

$$\begin{aligned}&(\hat{\psi}, t, s) \in \mathcal{B}_1'' \times \mathbb{R} \times \mathcal{C}_1 \to \left(x^0(t+s), y^0(t+s), \hat{I}^0(t+s), \hat{\theta}^0(\hat{\psi}, t)\right) = \\ &= \left(4\arctan(e^{\sqrt{A}(t+s)}), \frac{2\sqrt{A}}{\cosh(\sqrt{A}(t+s))}, \alpha_1, \alpha_2, \psi_1 + \frac{t}{\varepsilon}, \psi_2 + \frac{\beta t}{\varepsilon}\right).\end{aligned} \tag{1.1.22}$$

This will be a convenient choice for several reasons. Among them, we remark that the function $y^0 = y^0(t+s)$ exhibits a pole at $t + \mathrm{Re}\ s = 0$, $\mathrm{Im}\ s = \pm\dfrac{\pi}{2\sqrt{A}}$ and this fact will turn out to be crucial when the Main Theorem I will be proved.

On the other hand, once $s_0 \in \mathcal{C}_1$ is fixed, each one of the curves

$$\left\{\left(x^0(t+s_0), y^0(t+s_0), \hat{I}^0(t+s_0), \hat{\theta}^0(\hat{\psi}, t)\right) :\ \hat{\psi} = \hat{\psi}^0\right\}$$

is a (homoclinic) solution of the unperturbed system (1.1.20).

Remark 1.1.10 *The parameter s may be seen as a new (spatial) variable and the new equation $\dot{s} = 0$ may be added to the old ones given in (1.1.20) (in fact, this will be done at the beginning of Section* 1.3*).*

Let us denote by V the neighbourhood of $x = y = 0$ where the analytic change of coordinates $(q,p) = \varphi(x,y)$ introduced in (1.1.10) and Lemma 1.1.4 is defined. Let us also consider the real parameterization of the unperturbed separatrix given in (1.1.21) and choose T_0 and $\tilde{T}_0$ sufficiently large real numbers satisfying $(x^0(t), y^0(t)) \in V$, for every $t \in (-\infty, -\tilde{T}_0] \cup [T_0, \infty)$.

We observe that we may take V and σ (see the statement of Lemma 1.1.4) in a convenient way to get $\tilde{T}_0 = T_0$ and $\varphi(V) = B^2_\sigma = \{(q,p) \in \mathbb{C}^2 :\ |q| < \sigma, |p| < \sigma\}$.

Therefore,

$$\begin{aligned} \varphi(x^0(t), y^0(t)) &= \left(0, T' e^{\sqrt{A}t}\right), \quad \text{if } t \in (-\infty, -T_0] \\ \varphi(x^0(t), y^0(t)) &= \left(T' e^{-\sqrt{A}t}, 0\right), \quad \text{if } t \in [T_0, \infty), \end{aligned} \tag{1.1.23}$$

where $T' = \exp(-\sqrt{A}(T_0' - T_0))$ and $T_0' = \dfrac{1}{\sqrt{A}}\,|\ln \sigma|$.

Let us remark that we may modify φ in order to get $T' = 1$ (in fact, Lemma 1.1.4 still holds by multiplying the obtained Hamiltonian $\tilde{H} = \tilde{H}(q,p)$ by a positive real constant and, of course, this minor modification does not affect the global argument directed to prove the Main Theorem I).

Now, let us consider the complex parameterization of the unperturbed separatrix given in (1.1.22) and, for any $s \in \mathcal{C}_1$, let us choose sufficiently large real values of time $T^-(s)$ and $T^+(s)$ for which

$$(x^0(t+s), y^0(t+s)) \in V, \quad \text{for every } t \in (-\infty, -T^-(s)] \cup [T^+(s), \infty).$$

Since φ is an analytic change of coordinates there exist analytic functions

$$s \in \mathcal{C}_1 \to s^-(s) \qquad s \in \mathcal{C}_1 \to s^+(s)$$

satisfying

$$\begin{aligned} \varphi(x^0(t+s), y^0(t+s)) &= \left(0, e^{\sqrt{A}(t+s^-)}\right), \quad \text{if } t \in (-\infty, -T^-(s)] \\ \varphi(x^0(t+s), y^0(t+s)) &= \left(e^{-\sqrt{A}(t+s^+)}, 0\right), \quad \text{if } t \in [T^+(s), \infty). \end{aligned}$$

These last parameterizations in coordinates (q,p) are obtained, as for the already discussed real case, by using the expression of the unperturbed system ($\mu=0$) given in (1.1.11), together with the fact that $F_J=0$ if $p=0$ or $q=0$. We point out that the equations (1.1.23) imply $s^+(0)=s^-(0)=0$ and, furthermore, it is easy to see that

$$T^+(s)+\text{Re } s^+ = T^-(s)-\text{Re } s^- = T_0'.$$

Moreover, since for $s_1,s_2\in\mathcal{C}_1$ with Im $s_1=$ Im s_2, the curves

$$t\in\mathbb{R}\to(x^0(t+s_1),y^0(t+s_1)),\qquad t\in\mathbb{R}\to(x^0(t+s_2),y^0(t+s_2))$$

coincide, we also have

$$T^+(s)+\text{Re } s = ctant,\qquad T^-(s)-\text{Re } s = ctant.$$

Hence,

$$\text{Re } s^+ = \text{Re } s + ctant,\qquad \text{Re } s^- = \text{Re } s + ctant.$$

Therefore, since s^+ and s^- are analytic and $s^+(0)=s^-(0)=0$, we finally deduce

$$s^+(s)=s^-(s)=s,\quad \text{for every } s\in\mathcal{C}_1.$$

In this way, by applying the change of coordinates φ to convenient pieces (those ones contained in V) of the unperturbed separatrix given in (1.1.22), we get their following parameterizations in $(q,p,\hat{I},\hat{\theta})$ coordinates

$$\begin{aligned}(\hat{\psi},t,s)\in\mathcal{B}_1''\times(-\infty,-T_0-\text{Re } s]\times\mathcal{C}_1\to\left(q^0(t+s),p^0(t+s),\hat{I}^0(t+s),\hat{\theta}^0(\hat{\psi},t)\right)=\\=\left(0,e^{\sqrt{A}(t+s)},\alpha_1,\alpha_2,\psi_1+\frac{t}{\varepsilon},\psi_2+\frac{\beta t}{\varepsilon}\right)\end{aligned}\tag{1.1.24}$$

for the local unstable case, and

$$\begin{aligned}(\hat{\psi},t,s)\in\mathcal{B}_1''\times[T_0-\text{Re } s,\infty)\times\mathcal{C}_1\to\left(q^0(t+s),p^0(t+s),\hat{I}^0(t+s),\hat{\theta}^0(\hat{\psi},t)\right)=\\=\left(e^{-\sqrt{A}(t+s)},0,\alpha_1,\alpha_2,\psi_1+\frac{t}{\varepsilon},\psi_2+\frac{\beta t}{\varepsilon}\right)\end{aligned}\tag{1.1.25}$$

for the local stable one.

The parameterizations given in (1.1.24) and (1.1.25) will allow us to construct suitable parameterizations of the local invariant manifolds of T_{α_1,α_2} for the perturbed system (1.1.6). These parameterizations are given by the next lemma. Before proving this result, let us comment that the parameter (or new variable) s gives, once more, the initial conditions for the (asymptotic) solutions of the unperturbed system ($\mu=0$) introduced in (1.1.11).

Lemma 1.1.11 *For every positive parameters* β, ε, μ *with* $\mu \in (0, \varepsilon^m)$, $m = m(b, N)$ *large enough and for a sufficiently large* $T_0 \in \mathbb{R}$, *the local perturbed stable and unstable manifolds of* T_{α_1,α_2} *can be written, respectively, in the following way:*

$$(\hat{\psi}, t, s) \in \mathcal{B}_1'' \times [T_0 - \text{Re } s, \infty) \times \mathcal{C}_1 \to \Big(x^+(\hat{\psi}, t, s), y^+(\hat{\psi}, t, s), \hat{I}^+(\hat{\psi}, t, s), \hat{\theta}^0(\hat{\psi}, t) \Big)$$

and

$$(\hat{\psi}, t, s) \in \mathcal{B}_1'' \times (-\infty, -T_0 - \text{Re } s] \times \mathcal{C}_1 \to \Big(x^-(\hat{\psi}, t, s), y^-(\hat{\psi}, t, s), \hat{I}^-(\hat{\psi}, t, s), \hat{\theta}^0(\hat{\psi}, t) \Big),$$

in such a way that the following properties hold:

1. *By denoting*

$$\mathcal{U}^+ = \mathcal{B}_1'' \times [T_0 - \text{Re } s, \infty) \times \mathcal{C}_1 \quad \textit{and} \quad \mathcal{U}^- = \mathcal{B}_1'' \times (-\infty, -T_0 - \text{Re } s] \times \mathcal{C}_1,$$

   it follows that

$$\left\| (x^*, y^*, \hat{I}^*, \hat{\theta}^0) - (x^0, y^0, \hat{I}^0, \hat{\theta}^0) \right\|_{\mathcal{U}^*} \leq \textit{ctant } \mu \left| \ln \mu \right| \varepsilon^{-b(N+3)},$$

   where $(x^0, y^0, \hat{I}^0, \hat{\theta}^0)$ *is the parameterization given in (1.1.22) and* $*$ *stands for* $-$ *or* $+$.

2. *If we denote by*

$$\mathcal{U}_1^+ = \mathcal{B}_1'' \times [T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}_1, \quad \mathcal{U}_1^- = \mathcal{B}_1'' \times [-2T_0 - \text{Re } s, -T_0 - \text{Re } s] \times \mathcal{C}_1,$$

   then it follows that

$$\left\| (x^*, y^*, \hat{I}^*, \hat{\theta}^0) - (x^0, y^0, \hat{I}^0, \hat{\theta}^0) \right\|_{\mathcal{U}_1^*} \leq \textit{ctant } \mu \varepsilon^{-b(N+3)},$$

   where $*$ *stands for* $-$ *or* $+$.

3. *Once* $(\hat{\psi}, s) \in \mathcal{B}_1'' \times \mathcal{C}_1$ *is fixed, the curves*

$$\begin{aligned} t \in [T_0 - \text{Re } s, \infty) &\to \Big(x^+(\hat{\psi}, t, s), y^+(\hat{\psi}, t, s), \hat{I}^+(\hat{\psi}, t, s), \hat{\theta}^0(\hat{\psi}, t) \Big) \\ t \in (-\infty, -T_0 - \text{Re } s] &\to \Big(x^-(\hat{\psi}, t, s), y^-(\hat{\psi}, t, s), \hat{I}^-(\hat{\psi}, t, s), \hat{\theta}^0(\hat{\psi}, t) \Big) \end{aligned}$$

   are solutions of the perturbed system (1.1.6).

Proof

Let us begin by observing that, once $\theta_1 \in \mathbb{C}$ with $|\text{Im } \theta_1| < -\ln(a\varepsilon^p) - 2\varepsilon^b$ is fixed (say $\theta_1 = \theta_1^0$), Theorem 1.1.8 provides three functions, which were denoted by $A^+_{\beta,\varepsilon,\mu}$, $B^{+,1}_{\beta,\varepsilon,\mu}$ and $B^{+,2}_{\beta,\varepsilon,\mu}$, completely determining the local stable manifold of

$$\mathcal{T}_{\alpha_1,\alpha_2}(\theta_1^0) = \left\{ (q, p, \hat{I}, \hat{\theta}) \in T_{\alpha_1,\alpha_2} : \ \theta_1 = \theta_1^0 \right\}.$$

Of course, those functions $A^+_{\beta,\varepsilon,\mu}$, $B^{+,1}_{\beta,\varepsilon,\mu}$ and $B^{+,2}_{\beta,\varepsilon,\mu}$ also depend on the variable θ_1 (for different values of θ_1 those functions are different), in such a way that, henceforth, we will denote by $\tilde{A}^+_{\beta,\varepsilon,\mu}$, $\tilde{B}^{+,1}_{\beta,\varepsilon,\mu}$ and $\tilde{B}^{+,2}_{\beta,\varepsilon,\mu}$ the (new) functions defined on (see Lemma 1.1.4 for the definition of σ)

$$\left\{(q,\hat{\theta}):\ |q|<\sigma,\hat{\theta}\in\mathcal{B}_1''\right\}$$

by

$$\tilde{A}^+_{\beta,\varepsilon,\mu}(q,\hat{\theta})=\tilde{A}^+_{\beta,\varepsilon,\mu}(q,\theta_1,\theta_2)=A^+_{\beta,\varepsilon,\mu}(q,\theta_2), \tag{1.1.26}$$

where $A^+_{\beta,\varepsilon,\mu}$ is the (first) function given by Theorem 1.1.8, furnishing the local stable manifold of $\mathcal{T}_{\alpha_1,\alpha_2}(\theta_1)$. The functions $\tilde{B}^{+,1}_{\beta,\varepsilon,\mu}$ and $\tilde{B}^{+,2}_{\beta,\varepsilon,\mu}$ are defined in an equivalent way.

Let us define

$$\mathcal{P}:(\hat{\psi},s)\in\mathcal{B}_1''\times\mathcal{C}_1\to\mathcal{P}(\hat{\psi},s)$$

the map whose components in $(q,p,\hat{I},\hat{\theta})$-coordinates are given by

$$\begin{aligned}
\mathcal{P}_1(\hat{\psi},s) &= e^{-\sqrt{A}(T_0-\text{Re }s+s)}=e^{-\sqrt{A}(T_0+\sqrt{-1}\text{Im }s)}=h(s)\\
\mathcal{P}_2(\hat{\psi},s) &= \mu\varepsilon^{-b(N+3)}h(s)\tilde{A}^+_{\beta,\varepsilon,\mu}\left(h(s),\mathcal{P}_5(\hat{\psi},s),\mathcal{P}_6(\hat{\psi},s)\right)\\
\mathcal{P}_{2+i}(\hat{\psi},s) &= \alpha_i+\mu\varepsilon^{-b(N+3)}h(s)\tilde{B}^{+,i}_{\beta,\varepsilon,\mu}\left(h(s),\mathcal{P}_5(\hat{\psi},s),\mathcal{P}_6(\hat{\psi},s)\right),\ i=1,2\\
\mathcal{P}_5(\hat{\psi},s) &= \psi_1+\frac{T_0-\text{Re }s}{\varepsilon},\quad \mathcal{P}_6(\hat{\psi},s)=\psi_2+\frac{\beta(T_0-\text{Re }s)}{\varepsilon}.
\end{aligned}$$

The graph of $\mathcal{P}$ is contained, according to Theorem 1.1.8 and (1.1.25), in $W^+_{loc}(T_{\alpha_1,\alpha_2})$. Moreover, the function $\mathcal{P}$ *detects* the parametric family $s\in\mathcal{C}_1\to\mathcal{P}(\cdot,s)$ of *frontier* tori shaping the "boundary" of the local stable perturbed manifold of T_{α_1,α_2}, i.e., those tori delimiting the region of the local stable perturbed manifold which we are going to explore. Now, for every $(\hat{\psi},s)\in\mathcal{B}_1''\times\mathcal{C}_1$ we may parameterize by

$$t\in[T_0-\text{Re }s,\infty)\to\left(q^+(\hat{\psi},t,s),p^+(\hat{\psi},t,s),\hat{I}^+(\hat{\psi},t,s),\hat{\theta}^0(\hat{\psi},t)\right)$$

the solution of (1.1.11) passing through $\mathcal{P}(\hat{\psi},s)$ at time $t=T_0-\text{Re }s$.

Remark 1.1.12 *We point out that, for any $s\in\mathcal{C}_1$ fixed, the frontier tori $\mathcal{P}(\hat{\psi},s)$ has dimension two. Therefore, the above parameterization in $(q,p,\hat{I},\hat{\theta})$-variables yields the whole local perturbed manifold of T_{α_1,α_2}, see also Remark 1.1.9.*

Let us again consider the analytic change of variables φ defined by (1.1.10) and Lemma 1.1.4 and take for every $(\hat{\psi},t,s)\in\mathcal{B}_1''\times[T_0-\text{Re }s,\infty)\times\mathcal{C}_1$,

$$\left(x^+(\hat{\psi},t,s),y^+(\hat{\psi},t,s)\right)=\varphi^{-1}\left(q^+(\hat{\psi},t,s),p^+(\hat{\psi},t,s)\right)$$

and observe that the parameterization

$$(\hat{\psi},t,s)\in\mathcal{B}_1''\times[T_0-\text{Re }s,\infty)\times\mathcal{C}_1\to\left(x^+(\hat{\psi},t,s),y^+(\hat{\psi},t,s),\hat{I}^+(\hat{\psi},t,s),\hat{\theta}^0(\hat{\psi},t)\right)$$

satisfies the third property announced in the statement of Lemma 1.1.11.

Let us prove the first two conclusions of Lemma 1.1.11: According to (1.1.11), for each $(\hat{\psi}, s) \in \mathcal{B}_1'' \times \mathcal{C}_1$, the map $t \in [T_0 - \text{Re } s, \infty) \to q^+ = q^+(\hat{\psi}, \cdot, s)$ is the solution of

$$\begin{cases} \dot{q} = -\sqrt{A}q(1 + F_J) + \mu f(q, p, \hat{\theta}) \\ q(\hat{\psi}, T_0 - \text{Re } s, s) = h(s) = q^0(T_0 - \text{Re } s + s) \end{cases} \tag{1.1.27}$$

where $q^0 = q^0(t + s) = e^{-\sqrt{A}(t+s)}$ is a solution of $\dot{q} = -\sqrt{A}q$ and the evolution in $[T_0 - \text{Re } s, \infty)$ of the variables $\hat{\theta}$ and p is given, according to (1.1.11) and Theorem 1.1.8, by

$$\hat{\theta} = \hat{\theta}(t) = \hat{\psi} + \left(\frac{t}{\varepsilon}, \frac{\beta t}{\varepsilon}\right), \quad p = p(t) = \mu\varepsilon^{-b(N+3)} q(t) \tilde{A}^+_{\beta,\varepsilon,\mu}(q(t), \hat{\theta}(t)),$$

and the map $\tilde{A}^+_{\beta,\varepsilon,\mu}$ was defined at (1.1.26).

We are going to prove that, for every $t \geq T_0 - \text{Re } s$,

$$\left| q^+(\hat{\psi}, t, s) - q^0(t + s) \right| \leq ctant\ \mu \left| \ln \mu \right| \varepsilon^{-b(N+3)}. \tag{1.1.28}$$

To this end, let us observe that, see (1.1.11), the function f is defined by

$$f(q, p, \hat{\theta}) = -\frac{(q^X - q^Y)}{\sqrt{2}} \sin \tilde{x} M_1(\hat{\theta}) - \frac{(q^X + q^Y)}{\sqrt{2A}} \tilde{y} \cos \tilde{x} M_1(\hat{\theta}).$$

Hence, since Theorem 1.1.8 implies

$$p^+(\hat{\psi}, t, s) = \mu\varepsilon^{-b(N+3)} q^+(\hat{\psi}, t, s) \tilde{A}^+_{\beta,\varepsilon,\mu}(q^+(\hat{\psi}, t, s), \hat{\theta}^0(\hat{\psi}, t)),$$

for every $(\hat{\psi}, s) \in \mathcal{B}_1'' \times \mathcal{C}_1$, we may use (1.1.14) to consider the (well-defined) function

$$g(q^+(\hat{\psi}, t, s), p^+(\hat{\psi}, t, s), \hat{\theta}^0(\hat{\psi}, t)) = \frac{f(q^+(\hat{\psi}, t, s), p^+(\hat{\psi}, t, s), \hat{\theta}^0(\hat{\psi}, t))}{q^+(\hat{\psi}, t, s)}. \tag{1.1.29}$$

Now, from (1.1.27) and (1.1.29), we may write

$$\frac{d}{dt}\left(\left| q^+(\hat{\psi}, t, s) \right| \right) = \left(-\sqrt{A}(1 + \text{Re } F_J) + \mu \text{ Re } g(q^+, p^+, \hat{\theta}^0) \right) \left| q^+(\hat{\psi}, t, s) \right| \tag{1.1.30}$$

and

$$\frac{d}{dt}(\arg(q^+(\hat{\psi}, t, s)) = -\sqrt{A}\text{Im } F_J + \mu \text{ Im } g(q^+, p^+, \hat{\theta}^0). \tag{1.1.31}$$

Moreover, according to Lemma 1.1.1, the function g introduced in (1.1.29) satisfies

$$\left| g(q^+(\hat{\psi}, t, s), p^+(\hat{\psi}, t, s), \hat{\theta}^0(\hat{\psi}, t)) \right| \leq ctant\ \varepsilon^{-b(N+2)} \tag{1.1.32}$$

and, on the other hand, Lemma 1.1.4 implies

$$|F_J| \leq ctant\ \mu\varepsilon^{-b(N+3)}. \tag{1.1.33}$$

Therefore, equation (1.1.30), together with (1.1.32) and (1.1.33), yields

$$\begin{aligned} \exp\left\{(-\sqrt{A} - ctant\ \mu\varepsilon^{-b(N+3)})(t + \text{Re } s)\right\} \leq \left|q^+(\hat{\psi}, t, s)\right| \leq \\ \leq \exp\left\{(-\sqrt{A} + ctant\ \mu\varepsilon^{-b(N+3)})(t + \text{Re } s)\right\} \end{aligned} \tag{1.1.34}$$

and, since $|q^0(t+s)| = e^{-\sqrt{A}(t+\text{Re } s)}$, one easily gets (1.1.28) when $t + \text{Re } s > \dfrac{2}{\sqrt{A}} |\ln \mu|$. Hence, let us assume $t + \text{Re } s \leq \dfrac{2}{\sqrt{A}} |\ln \mu|$. In this case, (1.1.34) implies

$$\left|\left|q^+(\hat{\psi}, t, s)\right| - \left|q^0(t+s)\right|\right| \leq ctant\ \mu\, |\ln \mu|\, \varepsilon^{-b(N+3)}.$$

Moreover, since (recall that we always work with real values of time t)

$$\frac{d}{dt}(\arg(q^0(t+s))) = 0,$$

equations (1.1.31), (1.1.32) and (1.1.33) give

$$\left|\arg(q^+(\hat{\psi}, t, s)) - \arg(q^0(t+s))\right| \leq ctant\ \mu\, |\ln \mu|\, \varepsilon^{-b(N+3)},$$

for every $t \in [T_0 - \text{Re } s, \frac{2}{\sqrt{A}} |\ln \mu| - \text{Re } s]$. In this way, (1.1.28) is proved.

Now, Theorem 1.1.8 implies

$$\left|p^+(\hat{\psi}, t, s)\right| \leq ctant\ \mu\varepsilon^{-b(N+3)} \left|q^+(\hat{\psi}, t, s)\right| \leq ctant\ \mu\varepsilon^{-b(N+3)}.$$

Hence, using that the change of variables $(q, p) = \varphi(x, y)$ defined by (1.1.10) and Lemma 1.1.4 is analytic, we conclude

$$\left|x^+(\hat{\psi}, t, s) - x^0(t+s)\right| \leq ctant\ \mu\, |\ln \mu|\, \varepsilon^{-b(N+3)}$$

and

$$\left|y^+(\hat{\psi}, t, s) - y^0(t+s)\right| \leq ctant\ \mu\, |\ln \mu|\, \varepsilon^{-b(N+3)}.$$

Finally, Theorem 1.1.8 also gives

$$\left|I_i^+(\hat{\psi}, t, s) - \alpha_i\right| \leq ctant\ \mu\varepsilon^{-b(N+3)} \left|q^+(\hat{\psi}, t, s)\right|$$

and thus the first statement of the lemma is proven.

Now, the second statement follows as the first one by taking into account that Gronwall's estimates give (compare with (1.1.28))

$$\left|q^+(\hat{\psi},t,s)-q^0(t+s)\right|\leq ctant\ \mu\varepsilon^{-b(N+3)},$$

for every $(\hat{\psi},s)\in\mathcal{B}_1''\times\mathcal{C}_1$ and every $t\in[T_0-\mathrm{Re}\ s,2T_0-\mathrm{Re}\ s]$.

Therefore, Lemma 1.1.11 is proven for the stable manifold case. The unstable manifold one can be obtained by using similar arguments. □

Remark 1.1.13 *If one takes the parameterization in the $(q,p,\hat{I},\hat{\theta})$ coordinates of the unperturbed local unstable manifold (see (1.1.24)) and writes the* frontier *unperturbed tori (i.e., those tori delimiting the local unperturbed unstable manifold)*

$$(\hat{\psi},s)\to\left(0,\tilde{h}(s),\alpha_1,\alpha_2,\psi_1-\frac{T_0+\mathrm{Re}\ s}{\varepsilon},\psi_2-\beta\frac{T_0+\mathrm{Re}\ s}{\varepsilon}\right),$$

where

$$\tilde{h}(s)=e^{\sqrt{A}(-T_0+\sqrt{-1}\mathrm{Im}\ s)},$$

then, Theorem 1.1.8 (in the local unstable manifold version) furnishes the frontier *perturbed tori (those tori delimiting the local perturbed stable manifold)*

$$\Big(\mu\varepsilon^{-b(N+3)}\tilde{h}(s)\tilde{A}^-_{\beta,\varepsilon,\mu}(\tilde{h}(s),\hat{\theta}(\hat{\psi},s)),\tilde{h}(s),\alpha_1+\mu\varepsilon^{-b(N+3)}\tilde{h}(s)\tilde{B}^{-,1}_{\beta,\varepsilon,\mu}(\tilde{h}(s),\hat{\theta}(\hat{\psi},s)),$$
$$\alpha_2+\mu\varepsilon^{-b(N+3)}\tilde{h}(s)\tilde{B}^{-,2}_{\beta,\varepsilon,\mu}(\tilde{h}(s),\hat{\theta}(\hat{\psi},s)),\hat{\theta}(\hat{\psi},s)\Big),$$

and

$$\hat{\theta}(\hat{\psi},s)=\left(\psi_1-\frac{T_0+\mathrm{Re}\ s}{\varepsilon},\psi_2-\beta\frac{T_0+\mathrm{Re}\ s}{\varepsilon}\right)$$

for the perturbed local unstable manifold.

Let us observe that the points belonging to these tori are used as initial conditions to construct a suitable parameterization

$$\left(q^-(\hat{\psi},t,s),p^-(\hat{\psi},t,s),\hat{I}^-(\hat{\psi},t,s),\hat{\theta}^0(\hat{\psi},t)\right)$$

of the local unstable perturbed manifold and we want to point out that the components of this parameterization are real whenever $(s,\psi_1,\psi_2)\in\mathbb{R}^3$. The same holds for the local stable manifold.

Once parameterizations $(x^*,y^*,\hat{I}^*,\hat{\theta}^0)$ for the local invariant perturbed manifolds of T_{α_1,α_2} have been achieved, we can finish this section by introducing the Extension Theorem I which, by using the second statement of Lemma 1.1.11, establishes that the local unstable perturbed manifold of every invariant torus T_{α_1,α_2} can be extended until it reaches again the domain where the normal form coordinates (q,p), used along this section, are defined.

In fact, we will get fine properties of the extension to the time interval

$$[-T_0 - \text{Re } s, 2T_0 - \text{Re } s]$$

of any solution $(x(t), y(t), \hat{I}(t), \hat{\theta}(t))$ of (1.1.6) satisfying

$$\begin{aligned} \left|x(t_0) - x^0(t_0 + s)\right| &\leq C_1 \mu \varepsilon^{-b(N+3)}, \quad \left|y(t_0) - y^0(t_0 + s)\right| \leq C_1 \mu \varepsilon^{-b(N+3)}, \\ |I_i(t_0) - \alpha_i| &\leq C_1 \mu \varepsilon^{-b(N+3)}, \quad i = 1, 2, \quad (\theta_1(t_0), \theta_2(t_0)) \in \mathcal{B}_1'', \end{aligned} \tag{1.1.35}$$

for $t_0 = -T_0 - \text{Re } s$, some positive constant C_1 and some $s \in \mathcal{C}_1'$, where

$$\mathcal{C}_1' = \left\{ s \in \mathbb{C} : \ |\text{Im } s| \leq \frac{\pi}{2\sqrt{A}} - \varepsilon^b \right\}. \tag{1.1.36}$$

Let us recall that we have already defined

$$\mathcal{B}_1'' = \{(\theta_1, \theta_2) \in \mathbb{C}^2 : \ |\text{Im } \theta_i| \leq -\ln(a\varepsilon^p) - 2\varepsilon^b, \ i = 1, 2\}.$$

Theorem 1.1.14 (The Extension Theorem I) *There exists a positive constant C_1' such that, if $\mu \in (0, \varepsilon^m)$, with $m > b(N + 6)$, then every solution of (1.1.6) verifying (1.1.35) can be extended to the time interval*

$$[-T_0 - \text{Re } s, 2T_0 - \text{Re } s]$$

and, for every $t \in [-T_0 - \text{Re } s, 2T_0 - \text{Re } s]$, it holds that

$$\begin{aligned} \left|x(t) - x^0(t + s)\right| &\leq C_1' \mu \varepsilon^{-b(N+5)}, \quad \left|y(t) - y^0(t + s)\right| \leq C_1' \mu \varepsilon^{-b(N+5)}, \\ |I_i(t) - \alpha_i| &\leq C_1' \mu \varepsilon^{-b(N+5)}, \quad i = 1, 2. \end{aligned}$$

The proof of the Extension Theorem I is given in Chapter 3.

Remark 1.1.15 *Let us show how the Extension Theorem ensures the existence of homoclinic orbits for the complete perturbed system associated to the Hamiltonian $H_{\varepsilon,\beta,\mu}$ given in (1.0.1). Firstly, let us point out that*

$$H_{\varepsilon,\beta,\mu}(x, y, \hat{I}, \hat{\theta}) = H_{\varepsilon,\beta,\mu}(-x, y, \hat{I}, -\hat{\theta}).$$

Hence, $\dot{H}_{\varepsilon,\beta,\mu}(-x(-t), y(-t), \hat{I}(-t), -\hat{\theta}(-t)) = 0$ if $\dot{H}_{\varepsilon,\beta,\mu}(x(t), y(t), \hat{I}(t), \hat{\theta}(t)) = 0$.

Therefore, to obtain homoclinic orbits it will be enough to get initial conditions on the unstable manifold $W^-(T_{\alpha_1,\alpha_2})$ satisfying

$$x(t_0) = -x(-t_0), \ y(t_0) = y(-t_0), \ \hat{I}(t_0) = \hat{I}(-t_0), \ \hat{\theta}(t_0) = -\hat{\theta}(-t_0).$$

By taking $t_0 = 0$ all the above equalities are satisfied when $(\psi_1, \psi_2) \in \{0, \pi\} \times \{0, \pi\}$ and $x(0) = \pi$ (this equality can be achieved because the unperturbed separatrix (1.1.21) transversally intersects the section $x = \pi$ and therefore, using the Extension Theorem, *it becomes evident that the same holds for the unstable perturbed manifold* $W^-(T_{\alpha_1,\alpha_2})$*, whenever* μ *is small enough).*

Furthermore, let us take $x^-(\hat{\psi},t,s)$ *the first component of the parameterization of the local unstable manifold of* T_{α_1,α_2} *given in Lemma 1.1.11. For* $\hat{\psi}$ *and* s *fixed, let us consider* $x^-(\hat{\psi},\cdot,s)$ *the first component of the solution of (1.1.6) passing through*

$$\Big(x^-(\hat{\psi},-T_0-\text{Re } s,s), y^-(\hat{\psi},-T_0-\text{Re } s,s), \hat{I}^-(\hat{\psi},-T_0-\text{Re } s,s), \hat{\theta}^0(\hat{\psi},-T_0-\text{Re } s)\Big)$$

at time $t=-T_0-\text{Re } s$*. Then, the Extension Theorem I and the equality* $x^0(t+s)=\pi$ *when* $t=s=0$ *imply that, for every* μ *sufficiently small and for each one of the four initial choices* $(\psi_1,\psi_2)=(\overline{\psi}_1,\overline{\psi}_2)\in\{0,\pi\}\times\{0,\pi\}$*, we can make a time translation (close to the identity) in such a way that* $x^-(\hat{\psi},t,s)=\pi$ *when* $\hat{\psi}=(\overline{\psi}_1,\overline{\psi}_2)$ *and* $t=s=0$*. Therefore,*

$$\left\{\Big(x^-(\hat{\psi},t,s), y^-(\hat{\psi},t,s), \hat{I}^-(\hat{\psi},t,s), \hat{\theta}^0(\hat{\psi},t)\Big) : \ \hat{\psi}=(\overline{\psi}_1,\overline{\psi}_2), s=0\right\}$$

is a homoclinic orbit for the perturbed system.

It is clear that, at some moment, we must choose the homoclinic orbit along which the splitting size $\Upsilon=\Upsilon(\overline{\psi}_1,\overline{\psi}_2)$ (see the statement of Theorem 0.0.2) is going to be estimated. Nevertheless, as we will see in Subsection 1.3.1 (see Lemma 1.3.13), this choice only depends on the value of ε and, more concretely, for any ε in the open set $\mathcal{U}_\varepsilon$ in which the Main Theorem I will hold, we are going to give a detailed algorithm directed to select the above mentioned specific homoclinic orbit among the four ones described in Remark 1.1.15.

1.2 The renormalized Melnikov functions

In this section we introduce the renormalized Melnikov functions associated to our Hamiltonian (1.0.1).

These renormalized Melnikov functions are going to be denoted by

$$\mathcal{M}_j=\mathcal{M}_j(s,\hat{\psi}),\quad j=1,2,3$$

although they will also depend on the parameters β, ε and μ. These computable functions are going to give a suitable approximation for the intersection angle between the perturbed manifolds along the homoclinic orbits (see Remark 1.1.15) of the perturbed system.

The most difficult part of the proof of this last *suitable approximation* is going to be developed in the next section and, more concretely, it will be mostly established along the second part of the proof of the Main Theorem I (see Subsection 1.3.2).

Essentially, in the present section we pursue two objectives: First, we emphasize the role of the renormalized Melnikov functions by proving Lemma 1.2.1, which is stated below. Second, we obtain analytic expressions for each renormalized Melnikov function.

Lemma 1.2.1 ensures that the differences of the unperturbed energies

$$\begin{aligned}\mathcal{Q}_1(x,y,\hat{I},\hat{\theta}) &= H_1(x,y)=\frac{y^2}{2}+A(\cos x-1)\\ \mathcal{Q}_{1+j}(x,y,\hat{I},\hat{\theta}) &= I_j, \quad j=1,2,\end{aligned}$$

evaluated at a point in some special piece of the local stable perturbed manifold and a point in another special piece of the unstable perturbed manifold essentially coincide with the Melnikov functions $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$, respectively.

These special pieces of invariant manifolds are those defined on

$$\mathcal{B}_1'' \times [T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}_1'$$

by

$$\left(x^*(\hat{\psi},t,s), y^*(\hat{\psi},t,s), \hat{I}^*(\hat{\psi},t,s), \hat{\theta}^0(\hat{\psi},t)\right)$$

where $*$, as usual, stands for $+$ or $-$. These parameterizations are furnished by Lemma 1.1.11 (in the case of the stable manifold) and the Extension Theorem (in the unstable manifold case).

On the other hand, if we denote by

$$\mathcal{Q}_j^*(\hat{\psi},t,s)=\mathcal{Q}_j(x^*(\hat{\psi},t,s), y^*(\hat{\psi},t,s), \hat{I}^*(\hat{\psi},t,s), \hat{\theta}^0(\hat{\psi},t)),$$

then the above mentioned differences of energies are given by the functions

$$\mathcal{Q}_j^-(\hat{\psi},t,s)-\mathcal{Q}_j^+(\hat{\psi},t,s), \quad j=1,2,3.$$

Roughly speaking, it seems clear that a good control on the differences of unperturbed energies between points in the stable manifold and points in the unstable one would be fruitful for measuring the *distance* between both manifolds. We refer the reader to formulae (1.3.90) and (1.3.91), where the above mentioned differences and the respective splitting functions (those ones taking part in the definition of the transversality (1.3.103)) are related.

1.2.1 The role of the renormalized Melnikov functions

Let us recall that, given an orbit $\Theta(t)=(x,y,\hat{I},\hat{\theta})(t)$ of the dynamical system associated to the perturbed Hamiltonian $H_{\varepsilon,\beta,\mu}$ introduced in (1.0.1), then, for $j=1,2,3$,

$$\dot{\mathcal{Q}}_j(\Theta(t))=\{\mathcal{Q}_j,H_{\varepsilon,\beta,\mu}\}(\Theta(t))=\{\mathcal{Q}_j,H_{\varepsilon,\beta,\mu}\}(x,y,\hat{I},\hat{\theta})(t),$$

where, as usual, $\{\cdot,\cdot\}$ denotes the Poisson brackets and the derivative is taken with respect to the time t.

Then, for any $(\hat{\psi}, t, s) \in \mathcal{B}_1'' \times [T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}_1'$ and $j = 1, 2, 3$, we obtain

$$\mathcal{Q}_j^-(\hat{\psi}, t, s) - \mathcal{Q}_j^+(\hat{\psi}, t, s) = \int_{-\infty}^{t} \{\mathcal{Q}_j, H_{\varepsilon,\beta,\mu}\}(x^-, y^-, \hat{I}^-, \hat{\theta}^0)(\hat{\psi}, \gamma, s) d\gamma + \\ + \int_{t}^{\infty} \{\mathcal{Q}_j, H_{\varepsilon,\beta,\mu}\}(x^+, y^+, \hat{I}^+, \hat{\theta}^0)(\hat{\psi}, \gamma, s) d\gamma,$$

where we have used the third statement of Lemma 1.1.11.

Unfortunately, we do not know any expression for the solutions of the perturbed systems. However, Lemma 1.1.11 and the Extension Theorem I tell us that those special perturbed solutions are close to the respective (homoclinic) ones for the unperturbed system. Hence, it seems wise to write

$$\mathcal{Q}_j^-(\hat{\psi}, t, s) - \mathcal{Q}_j^+(\hat{\psi}, t, s) \approx \int_{\mathbb{R}} \{\mathcal{Q}_j, H_{\varepsilon,\beta,\mu}\}(x^0, y^0, \hat{I}^0, \hat{\theta}^0)(\hat{\psi}, \gamma, \Gamma) d\gamma,$$

where $\Gamma = \gamma + s$ and $(x^0, y^0, \hat{I}^0, \hat{\theta}^0)(\hat{\psi}, t, t + s)$ is the *known* parameterization of the unperturbed manifold introduced in (1.1.22).

Explicit expressions for each one of the renormalized (complex) Melnikov functions

$$\mathcal{M}_j(s, \hat{\psi}) = \int_{\mathbb{R}} \{\mathcal{Q}_j, H_{\varepsilon,\beta,\mu}\}(x^0, y^0, \hat{I}^0, \hat{\theta}^0)(\hat{\psi}, \gamma, \Gamma) d\gamma, \quad j = 1, 2, 3 \tag{1.2.37}$$

are obtained in the next subsection.

Now, in order to prove Lemma 1.2.1, we need to introduce some definitions: Once a complex number $s \in \mathcal{C}_1'$ is fixed, let us define the real function

$$\tau(t) = \left| t + s - \frac{\sqrt{-1}\pi}{2\sqrt{A}} \right|. \tag{1.2.38}$$

Furthermore, given a real interval $[t_0, t]$, let us define, for every $z \in \mathbb{Z}$,

$$\rho_{[t_0,t]}(z) = \begin{cases} \sup_{\sigma \in [t_0,t]} \dfrac{1}{\tau^z(\sigma)}, & \text{if } z \neq 0, \\ \sup_{\sigma \in [t_0,t]} |\ln(\tau(\sigma))|, & \text{if } z = 0. \end{cases}$$

Let us observe that, if we consider a time interval $[t_0, t] \subset [-T_0 - \text{Re } s, 2T_0 - \text{Re } s]$, then it follows that $-T_0 \leq t_0 + \text{Re } s \leq t + \text{Re } s \leq 2T_0$.

Hence, see Lemma 10 in [7] for details, for every $z \in \mathbb{R}$ we deduce the existence of some positive constant $\tilde{K}_1 = \tilde{K}_1(T_0, z)$ such that, if $[t_0, t] \subset [-T_0 - \text{Re } s, 2T_0 - \text{Re } s]$, then

$$\int_{t_0}^{t} \frac{1}{\tau^z(\gamma)} d\gamma \leq \tilde{K}_1 \rho_{[t_0,t]}(z - 1). \tag{1.2.39}$$

Moreover, since the definition of $\mathcal{C}_1'$ (see (1.1.36)) implies $\tau(\gamma) \geq \varepsilon^b$, for every $\gamma \in [t_0, t]$, we also conclude that

$$\int_{t_0}^{t} \frac{1}{\tau^z(\gamma)} d\gamma \leq \tilde{K}_1 \varepsilon^{-b(z-1)}, \tag{1.2.40}$$

whenever $z \neq 0$.

Lemma 1.2.1 *For $j = 1, 2, 3$ it follows that*

$$\left| \mathcal{Q}_j^-(\hat{\psi}, t, s) - \mathcal{Q}_j^+(\hat{\psi}, t, s) - \mathcal{M}_j(s, \hat{\psi}) \right| \leq ctant\ \mu^2 \left| \ln \mu \right| \varepsilon^{-2b(N+5)},$$

for every $(\hat{\psi}, t, s) \in \mathcal{B}_1'' \times [T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}_1'$.

Proof

From the definition of $\mathcal{M}_j$ (see (1.2.37)) it is clear that Lemma 1.2.1 follows if we get suitable bounds for the functions $\mathcal{R}_j^-$ and $\mathcal{R}_j^+$, $j = 1, 2, 3$, defined on

$$\mathcal{B}_1'' \times [T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}_1'$$

by

$$\mathcal{R}_j^-(\hat{\psi}, t, s) = \int_{-\infty}^{t} \mathcal{D}_j^-(\hat{\psi}, \gamma, s) d\gamma, \quad \mathcal{R}_j^+(\hat{\psi}, t, s) = \int_{t}^{\infty} \mathcal{D}_j^+(\hat{\psi}, \gamma, s) d\gamma,$$

where, for $\Gamma = \gamma + s$,

$$\mathcal{D}_j^*(\hat{\psi}, \gamma, s) = \{\mathcal{Q}_j, H_{\varepsilon,\beta,\mu}\}(x^*, y^*, \hat{I}^*, \hat{\theta}^0)(\hat{\psi}, \gamma, s) - \{\mathcal{Q}_j, H_{\varepsilon,\beta,\mu}\}(x^0, y^0, \hat{I}^0, \hat{\theta}^0)(\hat{\psi}, \gamma, \Gamma)$$

and $*$ stands for $-$ or $+$.

Let us start by considering the functions $\mathcal{R}_j^-$: Since $t \in [T_0 - \text{Re } s, 2T_0 - \text{Re } s]$, we can decompose the functions $\mathcal{R}_j^-$ in the following way

$$\mathcal{R}_j^-(\hat{\psi}, t, s) = \mathcal{R}_j^-(\hat{\psi}, -T_0 - \text{Re } s, s) + \tilde{\mathcal{R}}_j^-(\hat{\psi}, t, s)$$

where

$$\tilde{\mathcal{R}}_j^-(\hat{\psi}, t, s) = \int_{-T_0 - \text{Re } s}^{t} \mathcal{D}_j^-(\hat{\psi}, \gamma, s) d\gamma.$$

In order to bound $\mathcal{R}_j^-(\hat{\psi}, -T_0 - \text{Re } s, s)$ and the functions $\tilde{\mathcal{R}}_j^-$ for the case in which $j = 2, 3$, it will be useful to observe that, from the definition of Poisson brackets, we have

$$\{I_j, H_{\varepsilon,\beta,\mu}\}(\Theta(t)) = -\mu y(t) \sin x(t) \frac{\partial M_1}{\partial \theta_j}(\hat{\theta}(t)),$$

for $j = 1, 2$. Then, recalling that $\mathcal{Q}_2 = I_1$ and $\mathcal{Q}_3 = I_2$, we may write, for any $\gamma \in \mathbb{R}$,

$$\begin{aligned} \left| \mathcal{D}_j^-(\hat{\psi}, \gamma, s) \right| \leq ctant\ \mu \varepsilon^{-b(N+3)} \Big(& \left| y^-(\hat{\psi}, \gamma, s) - y^0(\Gamma) \right| \left| \sin(x^-(\hat{\psi}, \gamma, s)) \right| + \\ & + \left| y^0(\Gamma) \right| \left| \sin(x^-(\hat{\psi}, \gamma, s)) - \sin(x^0(\Gamma)) \right| \Big), \ j = 2, 3, \end{aligned} \tag{1.2.41}$$

where we have also used (1.1.8).

1. *Bounds for* $\mathcal{R}_j^-(\hat{\psi}, -T_0 - \mathrm{Re}\ s, s)$, $j = 2, 3$.

Let us assume $\gamma \le -T_0 - \mathrm{Re}\ s$. In this case, the first statement of Lemma 1.1.11 yields

$$\left|\sin(x^-(\hat{\psi}, \gamma, s)) - \sin(x^0(\Gamma))\right| \le ctant\ \mu\, |\ln \mu|\, \varepsilon^{-b(N+3)}$$

and

$$\left|y^-(\hat{\psi}, \gamma, s) - y^0(\Gamma)\right| \le ctant\ \mu\, |\ln \mu|\, \varepsilon^{-b(N+3)}.$$

Moreover, since $|\gamma|$ can be assumed to be large enough, it is also easy to check that

$$\left|\sin(x^-(\hat{\psi}, \gamma, s))\right| \le ctant\ \left|x^-(\hat{\psi}, \gamma, s)\right|.$$

Now, taking into account the properties of the change of coordinates $(q, p) = \varphi(x, y)$, defined in (1.1.10) and Lemma 1.1.4, one has

$$|x| \le ctant\ (|p| + |q|).$$

Thus, by means of Theorem 1.1.8 (in the local unstable manifold case) we obtain, for any $\gamma \le -T_0 - \mathrm{Re}\ s$, that

$$\left|x^-(\hat{\psi}, \gamma, s)\right| \le ctant\ \left|p^-(\hat{\psi}, \gamma, s)\right|.$$

Furthermore, in the same way as (1.1.34) was deduced, we get

$$\left|p^-(\hat{\psi}, \gamma, s)\right| \le \exp\left\{(\sqrt{A} - ctant\ \mu\varepsilon^{-b(N+3)})(\gamma + \mathrm{Re}\ s)\right\},$$

for every $\gamma \in (-\infty, -T_0 - \mathrm{Re}\ s]$. Hence,

$$\int_{-\infty}^{-T_0 - \mathrm{Re}\ s} \left|\sin(x^-(\hat{\psi}, \gamma, s))\right| d\gamma \le ctant.$$

Moreover, since we also have (see (1.1.24))

$$\left|y^0(\Gamma)\right| \le ctant\ \left(\left|p^0(\gamma + s)\right| + \left|q^0(\gamma + s)\right|\right) = ctant\ e^{\sqrt{A}(\gamma + \mathrm{Re}\ s)}$$

it holds that

$$\int_{-\infty}^{-T_0 - \mathrm{Re}\ s} \left|y^0(\Gamma)\right| d\gamma \le ctant.$$

Therefore, using (1.2.41), we obtain

$$\left|\mathcal{R}_j^-(\hat{\psi}, -T_0 - \mathrm{Re}\ s, s)\right| \le ctant\ \mu^2\, |\ln \mu|\, \varepsilon^{-2b(N+3)}, \tag{1.2.42}$$

for $j = 2, 3$.

2. *Bounds for the functions* $\tilde{\mathcal{R}}_j^-$, $j = 2, 3$.

Let us now assume that $\gamma \in [-T_0 - \text{Re } s, t] \subset [-T_0 - \text{Re } s, 2T_0 - \text{Re } s]$. In this case, we can not use Lemma 1.1.11 but the Extension Theorem I gives

$$\left| x^-(\hat{\psi}, \gamma, s) - x^0(\Gamma) \right| \le C_1' \mu \varepsilon^{-b(N+5)}, \quad \left| y^-(\hat{\psi}, \gamma, s) - y^0(\Gamma) \right| \le C_1' \mu \varepsilon^{-b(N+5)}. \quad (1.2.43)$$

Hence, if we take $\mu \in (0, \varepsilon^m)$ with $m > b(N+5)$, we also have

$$\left| \sin(x^-(\hat{\psi}, \gamma, s)) - \sin(x^0(\Gamma)) \right| \le ctant\ \mu \varepsilon^{-b(N+5)} \tau^{-2}(\gamma),$$
$$\left| \sin(x^-(\hat{\psi}, \gamma, s)) \right| \le ctant\ \tau^{-2}(\gamma) \quad \text{and} \quad \left| y^0(\Gamma) \right| \le ctant\ \tau^{-1}(\gamma),$$

where these bounds follow from the fact that the function y^0 has a simple pole at $\pm \dfrac{\sqrt{-1}\pi}{2\sqrt{A}}$ while the functions $\sin x^0$ and $\cos x^0$ have a double pole at this point (recall also the definition of the function τ given in (1.2.38)).

Therefore, using also (1.2.41), we get

$$\left| \mathcal{D}_j^-(\hat{\psi}, \gamma, s) \right| \le ctant\ \mu^2 \varepsilon^{-2b(N+4)} \tau^{-3}(\gamma).$$

Hence, from (1.2.40)

$$\left| \tilde{\mathcal{R}}_j^-(\hat{\psi}, t, s) \right| \le \int_{-T_0 - \text{Re } s}^{t} \left| \mathcal{D}_j^-(\hat{\psi}, \gamma, s) \right| d\gamma \le ctant\ \mu^2 \varepsilon^{-2b(N+5)}$$

and, using also (1.2.42), we then deduce

$$\left\| \mathcal{R}_j^- \right\| \le ctant\ \mu^2 \left| \ln \mu \right| \varepsilon^{-2b(N+5)},$$

for $j = 2, 3$.

In order to bound $\left\| \mathcal{R}_j^+ \right\|$, for $j = 2, 3$, we proceed as above by using the first statement of Lemma 1.1.11 to replace the bounds obtained in (1.2.43) by the following ones

$$\left| x^+(\hat{\psi}, \gamma, s) - x^0(\Gamma) \right| \le ctant\ \mu \left| \ln \mu \right| \varepsilon^{-b(N+3)},$$
$$\left| y^+(\hat{\psi}, \gamma, s) - y^0(\Gamma) \right| \le ctant\ \mu \left| \ln \mu \right| \varepsilon^{-b(N+3)},$$

for every $\gamma > T_0 - \text{Re } s$.

Finally, by denoting

$$H_1(x, y) = \frac{y^2}{2} + A(\cos x - 1) \quad \text{and} \quad H_2(x, y, \hat{\theta}) = y M_1(\hat{\theta}) \sin x$$

it is easy to see that

$$\{H_1, H_{\varepsilon,\beta,\mu}\} = \mu \left(\frac{\partial H_1}{\partial x} \frac{\partial H_2}{\partial y} - \frac{\partial H_1}{\partial y} \frac{\partial H_2}{\partial x} \right).$$

Therefore, recalling that $\mathcal{Q}_1 = H_1$, for every $\gamma \in \mathbb{R}$ we have

$$\left|\mathcal{D}_1^-(\hat{\psi},\gamma,s)\right| \le \mu \left|M_1(\hat{\theta}^0)\right| \Big\{A\left|\sin^2(x^0(\Gamma)) - \sin^2(x^-(\hat{\psi},\gamma,s))\right| + \\ \left|(y^0(\Gamma))^2 - (y^-(\hat{\psi},\gamma,s))^2\right| \left|\cos(x^-(\hat{\psi},\gamma,s))\right| + \left|y^0(\Gamma)\right|^2 \left|\cos(x^0(\Gamma)) - \cos(x^-(\hat{\psi},\gamma,s))\right|\Big\}.$$

Thus, it is enough to write

$$\left|\sin^2(x^0(\Gamma)) - \sin^2(x^-(\hat{\psi},\gamma,s))\right| = 2\left|\sin(\hat{x}(\hat{\psi},\gamma,s))\cos(\hat{x}(\hat{\psi},\gamma,s))\right| \left|x^0(\Gamma) - x^-(\hat{\psi},\gamma,s)\right|$$

and

$$\left|(y^0(\Gamma))^2 - (y^-(\hat{\psi},\gamma,s))^2\right| = 2\left|\hat{y}(\hat{\psi},\gamma,s)\right| \left|y^0(\Gamma) - y^-(\hat{\psi},\gamma,s)\right|,$$

take into account that

$$\left|\hat{x}(\hat{\psi},\gamma,s)\right| \le \left|x^0(\Gamma)\right| + \left|x^0(\Gamma) - x^-(\hat{\psi},\gamma,s)\right|$$

and

$$\left|\hat{y}(\hat{\psi},\gamma,s)\right| \le \left|y^0(\Gamma)\right| + \left|y^0(\Gamma) - y^-(\hat{\psi},\gamma,s)\right|$$

to get, applying the same as the one used to bound $\mathcal{R}_2^*$ and $\mathcal{R}_3^*$, that

$$\max\left\{\left\|\mathcal{R}_1^-\right\|, \left\|\mathcal{R}_1^+\right\|\right\} \le ctant\ \mu^2 \left|\ln\mu\right| \varepsilon^{-2b(N+5)}$$

and, therefore, Lemma 1.2.1 is proven. □

1.2.2 Computing the renormalized Melnikov functions

In this subsection we will obtain the explicit expressions for each renormalized Melnikov function (see (1.2.37))

$$(s,\hat{\psi}) \in \mathcal{C}_1' \times \mathcal{B}_1'' \to \mathcal{M}_j(s,\hat{\psi}) = \int_{\mathbb{R}} \{\mathcal{Q}_j, H_{\varepsilon,\beta,\mu}\}(\Delta^0(\hat{\psi},\gamma,\Gamma))d\gamma, \qquad j = 1,2,3,$$

where $\mathcal{Q}_1 = H_1$, $\mathcal{Q}_2 = I_1$, $\mathcal{Q}_3 = I_2$, $\Gamma = \gamma + s$ and (see also (1.1.22))

$$\begin{aligned}\Delta^0(\hat{\psi},t,t+s) &= (x^0(t+s), y^0(t+s), \hat{I}^0(t+s), \hat{\theta}^0(\hat{\psi},t)) = \\ &= \left(4\arctan(e^{\sqrt{A}(t+s)}), \frac{2\sqrt{A}}{\cosh(\sqrt{A}(t+s))}, \alpha_1, \alpha_2, \psi_1 + \frac{t}{\varepsilon}, \psi_2 + \frac{\beta t}{\varepsilon}\right)\end{aligned}$$

denotes the parameterization of the unperturbed homoclinic manifold.

During the proof of Lemma 1.2.1 we have already computed the Poisson bracket functions $\{\mathcal{Q}_j, H_{\varepsilon,\beta,\mu}\}$, $j = 1,2,3$ in such a way that we may write

$$\mathcal{M}_1(s,\hat{\psi}) = \mu \int_{\mathbb{R}} \left(\frac{\partial H_1}{\partial x}\frac{\partial H_2}{\partial y} - \frac{\partial H_1}{\partial y}\frac{\partial H_2}{\partial x}\right)(\Delta^0(\hat{\psi},\gamma,\Gamma))d\gamma$$

and

$$\mathcal{M}_{j+1}(s,\hat{\psi}) = -\mu \int_{\mathbb{R}} \frac{\partial H_2}{\partial \theta_j}(\Delta^0(\hat{\psi},\gamma,\Gamma))d\gamma, \quad j=1,2,$$

where

$$H_1(x,y) = \frac{y^2}{2} + A(\cos x - 1) \quad \text{and} \quad H_2(x,y,\hat{\theta}) = yM_1(\hat{\theta})\sin x.$$

Now, using that for any $(s,\hat{\psi}) \in \mathcal{C}_1' \times \mathcal{B}_1''$ fixed, the curve $t \in \mathbb{R} \to \Delta^0(\hat{\psi},t,t+s)$ is a solution of the unperturbed system (1.1.20), we deduce that

$$\cos(x^0(t+s)) = 1 - \frac{2}{\cosh^2(\sqrt{A}(t+s))}$$

and therefore

$$\sin(x^0(t+s)) = -\frac{2\sinh(\sqrt{A}(t+s))}{\cosh^2(\sqrt{A}(t+s))}.$$

Thus, using again the above expression of the homoclinic manifold Δ^0, we obtain

$$\mathcal{M}_1(s,\hat{\psi}) = \mu A \int_{\mathbb{R}} \left(\frac{12}{\cosh^4(\sqrt{A}(t+s))} - \frac{8}{\cosh^2(\sqrt{A}(t+s))} \right) M_1(\hat{\theta}^0(t))dt \qquad (1.2.44)$$

and, for $j=1,2$,

$$\mathcal{M}_{j+1}(s,\hat{\psi}) = 4\mu\sqrt{A} \int_{\mathbb{R}} \frac{\sinh(\sqrt{A}(t+s))}{\cosh^3(\sqrt{A}(t+s))} \frac{\partial M_1}{\partial \theta_j}(\hat{\theta}^0(t))dt, \qquad (1.2.45)$$

for every $(s,\hat{\psi}) = (s,\psi_1,\psi_2) \in \mathcal{C}_1' \times \mathcal{B}_1''$, where $\mathcal{C}_1' = \left\{ s \in \mathbb{C} : \ |\mathrm{Im}\ s| < \frac{\pi}{2\sqrt{A}} - \varepsilon^b \right\}$ and $\mathcal{B}_1'' = \big\{ (\psi_1,\psi_2) \in \mathbb{C}^2 : \ |\mathrm{Im}\ \psi_i| < -\ln(a\varepsilon^p) - 2\varepsilon^b, \ i=1,2 \big\}$.

Now, recalling the definition of the function M_1 given in (1.0.2), and defining

$$C_{\hat{k}}^{(j+1)} = \frac{k_j}{f(\hat{k})} \qquad (1.2.46)$$

for $j=1,2$, $\hat{k}=(k_1,k_2) \in \Lambda$, we may write

$$\begin{aligned} \frac{\partial M_1}{\partial \theta_j}(\hat{\theta}^0(t)) &= \sum_{\hat{k}\in\Lambda} C_{\hat{k}}^{(j+1)}(a\varepsilon^p)^{|\hat{k}|} \cos(k_1(\psi_1 + \frac{t}{\varepsilon}) + k_2(\psi_2 + \frac{\beta t}{\varepsilon})) = \\ &= \sum_{\hat{k}\in\Lambda} C_{\hat{k}}^{(j+1)}(a\varepsilon^p)^{|\hat{k}|} \cos(\beta_{\hat{k}}\varepsilon^{-1}(t+s)) \cos(k_1(\psi_1 - \frac{s}{\varepsilon}) + k_2(\psi_2 - \frac{\beta s}{\varepsilon})) - \\ &\quad - \sum_{\hat{k}\in\Lambda} C_{\hat{k}}^{(j+1)}(a\varepsilon^p)^{|\hat{k}|} \sin(\beta_{\hat{k}}\varepsilon^{-1}(t+s)) \sin(k_1(\psi_1 - \frac{s}{\varepsilon}) + k_2(\psi_2 - \frac{\beta s}{\varepsilon})), \end{aligned}$$

where $\beta_{\hat{k}} = k_1 + k_2\beta = \hat{k}\omega$, $\omega = (1,\beta)$.

Hence, according to (1.2.45), we will get a new expression for $\mathcal{M}_{j+1}$, $j = 1, 2$, by calculating

$$\int_{\mathbb{R}} \frac{\sinh(\sqrt{A}(t+s))}{\cosh^3(\sqrt{A}(t+s))} \cos(\beta_{\hat{k}}\varepsilon^{-1}(t+s))dt, \quad \int_{\mathbb{R}} \frac{\sinh(\sqrt{A}(t+s))}{\cosh^3(\sqrt{A}(t+s))} \sin(\beta_{\hat{k}}\varepsilon^{-1}(t+s))dt.$$

Let us observe that, since $s \in \mathcal{C}_1'$, the above integrals coincide with the following ones

$$\int_{\mathbb{R}} \frac{\sinh(\sqrt{A}t)}{\cosh^3(\sqrt{A}t)} \cos(\beta_{\hat{k}}\varepsilon^{-1}t)dt = 0, \quad \int_{\mathbb{R}} \frac{\sinh(\sqrt{A}t)}{\cosh^3(\sqrt{A}t)} \sin(\beta_{\hat{k}}\varepsilon^{-1}t)dt = \frac{1}{\sqrt{A}} T_3(\overline{\beta}_{\hat{k}}),$$

where

$$\overline{\beta}_{\hat{k}} = \frac{\beta_{\hat{k}}}{\varepsilon\sqrt{A}} = \frac{\hat{k}\omega}{\varepsilon\sqrt{A}}$$

and, for any $r, m \in \mathbb{N}$ with $r \geq 2$, $m \geq 1$,

$$T_r(\rho) = \int_{\mathbb{R}} \frac{\sinh u \sin(\rho u)}{\cosh^r u} du = \frac{\rho}{r-1} I_{r-1}(\rho), \quad I_m(\rho) = \int_{\mathbb{R}} \frac{\cos(\rho u)}{\cosh^m u} du. \tag{1.2.47}$$

Therefore, from (1.2.45), we conclude that

$$\mathcal{M}_{j+1}(s, \psi_1, \psi_2) = \sum_{\hat{k} \in \Lambda} M_{\hat{k}}^{(j+1)} \sin(k_1(\psi_1 - \frac{s}{\varepsilon}) + k_2(\psi_2 - \frac{\beta s}{\varepsilon})), \tag{1.2.48}$$

where

$$M_{\hat{k}}^{(j+1)} = -4\mu C_{\hat{k}}^{(j+1)} (a\varepsilon^p)^{|\hat{k}|} T_3(\overline{\beta}_{\hat{k}}).$$

Remark 1.2.2 *Let us observe that, if $\hat{k}\omega = 0$ for some $\hat{k} = (k_1, k_2)$, i.e., β is a rational number, then $\overline{\beta}_{\hat{k}} = 0$ and hence $M_{\hat{k}}^{(j+1)} = 0$. Therefore, we are going to restrict the task of the computation of the coefficients $M_{\hat{k}}^{(j+1)}$ for those indices $\hat{k} = (k_1, k_2)$ satisfying $\hat{k}\omega \neq 0$.*

In order to compute the coefficients $M_{\hat{k}}^{(j+1)}$ we begin by observing that, for $m \geq 1$,

$$I_m(\rho) = \frac{2^{m+2\lambda-2}\pi}{(m-1)!} \rho^{1-2\lambda} \hat{s}^{2\lambda} \hat{c}^{1-2\lambda} \prod_{k=1}^{[\frac{m-1}{2}]} \left(\frac{\rho^2}{4} + (k-\lambda)^2 \right),$$

with $\lambda = 1/2$ if m is odd, $\lambda = 0$ if m is even, $\hat{s} = \cosh^{-1}\left(\frac{\rho\pi}{2}\right)$ and $\hat{c} = \sinh^{-1}\left(\frac{\rho\pi}{2}\right)$.

Consequently,

$$T_3(\overline{\beta}_{\hat{k}}) = \frac{\pi(\hat{k}\omega)^2}{2\varepsilon^2 A \sinh\left(\dfrac{\pi(\hat{k}\omega)}{2\varepsilon\sqrt{A}}\right)}.$$

Now, we write

$$\frac{\hat{k}\omega}{\sinh\left(\dfrac{\pi(\hat{k}\omega)}{2\varepsilon\sqrt{A}}\right)} = H(\hat{k}\omega)\left|\hat{k}\omega\right| \exp\left(-\frac{\pi\left|\hat{k}\omega\right|}{2\varepsilon\sqrt{A}}\right), \tag{1.2.49}$$

where

$$H(\hat{k}\omega) = \frac{2}{1-\exp\left(-\dfrac{\pi\left|\hat{k}\omega\right|}{\varepsilon\sqrt{A}}\right)}. \tag{1.2.50}$$

Hence, we may express the coefficients $M_{\hat{k}}^{(j+1)}$ taking part in the expansion (1.2.48) in the following way

$$M_{\hat{k}}^{(j+1)} = B_{\hat{k}}^{(j+1)}\mathcal{E}_{\hat{k}}, \tag{1.2.51}$$

where

$$\mathcal{E}_{\hat{k}} = \exp\left(-\frac{\pi\left|\hat{k}\omega\right|}{2\varepsilon\sqrt{A}}\right)(a\varepsilon^p)^{|\hat{k}|} \tag{1.2.52}$$

and

$$B_{\hat{k}}^{(j+1)} = -2\mu\varepsilon^{-2}\left|\hat{k}\omega\right| H(\hat{k}\omega)\pi A^{-1}C_{\hat{k}}^{(j+1)}(\hat{k}\omega). \tag{1.2.53}$$

Now, using (1.2.44), the expression of M_1 given in (1.0.2) and the fact that $\hat{\theta}^0(t) = (\psi_1 + \varepsilon^{-1}t, \psi_2 + \beta\varepsilon^{-1}t)$, we get

$$\mathcal{M}_1(s,\psi_1,\psi_2) = \sum_{\hat{k}\in\Lambda} M_{\hat{k}}^{(1)} \sin(k_1(\psi_1 - \frac{s}{\varepsilon}) + k_2(\psi_2 - \frac{\beta s}{\varepsilon})) \tag{1.2.54}$$

with $M_{\hat{k}}^{(1)} = \sqrt{A}\mu C_{\hat{k}}^{(1)}(a\varepsilon^p)^{|\hat{k}|}\left(12I_4(\overline{\beta}_{\hat{k}}) - 8I_2(\overline{\beta}_{\hat{k}})\right)$.

Now, the coefficients $C_{\hat{k}}^{(1)}$ are given by $C_{\hat{k}}^{(1)} = (f(\hat{k}))^{-1}$ and the integral formulae $I_m(\rho)$ were defined in (1.2.47).

Thus, using also (1.2.49), we conclude that $M_{\hat{k}}^{(1)} = B_{\hat{k}}^{(1)}\mathcal{E}_{\hat{k}}$, with $\mathcal{E}_{\hat{k}}$ given in (1.2.52) and

$$B_{\hat{k}}^{(1)} = 2\mu\varepsilon^{-3}\left|\hat{k}\omega\right| H(\hat{k}\omega)\pi A^{-1}C_{\hat{k}}^{(1)}(\hat{k}\omega)^2.$$

We finish this section by pointing out that the arguments used to prove Lemma 1.2.1 also apply to prove the following lemma:

Lemma 1.2.3 *For $i = 1,2,3$ and every $(s,\hat{\psi}) \in \mathcal{C}_1' \times \mathcal{B}_1''$, it follows that*

$$\left|\mathcal{M}_i(s,\hat{\psi})\right| \le ctant\ \mu\varepsilon^{-b(N+5)}.$$

1.3 The measure of the splitting

In Section 1.1 we proved the existence of (local) invariant perturbed manifolds for each one of the invariant whiskered tori $T_{\alpha_1,\alpha_2} = \left\{(x, y, \hat{I}, \hat{\theta}) : \ x = y = 0, I_i = \alpha_i, \ i = 1, 2\right\}$ associated to the system given in (1.1.6). Furthermore, by means of Lemma 1.1.11, we have shown that these invariant manifolds can be parameterized in the $(x, y, \hat{I}, \hat{\theta})$ variables by

$$(\hat{\psi}, t, s) \in \mathcal{B}_1'' \times (-\infty, -T_0 - \text{Re } s] \times \mathcal{C}_1 \to \left(x^-(\hat{\psi}, t, s), y^-(\hat{\psi}, t, s), \hat{I}^-(\hat{\psi}, t, s), \hat{\theta}^0(\hat{\psi}, t)\right)$$

$$(\hat{\psi}, t, s) \in \mathcal{B}_1'' \times [T_0 - \text{Re } s, \infty) \times \mathcal{C}_1 \to \left(x^+(\hat{\psi}, t, s), y^+(\hat{\psi}, t, s), \hat{I}^+(\hat{\psi}, t, s), \hat{\theta}^0(\hat{\psi}, t)\right)$$

where $\hat{\psi} = (\psi_1, \psi_2)$, $\hat{I}(\hat{\psi}, t, s) = (I_1(\hat{\psi}, t, s), I_2(\hat{\psi}, t, s))$ and $\hat{\theta}^0(\hat{\psi}, t) = (\theta_1^0(\psi_1, t), \theta_2^0(\psi_2, t))$ $= (\psi_1 + \varepsilon^{-1}t, \psi_2 + \beta\varepsilon^{-1}t)$. In Section 1.1 we also stated the Extension Theorem I which implies that, if $s \in \mathcal{C}_1'$, with (see (1.1.36))

$$\mathcal{C}_1' = \left\{s \in \mathbb{C} : \ |\text{Im } s| \leq \frac{\pi}{2\sqrt{A}} - \varepsilon^b\right\},$$

then the trajectories of the perturbed system with initial conditions in the local perturbed unstable manifold remain close enough to the homoclinic manifold of the unperturbed system during a sufficiently large period of time in order to guarantee that they come back to the domain where the coordinates (q, p) are defined. Therefore, we may also consider the piece of unstable manifold

$$(\hat{\psi}, t, s) \in \mathcal{B}_1'' \times [T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}_1' \to \left(x^-(\hat{\psi}, t, s), y^-(\hat{\psi}, t, s), \hat{I}^-(\hat{\psi}, t, s), \hat{\theta}^0(\hat{\psi}, t)\right)$$

which is contained, just as the two ones above, in the neighbourhood V of $x = y = 0$ where the analytic change of variables $(q, p) = \varphi(x, y)$ (see, once more, (1.1.10) and Lemma 1.1.4) is defined.

Hence, we may consider the parameterizations

$$\begin{aligned}\left(q^*(\hat{\psi}, t, s), p^*(\hat{\psi}, t, s), \hat{I}^*(\hat{\psi}, t, s), \hat{\theta}^0(\hat{\psi}, t)\right) &= \\ = \left(\varphi(x^*(\hat{\psi}, t, s), y^*(\hat{\psi}, t, s)), \hat{I}^*(\hat{\psi}, t, s), \hat{\theta}^0(\hat{\psi}, t)\right)&\end{aligned}$$

defined on $\mathcal{B}_1'' \times [T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}_1'$, where $*$ stands for $+$ or $-$.

We also recall the parameterization of the piece of unperturbed separatrix

$$(\hat{\psi}, t, s) \in \mathcal{B}_1'' \times [T_0 - \text{Re } s, \infty) \times \mathcal{C}_1 \to \left(q^0(t + s), p^0(t + s), \hat{I}^0(t + s), \hat{\theta}^0(\hat{\psi}, t)\right)$$

given in (1.1.25) in order to point out that the second statement of Lemma 1.1.11 and the Extension Theorem I respectively imply that

$$\begin{aligned}\left\|(q^+, p^+, \hat{I}^+, \hat{\theta}^0) - (q^0, p^0, \hat{I}^0, \hat{\theta}^0)\right\| &< ctant \ \mu\varepsilon^{-b(N+3)}, \\ \left\|(q^-, p^-, \hat{I}^-, \hat{\theta}^0) - (q^0, p^0, \hat{I}^0, \hat{\theta}^0)\right\| &< ctant \ \mu\varepsilon^{-b(N+5)},\end{aligned} \tag{1.3.55}$$

where both norms are evaluated on the domain $\mathcal{B}_1'' \times [T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}_1'$.

Therefore, for every $(\hat{\psi}, t, s) \in \mathcal{B}_1'' \times [T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}_1'$, we may assume $\left| q^*(\hat{\psi}, t, s) \right| < \sigma$, where σ is the constant introduced in Lemma 1.1.4. Let us also choose $\sigma' \in (0, \sigma)$ satisfying $\sigma' < \left| q^*(\hat{\psi}, t, s) \right|$ for every $(\hat{\psi}, t, s)$ in the above domain.

On the other hand, we recall that, up to now we do not need to impose any condition on the real part of the complex parameter s. Nevertheless, for proving the next result (Lemma 1.3.1) we must assume the real part of s to be sufficiently small. This is the reason why, from now on, we redefine the domain $\mathcal{C}_1'$ in order to write

$$\mathcal{C}_1' = \left\{ s \in \mathbb{C} : \ |\text{Re } s| \leq \varepsilon, \ |\text{Im } s| \leq \frac{\pi}{2\sqrt{A}} - \varepsilon^b \right\}. \tag{1.3.56}$$

The Main Theorem I, see Theorem 0.0.2, gives estimates for the splitting between the unstable and the local stable manifolds along one homoclinic orbit associated to the invariant perturbed tori T_{α_1,α_2}.

For proving this main result we need to define some special coordinates (which are going to be called flow-box coordinates) in some domain $\mathcal{U} = \mathcal{U}(\mu, \varepsilon)$ containing the pieces of invariant manifolds

$$(\hat{\psi}, t, s) \in \mathcal{B}_1'' \times [T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}_1' \to \left(q^*(\hat{\psi}, t, s), p^*(\hat{\psi}, t, s), \hat{I}^*(\hat{\psi}, t, s), \hat{\theta}^0(\hat{\psi}, t) \right)$$

where $*$ stands for $+$ or $-$.

We need the change of variables transforming the old coordinates $(q, p, \hat{I}, \hat{\theta})$ into the flow-box ones to be holomorphic. This is the reason why (see Remark 1.3.2) we need to introduce s as a new spatial variable and consider (independently of whether $\mu = 0$ or not) $\dot{s} = 0$ as a new equation of motion, see also Remark 1.1.10 where those considerations were already announced.

In this context, the above pieces of invariant manifolds are extended to the following ones

$$(\hat{\psi}, t, s) \in \mathcal{B}_1'' \times [T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}_1' \to Q^*(\hat{\psi}, t, s), \tag{1.3.57}$$

where, from now on, we take the notation

$$Q^*(\hat{\psi}, t, s) = \left(q^*(\hat{\psi}, t, s), p^*(\hat{\psi}, t, s), \hat{I}^*(\hat{\psi}, t, s), \hat{\theta}^0(\hat{\psi}, t), s \right).$$

On the other hand, from (1.1.25), it is clear that, by denoting

$$q^0(t+s) = e^{-\sqrt{A}(t+s)} = q_1^0(t+s) + \sqrt{-1}\, q_2^0(t+s),$$

we get

$$\frac{q_2^0(t+s)}{q_1^0(t+s)} = \tan(-\sqrt{A}\text{Im } s).$$

This means that the local stable manifold of the unperturbed system is confined to a very specific region in such a way that, using (1.3.55), we guarantee that both above considered pieces of perturbed manifolds are contained in (recall that $p^0 \equiv 0$)

$$\begin{aligned} \mathcal{U}' &= \mathcal{U}'(\mu,\varepsilon) = \Big\{ (q,p,\hat{I},\hat{\theta},s) : \ \sigma' < |q| < \sigma, \ \left| \arctan \frac{q_2}{q_1} + \sqrt{A} \mathrm{Im}\ s \right| < \mu \varepsilon^{-b(N+7)}, \\ & \qquad |p| < \mu\varepsilon^{-b(N+7)}, \ |\mathrm{Im}\ \theta_i| < -\ln(a\varepsilon^p) - 2\varepsilon^b, \ i = 1,2, \ s \in \mathcal{C}'_1 \Big\}, \end{aligned}$$

where we have denoted $q = q_1 + \sqrt{-1} q_2$.

We want to point out that the condition

$$\left| \arctan \frac{q_2}{q_1} + \sqrt{A} \mathrm{Im}\ s \right| < \mu \varepsilon^{-b(N+7)}$$

could be replaced by $|q - q^0(t+s)| < \mu\varepsilon^{-b(N+7)}$. Nevertheless (see equation (1.3.67)), only the first one will be used in a direct way.

The above mentioned flow-box coordinates will be defined on

$$\begin{aligned} \mathcal{U} &= \mathcal{U}(\mu,\varepsilon) = \Big\{ (q,p,\hat{I},\hat{\theta},s) : \ \sigma' < |q| < \sigma_2, \ \left| \arctan \frac{q_2}{q_1} + \sqrt{A} \mathrm{Im}\ s \right| < \mu \varepsilon^{-b(N+6)}, \\ & \quad |p| < \mu\varepsilon^{-b(N+6)}, \ |\mathrm{Im}\ \theta_i| \le -\ln(a\varepsilon^p) - 3\varepsilon^b, \ i = 1,2, \ s \in \mathcal{C}'_1 \Big\} \subset \mathcal{U}', \end{aligned} \tag{1.3.58}$$

by means of Lemma 1.3.1. The constant $\sigma_2 \in (\sigma', \sigma)$ taking part in the definition of $\mathcal{U}$ is taken to verify

$$\left(\frac{\sigma_2}{\sigma'} \right)^3 \sigma_2 < \frac{\sigma' + \sigma}{2}. \tag{1.3.59}$$

On the other hand, from now on, we will restrict the variation of the initial phases (ψ_1, ψ_2) to the complex strip

$$\mathcal{B}'''_1 = \left\{ (\psi_1,\psi_2) \in \mathbb{C}^2 : \ |\mathrm{Im}\ \psi_i| \le -\ln(a\varepsilon^p) - 3\varepsilon^b, \ i = 1,2 \right\}$$

in such a way that the pieces of invariant manifolds

$$(\hat{\psi}, t, s) \in \mathcal{B}'''_1 \times [T_0 - \mathrm{Re}\ s, 2T_0 - \mathrm{Re}\ s] \times \mathcal{C}'_1 \to Q^*(\hat{\psi}, t, s)$$

are contained in $\mathcal{U}$.

In order to state Lemma 1.3.1, let us denote by $Q = (q,p,\hat{I},\hat{\theta},s)$ the points in $\mathcal{U}'$ and take, for $i = 1,2$, the functions

$$\mathcal{J}_i^0(Q) = I_i$$

and

$$\mathcal{K}^0(Q) = \mathcal{K}^0(q,p) = \tilde{H}(q,p) = -\sqrt{A}\,(pq + F(pq)) \tag{1.3.60}$$

the Hamiltonian given in Lemma 1.1.4.

Let us also consider the analytic function $\mathcal{S}^0$ defined on $\mathcal{U}'$ by

$$\mathcal{S}^0(Q) = -\frac{\ln q}{\sqrt{A}(1+F_J)},$$

where $F_J = F'(J)$, $J = pq$, was introduced in the definition of the perturbed system (1.1.11).

Therefore, one may check easily that the analytic (holomorphic) change of variables

$$V_0 : Q \in \mathcal{U}' \to \Big(\mathcal{S}^0(Q), \mathcal{K}^0(Q), \mathcal{J}_1^0(Q), \mathcal{J}_2^0(Q), \hat{\theta}, s\Big)$$

transforms the unperturbed vector field

$$\dot{Q} = g_0(Q) = \Big(-\sqrt{A}q(1+F_J), \sqrt{A}p(1+F_J), 0, 0, \frac{1}{\varepsilon}, \frac{\beta}{\varepsilon}, 0\Big)$$

into the flow-box system

$$\dot{\mathcal{S}}^0 = 1, \quad \dot{\mathcal{K}}^0 = 0, \quad \dot{\mathcal{J}}_1^0 = 0, \quad \dot{\mathcal{J}}_2^0 = 0, \quad \dot{\hat{\theta}} = \tilde{\omega}, \quad \dot{s} = 0,$$

where $\tilde{\omega} = (\varepsilon^{-1}, \beta\varepsilon^{-1})$.

Now, let us take the parametric family of perturbed systems

$$\dot{Q} = g_\mu(Q)$$

each one of them formed by the six equations given at (1.1.11) together with the new one $\dot{s} = 0$.

The next result ensures that, for each sufficiently small μ, we may find a complex analytic (holomorphic) change of variables transforming the vector field $\dot{Q} = g_\mu(Q)$ into a flow-box system and, moreover, that these changes depend in a continuous way on μ.

Lemma 1.3.1 (Flow-box coordinates) *For every $\mu \in (0, \varepsilon^m)$ with $m > b(N+7)+1$, there exists an analytic change of variables*

$$V_\mu : Q \in \mathcal{U} \to \Big(\mathcal{S}^\mu(Q), \mathcal{K}^\mu(Q), \mathcal{J}_1^\mu(Q), \mathcal{J}_2^\mu(Q), \hat{\theta}, s\Big)$$

transforming the vector field $\dot{Q} = g_\mu(Q)$ into the flow-box system

$$\dot{\mathcal{S}}^\mu = 1, \quad \dot{\mathcal{K}}^\mu = 0, \quad \dot{\mathcal{J}}_1^\mu = 0, \quad \dot{\mathcal{J}}_2^\mu = 0, \quad \dot{\hat{\theta}} = \tilde{\omega}, \quad \dot{s} = 0.$$

Moreover,

$$\|V_\mu - V_0\|_{\mathcal{U}} \leq ctant\ \mu\varepsilon^{-b(N+3)}.$$

Proof

Let us denote, for every μ small enough, by $\phi(t, Q, \mu)$ the flow associated to the vector field $\dot{Q} = g_\mu(Q)$. During the proof of the lemma we need to consider the flows $\phi(\cdot, Q, \mu)$ defined for complex values of time $T = t + \sqrt{-1}t'$. Of course, by taking into account that we are dealing with analytic vector fields, we may write

$$\phi(T, Q, \mu) = \phi\left(\sqrt{-1}t', \phi(t, Q, \mu), \mu\right).$$

Therefore, we must only discuss what the flow associated to the differential equation $\dot{Q} = g_\mu(Q)$ means for pure imaginary values of time. However, it is clear that, taking $v = \sqrt{-1}t$, $t \in \mathbb{R}$, then $\tilde{x} = \tilde{x}(v)$ is a solution of

$$\frac{dx}{dv} = f(x(v)),$$

whenever $\tilde{x}(v) = \overline{y}(v/\sqrt{-1})$ with $\overline{y} = \overline{y}(t)$ a solution of

$$\frac{dy}{dt} = \sqrt{-1}f(y(t)).$$

This means that, in order to compute solutions for pure imaginary values of time, we must solve (in a standard way) the respective vector field $\dot{Q} = \sqrt{-1}g_\mu(Q)$. Therefore, by denoting $\Psi(t, Q, \mu)$ the flow associated to this last family of differential equations, we deduce that, if $T = t + \sqrt{-1}t'$, then

$$\phi(T, Q, \mu) = \Psi(t', \phi(t, Q, \mu), \mu).$$

The global strategy directed to find an analytic change V_μ satisfying the required properties is based on the existence, for every μ small enough, of a complex analytic (holomorphic) conjugation between the vector fields $\dot{Q} = g_\mu(Q)$ and $\dot{Q} = g_0(Q)$. Since this conjugation must be analytic, it will be constructed by using an analytic (complex) time T^0 defined on $\mathcal{U}(\mu, \varepsilon)$. Of course, there exists an standard way to produce conjugations (defined in $\mathcal{U}$) between the vector fields $\dot{Q} = g_\mu(Q)$ and $\dot{Q} = g_0(Q)$, but this standard method uses real (non-constant) time functions and therefore it is not enough for our purposes.

More concretely, let us define the family of analytic maps

$$h^*_\mu : \mathcal{U} \to h^*_\mu(\mathcal{U})$$

by the formula

$$\phi(-T^0(Q), Q, 0) = \phi(-T^0(Q), h^*_\mu(Q), \mu), \tag{1.3.61}$$

where

$$T^0(Q) = -\frac{\ln(q/\sigma_2)}{\sqrt{A}(1 + F_J)} - s, \tag{1.3.62}$$

with σ_2 the constant used to define $\mathcal{U}$ (see (1.3.58)).

We point out that, from the choice of the definition domain $\mathcal{U}$, the imaginary part of the time-function $T^0(Q)$ will be much smaller (see (1.3.68)) than ε. This is one of the needed properties for the function T^0, because the imaginary part of the variables θ_1 and θ_2 move extremely fast (recall the equations $\dot\theta_1 = \varepsilon^{-1}$, $\dot\theta_2 = \beta\varepsilon^{-1}$) by the flow of the system $\dot Q = \sqrt{-1}g_0(Q)$ (observe that, now, $\dot\theta_1 = \sqrt{-1}\varepsilon^{-1}$ and $\dot\theta_2 = \sqrt{-1}\beta\varepsilon^{-1}$ are equations of such system). Nevertheless, by using the above time-function $T^0(Q)$ we will be able to prove that, if $Q \in \mathcal{U}$ (see (1.3.58)), then $\phi(-t, Q, 0) \in \mathcal{U}'$, for every $t \in [0, \mathrm{Re}\, T^0(Q)]^*$ (here we take the notation $[a, b]^* = [a, b]$, if $a \le b$; $[a, b]^* = [b, a]$, if $b \le a$) and $\Psi(-t', \phi(-\mathrm{Re}\, T^0(Q), Q, 0), 0) \in \mathcal{U}'$, for every $t' \in [0, \mathrm{Im}\, T^0(Q)]^*$. These properties (which also hold true when replacing 0 by μ, μ sufficiently small) will be essential for applying Gronwall's results to get the required estimate

$$\|V_\mu - V_0\|_{\mathcal{U}} \le ctant\ \mu\varepsilon^{-b(N+3)}.$$

Moreover, we also observe that, since

$$T^0(\phi(t, Q, 0)) = T^0(Q) + t \tag{1.3.63}$$

for any real value of t, then

$$\begin{aligned}
&\phi(-T^0(Q) - t, \phi(t, h_\mu^*(Q), \mu), \mu) = \phi(-T^0(Q), h_\mu^*(Q), \mu) = \phi(-T^0(Q), Q, 0) = \\
&= \phi(-T^0(Q) - t, \phi(t, Q, 0), 0) = \phi\left(-T^0(\phi(t, Q, 0)), \phi(t, Q, 0), 0\right) = \\
&= \phi\left(-T^0(\phi(t, Q, 0)), h_\mu^*(\phi(t, Q, 0)), \mu\right) = \phi\left(-T^0(Q) - t, h_\mu^*(\phi(t, Q, 0)), \mu\right).
\end{aligned}$$

Hence, we deduce that

$$\phi(t, h_\mu^*(Q), \mu) = h_\mu^*(\phi(t, Q, 0)).$$

This means that h_μ^* is a family of analytic conjugations between the vector fields $\dot Q = g_0(Q)$ and $\dot Q = g_\mu(Q)$.

Now, we want to apply Gronwall's estimates to deduce that

$$\left\|h_\mu^* - I\right\|_{\mathcal{U}} \le ctant\ \mu\varepsilon^{-b(N+3)}. \tag{1.3.64}$$

To this end, we begin by proving that $\phi(-t, Q, 0) \in \mathcal{U}'$, for every $Q \in \mathcal{U}$ and every $t \in [0, T_1(Q)]^*$, where we have taken the notation

$$T^0(Q) = T_1(Q) + \sqrt{-1}T_2(Q).$$

This can be done by taking into account the following considerations: Since $F_J = F'(J)$ and, see Lemma 1.1.4, $F(J) = \mathrm{O}(J^2)$, we may write

$$|F_J| \le ctant\ |p|\,|q| < ctant\ \mu\varepsilon^{-b(N+6)}. \tag{1.3.65}$$

Then, for ε (and μ) small enough, we have (recall the new definition of $\mathcal{C}_1'$ given in (1.3.56))

$$|T_1(Q)| < \frac{2}{\sqrt{A}} \ln \frac{\sigma_2}{\sigma'}. \tag{1.3.66}$$

On the other hand, the equation $\dot{q} = -\sqrt{A}q(1+F_J)$ implies that $u(t) = |q(t)|$ is solution of

$$\dot{u} = -\sqrt{A}(1 + \text{Re } F_J)u$$

and therefore, for every $Q \in \mathcal{U}$ and every $t \in [0, T_1(Q)]^*$, we obtain

$$|q(-t)| < \left(\frac{\sigma_2}{\sigma'}\right)^{2(1+\text{Re } F_J)} \sigma_2 < \left(\frac{\sigma_2}{\sigma'}\right)^3 \sigma_2.$$

Thus, it is clear that (1.3.59) implies the required property: $\phi(-t, Q, 0) \in \mathcal{U}'$ whenever $Q \in \mathcal{U}$ and $t \in [0, T_1(Q)]^*$.

Now we are going to prove that $\Psi(-t', \phi(-T_1(Q), Q, 0), 0) \in \mathcal{U}'$ whenever $Q \in \mathcal{U}$ and $t' \in [0, T_2(Q)]^*$.

To this end, we begin by observing that the definition of the time-function

$$T^0(Q) = -\frac{\ln(q/\sigma_2)}{\sqrt{A}(1 + F_J)} - s = T_1(Q) + \sqrt{-1}T_2(Q)$$

together with (1.3.65) give

$$\left| T_2(Q) + \frac{1}{\sqrt{A}} \arctan \frac{q_2}{q_1} + \text{Im } s \right| \leq ctant\ \mu\varepsilon^{-b(N+6)}. \tag{1.3.67}$$

Thus, the definition of $\mathcal{U}$ implies that

$$|T_2(Q)| \leq ctant\ \mu\varepsilon^{-b(N+6)}. \tag{1.3.68}$$

Remark 1.3.2 *Without using the* variable s *we are not able to find an analytic complex time-function* $T^0 = T^0(Q)$ *satisfying, see (1.3.63),*

$$T^0(\phi(t, Q, 0)) = T^0(Q) + t$$

and also, see (1.3.68),

$$\left|\text{Im } T^0(Q)\right| = |T_2(Q)| \leq ctant\ \mu\varepsilon^{-b(N+6)}.$$

These two properties are essential in our proof of Lemma 1.3.1.

Let us continue with the proof of Lemma 1.3.1 by denoting, for every $t' \in [0, T_2(Q)]^*$, by

$$\tilde{\theta}_i(-t') = \theta_i(-t', \phi(-T_1(Q), Q, 0), 0), \quad i = 1, 2$$

the θ_i-component (at time $-t'$) of $\Psi(-t', \phi(-T_1(Q), Q, 0), 0)$. Then, the equations

$$\dot{\theta}_1 = \sqrt{-1}\varepsilon^{-1}, \quad \dot{\theta}_2 = \sqrt{-1}\beta\varepsilon^{-1}$$

of the system $\dot{Q} = \sqrt{-1}g_0(Q)$, together with (1.3.68), allow us to write

$$\left|\text{Im } \tilde{\theta}_i(-t') - \text{Im } \tilde{\theta}_i(0)\right| \leq ctant\ \mu\varepsilon^{-b(N+6)-1}, \tag{1.3.69}$$

for every $t' \in [0, T_2(Q)]^*$.

Now, recalling the notation $Q = (q, p, \hat{I}, \hat{\theta}, s)$ and bearing in mind that, for $i = 1, 2$,

$$\text{Im } \tilde{\theta}_i(0) = \text{Im } \theta_i(0, \phi(-T_1(Q), Q, 0), 0) = \text{Im } \theta_i,$$

we may use that $Q \in \mathcal{U}$ (and therefore, according to (1.3.58), $\left|\text{Im } \tilde{\theta}_i(0)\right| = |\text{Im } \theta_i| \leq -\ln(a\varepsilon^p) - 3\varepsilon^b$) to conclude from (1.3.69) that

$$\left|\text{Im } \tilde{\theta}_i(-t')\right| \leq -\ln(a\varepsilon^p) - 3\varepsilon^b + ctant\ \mu\varepsilon^{-b(N+6)-1} \leq -\ln(a\varepsilon^p) - 2\varepsilon^b,$$

whenever $\mu \in (0, \varepsilon^m)$ with $m > b(N+7)+1$. Therefore, we already obtain the (second) required property: $\Psi(-t', \phi(-T_1(Q), Q, 0), 0) \in \mathcal{U}'$ whenever $Q \in \mathcal{U}$ and $t' \in [0, T_2(Q)]^*$.

Now, the expression of the perturbed system (1.1.11), together with (1.1.8), allow us to deduce that, for every small enough μ, not only

1. $\phi(-t, Q, \mu) \in \mathcal{U}'$ whenever $Q \in \mathcal{U}$ and $t \in [0, T_1(Q)]^*$
2. $\Psi(-t', \phi(-T_1(Q), Q, \mu), \mu) \in \mathcal{U}'$ whenever $Q \in \mathcal{U}$ and $t' \in [0, T_2(Q)]^*$

but also (1.3.64) holds true.

Therefore, Lemma 1.1.2 implies $\left\|Dh^*_\mu - I\right\|_{\mathcal{U}} \leq ctant\ \mu\varepsilon^{-b(N+4)}$. Hence, we may construct a family $h_\mu = (h^*_\mu)^{-1}$ of analytic conjugations between the systems $\dot{Q} = g_\mu(Q)$ and $\dot{Q} = g_0(Q)$. Then, we also have

$$\|h_\mu - I\|_{\mathcal{U}} \leq ctant\ \mu\varepsilon^{-b(N+3)} \tag{1.3.70}$$

and

$$\|Dh_\mu - I\|_{\mathcal{U}} \leq ctant\ \mu\varepsilon^{-b(N+4)}. \tag{1.3.71}$$

Let us define

$$\mathcal{K}^\mu(Q) = \mathcal{K}^0(h_\mu(Q)), \quad \mathcal{J}^\mu_i(Q) = \mathcal{J}^0_i(h_\mu(Q)) \quad \text{and} \quad \mathcal{S}^\mu(Q) = \mathcal{S}^0(h_\mu(Q)). \tag{1.3.72}$$

Thus, in the new coordinates

$$V_\mu(Q) = \left(\mathcal{S}^\mu(Q), \mathcal{K}^\mu(Q), \mathcal{J}^\mu_1(Q), \mathcal{J}^\mu_2(Q), \hat{\theta}, s\right),$$

our perturbed system takes the expression

$$\frac{d}{dt}\mathcal{K}^\mu = \frac{d}{dt}\mathcal{J}^\mu_i = 0, \quad \frac{d}{dt}\mathcal{S}^\mu = 1, \quad \frac{d}{dt}\hat{\theta} = \tilde{\omega}, \quad \frac{d}{dt}s = 0.$$

Moreover, from (1.3.70), it also follows that

$$\left|\mathcal{K}^\mu(Q) - \mathcal{K}^0(Q)\right| = \left|\mathcal{K}^0(h_\mu(Q)) - \mathcal{K}^0(Q)\right| \leq ctant\ \mu\varepsilon^{-b(N+3)}$$

and, taking into account how the domain $\mathcal{U}$ was defined (the q-coordinate is far from being zero), we also obtain

$$\left|\mathcal{J}_i^\mu(Q) - \mathcal{J}_i^0(Q)\right| \le ctant\ \mu\varepsilon^{-b(N+3)} \quad \text{and} \quad \left|\mathcal{S}^\mu(Q) - \mathcal{S}^0(Q)\right| \le ctant\ \mu\varepsilon^{-b(N+3)}.$$

Consequently,

$$\|V_\mu - V_0\|_{\mathcal{U}} \le ctant\ \mu\varepsilon^{-b(N+3)}.$$

Therefore, Lemma 1.3.1 is proved. □

Once the flow-box coordinates are defined in $\mathcal{U}$, we can evaluate them on the pieces of the invariant manifolds

$$(\hat{\psi}, t, s) \in \mathcal{B}_1''' \times [T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}_1' \to Q^*(\hat{\psi}, t, s),$$

giving rise to four functions (the first three ones called splitting functions)

$$\begin{aligned} \mathcal{K}_u^\mu(s, \psi_1, \psi_2) &= \mathcal{K}^\mu(Q^-(\hat{\psi}, t, s)) - \mathcal{K}^\mu(Q^+(\hat{\psi}, t, s)) \\ \mathcal{J}_{i,u}^\mu(s, \psi_1, \psi_2) &= \mathcal{J}_i^\mu(Q^-(\hat{\psi}, t, s)) - \mathcal{J}_i^\mu(Q^+(\hat{\psi}, t, s)), \quad i = 1, 2 \\ \mathcal{S}_u^\mu(s, \psi_1, \psi_2) &= \mathcal{S}^\mu(Q^-(\hat{\psi}, t, s)) - t \end{aligned} \tag{1.3.73}$$

defined on $\mathcal{C}_1' \times \mathcal{B}_1'''$.

Remark 1.3.3 *By definition of $\mathcal{K}^\mu$, we have*

$$\mathcal{K}^\mu(Q^-(\hat{\psi}, t, s)) - \mathcal{K}^\mu(Q^+(\hat{\psi}, t, s)) = \mathcal{K}^0(h_\mu(Q^-(\hat{\psi}, t, s))) - \mathcal{K}^0(h_\mu(Q^+(\hat{\psi}, t, s))).$$

Then, once $(s, \hat{\psi}) \in \mathcal{C}_1' \times \mathcal{B}_1'''$ is fixed, since $h_\mu(Q^-(\hat{\psi}, t, s))$ and $h_\mu(Q^+(\hat{\psi}, t, s))$ are orbits of the flow-box system (due to the fact that $Q^-(\hat{\psi}, t, s)$ and $Q^+(\hat{\psi}, t, s)$ are orbits of the system $\dot{Q} = g_\mu(Q)$ formed by the equations in (1.1.11) together with the new one $\dot{s} = 0$) we conclude that (recall the equation $\dot{\mathcal{K}}^0 = 0$) the function $\mathcal{K}_u^\mu$ does not depend on t. The same holds for the functions $\mathcal{J}_{i,u}^\mu$, $i = 1, 2$ and $\mathcal{S}_u^\mu$. In other words, we are able to define the above four functions only on variables $(s, \hat{\psi})$ (by skipping the dependence on t of the right-hand side terms of equations (1.3.73)) because those functions $\mathcal{K}_u^\mu$, $\mathcal{J}_{i,u}^\mu$ and $\mathcal{S}_u^\mu$ remain constant along the orbits of the flow-box system.

To prove the Main Theorem I it will be necessary to show that the functions $\mathcal{K}_u^\mu$ and $\mathcal{J}_{i,u}^\mu$ essentially coincide with the renormalized Melnikov functions $\mathcal{M}_j$, $j = 1, 2, 3$, explicitly obtained in Section 1.2. Moreover, the proof of the Main Theorem I strongly depends also on three more lemmas: Lemma 1.3.4 is a standard result related to the bounds (in a complex strip) for the Fourier coefficients of some kind of analytic functions. Lemma 1.3.5 states that, for a fixed $(\psi_1, \psi_2) \in \mathcal{B}_1'''$, the function $U(s) = \mathcal{S}_u^\mu(s, \psi_1, \psi_2)$ defined in (1.3.73) is invertible. This lemma will be used to establish a natural geometrical way for measuring the splitting functions. Finally, Lemma 1.3.10 deals with the leading order behaviour of some specific series. It will be used to conclude the proof of the Main Theorem by establishing the existence of two dominant terms in the numerical series

$$\frac{\partial \overline{\mathcal{M}}_i}{\partial \psi_j}(\overline{\psi}_1, \overline{\psi}_2), \qquad \left(\overline{\mathcal{M}}_i(\psi_1, \psi_2) = \mathcal{M}_i(0, \psi_1, \psi_2)\right)$$

for $j = 1, 2$, where $\mathcal{M}_i = \mathcal{M}_i(s, \psi_1, \psi_2)$, $i = 1, 2, 3$ are the renormalized Melnikov functions computed in Section 1.2 and $\overline{\psi}_i = \overline{\psi}_i(\varepsilon)$ represents an adequate homoclinic orbit (selected by means of Lemma 1.3.13). This control on the above series will allow us to obtain estimates for the transversality (see (1.3.103)) of the splittings. The proof of Lemma 1.3.10 will be developed in Chapter 4.

For every positive constants ρ, ρ_1 and ρ_2 let us define

$$D(\overline{\rho}, \rho, \rho_1, \rho_2) = \left\{(S, \psi_1, \psi_2) \in \mathbb{C}^3;\ |\text{Re } S| \le \overline{\rho},\ |\text{Im } S| \le \rho, |\text{Im } \psi_i| \le \rho_i,\ i = 1, 2\right\}. \tag{1.3.74}$$

Let us fix $\beta > 0$. For every positive constant ρ^* let us denote by $\mathcal{A}(\overline{\rho}, \rho, \rho_1, \rho_2, \rho^*)$ the set of analytic functions

$$G = G(S, \psi_1, \psi_2) = F\left(\psi_1 - \frac{S}{\varepsilon}, \psi_2 - \frac{\beta S}{\varepsilon}\right) \tag{1.3.75}$$

defined on $D(\overline{\rho}, \rho, \rho_1, \rho_2)$, which are 2π-periodic in (ψ_1, ψ_2) and satisfy

$$\|G\| = \sup_{(S, \psi_1, \psi_2) \in D(\rho, \rho_1, \rho_2)} |G(S, \psi_1, \psi_2)| \le \rho^*.$$

Let us observe that, from (1.2.48), (1.2.54) and Lemma 1.2.3, we deduce

$$\mathcal{M}_i \in \mathcal{A}\left(\overline{\rho}, \frac{\pi}{2\sqrt{A}} - \varepsilon^b, -\ln(a\varepsilon^p) - 2\varepsilon^b, -\ln(a\varepsilon^p) - 2\varepsilon^b, ctant\ \mu\varepsilon^{-b(N+5)}\right), \tag{1.3.76}$$

for $i = 1, 2, 3$ and any arbitrary positive constant $\overline{\rho}$.

If we restrict to the set $\mathbb{R}^3$ the definition domain of one arbitrary function $G \in \mathcal{A}(\overline{\rho}, \rho, \rho_1, \rho_2, \rho^*)$ and denote

$$\psi_i^* = \psi_i - \frac{\beta^{i-1} S}{\varepsilon},$$

for $i = 1, 2$, then $F = F(\psi_1^*, \psi_2^*)$ is 2π-periodic in (ψ_1^*, ψ_2^*) and we may compute the Fourier coefficients

$$F_{k_1, k_2} = \frac{1}{4\pi^2} \int_{\mathbb{T}^2} F(\psi_1^*, \psi_2^*) \exp\left(-\sqrt{-1}(k_1\psi_1^* + k_2\psi_2^*)\right) d\psi_1^* d\psi_2^*$$

of F, for every $(k_1, k_2) \in \mathbb{Z}^2$.

Lemma 1.3.4 *If $G \in \mathcal{A}(\rho, \rho_1, \rho_2, \rho^*)$, then*

$$|F_{k_1, k_2}| \le \rho^* \exp\left(-|k_1|\rho_1 - |k_2|\rho_2 - \rho\frac{|k_1 + \beta k_2|}{\varepsilon}\right),$$

for every $(k_1, k_2) \in \mathbb{Z}^2$.

Proof

Let us assume $k_1 > 0$ and $k_2 > 0$. Due to Cauchy's integral formula we get

$$F_{k_1,k_2} = \frac{-1}{4\pi^2} \int_{\mathbb{T}^2} F(\tilde{\psi}_1, \tilde{\psi}_2) \exp\left(-\sqrt{-1}(k_1\tilde{\psi}_1 + k_2\tilde{\psi}_2)\right) d\psi_1^* d\psi_2^*,$$

with

$$\tilde{\psi}_i = \psi_i^* - \sqrt{-1}\left(\rho_i + \frac{\beta^{i-1}\rho}{\varepsilon}\right), \quad i = 1, 2.$$

Hence, the lemma follows by taking into account that

$$\left|\exp\left(-\sqrt{-1}(k_1\tilde{\psi}_1 + k_2\tilde{\psi}_2)\right)\right| = \exp\left(-k_1\rho_1 - k_2\rho_2 - \rho\frac{k_1 + \beta k_2}{\varepsilon}\right).$$

The rest of the cases can be treated in the same way by taking

$$\tilde{\psi}_i = \psi_i^* - \sqrt{-1}\left(\frac{k_i}{|k_i|}\rho_i + \frac{k_1 + \beta k_2}{|k_1 + \beta k_2|}\frac{\beta^{i-1}\rho}{\varepsilon}\right), \quad i = 1, 2.$$

□

To state the next result, let us recall the definition of the function

$$(s, \psi_1, \psi_2) \in \mathcal{C}_1' \times \mathcal{B}_1''' \to \mathcal{S}_u^\mu(s, \psi_1, \psi_2) = \mathcal{S}^\mu(Q^-(\hat{\psi}, t, s)) - t$$

given in (1.3.73) and let us introduce the complex subset of $\mathcal{C}_1'$ given by

$$\mathcal{C}_1'' = \left\{ s \in \mathbb{C} : \; |\text{Re } s| \leq \varepsilon, \; |\text{Im } s| \leq \frac{\pi}{2\sqrt{A}} - 2\varepsilon^b \right\}.$$

Lemma 1.3.5 *If $\mu \in (0, \varepsilon^m)$, $m > b(N + 7) + 1$, and once $(\psi_1, \psi_2) \in \mathcal{B}_1'''$ is fixed, the function*

$$U(s) = \mathcal{S}_u^\mu(s, \psi_1, \psi_2)$$

is invertible on $\mathcal{C}_1''$ and its inverse, denoted by $s = U^{-1}(v)$ $(v = U(s))$, satisfies

$$\left|U^{-1}(v) - v\right| \leq ctant\; \mu\varepsilon^{-b(N+5)}.$$

Proof

Let us fix $(\psi_1, \psi_2) \in \mathcal{B}_1'''$ and apply Lemma 1.3.1 to write

$$|U(s) - s| = \left|\mathcal{S}^\mu(Q^-(\hat{\psi}, t, s)) - t - s\right| \leq \left|\mathcal{S}^0(Q^-(\hat{\psi}, t, s)) - t - s\right| + ctant\; \mu\varepsilon^{-b(N+3)}.$$

Moreover, from the Extension Theorem 1.1.14 (see, in particular, (1.3.55)) and the fact that the function

$$\mathcal{S}^0 = \mathcal{S}^0(Q) = -\frac{\ln q}{\sqrt{A}(1 + F_J)}$$

has bounded derivatives in its definition domain $\mathcal{U}$ (see (1.3.58)), we deduce that

$$|U(s)-s| \leq \left|\mathcal{S}^0(Q^0(t+s))-t-s\right| + ctant\ \mu\varepsilon^{-b(N+5)},$$

where

$$Q^0(t+s) = \left(q^0(t+s), p^0(t+s), \hat{I}^0(t+s), \hat{\theta}^0(\hat{\psi},t), s\right)$$

represents the homoclinic separatrix of the unperturbed system.

On the other hand, since (1.1.25) gives

$$q^0(t+s) = e^{-\sqrt{A}(t+s)} \quad \text{and} \quad p^0(t+s) = 0,$$

it follows that $F_J(Q^0(t+s)) \equiv 0$ and thus

$$\mathcal{S}^0(Q^0(t+s)) = t+s.$$

Hence,

$$|U(s)-s| \leq ctant\ \mu\varepsilon^{-b(N+5)}, \tag{1.3.77}$$

for every $s \in \mathcal{C}_1'$. Therefore, bearing in mind that the function U is analytic in $\mathcal{C}_1'$, we may apply Lemma 1.1.2 to get

$$\left|\frac{d}{ds}(U(s))-1\right| \leq ctant\ \mu\varepsilon^{-b(N+6)}$$

for every $s \in \mathcal{C}_1''$. Thus, since $\mu \in (0, \varepsilon^m)$ with $m > b(N+6)$, the function U can be inverted to obtain an analytic function $s = U^{-1}(v)$. Moreover, from (1.3.77),

$$\left|U^{-1}(v)-v\right| \leq ctant\ \mu\varepsilon^{-b(N+5)}.$$

□

Remark 1.3.6 *If we make $s=0$, then the function*

$$v^* = v^*(\psi_1,\psi_2) = U(0) = \mathcal{S}_u^\mu(0,\psi_1,\psi_2) = \mathcal{S}^\mu(Q^-(\hat{\psi},t,0)) - t$$

is real when it is restricted to $\mathbb{R}^2$. This fact will be used at the fourth step of the second part of the proof of the Main Theorem I and it is a consequence of the following considerations:

1. *From Remark 1.1.13 we know that the components of $Q^-(\hat{\psi},t,s)$ are real, whenever $\hat{\psi} \in \mathbb{R}^2$ and $s=0$.*

2. *The time-function*

$$T^0 = T^0(Q) = -\frac{\ln(q/\sigma_2)}{\sqrt{A}(1+F_J)} - s,$$

defined at (1.3.62), is real whenever the components of Q are real and $s=0$.

3. From (1.3.72) we have $\mathcal{S}^{\mu}(Q^{-}(\hat{\psi},t,s)) = \mathcal{S}^0(h_\mu(Q^{-}(\hat{\psi},t,s)))$*, with*

$$\mathcal{S}^0(Q) = -\frac{\ln q}{\sqrt{A}(1+F_J)}, \quad Q = (q,p,\hat{I},\hat{\theta},s)$$

and h_μ *a family of analytic transformations defined by (see (1.3.61) and recall that* $h_\mu = (h_\mu^*)^{-1}$*)*

$$\phi(-T^0(Q), h_\mu(Q), 0) = \phi(-T^0(Q), Q, \mu).$$

Therefore, it is easy to see that, if $T^0(Q)$ *and the components of* Q *are real, then not only the components of* $h_\mu(Q)$ *are real, but also* $\mathcal{S}^0(h_\mu(Q))$ *is real.*

We finish this subsection by introducing the Main Lemma I, which furnishes the leading order terms of some kind of numerical series. We will be considering a series S of the following form

$$S = \sum_{\hat{k}\in\mathbb{Z}^2\setminus\{(0,0)\}} S_{\hat{k}}\hat{\mathcal{E}}_{\hat{k}},$$

where, for $\omega = (1,\beta)$ and every $\hat{k} = (k_1,k_2) \in \mathbb{Z}^2\setminus\{(0,0)\}$,

$$\hat{\mathcal{E}}_{\hat{k}} = \hat{\mathcal{E}}_{\hat{k}}(c,l,d,\varepsilon,\beta) = \exp\left(-\left|\hat{k}\right|(c\left|\ln\varepsilon\right| + l) - \frac{d\left|\hat{k}\omega\right|}{\varepsilon}\right), \tag{1.3.78}$$

with c, l, d, ε and β positive parameters, ε sufficiently small, and $\left|\hat{k}\omega\right| = |k_1 + \beta k_2|$.

Definition 1.3.7 *We say that* $S \in \mathcal{S}_1(c,l,d,\varepsilon,\beta)$ *if the coefficients* $S_{\hat{k}}$ *of* S *do not increase faster than some finite power of* $\left|\hat{k}\right| = |k_1| + |k_2|$*, i.e., there exist positive constants* $\mathcal{X}_1$ *and* W_1 *such that, for every* $\hat{k} \in \mathbb{Z}^2\setminus\{(0,0)\}$*,*

$$|S_{\hat{k}}| \le W_1 \left|\hat{k}\right|^{\mathcal{X}_1}.$$

Definition 1.3.8 *We say that* $S \in \mathcal{S}_2(c,l,d,\varepsilon,\beta)$ *if the coefficients* $S_{\hat{k}}$ *of* S *satisfy the following property: There exist positive constants* $\mathcal{X}_2$ *and* W_2 *such that*

$$|S_{\hat{k}^{(j)}}| \ge W_2 \left|\hat{k}^{(j)}\right|^{-\mathcal{X}_2},$$

whenever $\hat{k}^{(j)} = (k_1^{(j)}, k_2^{(j)})$*,* $k_1^{(j)}/k_2^{(j)}$ *is a best approximation to the golden mean with*

$$k_2^{(j)} \in \left(\varepsilon^{-3/8}, \varepsilon^{-1/2}\left|\ln\varepsilon\right|^{-1/4}\right).$$

Let us point out that, in the appendix of this chapter, we introduce the definition of best approximations as well as basic properties of Continued Fraction Theory.

Definition 1.3.9 *We say that* $S \in \mathcal{S}_3(c,l,d,\varepsilon,\beta)$ *if the coefficients* $S_{\hat{k}}$ *of* S *satisfy* $S_{\hat{k}} = S_{-\hat{k}}$*, for every* $\hat{k} \in \mathbb{Z}^2 \backslash \{(0,0)\}$.

In particular, if $S \in \mathcal{S}_3(c,l,d,\varepsilon,\beta)$, then we may write

$$S = 2 \sum_{\hat{k} \in \mathbb{Z}^2_+} S_{\hat{k}} \hat{\mathcal{E}}_{\hat{k}},$$

where

$$\mathbb{Z}^2_+ = \{(k_1,k_2) \in \mathbb{Z}^2 : \ k_2 > 0\} \cup \{(k_1,0) \in \mathbb{Z}^2 : \ k_1 > 0\}.$$

Let us define

$$\mathcal{S}(c,l,d,\varepsilon,\beta) = \bigcap_{i=1}^{3} \mathcal{S}_i(c,l,d,\varepsilon,\beta).$$

Lemma 1.3.10 yields the two leading order terms of any numerical series

$$S = \sum_{\hat{k} \in \mathbb{Z}^2 \backslash \{(0,0)\}} S_{\hat{k}} \hat{\mathcal{E}}_{\hat{k}}$$

in $\mathcal{S}(c,l,d,\varepsilon,\beta)$ also satisfying the extra assumption $S_{\hat{k}(n^0)} S_{\hat{k}(n^1)} > 0$. Those special indices n^0 and n^1 are furnished by the Main Lemma I (they depend, see Remark 4.1.5 for details, on c, l, d and ε but not on the series S) and, if we write $\hat{k}^{(n^\nu)} = (k_1^{(n^\nu)}, k_2^{(n^\nu)})$, then $k_1^{(n^\nu)}/k_2^{(n^\nu)}$ are best approximations to the golden mean, for $\nu = 0,1$.

Let us comment that, for getting the leading terms of the series S, the hypothesis $S \in \mathcal{S}_3(c,l,d,\varepsilon,\beta)$ could be avoided. The property on the symmetry of the coefficients $S_{\hat{k}}$ is not necessary to obtain the leading terms of the whole series. Nevertheless, this hypothesis permits us to work with only two dominant terms (see the statement of Lemma 1.3.10) instead of the four ones which could be necessary in case that the considered series S would not belong to $\mathcal{S}_3(c,l,d,\varepsilon,\beta)$.

In order to state our first Main Lemma, let us recall that by $\mathcal{L}$ we denote the Lebesgue measure on $\mathbb{R}$. Let us also introduce the map

$$F : \varepsilon \in \mathbb{R}^+ \to F(\varepsilon) = \varepsilon \left| \ln \varepsilon \right| .$$

Lemma 1.3.10 (Main Lemma I) *Once three positive constants* c*,* l *and* d *are fixed, there exist some* $\varepsilon_0 \in (0,1)$ *and a real open subset* $\mathcal{U}_\varepsilon \subset (0,\varepsilon_0]$*,* $\mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon(c,l,d)$*, with*

$$ctant\ \varepsilon_0^{8/3} \left| \ln \varepsilon_0 \right|^{3/2} \leq \mathcal{L}(F(\mathcal{U}_\varepsilon)) \leq ctant\ \varepsilon_0^{8/3} \left| \ln \varepsilon_0 \right|^{8/3}$$

satisfying the following property: For every $\varepsilon \in \mathcal{U}_\varepsilon$ *there exist two natural numbers* n^0 *and* n^1*, depending on* ε*,* c*,* d *and* l*, with* $|n^1 - n^0| = 1$*,* $k_1^{(n^0)}/k_2^{(n^0)}$*,* $k_1^{(n^1)}/k_2^{(n^1)}$ *best approximations to the golden mean* $\tilde{\beta} = \dfrac{\sqrt{5}+1}{2}$ *with*

$$ctant\ \varepsilon^{-1/2} \left| \ln \varepsilon \right|^{-1/2} \leq k_j^{(t)} \leq ctant\ \varepsilon^{-1/2} \left| \ln \varepsilon \right|^{-1/2}, \quad j = 1,2 \quad t = n^0, n^1$$

such that, for every β in a golden mean open neighbourhood $I_{\tilde{\beta}} = I_{\tilde{\beta}}(\varepsilon)$ with

$$\frac{1}{100}\varepsilon^{5/3}\left|\ln\varepsilon\right|^{1/2} \leq \mathit{length}(I_{\tilde{\beta}}) \leq \frac{1}{2}\varepsilon^{5/3}\left|\ln\varepsilon\right|^{1/2}$$

and for any numerical series

$$S = \sum_{\hat{k}\in\mathbb{Z}^2\setminus\{(0,0)\}} S_{\hat{k}}\hat{\mathcal{E}}_{\hat{k}} \in \mathcal{S}(c,l,d,\varepsilon,\beta)$$

also satisfying

$$S_{\hat{k}(n^0)}S_{\hat{k}(n^1)} > 0,$$

it holds that

$$S = 2\left(S_{\hat{k}(n^0)}\hat{\mathcal{E}}_{\hat{k}(n^0)} + S_{\hat{k}(n^1)}\hat{\mathcal{E}}_{\hat{k}(n^1)}\right)\left[1 + \mathrm{O}\left(\exp\left(-\frac{\left|\ln\varepsilon\right|^{1/5}}{\varepsilon^{1/2}}\right)\right)\right].$$

The proof of the Main Lemma I is given in Chapter 4, where we also prove the following result which will be used in the fourth step of the second part of the Main Theorem proof:

Lemma 1.3.11 (First Perturbing Lemma) *Let c, l and d be positive constants and $\mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon(c,l,d)$ the set of values of ε furnished by the Main Lemma I. Let*

$$S' = \sum_{\hat{k}\in\mathbb{Z}^2\setminus\{(0,0)\}} S_{\hat{k}}\hat{\mathcal{E}}'_{\hat{k}}$$

be a numerical series with

$$S' \in \mathcal{S}(c,l',d',\varepsilon,\beta)$$

for some $\varepsilon \in \mathcal{U}_\varepsilon$, $\beta \in I_{\tilde{\beta}}(\varepsilon)$ and positive parameters l', d' satisfying

$$\max\{\left|l-l'\right|, \left|d-d'\right|\} < \mathit{ctant}\ \varepsilon^{\alpha}$$

for some constant $\alpha > 5/8$. Let n^0 and n^1 be those natural numbers (depending on ε, c, d and l) given by the Main Lemma I. Then, it follows that

$$S' = 2\left(S_{\hat{k}(n^0)}\hat{\mathcal{E}}'_{\hat{k}(n^0)} + S_{\hat{k}(n^1)}\hat{\mathcal{E}}'_{\hat{k}(n^1)}\right)\left[1 + \mathrm{O}\left(\exp\left(-\frac{\left|\ln\varepsilon\right|^{1/5}}{\varepsilon^{1/2}}\right)\right)\right],$$

whenever $S_{\hat{k}(n^0)}S_{\hat{k}(n^1)} > 0$.

Remark 1.3.12 *The open set $\mathcal{U}_\varepsilon$ and the neighbourhood $I_{\tilde{\beta}}$ for which the conclusions of the Main Theorem I (see Theorem 0.0.2) are valid coincide with those ones given in the statement of the Main Lemma I by choosing $c = p$, $l = -\ln a$ and $d = \dfrac{\pi}{2\sqrt{A}}$. Some* *considerations on the length of the neighbourhood $I_{\tilde{\beta}}$ have to be taken: In fact, one can check that the whole argument directed to prove the Main Lemma I still works when the frequency value β belongs to some neighbourhood $\hat{I}_{\tilde{\beta}}$ of the golden mean value with*

$$ctant\ \varepsilon^{3/2+\alpha_1} \leq length(\hat{I}_{\tilde{\beta}}) \leq ctant\ \varepsilon^{3/2+\alpha_1},$$

where α_1 is any arbitrarily small positive constant.

Nevertheless, in order to extend our results to $\hat{I}_{\tilde{\beta}}$, some extra tedious notation has to be implemented and therefore, we are not going to give the details to achieve this small improvement of the main result.

1.3.1 Proof of the Main Theorem I. First part: Selecting the homoclinic orbit

Let us start the proof of the Main Theorem I by giving an algorithm directed to select the homoclinic orbit of the perturbed system (1.1.6) along which we are going to obtain the required bounds for the transversality of the splitting. The choice of such homoclinic orbit is related to the following facts: In order to get estimates for the transversality of the splitting, we have to apply the Main Lemma I for getting suitable expressions for the numerical series

$$\frac{\partial \overline{\mathcal{M}}_i}{\partial \psi_j}(\overline{\psi}_1, \overline{\psi}_2), \qquad i = 2, 3, \quad j = 1, 2,$$

where $\overline{\mathcal{M}}_i(\psi_1, \psi_2) = \mathcal{M}_i(0, \psi_1, \psi_2)$, $\mathcal{M}_i = \mathcal{M}_i(s, \psi_1, \psi_2)$ the renormalized Melnikov functions obtained in Section 1.2 and $(\overline{\psi}_1, \overline{\psi}_2) \in \{0, \pi\} \times \{0, \pi\}$ giving rise (see Remark 1.1.15) to homoclinic orbits for the perturbed system (1.1.6). We are going to describe how to choose $(\overline{\psi}_1, \overline{\psi}_2)$ in $\{0, \pi\} \times \{0, \pi\}$ in order that the Main Lemma I (see Lemma 1.3.10) could be applied to obtain an asymptotic formula to the series

$$\frac{\partial \overline{\mathcal{M}}_i}{\partial \psi_j}(\overline{\psi}_1, \overline{\psi}_2), \qquad i = 2, 3, \quad j = 1, 2.$$

Let us recall that, according to (1.2.48), the functions $\overline{\mathcal{M}}_i$ can be written in the following way

$$\overline{\mathcal{M}}_i(\psi_1, \psi_2) = \sum_{\hat{k} \in \Lambda} M_{\hat{k}}^{(j+1)} \sin(k_1\psi_1 + k_2\psi_2),\ i = 2, 3.$$

Furthermore, using (1.2.51) and (1.2.53), we obtain, for $i = 1, 2$, that

$$\overline{\mathcal{M}}_i(\psi_1, \psi_2) = -2\mu\varepsilon^{-2}\pi A^{-1} \sum_{\hat{k} \in \Lambda} \tilde{B}_{\hat{k}}^{(i)} \mathcal{E}_{\hat{k}} \sin(k_1\psi_1 + k_2\psi_2),$$

where

$$\tilde{B}_{\hat{k}}^{(i)} = \left|\hat{k}\omega\right| H(\hat{k}\omega) C_{\hat{k}}^{(i)}(\hat{k}\omega) \tag{1.3.79}$$

and, from (1.2.46), for $i = 2, 3$

$$C^{(i)}_{\hat{k}} = \frac{k_{i-1}}{f(\hat{k})}. \tag{1.3.80}$$

Therefore, we have, for $j = 1, 2$,

$$\frac{\partial \overline{\mathcal{M}}_i}{\partial \psi_j}(\overline{\psi}_1, \overline{\psi}_2) = -2\mu\varepsilon^{-2}\pi A^{-1} \sum_{\hat{k}\in\Lambda} k_j \tilde{B}^{(i)}_{\hat{k}} \mathcal{E}_{\hat{k}} \cos(k_1\overline{\psi}_1 + k_2\overline{\psi}_2).$$

Let us observe that, if we define

$$\hat{k} \in \mathbb{Z}^2 \backslash \{(0,0)\} \to S_{\hat{k}} = \begin{cases} k_j \tilde{B}^{(i)}_{\hat{k}} \cos(k_1\overline{\psi}_1 + k_2\overline{\psi}_2), & \text{if } \hat{k} \in \Lambda \\ 0, \quad \text{if } \hat{k} \in \mathbb{Z}^2 \backslash (\Lambda \cup \{(0,0)\}) \end{cases} \tag{1.3.81}$$

then we may write

$$\frac{\partial \overline{\mathcal{M}}_i}{\partial \psi_j}(\overline{\psi}_1, \overline{\psi}_2) = -2\mu\varepsilon^{-2}\pi A^{-1} \sum_{\hat{k}\in\mathbb{Z}^2\backslash\{(0,0)\}} S_{\hat{k}} \mathcal{E}_{\hat{k}} \tag{1.3.82}$$

with (see (1.3.78) and compare with (1.2.52))

$$\mathcal{E}_{\hat{k}} = \hat{\mathcal{E}}_{\hat{k}} \left(p, -\ln a, \frac{\pi}{2\sqrt{A}}, \varepsilon, \beta \right). \tag{1.3.83}$$

By taking $c = p$, $l = -\ln a$ and $d = \dfrac{\pi}{2\sqrt{A}}$, the Main Lemma I furnishes an open set of values of ε, $\mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon\left(p, -\ln a, \dfrac{\pi}{2\sqrt{A}}\right)$, and guarantees the existence of two natural numbers n^0 and n^1 (depending only on ε, p and a, see Remark 4.1.5 for details) for which one expects the equality

$$\overline{S} = \sum_{\hat{k}\in\mathbb{Z}^2\backslash\{(0,0)\}} S_{\hat{k}} \mathcal{E}_{\hat{k}} = 2\left(S_{\hat{k}(n^0)}\mathcal{E}_{\hat{k}(n^0)} + S_{\hat{k}(n^1)}\mathcal{E}_{\hat{k}(n^1)}\right)\left[1 + \mathrm{O}\left(\exp\left(-\frac{|\ln\varepsilon|^{1/5}}{\varepsilon^{1/2}}\right)\right)\right] \tag{1.3.84}$$

holds true whenever $\varepsilon \in \mathcal{U}_\varepsilon$ and β belongs to the respective neighbourhood $I_{\tilde{\beta}} = I_{\tilde{\beta}}(\varepsilon)$ of the golden mean value also given by the Main Lemma I.

Nevertheless, for applying the Main Lemma I to get (1.3.84), we must check that, for every $\varepsilon \in \mathcal{U}_\varepsilon\left(p, -\ln a, \dfrac{\pi}{2\sqrt{A}}\right)$ and any $\beta \in I_{\tilde{\beta}}(\varepsilon)$, the following two conditions are satisfied:

$$\overline{S} = \sum_{\hat{k}\in\mathbb{Z}^2\backslash\{(0,0)\}} S_{\hat{k}} \mathcal{E}_{\hat{k}} \in \mathcal{S}\left(p, -\ln a, \frac{\pi}{2\sqrt{A}}, \varepsilon, \beta\right) \quad \text{and} \quad S_{\hat{k}(n^0)} S_{\hat{k}(n^1)} > 0.$$

We now focus on the second condition because the first one can be obtained (see the fifth step of the second part of the proof of the Main Theorem I) independently of the value of the initial phases $\overline{\psi}_1$ and $\overline{\psi}_2$.

Lemma 1.3.13 *For every* $\varepsilon \in \mathcal{U}_\varepsilon\left(p, -\ln a, \dfrac{\pi}{2\sqrt{A}}\right)$ *there exist* $\overline{\psi}_i = \overline{\psi}_i(\varepsilon) \in \{0,\pi\}$, $i = 1, 2$, *such that, for every* $\beta \in I_{\tilde{\beta}}(\varepsilon)$,

$$S_{\hat{k}^{(n^0)}} S_{\hat{k}^{(n^1)}} > 0,$$

where the coefficients $S_{\hat{k}}$ *were defined at (1.3.81) and* n^0, n^1 *are those indices (depending on* ε, p *and* a*) given by the Main Lemma I for the special choice* $c = p$, $l = -\ln a$ *and* $d = \dfrac{\pi}{2\sqrt{A}}$.

Proof

Let us denote $\hat{k}^{(n^0)} = (k_1^{(n^0)}, k_2^{(n^0)})$ and $\hat{k}^{(n^1)} = (k_1^{(n^1)}, k_2^{(n^1)})$. We observe that, according to the Main Lemma I, $k_1^{(n^0)}/k_2^{(n^0)}$ and $k_1^{(n^1)}/k_2^{(n^1)}$ are best approximations to the golden mean satisfying

$$\text{ctant } \varepsilon^{-1/2} |\ln \varepsilon|^{-1/2} \le k_j^{(t)} \le \text{ctant } \varepsilon^{-1/2} |\ln \varepsilon|^{-1/2}, \quad j = 1, 2, \quad t = n^0, n^1.$$

Therefore, taking small enough values of ε, we may assume that

$$\mathcal{N}_i << \varepsilon^{-1/2} |\ln \varepsilon|^{-1/2} \quad \text{for } i = 1, 2, 3$$

where $\mathcal{N}_i$, $i = 1, 2, 3$, are those constants (see (1.0.3)) taking part in the definition of the set of indices Λ. Thus, if $t = n^0$, n^1, then $\hat{k}^{(t)} \in \Lambda$. Therefore, see also the definition of the coefficients $S_{\hat{k}}$ given at (1.3.81), the condition $S_{\hat{k}^{(n^0)}} S_{\hat{k}^{(n^1)}} > 0$ is equivalent to

$$\left(\cos(k_1^{(n^0)}\overline{\psi}_1 + k_2^{(n^0)}\overline{\psi}_2) k_j^{(n^0)} \tilde{B}^{(i)}_{\hat{k}^{(n^0)}}\right) \left(\cos(k_1^{(n^1)}\overline{\psi}_1 + k_2^{(n^1)}\overline{\psi}_2) k_j^{(n^1)} \tilde{B}^{(i)}_{\hat{k}^{(n^1)}}\right) > 0$$

or, in other words,

$$\text{sign}\left\{\cos(k_1^{(n^0)}\overline{\psi}_1 + k_2^{(n^0)}\overline{\psi}_2) k_j^{(n^0)} \tilde{B}^{(i)}_{\hat{k}^{(n^0)}}\right\} = \text{sign}\left\{\cos(k_1^{(n^1)}\overline{\psi}_1 + k_2^{(n^1)}\overline{\psi}_2) k_j^{(n^1)} \tilde{B}^{(i)}_{\hat{k}^{(n^1)}}\right\}. \tag{1.3.85}$$

We start by describing how to choose $(\overline{\psi}_1, \overline{\psi}_2)$ in $\{0,\pi\} \times \{0,\pi\}$ satisfying (1.3.85) by assuming that β coincides with the golden mean value $\tilde{\beta} = \dfrac{\sqrt{5}+1}{2}$ and, after this, we will explain why this choice of the initial phases can be taken independently of the value of the frequency β, whenever $\beta \in I_{\tilde{\beta}}(\varepsilon)$, $I_{\tilde{\beta}}(\varepsilon)$ the neighbourhood of $\tilde{\beta}$ given by the Main Lemma I.

Let us observe that, according to the Main Lemma I and independently of whether $\beta = \tilde{\beta}$ or not, we can restrict the study of the validity of the equality (1.3.85) for the case in which the entire numbers $k_1^{(n^0)}$ and $k_1^{(n^1)}$ (respectively $k_2^{(n^0)}$ and $k_2^{(n^1)}$) belong to the sequence $\{k_1^{(n)}\}_{n\in\mathbb{N}}$ (respectively $\{k_2^{(n)}\}_{n\in\mathbb{N}}$), defined at (1.4.120) in the appendix of this chapter, when

$$\beta = \tilde{\beta} = [1, 1, 1, \ldots] = \frac{\sqrt{5}+1}{2}.$$

Hence, $\{k_2^{(n)}\}_{n\in\mathbb{N}}$ is the sequence of Fibonacci numbers, i.e.,

$$k_2^{(1)} = 1, \quad k_2^{(2)} = 2, \quad k_2^{(n)} = k_2^{(n-2)} + k_2^{(n-1)}, \quad n > 2$$

and, moreover, for any $n \in \mathbb{N}$, $k_2^{(n)} = -k_1^{(n-1)}$. Hence, it trivially holds that, for $j = 1, 2$, $\mathrm{sign}(k_j^{(n^0)}) = \mathrm{sign}(k_j^{(n^1)})$ and, using also the expressions (1.3.79) and (1.3.80), one can easily check that

$$\mathrm{sign}(\tilde{B}^{(2)}_{\hat{k}^{(t)}}) = \mathrm{sign}(-\tilde{B}^{(3)}_{\hat{k}^{(t)}}), \quad t = n^0, n^1. \tag{1.3.86}$$

Thus, condition (1.3.85) can be replaced by the following one:

$$\mathrm{sign}\left\{\cos(k_1^{(n^0)}\overline{\psi}_1 + k_2^{(n^0)}\overline{\psi}_2)\tilde{B}^{(2)}_{\hat{k}^{(n^0)}}\right\} = \mathrm{sign}\left\{\cos(k_1^{(n^1)}\overline{\psi}_1 + k_2^{(n^1)}\overline{\psi}_2)\tilde{B}^{(2)}_{\hat{k}^{(n^1)}}\right\}. \tag{1.3.87}$$

Now, in order to finish the proof of Lemma 1.3.13, we need to recover some partial results derived from the proof of the Main Lemma I. The open set $\mathcal{U}_\varepsilon$ of good parameters ε announced by Lemma 1.3.10 is constructed, according to (4.1.10), as

$$\mathcal{U}_\varepsilon = \bigcup_{n\geq n^*} \mathcal{U}_n,$$

where n^* is a large enough natural number (see (4.1.8)). Moreover, $\{\mathcal{U}_n, n \geq n^*\}$ is a family of two by two disjoint open real intervals which are constructed in such a way that, if $\varepsilon \in \mathcal{U}_n$, then $n^0 = n$, $n^1 = n + 1$, where n^0 and n^1 are those indices for which the conclusions of the Main Lemma I hold.

Then, in order to select those $\overline{\psi}_i = \overline{\psi}_i(\varepsilon)$, $i = 1, 2$, for which the conclusions of Lemma 1.3.13 hold, we will distinguish between two cases:

A. If n is such that

$$\tilde{B}^{(2)}_{\hat{k}^{(n)}}\tilde{B}^{(2)}_{\hat{k}^{(n+1)}} > 0,$$

then we choose $\overline{\psi}_i = \overline{\psi}_i(\varepsilon) = 0$ for $i = 1, 2$ thus that (1.3.87) holds immediately.

B. Now, let us assume that n is such that

$$\tilde{B}^{(2)}_{\hat{k}^{(n)}}\tilde{B}^{(2)}_{\hat{k}^{(n+1)}} < 0.$$

Of course, in this case, we must choose $\overline{\psi}_1 = \overline{\psi}_1(\varepsilon)$ and $\overline{\psi}_2 = \overline{\psi}_2(\varepsilon)$ in $\{0, \pi\}$ satisfying

$$\mathrm{sign}(\cos(k_1^{(n)}\overline{\psi}_1 + k_2^{(n)}\overline{\psi}_2)) = \mathrm{sign}(-\cos(k_1^{(n+1)}\overline{\psi}_1 + k_2^{(n+1)}\overline{\psi}_2)).$$

We will still distinguish between three subcases:

$B1$. Let us assume $k_2^{(n)}$ is odd and $k_2^{(n+1)}$ is even. Then, the Fibonacci $k_2^{(n+2)}$ is odd and we take $\overline{\psi}_1 = \pi$ and $\overline{\psi}_2 = 0$ in order to obtain

$$\cos(k_1^{(n)}\overline{\psi}_1 + k_2^{(n)}\overline{\psi}_2) = (-1)^{k_1^{(n)}} = (-1)^{-k_1^{(n)}} = (-1)^{k_2^{(n+1)}} = 1$$

and

$$\cos(k_1^{(n+1)}\overline{\psi}_1 + k_2^{(n+1)}\overline{\psi}_2) = (-1)^{k_1^{(n+1)}} = (-1)^{k_2^{(n+2)}} = -1.$$

$B2$. If $k_2^{(n)}$ and $k_2^{(n+1)}$ are odd, then $k_2^{(n+2)}$ is even and therefore $\overline{\psi}_1 = \pi$, $\overline{\psi}_2 = 0$ is also a good choice.

$B3$. If $k_2^{(n)}$ is even and $k_2^{(n+1)}$ is odd, then we take $\overline{\psi}_1 = 0$, $\overline{\psi}_2 = \pi$ to get

$$\cos(k_1^{(n)}\overline{\psi}_1 + k_2^{(n)}\overline{\psi}_2) = (-1)^{k_2^{(n)}} = 1 = -(-1)^{k_2^{(n+1)}} = -\cos(k_1^{(n+1)}\overline{\psi}_1 + k_2^{(n+1)}\overline{\psi}_2).$$

Finally, we also observe that there are no two consecutive even Fibonacci numbers and, moreover, since we are assuming that β coincides with the golden mean, we have that

$$\tilde{B}^{(2)}_{\hat{k}^{(n)}}\tilde{B}^{(2)}_{\hat{k}^{(n+1)}} \neq 0$$

and therefore there are no more cases than the two (A and B) studied before.

Now, in order to see how to extend the above arguments to the neighbourhood $I_{\tilde{\beta}}(\varepsilon)$ of $\tilde{\beta}$ given by the Main Lemma I, let us remark that equations (1.4.124) and (1.4.128) in the appendix of this chapter imply that the two functions

$$\beta \in I_{\tilde{\beta}} \to \hat{k}^{(t)}\omega \in \mathbb{R}, \quad t = n^0, n^1, \quad \omega = (1, \beta) \tag{1.3.88}$$

do not vanish on $I_{\tilde{\beta}}$. This is crucial to state that, see (1.3.79), the sign of the coefficients $\tilde{B}^{(i)}_{\hat{k}}$ does not change when β belongs to $I_{\tilde{\beta}}$. Therefore, (1.3.85) holds for every $\varepsilon \in \mathcal{U}_\varepsilon$ and any $\beta \in I_{\tilde{\beta}}$, and therefore Lemma 1.3.13 is proved. □

Remark 1.3.14 *If we consider the case in which one of the two functions defined in (1.3.88) vanish, then we are not able to prove the Main Theorem I. In fact, although we can give estimates on the splitting when the frequency β is rational, this is not the case when such resonance appears too* soon *(the resonant cases solved in this book always satisfy $\beta = k_1/k_2$ with $k_2 - k_2^{(t)}$ a sufficiently large positive number, for $t = n^0, n^1$). This is the main reason why the length of our good set of frequencies, $I_{\tilde{\beta}}$, depends on ε. Of course, see Remark 2.3.11 for related details, it would be quite useful to extend our results to those values of β for which one of the two functions defined in (1.3.88) vanishes.*

The proof of Lemma 1.3.13 describes the way in which we select the homoclinic orbit of the perturbed system once a value of $\varepsilon \in \mathcal{U}_\varepsilon$ is fixed.

Moreover, during the second part of the proof of the Main Theorem I we will use the following property: Once the parameters a and p taking part in the definition of the Hamiltonians (1.0.1) are fixed, if we write the renormalized Melnikov functions coefficients as

$$\mathcal{E}_{\hat{k}} = \hat{\mathcal{E}}_{\hat{k}}\left(p, -\ln a, \frac{\pi}{2\sqrt{A}}, \varepsilon, \beta\right) = \exp\left(-\frac{\pi\left|\hat{k}\omega\right|}{2\varepsilon\sqrt{A}}\right)(a\varepsilon^p)^{|\hat{k}|} = \mathcal{E}_{\hat{k}}(\varepsilon,\beta),$$

then Corollary 4.1.8 implies that

$$\frac{1}{4} \leq \frac{\mathcal{E}_{\hat{k}(n^0)}(\varepsilon,\beta)}{\mathcal{E}_{\hat{k}(n^1)}(\varepsilon,\beta)} \leq 4 \tag{1.3.89}$$

whenever $\varepsilon \in \mathcal{U}_n$ (and thus $n^0 = n$, $n^1 = n+1$) and $\beta \in I_{\tilde{\beta}}(\varepsilon)$.

Therefore, for some constant $\varepsilon_0 \in (0,1)$, we have already obtained an open set $\mathcal{U}_\varepsilon$ contained in $(0, \varepsilon_0]$ depending on the parameters a and p (more concretely, see Lemma 1.3.10, $\mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon(p, -\ln a, \frac{\pi}{2\sqrt{A}})$), satisfying

$$ctant\ \varepsilon_0^{8/3}\left|\ln \varepsilon_0\right|^{3/2} \leq \mathcal{L}(F(\mathcal{U}_\varepsilon)) \leq ctant\ \varepsilon_0^{8/3}\left|\ln \varepsilon_0\right|^{8/3}$$

such that, if $\varepsilon \in \mathcal{U}_\varepsilon$, then there are two indices n^0, n^1 and two initial phases $\overline{\psi}_i = \overline{\psi}_i(\varepsilon) \in \{0, \pi\}$, $i = 1, 2$, for which (1.3.85) holds for every $\beta \in I_{\tilde{\beta}}$, $I_{\tilde{\beta}} = I_{\tilde{\beta}}(\varepsilon)$ a golden mean value neighbourhood with

$$\frac{1}{100}\varepsilon^{5/3}\left|\ln \varepsilon\right|^{1/2} \leq \mathrm{length}(I_{\tilde{\beta}}) \leq \frac{1}{2}\varepsilon^{5/3}\left|\ln \varepsilon\right|^{1/2}.$$

Moreover, (1.3.89) is satisfied whenever $\varepsilon \in \mathcal{U}_\varepsilon$, $\beta \in I_{\tilde{\beta}}$ and $\mathcal{E}_{\hat{k}}$ are those coefficients (see (1.2.52)) taking part in the definition of the renormalized Melnikov functions computed in Section 1.2.

1.3.2 Proof of the Main Theorem I. Second part: Bounding the transversality

Once the set of values of ε, $\mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon(a, p)$ was found and the correspondent homoclinic orbit of the perturbed system was selected, the main purpose of this subsection is to obtain the lower and upper bounds announced by the Main Theorem I for the transversality $\Upsilon = \Upsilon(\overline{\psi}_1, \overline{\psi}_2)$ of the splitting along the respective homoclinic orbit. Once a value of $\varepsilon \in \mathcal{U}_\varepsilon$ is fixed, the estimates for the transversality have to be valid for every value of β in the respective neighbourhood $I_{\tilde{\beta}} = I_{\tilde{\beta}}(\varepsilon)$ of the golden mean given by the Main Lemma I. The whole required argument for getting those bounds is divided in eight steps.

First step. *Bounds for the error functions.*

Let us start by recalling that, at (1.3.57), we have introduced the notation

$$(\hat{\psi}, t, s) \in \mathcal{B}_1'' \times [T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}_1' \to Q^*(\hat{\psi}, t, s)$$

for denoting the points in certain special pieces of the invariant manifolds for each of the invariant whiskered tori T_{α_1,α_2} of the perturbed system (1.1.6). Let us now simplify the notation by taking

$$Q^* = Q^*(\hat{\psi}, t, s)$$

and, in order that Lemma 1.3.1 and Lemma 1.3.5 can be applied, we restrict the variation of the variable $(\hat{\psi}, t, s)$ to the set

$$\mathcal{B}_1''' \times [T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}_1''$$

where

$$\mathcal{B}_1''' = \left\{(\psi_1, \psi_2) \in \mathbb{C}^2 : \ |\text{Im } \psi_i| \leq -\ln(a\varepsilon^p) - 3\varepsilon^b, \ i = 1, 2\right\}$$

and

$$\mathcal{C}_1'' = \left\{ s \in \mathbb{C} : \ |\text{Re } s| \leq \varepsilon, |\text{Im } s| \leq \frac{\pi}{2\sqrt{A}} - 2\varepsilon^b \right\}.$$

Let us again consider the splitting functions introduced in (1.3.73) and recall that

$$\begin{aligned} \mathcal{K}_u^\mu(s, \psi_1, \psi_2) &= \mathcal{K}^\mu(Q^-) - \mathcal{K}^\mu(Q^+) = \mathcal{K}^0(h_\mu(Q^-)) - \mathcal{K}^0(h_\mu(Q^+)) = \\ &= \mathcal{K}^0(Q^-) - \mathcal{K}^0(Q^+) + \mathcal{K}^0(h_\mu(Q^-)) - \mathcal{K}^0(Q^-) - \mathcal{K}^0(h_\mu(Q^+)) + \mathcal{K}^0(Q^+), \end{aligned}$$

where we have used the family of conjugations h_μ implicitly given in the proof of Lemma 1.3.1 (see especially (1.3.72)).

Now, let us observe that there exist $\tilde{Q}^-$, $\tilde{Q}^+$ and $\hat{Q}$ such that

$$\begin{aligned} &\left|\mathcal{K}^0(h_\mu(Q^-)) - \mathcal{K}^0(Q^-) - \mathcal{K}^0(h_\mu(Q^+)) + \mathcal{K}^0(Q^+)\right| = \\ &= \left| D\mathcal{K}^0(\tilde{Q}^-)(h_\mu(Q^-) - Q^-) - D\mathcal{K}^0(\tilde{Q}^+)(h_\mu(Q^+) - Q^+) \right| \leq \\ &\leq \left\|D^2\mathcal{K}^0\right\| \left|\tilde{Q}^- - \tilde{Q}^+\right| \left|h_\mu(Q^-) - Q^-\right| + \left\|D\mathcal{K}^0\right\| \left\|Dh_\mu(\hat{Q}) - I\right\| \left|Q^+ - Q^-\right| \end{aligned}$$

with, according to (1.3.70) and the Extension Theorem I (see also (1.3.55)),

$$\begin{aligned} \left|\tilde{Q}^- - \tilde{Q}^+\right| &\leq \left|\tilde{Q}^- - Q^-\right| + \left|Q^- - Q^+\right| + \left|Q^+ - \tilde{Q}^+\right| \leq \left|h_\mu(Q^-) - Q^-\right| \\ &+ \left|Q^- - Q^+\right| + \left|Q^+ - h_\mu(Q^+)\right| \leq \mathit{ctant}\ \mu\varepsilon^{-b(N+5)}. \end{aligned}$$

Therefore, also using (1.3.71), we deduce

$$\left|\mathcal{K}^0(h_\mu(Q^-)) - \mathcal{K}^0(Q^-) - \mathcal{K}^0(h_\mu(Q^+)) + \mathcal{K}^0(Q^+)\right| \leq \mathit{ctant}\ \mu^2\varepsilon^{-b(2N+9)}.$$

Hence, since from (1.3.60) we have $\mathcal{K}^0(Q^*) = \tilde{H}(q^*, p^*)$ and recalling that the change of variables $(q, p) = \varphi(x, y)$ given by (1.1.10) and Lemma 1.1.4 transforms the Hamiltonian

$$H_1 = H_1(x, y) = \frac{y^2}{2} + A(\cos x - 1)$$

into the Hamiltonian

$$\tilde{H} = \tilde{H}(q,p) = -\sqrt{A}(pq + F(pq)),$$

we obtain $\mathcal{K}^0(Q^*) = H_1(x^*, y^*)$.

Therefore, we may definitively write

$$\left|\mathcal{K}^\mu_u(s,\psi_1,\psi_2) - H_1(x^-,y^-,\hat{I}^-,\hat{\theta}^0)(\hat{\psi},t,s) + H_1(x^+,y^+,\hat{I}^+,\hat{\theta}^0)(\hat{\psi},t,s)\right| \leq$$
$$\leq ctant\ \mu^2\varepsilon^{-b(2N+9)}, \tag{1.3.90}$$

for every $(\hat{\psi},t,s) \in \mathcal{B}_1''' \times [T_0 - \text{Re}\ s, 2T_0 - \text{Re}\ s] \times \mathcal{C}_1''$. In the same way, for $i = 1,2$, we deduce

$$\left|\mathcal{J}^\mu_{i,u}(s,\psi_1,\psi_2) - I_i(x^-,y^-,\hat{I}^-,\hat{\theta}^0)(\hat{\psi},t,s) + I_i(x^+,y^+,\hat{I}^+,\hat{\theta}^0)(\hat{\psi},t,s)\right| \leq$$
$$\leq ctant\ \mu^2\varepsilon^{-b(2N+9)}. \tag{1.3.91}$$

From these arguments and Lemma 1.2.1 it follows that, if we define the error functions

$$(s,\psi_1,\psi_2) \in \mathcal{C}_1'' \times \mathcal{B}_1''' \to E^\mu_i(s,\psi_1,\psi_2) = C^\mu_{i,u}(s,\psi_1,\psi_2) - \mathcal{M}_i(s,\psi_1,\psi_2) \tag{1.3.92}$$

for

$$C^\mu_{1,u} = \mathcal{K}^\mu_u, \quad C^\mu_{i,u} = \mathcal{J}^\mu_{i-1,u}, \quad i = 2,3,$$

then

$$|E^\mu_i(s,\psi_1,\psi_2)| \leq ctant\ \mu^2 \left|\ln\mu\right| \varepsilon^{-2b(N+5)}, \tag{1.3.93}$$

for every $(s,\psi_1,\psi_2) \in \mathcal{C}_1'' \times \mathcal{B}_1'''$.

Of course, if we also apply Lemma 1.2.3, then, for every $(s,\psi_1,\psi_2) \in \mathcal{C}_1'' \times \mathcal{B}_1'''$,

$$\left|C^\mu_{i,u}(s,\psi_1,\psi_2)\right| \leq ctant\ \mu\varepsilon^{-b(N+5)} \tag{1.3.94}$$

whenever $\mu \in (0,\varepsilon^m)$, with $m > \dfrac{5}{4}b(N+5)$ and $\mu^{1/5}\left|\ln\mu\right| < 1$ (we observe that, instead of $5/4$ we could work with any constant λ bigger than one by choosing only those values of μ for which $\mu^{\lambda-1}\left|\ln\mu\right| < 1$).

Second step. *Setting suitable coordinates.*

Let us observe that, once $(\psi_1,\psi_2) \in \mathcal{B}_1'''$ is fixed, we may use Lemma 1.3.5 to write, for any $s \in \mathcal{C}_1''$,

$$v = U(s) = \mathcal{S}^\mu_u(s,\psi_1,\psi_2) = \mathcal{S}^\mu(Q^-(\hat{\psi},t,s)) - t,$$

where the right-hand side term does not depend on t according to Remark 1.3.3. Moreover, Lemma 1.3.5 also implies that

$$|s - v| \leq ctant\ \mu\varepsilon^{-b(N+5)}$$

in such a way that, if we denote

$$\mathcal{C}_1''' = \left\{ s \in \mathbb{C} : \ |\text{Re } s| \leq \varepsilon, |\text{Im } s| \leq \frac{\pi}{2\sqrt{A}} - 3\varepsilon^b \right\},$$

then Cauchy estimates and (1.3.94) give

$$\left| C^{\mu}_{i,u}(s, \psi_1, \psi_2) - C^{\mu}_{i,u}(v, \psi_1, \psi_2) \right| \leq ctant\ \mu^2 \varepsilon^{-b(2N+11)}, \quad v = U(s), \tag{1.3.95}$$

for every $(s, \psi_1, \psi_2) \in \mathcal{C}_1''' \times \mathcal{B}_1'''$.

In the same way, from Lemma 1.2.3, we can also deduce

$$|\mathcal{M}_i(s, \psi_1, \psi_2) - \mathcal{M}_i(v, \psi_1, \psi_2)| \leq ctant\ \mu^2 \varepsilon^{-b(2N+11)}, \quad v = U(s)$$

and hence, from (1.3.92) and (1.3.93), it follows that

$$|E^{\mu}_i(v, \psi_1, \psi_2)| \leq ctant\ \mu^2 \left|\ln \mu\right| \varepsilon^{-b(2N+11)}, \tag{1.3.96}$$

for every $(v, \psi_1, \psi_2) \in \mathcal{C}_1''' \times \mathcal{B}_1'''$.

We point out that, in the (v, ψ_1, ψ_2) coordinates, our splitting functions display the following remarkable property:

Let us fix $(\psi_1, \psi_2) \in \mathcal{B}_1'''$ and take, once again, the function $U = U(s)$ given by Lemma 1.3.5 satisfying, for every $t \in [T_0 - \text{Re } s, 2T_0 - \text{Re } s]$, that

$$U(s) + t = v + t = \mathcal{S}^{\mu}(Q^{-}(\hat{\psi}, t, s))$$

being, as usual, $\mathcal{S}^{\mu}$ the first component of the analytic change of coordinates given by Lemma 1.3.1 transforming the vector field $\dot{Q} = g_{\mu}(Q)$ into the flow-box system.

Let us take, for every $(\psi_1, \psi_2) \in \mathcal{B}_1'''$, the notation

$$\mathcal{S}^{\mu} = \mathcal{S}^{\mu}(s, t) = \mathcal{S}^{\mu}(Q^{-}(\hat{\psi}, t, s)) = v + t. \tag{1.3.97}$$

We claim that, for every $(v, \psi_1, \psi_2) \in \mathcal{C}_1''' \times \mathcal{B}_1'''$, it holds that

$$C^{\mu}_{i,u}(v, \psi_1, \psi_2) = C^{\mu}_{i,u}(\mathcal{S}^{\mu} - t, \psi_1, \psi_2) = C^{\mu}_{i,u}\left(\mathcal{S}^{\mu}, \psi_1 + \frac{t}{\varepsilon}, \psi_2 + \frac{\beta t}{\varepsilon}\right) = C^{\mu}_{i,u}(\mathcal{S}^{\mu}, \theta_1, \theta_2),$$

where, as usual,

$$\theta_1 = \theta_1(t) = \psi_1 + \frac{t}{\varepsilon}, \quad \theta_2 = \theta_2(t) = \psi_2 + \frac{\beta t}{\varepsilon}. \tag{1.3.98}$$

Observe that the first and the third equalities in the above claim are evident. To check the second one it is enough to take into account (see Remark 1.3.3) that the splitting functions remain constant along the orbits of the flow-box system and this flow-box system contains, in particular, the equations $\dot{\mathcal{S}}^{\mu} = 1$, $\dot{\theta}_1 = \varepsilon^{-1}$ and $\dot{\theta}_2 = \beta\varepsilon^{-1}$.

Let us remark that , in spite of the functions $\mathcal{C}^{\mu}_{i,u}$ do not depend on t (see Remark 1.3.3), they can be defined on variables, i.e., $\mathcal{S}^{\mu}, \theta_1, \theta_2)$, which depend on time and this was done by first defining those functions on variables (v, ψ_1, ψ_2). Here is where the role of the variable v (and Lemma 1.3.5) becomes patent. Moreover, from equation

$$0 = \frac{d}{dt}\mathcal{C}^{\mu}_{i,u}(\mathcal{S}^{\mu}, \theta_1, \theta_2) = \frac{\partial \mathcal{C}^{\mu}_{i,u}}{\partial \mathcal{S}^{\mu}}(\mathcal{S}^{\mu}, \theta_1, \theta_2) + \frac{1}{\varepsilon}\frac{\partial \mathcal{C}^{\mu}_{i,u}}{\partial \theta_1}(\mathcal{S}^{\mu}, \theta_1, \theta_2) + \frac{\beta}{\varepsilon}\frac{\partial \mathcal{C}^{\mu}_{i,u}}{\partial \theta_2}(\mathcal{S}^{\mu}, \theta_1, \theta_2)$$

the Mean Value Theorem yields

$$C^{\mu}_{i,u}(\mathcal{S}^{\mu}, \theta_1, \theta_2) = C^{\mu}_{i,u}(\tilde{\mathcal{S}}^{\mu}, \tilde{\theta}_1, \tilde{\theta}_2) \tag{1.3.99}$$

whenever

$$\theta_1 - \frac{\mathcal{S}^{\mu}}{\varepsilon} = \tilde{\theta}_1 - \frac{\tilde{\mathcal{S}}^{\mu}}{\varepsilon}, \quad \theta_2 - \frac{\beta\mathcal{S}^{\mu}}{\varepsilon} = \tilde{\theta}_2 - \frac{\beta\tilde{\mathcal{S}}^{\mu}}{\varepsilon}.$$

Hence, in $(\mathcal{S}^{\mu}, \theta_1, \theta_2)$-coordinates we may write $C^{\mu}_{i,u}$ as functions which only depend on the arguments $\theta_1 - \frac{\mathcal{S}^{\mu}}{\varepsilon}$ and $\theta_2 - \frac{\beta\mathcal{S}^{\mu}}{\varepsilon}$; i.e.,

$$C^{\mu}_{i,u}(\mathcal{S}^{\mu}, \theta_1, \theta_2) = \tilde{F}^{\mu}_i\left(\theta_1 - \frac{\mathcal{S}^{\mu}}{\varepsilon}, \theta_2 - \frac{\beta\mathcal{S}^{\mu}}{\varepsilon}\right), \quad i = 1, 2, 3$$

or, in other words, using (1.3.97) and (1.3.98),

$$C^{\mu}_{i,u}(v, \psi_1, \psi_2) = \tilde{F}^{\mu}_i\left(\psi_1 - \frac{v}{\varepsilon}, \psi_2 - \frac{\beta v}{\varepsilon}\right). \tag{1.3.100}$$

Therefore, if we compare (1.3.100) with (1.3.75) and take into account that the splitting functions $C^{\mu}_{i,u} = C^{\mu}_{i,u}(v, \psi_1, \psi_2)$ are analytic in (see also (1.3.74))

$$D\left(\varepsilon, \frac{\pi}{2\sqrt{A}} - 3\varepsilon^b, -\ln(a\varepsilon^p) - 3\varepsilon^b, -\ln(a\varepsilon^p) - 3\varepsilon^b\right) =$$
$$\left\{(v, \psi_1, \psi_2) \in \mathbb{C}^3 : |\text{Re } v| < \varepsilon, |\text{Im } v| \leq \frac{\pi}{2\sqrt{A}} - 3\varepsilon^b, |\text{Im } \psi_i| \leq -\ln(a\varepsilon^p) - 3\varepsilon^b, i = 1, 2\right\}$$
$$= \mathcal{C}'''_1 \times \mathcal{B}'''_1,$$

we have

$$C^{\mu}_{i,u} \in \mathcal{A}\left(\varepsilon, \frac{\pi}{2\sqrt{A}} - 3\varepsilon^b, -\ln(a\varepsilon^p) - 3\varepsilon^b, -\ln(a\varepsilon^p) - 3\varepsilon^b, ctant\ \mu\varepsilon^{-b(N+5)}\right),$$

where we have also used (1.3.94), (1.3.95) and that $\mu \in (0, \varepsilon^m)$ with $m > b(N + 6)$.

This fact, together with (1.3.76), (1.3.92) and (1.3.96), lead to

$$E^{\mu}_i \in \mathcal{A}\left(\varepsilon, \frac{\pi}{2\sqrt{A}} - 3\varepsilon^b, -\ln(a\varepsilon^p) - 3\varepsilon^b, -\ln(a\varepsilon^p) - 3\varepsilon^b, ctant\ \mu^2\,|\ln\mu|\,\varepsilon^{-b(2N+11)}\right) \tag{1.3.101}$$

and, in particular, we may write the error functions E_i^μ as transformations which only depend on the arguments $\psi_1 - \frac{v}{\varepsilon}$ and $\psi_2 - \frac{\beta v}{\varepsilon}$:

$$E_i^\mu(v, \psi_1, \psi_2) = G_i^\mu\left(\psi_1 - \frac{v}{\varepsilon}, \psi_2 - \frac{\beta v}{\varepsilon}\right). \tag{1.3.102}$$

Henceforth, we are ready to apply Lemma 1.3.4 to these error functions E_i^μ.

Third step. *The transversality.*

In the first part of the proof (see especially Lemma 1.3.13) we gave an algorithm directed to select a homoclinic orbit for the perturbed system (1.1.6) located at $s = 0$ and $\hat{\psi} = (\psi_1, \psi_2)$ where $\psi_i = \overline{\psi}_i(\varepsilon) \in \{0, \pi\}$, $i = 1, 2$.

Moreover, by looking at the definition of the splitting functions given in (1.3.73), it follows that

$$\mathcal{K}_u^\mu(0, \overline{\psi}_1, \overline{\psi}_2) = \mathcal{J}_{i,u}^\mu(0, \overline{\psi}_1, \overline{\psi}_2) = 0, \qquad i = 1, 2.$$

Therefore, we can measure the transversality, or the size of the splitting of the perturbed manifolds along the considered homoclinic orbit, by defining, for $i = 2, 3$,

$$\begin{array}{rcl} \overline{\mathcal{J}}_{i-1,u}^\mu(\psi_1, \psi_2) & = & \mathcal{J}_{i-1,u}^\mu(0, \psi_1, \psi_2) \\ \overline{\mathcal{M}}_i(\psi_1, \psi_2) & = & \mathcal{M}_i(0, \psi_1, \psi_2) \\ \overline{E}_i^\mu(\psi_1, \psi_2) & = & E_i^\mu(0, \psi_1, \psi_2) \end{array}$$

and by estimating

$$\Upsilon = \Upsilon(\overline{\psi}_1, \overline{\psi}_2) = \det \begin{pmatrix} \dfrac{\partial \overline{\mathcal{J}}_{1,u}^\mu}{\partial \psi_1}(\overline{\psi}_1, \overline{\psi}_2) & \dfrac{\partial \overline{\mathcal{J}}_{1,u}^\mu}{\partial \psi_2}(\overline{\psi}_1, \overline{\psi}_2) \\ \dfrac{\partial \overline{\mathcal{J}}_{2,u}^\mu}{\partial \psi_1}(\overline{\psi}_1, \overline{\psi}_2) & \dfrac{\partial \overline{\mathcal{J}}_{2,u}^\mu}{\partial \psi_2}(\overline{\psi}_1, \overline{\psi}_2) \end{pmatrix}. \tag{1.3.103}$$

Now, we use the error functions introduced in (1.3.92) to write

$$\frac{\partial \overline{\mathcal{J}}_{i-1,u}^\mu}{\partial \psi_j}(\overline{\psi}_1, \overline{\psi}_2) = m^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2) + e^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2)$$

where

$$m^{i-1,j}(\psi_1, \psi_2) = \frac{\partial \overline{\mathcal{M}}_i}{\partial \psi_j}(\psi_1, \psi_2), \quad e^{i-1,j}(\psi_1, \psi_2) = \frac{\partial \overline{E}_i^\mu}{\partial \psi_j}(\psi_1, \psi_2), \tag{1.3.104}$$

for $j = 1, 2$ and $i = 2, 3$.

Fourth step. *Bounding* $e^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2)$.

As we have already seen in Remark 1.3.6, if we restrict the variation of (ψ_1,ψ_2) to $\mathbb{R}^2$, then the function

$$v^* = v^*(\psi_1,\psi_2) = U(0) = \mathcal{S}_u^\mu(0,\psi_1,\psi_2)$$

given by Lemma 1.3.5 is real. Moreover, since by definition

$$\overline{E}_i^\mu(\psi_1,\psi_2) = E_i^\mu(0,\psi_1,\psi_2) = E_i^\mu(v^*,\psi_1,\psi_2),$$

we may apply (1.3.102) to write

$$\overline{E}_i^\mu(\psi_1,\psi_2) = \sum_{\hat{k}\in\mathbb{Z}^2} G_{i,\hat{k}}^\mu \exp\left(\sqrt{-1}\left(k_1(\psi_1 - \frac{v^*}{\varepsilon}) + k_2(\psi_2 - \frac{\beta v^*}{\varepsilon})\right)\right),$$

where $G_{i,\hat{k}}^\mu$ are the Fourier coefficients of the functions G_i^μ and, as usual, $\hat{k} = (k_1,k_2)$.

Furthermore, taking derivatives, we have

$$e^{i-1,j}(\psi_1,\psi_2) = \sum_{\hat{k}\in\mathbb{Z}^2} G_{i,\hat{k}}^{j,\mu} \exp\left(\sqrt{-1}\left(k_1(\psi_1 - \frac{v^*}{\varepsilon}) + k_2(\psi_2 - \frac{\beta v^*}{\varepsilon})\right)\right)$$

where, for $j = 1,2$,

$$G_{i,\hat{k}}^{j,\mu} = \sqrt{-1}\left(k_j - \frac{k_1 + k_2\beta}{\varepsilon}\frac{\partial v^*}{\partial \psi_j}(\psi_1,\psi_2)\right) G_{i,\hat{k}}^\mu. \tag{1.3.105}$$

Hence, for $j = 1,2$, we obtain

$$G_{i,0,0}^{j,\mu} = 0$$

and thus

$$\left|e^{i-1,j}(\overline{\psi}_1,\overline{\psi}_2)\right| \leq \sum_{\hat{k}\in\mathbb{Z}^2\setminus\{(0,0)\}} \left|G_{i,\hat{k}}^{j,\mu}\right|.$$

On the other hand, since (see (1.3.72) and (1.3.73))

$$\begin{aligned} v^*(\psi_1,\psi_2) &= \mathcal{S}_u^\mu(0,\psi_1,\psi_2) = \mathcal{S}^\mu\left(q^-(\hat{\psi},t,0), p^-(\hat{\psi},t,0), \hat{I}^-(\hat{\psi},t,0), \hat{\theta}^0(\hat{\psi},t), 0\right) - t = \\ &= \mathcal{S}^0\left(h_\mu\left(q^-(\hat{\psi},t,0), p^-(\hat{\psi},t,0), \hat{I}^-(\hat{\psi},t,0), \hat{\theta}^0(\hat{\psi},t), 0\right)\right) - t \end{aligned}$$

is an analytic function in $\mathcal{B}_1'''$ and (see Lemma 1.3.5)

$$\|v^*\|_{\mathcal{B}_1'''} \leq ctant\ \mu\varepsilon^{-b(N+5)},$$

we may apply Lemma 1.1.2 to get

$$\left\|\frac{\partial v^*}{\partial \psi_j}\right\|_{\mathcal{B}_1''''} \leq ctant\ \mu\varepsilon^{-b(N+6)}, \quad j = 1,2,$$

where

$$\mathcal{B}_1'''' = \left\{(\psi_1, \psi_2) \in \mathbb{C}^2 : \ |\mathrm{Im}\ \psi_i| \leq -\ln(a\varepsilon^p) - 4\varepsilon^b,\ i = 1, 2\right\}.$$

Therefore, taking $\mu \in (0, \varepsilon^m)$, with $m > b(N+6)+1$, we have, from (1.3.105),

$$\left|G^{j,\mu}_{i,\hat{k}}\right| \leq ctant\ \left|\hat{k}\right| \left|G^{\mu}_{i,\hat{k}}\right|$$

from which we deduce

$$\left|e^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2)\right| \leq ctant \sum_{\hat{k}\in\mathbb{Z}^2\setminus\{(0,0)\}} \left|\hat{k}\right| \left|G^{\mu}_{i,\hat{k}}\right|.$$

Now, it will be necessary to furnish suitable bounds, for every $\hat{k} \in \mathbb{Z}^2\setminus\{(0,0)\}$, for each Fourier coefficient $G^{\mu}_{i,\hat{k}}$ of the error functions introduced in (1.3.102).

To this end, we use (1.3.101) and Lemma 1.3.4 to write, for every $\hat{k} \in \mathbb{Z}^2\setminus\{(0,0)\}$,

$$\left|G^{\mu}_{i,\hat{k}}\right| \leq ctant\ \mu^2 \left|\ln \mu\right| \varepsilon^{-b(2N+11)} \Lambda_{\hat{k}}(a, p, \varepsilon, \beta)$$

where, recalling the notation $\left|\hat{k}\omega\right| = k_1 + \beta k_2$,

$$\Lambda_{\hat{k}}(a, p, \varepsilon, \beta) \ = \ \exp\left\{-\left|\hat{k}\right| \left(p\left|\ln\varepsilon\right| - \ln a - 3\varepsilon^b\right) - \left(\frac{\pi}{2\sqrt{A}} - 3\varepsilon^b\right)\frac{\left|\hat{k}\omega\right|}{\varepsilon}\right\}.$$

Hence,

$$\left|e^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2)\right| \leq ctant\ \mu^2 \left|\ln\mu\right| \varepsilon^{-b(2N+11)} \sum_{\hat{k}\in\mathbb{Z}^2\setminus\{(0,0)\}} \left|\hat{k}\right| \Lambda_{\hat{k}}(a, p, \varepsilon, \beta). \tag{1.3.106}$$

Now, let us observe that, from (1.3.78), we have

$$\Lambda_{\hat{k}}(a, p, \varepsilon, \beta) = \hat{\mathcal{E}}_{\hat{k}}\left(p, -\ln a - 3\varepsilon^b, \frac{\pi}{2\sqrt{A}} - 3\varepsilon^b, \varepsilon, \beta\right).$$

Furthermore, in the first part of the proof, we chose an open set of values of ε

$$\mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon\left(p, -\ln a, \frac{\pi}{2\sqrt{A}}\right),$$

given by the Main Lemma I when taking $c = p$, $l = -\ln a$ and $d = \dfrac{\pi}{2\sqrt{A}}$, as the set of values of the parameter ε for which we are showing that the Main Theorem I holds. Of course, in order to get suitable bounds for $\left|e^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2)\right|$, we pretend, according to the estimate (1.3.106), to obtain the leading terms of the series

$$\tilde{S} = \sum_{\hat{k}\in\mathbb{Z}^2\setminus\{(0,0)\}} \left|\hat{k}\right| \Lambda_{\hat{k}}(a, p, \varepsilon, \beta) = \sum_{\hat{k}\in\mathbb{Z}^2\setminus\{(0,0)\}} \tilde{S}_{\hat{k}} \hat{\mathcal{E}}_{\hat{k}},$$

where

$$\tilde{S}_{\hat{k}} = \left|\hat{k}\right| \quad \text{and} \quad \hat{\mathcal{E}}_{\hat{k}} = \hat{\mathcal{E}}_{\hat{k}}\left(p, -\ln a - 3\varepsilon^b, \frac{\pi}{2\sqrt{A}} - 3\varepsilon^b, \varepsilon, \beta\right).$$

Nevertheless, a direct use of Lemma 1.3.10 would be unfruitful because, although one may check in an easy way that (observe that $\tilde{S}_{\hat{k}} = \left|\hat{k}\right|$)

$$\tilde{S} \in \mathcal{S}\left(p, -\ln a - 3\varepsilon^b, \frac{\pi}{2\sqrt{A}} - 3\varepsilon^b, \varepsilon, \beta\right),$$

we can not guarantee (in this first approach) that the conclusions of the Main Lemma I hold for such $\tilde{S}$ whenever $\varepsilon \in \mathcal{U}_\varepsilon\left(p, -\ln a, \frac{\pi}{2\sqrt{A}}\right)$ or, in summary, we can not check immediately that

$$\mathcal{U}_\varepsilon\left(p, -\ln a, \frac{\pi}{2\sqrt{A}}\right) \subset \mathcal{U}_\varepsilon\left(p, -\ln a - 3\varepsilon^b, \frac{\pi}{2\sqrt{A}} - 3\varepsilon^b\right).$$

Here, the reason why the first Perturbing Lemma (see Lemma 1.3.11) is useful becomes patent. Namely, it is enough to choose

$$b > \frac{5}{8} \tag{1.3.107}$$

in order to apply Lemma 1.3.11 to ensure that, for every $\varepsilon \in \mathcal{U}_\varepsilon\left(p, -\ln a, \frac{\pi}{2\sqrt{A}}\right)$ one has (observe that, in this case, the assumption $\tilde{S}_{\hat{k}(n^0)}\tilde{S}_{\hat{k}(n^1)} > 0$ holds trivially)

$$\left|e^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2)\right| \leq ctant\ \mu^2 \left|\ln\mu\right| \varepsilon^{-b(2N+11)} \left(\left|\hat{k}^{(n^0)}\right| \Lambda_{\hat{k}(n^0)} + \left|\hat{k}^{(n^1)}\right| \Lambda_{\hat{k}(n^1)}\right),$$

where n^0 and n^1 are those indices for which the Main Lemma I (for the particular choice $c = p$, $l = -\ln a$, $d = \frac{\pi}{2\sqrt{A}}$) holds (i.e., n^0 and n^1 are those indices for which (1.3.85) and (1.3.89) are satisfied), $k_1^{(n^0)}/k_2^{(n^0)}$, $k_1^{(n^1)}/k_2^{(n^1)}$ the corresponding best approximations to the golden mean number and being the last bound for the norm of the error functions (at homoclinic orbits) valid whenever $\beta \in I_{\tilde{\beta}}(\varepsilon)$, $I_{\tilde{\beta}}(\varepsilon)$ the neighbourhood of the golden mean value also furnished by the Main Lemma I when taking $c = p$, $l = -\ln a$ and $d = \frac{\pi}{2\sqrt{A}}$.

Moreover, we know that, for $t = n^0, n^1$, $r = 1, 2$,

$$\text{ctant}\ \varepsilon^{-1/2}\left|\ln\varepsilon\right|^{-1/2} \leq k_r^{(t)} \leq \text{ctant}\ \varepsilon^{-1/2}\left|\ln\varepsilon\right|^{-1/2}. \tag{1.3.108}$$

Hence, using (1.4.129), we also deduce

$$\left|k_1^{(t)} + \beta k_2^{(t)}\right| \leq ctant\ \left|k_2^{(t)}\right|^{-1} \leq ctant\ \varepsilon^{1/2}\left|\ln\varepsilon\right|^{1/2}$$

and this last bound, together with (1.3.108) and the fact that $b > 5/8$, allows us to write, for $t = n^0, n^1$,

$$\Lambda_{\hat{k}^{(t)}} \leq ctant\ \mathcal{E}_{\hat{k}^{(t)}},$$

where

$$\mathcal{E}_{\hat{k}} = \hat{\mathcal{E}}_{\hat{k}}\left(p, -\ln a, \frac{\pi}{2\sqrt{A}}, \varepsilon, \beta\right)$$

are the terms taking part in the coefficients of the renormalized Melnikov functions (see (1.2.52) and compare with (1.3.78)) computed in Section 1.2.

Therefore, for every $\varepsilon \in \mathcal{U}_\varepsilon\left(p, -\ln a, \frac{\pi}{2\sqrt{A}}\right)$, any $(\overline{\psi}_1, \overline{\psi}_2) \in \{0, \pi\} \times \{0, \pi\}$ and every $\beta \in I_{\tilde{\beta}}(\varepsilon)$, we finally obtain

$$\left|e^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2)\right| \leq ctant\ \mu^2 \left|\ln \mu\right| \varepsilon^{-b(2N+11)} \left(\left|\hat{k}^{(n^0)}\right| \mathcal{E}_{\hat{k}^{(n^0)}} + \left|\hat{k}^{(n^1)}\right| \mathcal{E}_{\hat{k}^{(n^1)}}\right), \qquad (1.3.109)$$

whenever $\mu \in (0, \varepsilon^m)$, $m > b(N+6)+1$ and $b > 5/8$.

Fifth step. *Suitable expressions for* $m^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2)$.

Let us remind that, at the end of the third step (see (1.3.104)), we have introduced the functions

$$m^{i-1,j}(\psi_1, \psi_2) = \frac{\partial \overline{\mathcal{M}}_i}{\partial \psi_j}(\psi_1, \psi_2), \quad i = 2, 3, \quad j = 1, 2,$$

where $\overline{\mathcal{M}}_i(\psi_1, \psi_2) = \mathcal{M}_i(0, \psi_1, \psi_2)$, $\mathcal{M}_i = \mathcal{M}_i(s, \psi_1, \psi_2)$ the renormalized Melnikov functions studied along Section 1.2.

Now, we want to apply the Main Lemma I in order to obtain suitable expressions for $m^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2)$.

Let us start by recalling that, according to the first part of the proof (see, in particular, equation (1.3.82)) we may write

$$m^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2) = -2\mu\varepsilon^{-2}\pi A^{-1}\overline{S}, \qquad (1.3.110)$$

with

$$\overline{S} = \sum_{\hat{k} \in \mathbb{Z}^2 \setminus \{(0,0)\}} S_{\hat{k}} \mathcal{E}_{\hat{k}}.$$

The coefficients $\mathcal{E}_{\hat{k}}$, according to (1.3.83), are given by

$$\mathcal{E}_{\hat{k}} = \hat{\mathcal{E}}_{\hat{k}}\left(p, -\ln a, \frac{\pi}{2\sqrt{A}}, \varepsilon, \beta\right)$$

while, from (1.3.81) and (1.3.79), we respectively obtain

$$S_{\hat{k}} = \begin{cases} k_j \tilde{B}^{(i)}_{\hat{k}} \cos(k_1\overline{\psi}_1 + k_2\overline{\psi}_2), & \text{if } \hat{k} \in \Lambda \\ 0, & \text{if } \hat{k} \in \mathbb{Z}^2 \setminus (\Lambda \cup \{(0,0)\}) \end{cases}$$

and, for $i = 2, 3$.

$$\tilde{B}_{\hat{k}}^{(i)} = \left|\hat{k}\omega\right| H(\hat{k}\omega) C_{\hat{k}}^{(i)}(\hat{k}\omega).$$

Let us prove that (recall Definition 1.3.7, Definition 1.3.8 and Definition 1.3.9)

$$\overline{S} \in \mathcal{S}\left(p, -\ln a, \frac{\pi}{2\sqrt{A}}, \varepsilon, \beta\right) = \bigcap_{i=1}^{3} \mathcal{S}_i\left(p, -\ln a, \frac{\pi}{2\sqrt{A}}, \varepsilon, \beta\right),$$

for every $\varepsilon \in \mathcal{U}_\varepsilon\left(p, -\ln a, \frac{\pi}{2\sqrt{A}}\right)$ and any $\beta \in I_{\tilde{\beta}}(\varepsilon)$.

To begin with, we point out that since from (1.3.80) we have

$$C_{\hat{k}}^{(i)} = \frac{k_{i-1}}{f(\hat{k})}, \quad i = 2, 3,$$

then, condition (1.0.4) leads to

$$\left|C_{\hat{k}}^{(i)}\right| \leq L_1^{-1} \left|\hat{k}\right|^{N+1}.$$

Furthermore, recalling the definition of the function $H = H(\hat{k}\omega)$ given in (1.2.50) it is easy to see that, if $\left|\hat{k}\omega\right| < 1$, then

$$\left|H(\hat{k}\omega)\right| \left|\hat{k}\omega\right| \leq ctant,$$

and, if $\left|\hat{k}\omega\right| > 1$, then

$$\left|H(\hat{k}\omega)\right| \leq ctant.$$

Hence, taking also into account that

$$\left|\hat{k}\omega\right| \leq ctant\ \left|\hat{k}\right|,$$

we finally obtain

$$|S_{\hat{k}}| \leq ctant\ \left|\hat{k}\right|^{N+4},$$

for every $\hat{k} \in \mathbb{Z}^2 \backslash \{(0,0)\}$. Thus, according to Definition 1.3.7, we have that

$$\overline{S} \in \mathcal{S}_1\left(p, -\ln a, \frac{\pi}{2\sqrt{A}}, \varepsilon, \beta\right).$$

Now, in order to prove that

$$\overline{S} \in \mathcal{S}_2\left(p, -\ln a, \frac{\pi}{2\sqrt{A}}, \varepsilon, \beta\right),$$

see Definition 1.3.8, we begin by taking $\hat{k} = (k_1^{(h)}, k_2^{(h)})$ such that $k_1^{(h)}/k_2^{(h)}$ is a best approximation to the golden mean number satisfying

$$k_2^{(h)} \in \left(\varepsilon^{-3/8}, \varepsilon^{-1/2} \left|\ln \varepsilon\right|^{-1/4}\right).$$

Of course, once $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}_3$ (see (1.0.3)) are fixed, we may take small enough values of ε in such a way that, if $k_2^{(h)} \geq \varepsilon^{-3/8}$ and $k_1^{(h)}/k_2^{(h)}$ is a best approximation to the golden mean number, then $(k_1^{(h)}, k_2^{(h)}) \in \Lambda$. Doing so, it is clear that $S_{\hat{k}^{(h)}} \neq 0$. On the other hand, from (1.4.129) in the appendix of this chapter we have, for ε small enough,

$$\left|\hat{k}^{(h)}\omega\right| \geq ctant \; \left|k_2^{(h)}\right|^{-1} \geq ctant \; \left|\hat{k}^{(h)}\right|^{-1}.$$

Therefore, since $\left|H(\hat{k}^{(h)}\omega)\right| > 1$, we use condition (1.0.4) to deduce

$$\left|\tilde{B}^{(i)}_{\hat{k}^{(h)}}\right| \geq ctant \; \left|\hat{k}^{(h)}\right|^{-N-2}.$$

Thus, using the fact that $(\overline{\psi}_1, \overline{\psi}_2) \in \{0, \pi\} \times \{0, \pi\}$, we have

$$|S_{\hat{k}^{(h)}}| \geq ctant \; \left|\hat{k}^{(h)}\right|^{-N-2}$$

and we deduce that

$$\overline{S} \in \mathcal{S}_2\left(p, -\ln a, \frac{\pi}{2\sqrt{A}}, \varepsilon, \beta\right),$$

for every $\varepsilon \in \mathcal{U}_\varepsilon\left(p, -\ln a, \frac{\pi}{2\sqrt{A}}\right)$ and any $\beta \in I_{\tilde{\beta}}(\varepsilon)$.

Finally, since (1.0.5) and (1.3.79) imply

$$\tilde{B}^{(i)}_{-\hat{k}} = -\tilde{B}^{(i)}_{\hat{k}}, \quad \text{for } i = 2, 3,$$

it follows easily that $S_{\hat{k}} = S_{-\hat{k}}$ and therefore (see Definition 1.3.9)

$$\overline{S} \in \mathcal{S}_3\left(p, -\ln a, \frac{\pi}{2\sqrt{A}}, \varepsilon, \beta\right), \tag{1.3.111}$$

for every $\varepsilon \in \mathcal{U}_\varepsilon\left(p, -\ln a, \frac{\pi}{2\sqrt{A}}\right)$ and any $\beta \in I_{\tilde{\beta}}(\varepsilon)$.

Therefore, we have

$$\overline{S} = \sum_{\hat{k} \in \mathbb{Z}^2 \setminus \{(0,0)\}} S_{\hat{k}} \mathcal{E}_{\hat{k}} \in \mathcal{S}\left(p, -\ln a, \frac{\pi}{2\sqrt{A}}, \varepsilon, \beta\right).$$

Moreover, by applying Lemma 1.3.13, we know that, for every $\varepsilon \in \mathcal{U}_\varepsilon$ there exist $\overline{\psi}_i = \overline{\psi}_i(\varepsilon) \in \{0, \pi\}$, $i = 1, 2$, for which

$$S_{\hat{k}(n^0)} S_{\hat{k}(n^1)} > 0$$

where the indices n^0 and n^1 are given by the Main Lemma I for the special choice $c = p$, $l = -\ln a$, $d = \dfrac{\pi}{2\sqrt{A}}$. Hence, these natural numbers n^0 and n^1 only depend on a, p and ε and, in fact, coincide with those indices for which (1.3.109) holds.

Finally, once the conditions $S_{\hat{k}(n^0)} S_{\hat{k}(n^1)} > 0$ (see Lemma 1.3.13) and

$$\overline{S} \in \mathcal{S}\left(p, -\ln a, \frac{\pi}{2\sqrt{A}}, \varepsilon, \beta\right)$$

have been checked, we may apply the Main Lemma I in order to write

$$\overline{S} = 2\left(\sum_{t=n^0,n^1} S_{\hat{k}(t)} \mathcal{E}_{\hat{k}(t)}\right)\left[1 + \mathrm{O}\left(\exp\left(-\frac{|\ln\varepsilon|^{1/5}}{\varepsilon^{1/2}}\right)\right)\right]$$

for every $\varepsilon \in \mathcal{U}_\varepsilon$ and any $\beta \in I_{\tilde{\beta}}(\varepsilon)$.

Therefore, see (1.3.110), we get

$$m^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2) = -4\mu\varepsilon^{-2}\pi A^{-1}\left(\sum_{t=n^0,n^1} S_{\hat{k}(t)} \mathcal{E}_{\hat{k}(t)}\right)\left[1 + \mathrm{O}\left(\exp\left(-\frac{|\ln\varepsilon|^{1/5}}{\varepsilon^{1/2}}\right)\right)\right].$$

Now, from (1.3.79) and (1.3.81), it is easy to see that

$$-2\mu\varepsilon^{-2}\pi A^{-1} S_{\hat{k}(t)} = k_j^{(t)} B_{\hat{k}(t)}^{(i)} \cos(k_1^{(t)}\overline{\psi}_1 + k_2^{(t)}\overline{\psi}_2), \quad t = n^0, n^1,$$

where $B_{\hat{k}}^{(i)}$ are the coefficients taken part in the renormalized Melnikov functions and were obtained in (1.2.53) as being

$$B_{\hat{k}}^{(i)} = -2\mu\varepsilon^{-2}\left|\hat{k}\omega\right| H(\hat{k}\omega)\pi A^{-1} C_{\hat{k}}^{(j+1)}(\hat{k}\omega). \tag{1.3.112}$$

Thus, we deduce that

$$m^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2) = 2A_{i,j}\left[1 + \mathrm{O}\left(\exp\left(-\frac{|\ln\varepsilon|^{1/5}}{\varepsilon^{1/2}}\right)\right)\right] \tag{1.3.113}$$

where

$$A_{i,j} = A_{i,j}(n^0, n^1) = P_0 k_j^{(n^0)} B_{\hat{k}(n^0)}^{(i)} \mathcal{E}_{\hat{k}(n^0)} + P_1 k_j^{(n^1)} B_{\hat{k}(n^1)}^{(i)} \mathcal{E}_{\hat{k}(n^1)}, \tag{1.3.114}$$

for $j = 1, 2$, $i = 2, 3$, n^0, n^1 those indices for which (1.3.109) holds and

$$P_\nu = \cos\left(k_1^{(n^\nu)}\overline{\psi}_1 + k_2^{(n^\nu)}\overline{\psi}_2\right),$$

for $\nu = 0, 1$.

Sixth step. *Decomposing the transversality* Υ.

We come back to the third step of the second part of the proof and recover the definition of the transversality Υ (see (1.3.103)) to write

$$\Upsilon = 4(1 + h(\varepsilon))^2 \Upsilon^* + 2(1 + h(\varepsilon))\tilde{\Upsilon} + \overline{\Upsilon}, \tag{1.3.115}$$

where

$$h(\varepsilon) = \mathrm{O}\left(\exp\left(-\frac{|\ln \varepsilon|^{1/5}}{\varepsilon^{1/2}}\right)\right)$$

is the function used to reach (1.3.113),

$$\begin{aligned}
\Upsilon^* &= A_{2,1}A_{3,2} - A_{2,2}A_{3,1} \\
\tilde{\Upsilon} &= A_{2,1}e^{2,2} + A_{3,2}e^{1,1} - A_{2,2}e^{2,1} - A_{3,1}e^{1,2} \\
\overline{\Upsilon} &= e^{1,1}e^{2,2} - e^{1,2}e^{2,1}
\end{aligned} \tag{1.3.116}$$

and we have denoted $e^{i-1,j} = e^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2)$, for $i = 2, 3$, $j = 1, 2$.

Seventh step. *Approximating the transversality.*

Now, we are going to obtain upper and lower bounds for

$$|\Upsilon^*| = |A_{2,1}A_{3,2} - A_{2,2}A_{3,1}|\,.$$

Let us begin by observing that (1.3.114) and the fact that $|P_\nu| = 1$, for $\nu = 0, 1$, imply

$$|\Upsilon^*| = \mathcal{E}_{\hat{k}^{(n^0)}}\mathcal{E}_{\hat{k}^{(n^1)}}\left|B^{(2)}_{\hat{k}^{(n^0)}}B^{(3)}_{\hat{k}^{(n^1)}} - B^{(2)}_{\hat{k}^{(n^1)}}B^{(3)}_{\hat{k}^{(n^0)}}\right|,$$

where we have also used Proposition 1.4.2 (recall that $|n^0 - n^1| = 1$) to obtain

$$\left|k_1^{(n^0)}k_2^{(n^1)} - k_2^{(n^0)}k_1^{(n^1)}\right| = 1.$$

This last equality, together with the expression of the coefficients $B^{(i)}_{\hat{k}}$ given in (1.3.112), also yields

$$\left|B^{(2)}_{\hat{k}^{(n^0)}}B^{(3)}_{\hat{k}^{(n^1)}} - B^{(2)}_{\hat{k}^{(n^1)}}B^{(3)}_{\hat{k}^{(n^0)}}\right| = \mathcal{L}_0\mathcal{L}_1$$

where, for $\nu = 0, 1$,

$$\mathcal{L}_\nu = \frac{2\mu\pi H(\hat{k}^{(n^\nu)}\omega)\left|\hat{k}^{(n^\nu)}\omega\right|^2}{A\varepsilon^2\left|f(\hat{k}^{(n^\nu)})\right|}.$$

Moreover, from (1.4.129) in the appendix, we know that

$$ctant\ \varepsilon^{1/2}\,|\ln\varepsilon|^{1/2} \le \left|\hat{k}^{(n^\nu)}\omega\right| \le ctant\ \varepsilon^{1/2}\,|\ln\varepsilon|^{1/2}$$

and, as we have already used in the fifth step, $H(\hat{k}^{(n^\nu)}\omega) > 1$ and

$$H(\hat{k}^{(n^\nu)}\omega)(\hat{k}^{(n^\nu)}\omega)^2 \leq ctant\ (\hat{k}^{(n^\nu)}\omega) \leq ctant\ \varepsilon^{1/2} \left|\ln\varepsilon\right|^{1/2}.$$

Finally, from (1.0.4),

$$ctant\ \varepsilon^{N/2} \leq L_2^{-1} \left|\hat{k}^{(n^\nu)}\right|^{-N} \leq \frac{1}{\left|f(\hat{k}^{(n^\nu)})\right|} \leq L_1^{-1} \left|\hat{k}^{(n^\nu)}\right|^{N} \leq ctant\ \varepsilon^{-N/2}.$$

Therefore,

$$ctant\ \mu\varepsilon^{\frac{N}{2}-1} \leq \mathcal{L}_\nu \leq ctant\ \mu\varepsilon^{-\frac{N+4}{2}},$$

from which we deduce

$$ctant\ \mu^2\varepsilon^{N-2}\mathcal{E}_{\hat{k}^{(n^0)}}\mathcal{E}_{\hat{k}^{(n^1)}} \leq |\Upsilon^*| \leq ctant\ \mu^2\varepsilon^{-N-4}\mathcal{E}_{\hat{k}^{(n^0)}}\mathcal{E}_{\hat{k}^{(n^1)}}.$$

Now, since $\varepsilon \in \mathcal{U}_\varepsilon\left(p, -\ln a, \dfrac{\pi}{2\sqrt{A}}\right)$, (1.3.89) gives

$$ctant\ \mu^2\varepsilon^{N-2}\mathcal{E}^2_{\hat{k}^{(n^0)}} \leq |\Upsilon^*| \leq ctant\ \mu^2\varepsilon^{-N-4}\mathcal{E}^2_{\hat{k}^{(n^0)}}. \tag{1.3.117}$$

Eight step. *Final bounds for the transversality.*

To finish the proof of the Main Theorem I it only remains to obtain upper bounds for the functions $\tilde{\Upsilon}$ and $\overline{\Upsilon}$ introduced in (1.3.116).

To this end, let us recall the expressions

$$A_{i,j} = P_0 k_j^{(n^0)} B^{(i)}_{\hat{k}^{(n^0)}}\mathcal{E}_{\hat{k}^{(n^0)}} + P_1 k_j^{(n^1)} B^{(i)}_{\hat{k}^{(n^1)}}\mathcal{E}_{\hat{k}^{(n^1)}}$$

given in (1.3.114) and observe that the Main Lemma I gives

$$\left|k_j^{(n^\nu)}\right| \leq ctant\ \varepsilon^{-1/2}\left|\ln\varepsilon\right|^{-1/2}, \quad \nu = 0, 1. \tag{1.3.118}$$

Then, we obtain

$$\left|k_j^{(n^\nu)} B^{(i)}_{\hat{k}^{(n^\nu)}}\right| \leq ctant\ \mu\varepsilon^{-\frac{N+5}{2}}, \quad \nu = 0,1,\ j = 1,2,\ i = 2,3.$$

To obtain this last bound it is also necessary to take into account the expression of the coefficients $B^{(i)}_{\hat{k}}$ given in (1.3.112), the inequalities

$$\left|\hat{k}^{(n^\nu)}\omega\right| \leq ctant\ \left|\hat{k}^{(n^\nu)}\right|^{-1} \leq ctant\ \varepsilon^{1/2}\left|\ln\varepsilon\right|^{1/2}, \quad \left|H(\hat{k}^{(n^\nu)}\omega)\right|\left|\hat{k}^{(n^\nu)}\omega\right| < ctant$$

and also the fact that (1.0.4) and (1.3.80) give

$$\left|C^{(i)}_{\hat{k}^{(n^\nu)}}\right| < ctant\ \left|k_{i-1}^{(n^\nu)}\right|^{N+1} \leq ctant\ \varepsilon^{-\frac{N+1}{2}}.$$

Now, using also (1.3.89), we may write

$$|A_{i,j}| \le ctant\ \mu\varepsilon^{-\frac{N+5}{2}}\mathcal{E}_{\hat{k}^{(n^0)}}.$$

On the other hand, from (1.3.109) we have

$$\left|e^{i-1,j}\right| \le ctant\ \mu^2 \left|\ln\mu\right| \varepsilon^{-b(2N+11)} \left(\left|\hat{k}^{(n^0)}\right| \mathcal{E}_{\hat{k}^{(n^0)}} + \left|\hat{k}^{(n^1)}\right| \mathcal{E}_{\hat{k}^{(n^1)}}\right).$$

Using (1.3.89) and (1.3.118), we obtain

$$\left|e^{i-1,j}\right| \le ctant\ \mu^2 \left|\ln\mu\right| \varepsilon^{-b(2N+11)-1/2}\mathcal{E}_{\hat{k}^{(n^0)}}.$$

Consequently (see (1.3.116)),

$$\left|\tilde{\Upsilon}\right| \le ctant\ \mu^3 \left|\ln\mu\right| \varepsilon^{-b(2N+11)-\frac{N+6}{2}}\mathcal{E}^2_{\hat{k}^{(n^0)}}$$

and

$$\left|\overline{\Upsilon}\right| \le ctant\ \mu^4 \left|\ln\mu\right|^2 \varepsilon^{-2b(2N+11)-1}\mathcal{E}^2_{\hat{k}^{(n^0)}}.$$

Thus, using the formula (1.3.115), we have

$$\left|\Upsilon - 4(1+h(\varepsilon))^2\Upsilon^*\right| \le ctant\ \mu^3 \left|\ln\mu\right| \varepsilon^{-b(2N+11)-\frac{N+6}{2}}\mathcal{E}^2_{\hat{k}^{(n^0)}},$$

whenever

$$\mu \in (0,\varepsilon^m) \quad \text{with} \quad m > \frac{5}{4}b(2N+11) \quad \text{and} \quad \mu^{1/5}\left|\ln\mu\right| < 1.$$

Therefore, using (1.3.117) and taking (see also (1.1.9))

$$m > \frac{5}{4}\left(\frac{3N}{2} + b(2N+11) + 1\right)$$

we finally deduce

$$ctant\ \mu^2\varepsilon^{N-2}\mathcal{E}^2_{\hat{k}^{(n^0)}} \le |\Upsilon| \le ctant\ \mu^2\varepsilon^{-N-4}\mathcal{E}^2_{\hat{k}^{(n^0)}}.$$

Now, the expressions (see (1.2.52) and the Main Lemma I)

$$\mathcal{E}_{\hat{k}^{(n^0)}} = \exp\left(-\left|\hat{k}^{(n^0)}\right|(p\left|\ln\varepsilon\right| - \ln a) - \frac{\pi\left|\hat{k}^{(n^0)}\omega\right|}{2\sqrt{A}\varepsilon}\right),$$

$$\text{ctant}\ \varepsilon^{-1/2}\left|\ln\varepsilon\right|^{-1/2} \le \left|\hat{k}^{(n^0)}\right| \le \text{ctant}\ \varepsilon^{-1/2}\left|\ln\varepsilon\right|^{-1/2}$$

together with (1.4.129), allow us to write

$$ctant\ \exp\left(-\frac{ctant\ \left|\ln\varepsilon\right|^{1/2}}{\varepsilon^{1/2}}\right) \le \mathcal{E}_{\hat{k}^{(n^0)}} \le ctant\ \exp\left(-\frac{ctant\ \left|\ln\varepsilon\right|^{1/2}}{\varepsilon^{1/2}}\right)$$

where, as usual, we denote by *ctant* several different constants not depending neither on ε nor on μ.

Hence, we conclude the existence of two positive constants b_1 and b_2, depending on N, p and a, but neither on ε nor on μ, such that

$$\mu^2 \exp\left(-\frac{b_1 |\ln \varepsilon|^{1/2}}{\varepsilon^{1/2}}\right) \le |\Upsilon| \le \mu^2 \exp\left(-\frac{b_2 |\ln \varepsilon|^{1/2}}{\varepsilon^{1/2}}\right), \tag{1.3.119}$$

for every $\varepsilon \in \mathcal{U}_\varepsilon\left(p, -\ln a, \frac{\pi}{2\sqrt{A}}\right) = \mathcal{U}_\varepsilon(a,p)$ and any $\mu \in (0, \varepsilon^m)$ with

$$m > \frac{5}{4}\left(\frac{3N}{2} + b(2N+11) + 1\right).$$

Let us finally observe that, instead of $5/4$, we could work with any constant λ bigger than one (and taking values of μ for which $\mu^{\lambda-1}|\ln \mu| < 1$). On the other hand, as we have explained in the fourth step of the proof, it is enough to assume $b > 5/8$ in order to apply the first Perturbing Lemma. Therefore, (1.3.119) holds whenever

$$\mu \in (0, \varepsilon^m) \quad \text{with} \quad m > 3N + 8.$$

Therefore, the Main Theorem I is proved. □

Remark 1.3.15 *In the case in which $N = 0$ and therefore the function f (see (1.0.4)) behaves as a constant, we deduce that the Main Theorem I works for every $\mu \in (0, \varepsilon^m)$ with $m > 8$. Even in this most favourable case, we did not get the optimum value of m. We think that it is more important to extend Arnold-like results from exponentially small values of μ to potentially small ones than to obtain the optimal value of m.*

1.4 Appendix: Continued Fraction Theory

As we have seen along the chapter, the *arithmetics* of the splitting associated to our family of Hamiltonian systems (1.0.1) strongly depends on the leading order behaviour of some numerical series of the following type:

$$S = \sum_{\hat{k}=(k_1,k_2)\in\mathbb{Z}^2\setminus\{(0,0)\}} S_{\hat{k}} \hat{\mathcal{E}}_{\hat{k}}$$

where the coefficients $S_{\hat{k}}$ essentially behave as a finite power of $\left|\hat{k}\right| = |k_1| + |k_2|$ (see Definition 1.3.7 and Definition 1.3.8) and $\hat{\mathcal{E}}_{\hat{k}}$ were given in (1.3.78).

By looking to the expression of the coefficients $\hat{\mathcal{E}}_{\hat{k}}$, one realizes that those indices $\hat{k} = (k_1, k_2)$ for which

$$\left|\hat{k}\omega\right| = |k_1 + k_2\beta|$$

is small will play a very important role in the global behaviour of the whole series.

Let us remark that, if we choose an irrational number β and due to the fact that $(k_1, k_2) \neq (0,0)$, then one can not expect to make $\left|\hat{k}\omega\right| = 0$. Hence, the most interesting choices of (k_1, k_2) will be those ones for which k_1/k_2 is a best approximation to β according to the following definition:

Definition 1.4.1 *We call a fraction k_1/k_2 where $-k_1$, $k_2 \in \mathbb{N}^*$, best approximation to β, if for any integer $0 < k_2' \leq k_2$ and all entire number k_1', not equal to k_1 if $k_2 = k_2'$, one has:*

$$\min_{k_1'} |k_1' + k_2'\beta| > |k_1 + k_2\beta| .$$

It is well-known that, for every β, there is an increasing sequence $\{k_2^{(j)}\}_{j\in\mathbb{N}}$ of positive integers with $k_2^{(0)} = 1$ and a sequence $\{k_1^{(j)}\}_{j\in\mathbb{N}}$ of integers (which are negative if β is positive) such that $k_1^{(j)}/k_2^{(j)}$ are best approximations to β, these sequences being finite for a rational β and infinite when it is irrational.

The computation of best approximations is closely related to the Continued Fraction Theory: For each real number $\beta > 0$ there exists a unique expansion into continued fraction

$$\beta = [a_0, a_1, \ldots] = a_0 + \frac{1}{[a_1, a_2, \ldots]} = a_0 + \frac{1}{a_1 + \dfrac{1}{[a_2, a_3, \ldots]}} =$$
$$= [a_0, \ldots, a_{j-1}, [a_j, a_{j+1}, \ldots]] ,$$

where $a_i \in \mathbb{N}$ for all $i \geq 0$.

Then, see [24] for details, one may check that the best approximations to β can be computed by using the following recurrent formulae:

$$k_1^{(-2)} = 0, \quad k_2^{(-2)} = 1, \quad k_1^{(-1)} = -1, \quad k_2^{(-1)} = 0,$$

and for $j \geq 0$,

$$k_1^{(j)} = a_j k_1^{(j-1)} + k_1^{(j-2)}, \quad k_2^{(j)} = a_j k_2^{(j-1)} + k_2^{(j-2)}. \tag{1.4.120}$$

Moreover, from [24], we also recover the following result:

Proposition 1.4.2 *For $j \geq -1$, it follows that*

$$k_2^{(j+1)} k_1^{(j)} - k_1^{(j+1)} k_2^{(j)} = (-1)^j .$$

Let us remark that Proposition 1.4.2 implies

$$k_1^{(j)} k_2^{(j+1)} + k_2^{(j)} k_1^{(j-1)} = k_1^{(j+1)} k_2^{(j)} + k_1^{(j)} k_2^{(j-1)} .$$

Hence, defining

$$z_j = [a_{j+1}, a_{j+2}, \ldots] = a_{j+1} + \frac{1}{z_{j+1}} \tag{1.4.121}$$

the expressions given in (1.4.120) allow us to conclude that the quotients

$$\frac{k_1^{(j)} z_j + k_1^{(j-1)}}{k_2^{(j)} z_j + k_2^{(j-1)}}$$

do not depend on j. Thus, since from Proposition 1.4.2 we get

$$\left| \frac{k_1^{(j)}}{k_2^{(j)}} - \frac{k_1^{(j)} z_j + k_1^{(j-1)}}{k_2^{(j)} z_j + k_2^{(j-1)}} \right| = \frac{1}{k_2^{(j)} (k_2^{(j)} z_j + k_2^{(j-1)})},$$

and $\left| k_1^{(j)} + \beta k_2^{(j)} \right|$ also goes to zero when j tends to infinity, we deduce that

$$\beta = -\frac{k_1^{(j)} z_j + k_1^{(j-1)}}{k_2^{(j)} z_j + k_2^{(j-1)}}, \tag{1.4.122}$$

for every $j \in \mathbb{N}$. Therefore

$$k_1^{(j)} + k_2^{(j)} \beta = \frac{(-1)^j}{k_2^{(j)} (z_j + x_j)}, \tag{1.4.123}$$

where we have introduced the notation

$$x_j = \frac{k_2^{(j-1)}}{k_2^{(j)}} = \frac{1}{a_j + x_{j-1}}.$$

It is also clear that

$$\left| \hat{k}^{(j)} \omega \right| = \frac{1}{k_2^{(j)} (z_j + x_j)} = \frac{A_j(\beta)}{k_2^{(j)}}, \tag{1.4.124}$$

where, for every $j \in \mathbb{N}$, we set

$$A_j(\beta) = \frac{1}{z_j + x_j}.$$

For the golden mean

$$\tilde{\beta} = \frac{\sqrt{5} + 1}{2} = [1, 1, 1, 1, \ldots]$$

one has

$$A_j(\tilde{\beta}) = \frac{k_2^{(j)}}{k_2^{(j-1)} + k_2^{(j)} \tilde{\beta}}.$$

Moreover, since for the golden mean case it is easy to see that $k_2^{(j)} = -k_1^{(j-1)}$, we may use (1.4.124) to obtain

$$\begin{aligned} \left| A_j(\tilde{\beta}) - \frac{1}{\tilde{\beta} + \tilde{\beta}^{-1}} \right| &= \frac{\left| k_1^{(j-1)} + k_2^{(j-1)} \tilde{\beta} \right|}{(k_2^{(j-1)} + k_2^{(j)} \tilde{\beta})(1 + \tilde{\beta}^2)} = \\ &= \frac{1}{(k_2^{(j-1)} + k_2^{(j)} \tilde{\beta})(k_2^{(j-2)} + k_2^{(j-1)} \tilde{\beta})(1 + \tilde{\beta}^2)} \leq \frac{1}{(k_2^{(j)})^2}. \end{aligned} \tag{1.4.125}$$

Hence, for j large enough (for instance, those $j \in \mathbb{N}$ such that $k_1^{(j)}/k_2^{(j)}$ is a best approximation to the golden mean with $k_2^{(j)} > \varepsilon^{-3/8}$)

$$\frac{1}{2(\tilde{\beta}+\tilde{\beta}^{-1})} \leq \left| A_j(\tilde{\beta}) \right| \leq \frac{2}{\tilde{\beta}+\tilde{\beta}^{-1}}.$$

Furthermore, along the first chapter we always work with frequency values β belonging to some neighbourhood $I_{\tilde{\beta}}$ of the golden mean value satisfying (see the statement of the Main Lemma I)

$$\frac{1}{100}\varepsilon^{5/3} \left|\ln\varepsilon\right|^{1/2} \leq \text{length}(I_{\tilde{\beta}}) \leq \frac{1}{2}\varepsilon^{5/3} \left|\ln\varepsilon\right|^{1/2}.$$

Among the properties displayed by the neighbourhood $I_{\tilde{\beta}}$ we make stand out here the following one: If $\beta \in I_{\tilde{\beta}}$, then (see Chapter 4 for details) the best approximations $k_1^{(j)}/k_2^{(j)}$ to β satisfying

$$k_2^{(j)} < \varepsilon^{-1/2} \left|\ln\varepsilon\right|^{-1/4}$$

coincide with the best approximations to the golden mean number $\tilde{\beta}$. Then, from (1.4.122) we may write

$$\beta = -\frac{k_1^{(j)} z_j(\beta) + k_1^{(j-1)}}{k_2^{(j)} z_j(\beta) + k_2^{(j-1)}}, \quad \tilde{\beta} = -\frac{k_1^{(j)} z_j(\tilde{\beta}) + k_1^{(j-1)}}{k_2^{(j)} z_j(\tilde{\beta}) + k_2^{(j-1)}}.$$

Hence, from Proposition 1.4.2,

$$\beta - \tilde{\beta} = \frac{(-1)^j (z_j(\tilde{\beta}) - z_j(\beta))}{\left(k_2^{(j)} z_j(\tilde{\beta}) + k_2^{(j-1)}\right)\left(k_2^{(j)} z_j(\beta) + k_2^{(j-1)}\right)}. \tag{1.4.126}$$

In this way, we get

$$\begin{aligned} A_j(\beta) - A_j(\tilde{\beta}) &= \frac{(z_j(\tilde{\beta}) - z_j(\beta))(k_2^{(j)})^2}{\left(k_2^{(j)} z_j(\tilde{\beta}) + k_2^{(j-1)}\right)\left(k_2^{(j)} z_j(\beta) + k_2^{(j-1)}\right)} = \\ &= (-1)^j (k_2^{(j)})^2 (\beta - \tilde{\beta}). \end{aligned} \tag{1.4.127}$$

Therefore, for every $\beta \in I_{\tilde{\beta}}$ and any $j \in \mathbb{N}$ for which $k_1^{(j)}/k_2^{(j)}$ is a best approximation to $\tilde{\beta}$ with

$$k_2^{(j)} < \varepsilon^{-1/2} \left|\ln\varepsilon\right|^{-1/4},$$

one has

$$\left| A_j(\beta) - A_j(\tilde{\beta}) \right| \leq ctant\ \varepsilon^{2/3}.$$

Thus, for every $\beta \in I_{\tilde{\beta}}$ and every $j \in \mathbb{N}$ for which $k_1^{(j)}/k_2^{(j)}$ is a best approximation to $\tilde{\beta}$ with

$$k_2^{(j)} \in (\varepsilon^{-3/8}, \varepsilon^{-1/2} \left|\ln\varepsilon\right|^{-1/4}),$$

it follows that

$$\frac{1}{4(\tilde{\beta}+\tilde{\beta}^{-1})} \leq |A_j(\beta)| \leq \frac{4}{\tilde{\beta}+\tilde{\beta}^{-1}} \qquad (1.4.128)$$

in such a way that (1.4.124) leads to

$$ctant \ \left|k_2^{(j)}\right|^{-1} \leq \left|\hat{k}^{(j)}\omega\right| \leq ctant \ \left|k_2^{(j)}\right|^{-1}, \qquad (1.4.129)$$

with

$$\left|\hat{k}^{(j)}\omega\right| = \left|k_1^{(j)} + \beta k_2^{(j)}\right|.$$

We also point out that, since the best approximations to the golden mean, $k_1^{(n^0)}/k_2^{(n^0)}$ and $k_1^{(n^1)}/k_2^{(n^1)}$, used during the proof of the Main Theorem I satisfy

$$k_2^{(t)} \in \left(\varepsilon^{-3/8}, \varepsilon^{-1/2}\left|\ln\varepsilon\right|^{-1/4}\right), \quad t = n^0, n^1,$$

we have that (1.4.129) holds for $j = n^0, n^1$.

Chapter 2

Arnold's diffusion

In this chapter we will prove the existence of micro-diffusion (as was defined in the Introduction) in the phase space associated to the following family of Hamiltonian systems (see (0.0.4)):

$$H_{\varepsilon,\mu}(x, y, I_1, I_2, \theta_1, \theta_2) = I_1 + \frac{I_2^2}{2\sqrt{\varepsilon}} + \frac{y^2}{2} + \varepsilon(\cos x - 1)\left(1 + \mu m(\theta_1, \theta_2)\right), \qquad (2.0.1)$$

where ε and μ are small positive parameters (μ the perturbing one) and the function m is introduced below. As we said in the Introduction, we will only need to impose that $\mu \in (0, \varepsilon^w)$ (w some positive constant essentially depending on the properties of the function m) to prove the Main Theorem II (see Theorem 0.0.4). The Main Theorem II establishes the existence of micro-diffusion in certain region of the phase space contained in

$$\left\{(x, y, I_1, I_2, \theta_1, \theta_2) \in [0, 2\pi] \times \mathbb{R}^3 \times \mathbb{T}^2 : \; \left| I_2 - \tilde{\beta} \right| \leq \varepsilon^{5/6} \right\},$$

where $\tilde{\beta} = \dfrac{\sqrt{5} + 1}{2}$.

The function m taking part in the perturbing term is given by

$$m(\theta_1, \theta_2) = \sum_{(k_1,k_2)\in\Lambda} m_{k_1,k_2} \cos(k_1\theta_1 + k_2\theta_2).$$

Let us observe that, unlike in the first chapter (where we considered odd perturbations on the angles), in the present chapter we choose an even perturbation on the angles. The reason to do so is related to the symmetry properties that allow us to locate in an easy way homoclinic orbits for the perturbed system (see Remark 2.1.5 for details).

On the subset of indexes Λ used to define the function m we impose the same assumption that the respective one given in the first chapter (see (1.0.3)), i.e., Λ can be any subset of $\mathbb{Z}^2$ which could be written as

$$\Lambda = \Lambda(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3) = \bigcup_{i=1}^{3} \Lambda_i(\mathcal{N}_i) \qquad (2.0.2)$$

where, for some positive constants $\mathcal{N}_i$,

$$\begin{aligned}\Lambda_i(\mathcal{N}_i) &= \{(k_1,k_2)\in\mathbb{Z}^2 : |k_i|\geq\mathcal{N}_i,\ k_{3-i}=0\}, \quad i=1,2\\ \Lambda_3(\mathcal{N}_3) &= \{(k_1,k_2)\in\mathbb{Z}^2 : |k_1|+|k_2|\geq\mathcal{N}_3,\ k_1k_2\neq 0\}.\end{aligned}$$

We also assume that the Fourier coefficients of m satisfy

$$m_{k_1,k_2} = g(k_1,k_2)\delta(k_1,k_2), \tag{2.0.3}$$

for every $(k_1,k_2)\in\Lambda$, that there exist two positive constants r_1 and r_2 for which

$$\sup_{(k_1,k_2)\in\Lambda} |g(k_1,k_2)\exp(r_1|k_1|+r_2|k_2|)| < \infty \tag{2.0.4}$$

and that there exist positive constants L_1, L_2 and N such that

$$L_1(|k_1|+|k_2|)^{-N} \leq |\delta(k_1,k_2)| \leq L_2(|k_1|+|k_2|)^N, \tag{2.0.5}$$

for every $(k_1,k_2)\in\Lambda$.

In order to get useful properties for the renormalized Melnikov functions (see, in particular, how (2.3.80) is obtained) we also assume that

$$m_{k_1,k_2} = m_{-k_1,-k_2} \tag{2.0.6}$$

for every $(k_1,k_2)\in\Lambda$.

Finally, we state the last assumption on the Fourier coefficients m_{k_1,k_2} of the function m: There exists $n_0\in\mathbb{N}$ such that, for every best approximation (see Definition 1.4.1) $k_1^{(j)}/k_2^{(j)}$ to the golden mean number with $j\geq n_0$, it follows that

$$\left|g(k_1^{(j)},k_2^{(j)})\right| \geq ctant\ \exp\left(-\left|k_1^{(j)}\right|r_1-\left|k_2^{(j)}\right|r_2\right), \tag{2.0.7}$$

where r_1 and r_2 are the constants for which (2.0.4) holds.

Assumption (2.0.7) implies the existence of a maximal complex strip

$$\mathcal{B}_2 = \left\{(\theta_1,\theta_2)\in\mathbb{C}^2 :\ |\mathrm{Im}\ \theta_i| < r_i,\ i=1,2\right\}$$

in which the function m is analytic (the function m can not be prolonged analytically onto a larger strip). See also [7] where the same kind of assumptions are imposed to the considered perturbation, in the case in which $N=0$.

Immediately, we are going to introduce new coordinates in order to put our Hamiltonian (2.0.1) in a more convenient way. In particular, let us consider a new set of variables $(X,Y,J_1,J_2,\theta_1,\theta_2)$ where

$$X=x, \quad Y=\frac{y}{\sqrt{\varepsilon}}, \quad J_1=\frac{I_1}{\sqrt{\varepsilon}}I_1, \quad J_2=\frac{I_2}{\sqrt{\varepsilon}}.$$

Notice that, although this change of variables is not symplectic, the (new) equations of motion are still canonical and they are associated to the Hamiltonian

$$\hat{H}_{\varepsilon,\mu}(X,Y,J_1,J_2,\theta_1,\theta_2) = J_1 + \frac{J_2^2}{2} + \sqrt{\varepsilon}\left(\frac{Y^2}{2} + (\cos X - 1)(1+\mu m(\theta_1,\theta_2))\right).$$

Let us make a rescaling of time in order to express the above Hamiltonian in the following way

$$H_{\varepsilon,\mu}(x,y,I_1,I_2,\theta_1,\theta_2) = \frac{I_1}{\sqrt{\varepsilon}} + \frac{I_2^2}{2\sqrt{\varepsilon}} + \frac{y^2}{2} + (\cos x - 1)\left(1+\mu m(\theta_1,\theta_2)\right), \qquad (2.0.8)$$

where, in order to make the exposition more clear, we have recovered the old notation $(x,y,I_1,I_2,\theta_1,\theta_2)$ for the definitive variables.

Using the notation $\hat{\theta} = (\theta_1,\theta_2)$, $\hat{I} = (I_1,I_2)$, we write the dynamical system generated by the Hamiltonian (2.0.8):

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \sin x(1+\mu m(\hat{\theta})) \\ \dot{I}_j &= -\mu(\cos x - 1)\frac{\partial m}{\partial \theta_j}(\hat{\theta}), \quad j=1,2 \\ \dot{\theta}_1 &= \frac{1}{\sqrt{\varepsilon}}, \quad \dot{\theta}_2 = \frac{I_2}{\sqrt{\varepsilon}}. \end{aligned} \qquad (2.0.9)$$

It is clear that, when $\mu = 0$, we have an integrable Hamiltonian system exhibiting a two-parameter family of invariant two-dimensional tori

$$T_{\beta_1,\beta_2} = \left\{(x,y,I_1,I_2,\theta_1,\theta_2):\ x=y=0,\ \hat{I} = (\beta_1,\beta_2)\right\}.$$

Due to the choice of our Hamiltonian family $H_{\varepsilon,\mu}$, it follows that, as in the first chapter, these invariant tori survive the perturbation: They are still invariant for the perturbed systems ($\mu \neq 0$).

In the following section we are going to prove that, for μ sufficiently small, those invariant tori have unstable ($W^-(T_{\beta_1,\beta_2})$) and stable ($W^+(T_{\beta_1,\beta_2})$) manifolds.

We finish this subsection by pointing out that the main goal of this chapter is to prove the Main Theorem II (see Theorem 0.0.4) which asserts that, under the above introduced assumptions on the function $m = m(\hat{\theta})$, there exist *distant* tori $T_{\beta_1^0,\beta_2^0}$ and $T_{\beta_1^n,\beta_2^n}$ which are *connected* and moreover, this fact holds whenever ε belongs to some open real subset and $\mu \in (0,\varepsilon^w)$, with $w > \frac{3N}{2} + 7$ (see (2.0.5) for the definition of the constant N).

Here "distant" means that

$$\left|\beta_2^n - \beta_2^0\right| > ctant\ \varepsilon^{5/6}$$

and "connected" means

$$W^-(T_{\beta_1^0,\beta_2^0}) \subset \overline{W^-(T_{\beta_1^n,\beta_2^n})}.$$

Of course, one of the greatest inherent difficulties for proving this result is that it does not hold for $\mu = 0$.

The way in which the Main Theorem II is proved in this chapter strongly depends on the conclusions of Theorem 0.0.6. When $\mu = 0$ each of the invariant tori T_{β_1,β_2} possesses a three-dimensional homoclinic manifold and Theorem 0.0.6 gives lower estimates on the splitting of this unperturbed manifold when the perturbation is considered. These events were described in the first chapter thus that we are not going to detail them here.

The idea to prove Theorem 0.0.6 is to follow the scheme that we have used to prove the Main Theorem I. Nevertheless, some new considerations have to be taken: Let us again consider the dynamical system (2.0.9) and observe that, for the perturbed and the unperturbed cases, the dynamics on the invariant tori T_{β_1,β_2} are very similar to that of the invariant tori T_{α_1,α_2} for the Hamiltonian systems studied in the first chapter (see (1.1.6)). However, outside those invariant tori the behaviour of the orbits of the system (2.0.9) is more complicated (even when $\mu = 0$) and this essentially happens due to the fact that the respective equations on $\dot\theta_2$ (compare (1.1.6) with (2.0.9)) are substantially different. These differences obligate us to introduce, once more, the guidelines used to prove the main result of the first chapter in order to emphasize those partial results in which extra arguments are needed for proving Theorem 0.0.6.

Remark 2.0.3 *Since we pretend to show the existence of transition chains in the phase space of (2.0.1), we need, in addition to the results stated in the first chapter, a new one: The Inclination Lemma (see Lemma 2.3.12). The Inclination Lemma, together with the estimates of the splitting size given by Theorem 0.0.6, will imply the Main Theorem II, see Theorem 0.0.4.*

Preliminaries. Setting new coordinates

As in the previous chapter, the study of the system (2.0.9) has to be carried on by working with complex values of the initial phases $\psi_i = \theta_i(0)$, $i = 1, 2$, and therefore, at least for the perturbed case ($\mu \neq 0$), all the variables x, y, I_1, I_2, θ_1 and θ_2 have to be considered as complex variables.

The existence of the positive constants r_1 and r_2 for which (2.0.4) holds, implies that, if we define

$$\mathcal{B}_2' = \{(\theta_1, \theta_2) \in \mathbb{C}^2 : |\text{Im } \theta_i| \leq r_i - \varepsilon^b,\ i = 1, 2\}, \tag{2.0.10}$$

with b some positive constant which will be fixed along the arguments (see condition (2.3.75)), then, in the same way as in the proof of Lemma 1.1.1, we get

$$\|m\|_{\mathcal{B}_2'} = \sup_{(\theta_1,\theta_2)\in\mathcal{B}_2'} |m(\theta_1, \theta_2)| \leq ctant\ \varepsilon^{-b(N+2)}. \tag{2.0.11}$$

Therefore, applying Cauchy estimates (see Lemma 1.1.2), we also conclude that

$$\left\| \frac{\partial m}{\partial \theta_j} \right\|_{\mathcal{B}_2''} \leq ctant\ \varepsilon^{-b(N+3)}, \tag{2.0.12}$$

where

$$\mathcal{B}_2'' = \left\{ (\theta_1, \theta_2) \in \mathbb{C}^2 : \ |\mathrm{Im}\ \theta_i| \leq r_i - 2\varepsilon^b,\ i = 1, 2 \right\}.$$

From the above considerations, and taking $\mu \in (0, \varepsilon^w)$, with $w \geq b(N+3)$, we conclude that (2.0.9) is a small perturbation of an integrable Hamiltonian system.

We will make a change of variables in order to put the integrable part of (2.0.9) in a convenient way. To this end, we follow the same ideas as in Chapter 1 (see (1.1.10)) to define

$$\xi = \frac{1}{\sqrt{2}}\,(y - x)\,, \quad \nu = \frac{1}{\sqrt{2}}\,(y + x)\,. \tag{2.0.13}$$

Then, the new equations of motion in coordinates (ξ, ν) are

$$\dot{\xi} = -\xi + \frac{f'(x)}{\sqrt{2}} + \frac{\mu}{\sqrt{2}} m(\hat{\theta}) \sin x, \quad \dot{\nu} = \nu + \frac{f'(x)}{\sqrt{2}} + \frac{\mu}{\sqrt{2}} m(\hat{\theta}) \sin x$$

where $f(x) = -\cos x + 1 - \dfrac{1}{2}x^2$ and $x = x(\xi, \nu)$. The unperturbed equations

$$\dot{\xi} = -\xi + \frac{1}{\sqrt{2}} f'(x), \quad \dot{\nu} = \nu + \frac{1}{\sqrt{2}} f'(x)$$

are associated to the Hamiltonian

$$H(\xi, \nu) = -\xi\nu + f(x(\xi, \nu)).$$

Therefore, Lemma 1.1.4 guarantees the existence of an analytic change of variables $\xi = \xi(q, p)$, $\nu = \nu(q, p)$ defined on

$$B_\sigma^2 = \left\{ (q, p) \in \mathbb{C}^2 : |q| < \sigma, |p| < \sigma \right\},$$

with σ some small positive constant such that the Hamiltonian $H = H(\xi, \nu)$ takes the form

$$\tilde{H}(q, p) = -(qp + F(qp)). \tag{2.0.14}$$

As in the first chapter, we will denote by $(q, p) = \varphi(x, y)$ the change of coordinates transforming the initial Hamiltonian

$$H_1(x, y) = \frac{y^2}{2} + \cos x - 1$$

into the Hamiltonian $\tilde{H} = \tilde{H}(q, p)$ given in (2.0.14).

Then, the whole initial Hamiltonian dynamical system (2.0.9) can be written in coordinates $(q, p, \hat{I}, \hat{\theta})$ in the following way

$$\begin{aligned}
\dot{q} &= -q(1+F_J) + (q^\xi + q^\nu)\frac{\mu}{\sqrt{2}} m(\hat{\theta}) \sin\tilde{x} \\
\dot{p} &= p(1+F_J) + (p^\xi + p^\nu)\frac{\mu}{\sqrt{2}} m(\hat{\theta}) \sin\tilde{x} \\
\dot{I}_j &= -\mu\,(\cos\tilde{x} - 1)\,\frac{\partial m}{\partial \theta_j}(\hat{\theta}), \quad j = 1, 2 \\
\dot{\theta}_1 &= \frac{1}{\sqrt{\varepsilon}}, \qquad \dot{\theta}_2 = \frac{I_2}{\sqrt{\varepsilon}},
\end{aligned} \tag{2.0.15}$$

where $q^\xi = q^\xi(q,p) = \dfrac{\partial q}{\partial \xi}(\xi(q,p), \nu(q,p))$ (the equivalent for q^ν, p^ξ and p^ν), $\tilde{x} = \tilde{x}(q,p) = x(\xi(q,p), \nu(q,p))$ and $F_J = F'(J), \quad J = qp$.

Here we refer the reader to Remark 1.1.5 in order to recall that, although equations (2.0.15) do not correspond to a Hamiltonian system, they will be used to give topological properties for the invariant manifolds of T_{β_1,β_2}.

2.1 Whiskered tori

Let us consider the set

$$B^2_\sigma \times \mathbb{C}^2 \times \mathcal{B}''_2 = \left\{ (q, p, I_1, I_2, \theta_1, \theta_2) : \ |q| < \sigma, |p| < \sigma, |\mathrm{Im}\ \theta_i| \le r_i - 2\varepsilon^b, \ i = 1, 2 \right\},$$

where σ is the constant given by Lemma 1.1.4 and let us denote by ϕ_t the flow associated to the system (2.0.15).

The subsets of $B^2_\sigma \times \mathbb{C}^2 \times \mathcal{B}''_2$ defined by

$$T_{\beta_1,\beta_2} = \left\{ (q, p, I_1, I_2, \theta_1, \theta_2) : \ p = q = 0, \hat{I} = (\beta_1, \beta_2) \right\}$$

are invariant tori for the flow ϕ_t and, in the present section, we are going to obtain, for every μ small enough, analytic expressions for their non-trivial invariant stable and unstable manifolds.

As in the first chapter, these invariant manifolds, $W^+(T_{\beta_1,\beta_2})$ and $W^-(T_{\beta_1,\beta_2})$, are going to be obtained by using Poincaré transformations associated to the flow ϕ_t.

Let $(q^0, p^0) \in B^2_\sigma = \{(q,p) \in \mathbb{C}^2 : \ |q| < \sigma, |p| < \sigma\}$ and consider $(\tilde{q}, \tilde{p})$ the solution of

$$\begin{cases} \dot{q} = -q(1+F_J), \quad \dot{p} = p(1+F_J) \\ \tilde{q}(0) = q^0, \quad \tilde{p}(0) = p^0. \end{cases}$$

Then, if we consider $r(t) = (q(t), p(t), I_1(t), I_2(t), \theta_1(t), \theta_2(t))$ the solution of the system (2.0.15) satisfying the initial condition $r(0) = (q^0, p^0, I^0_1, I^0_2, \theta^0_1, \theta^0_2) \in B^2_\sigma \times \mathbb{C}^2 \times \mathcal{B}''_2$, we have that

$$
\begin{aligned}
q(2\pi\sqrt{\varepsilon}) &= q^0\exp(-2\pi\sqrt{\varepsilon}(1+F_J)) + \frac{\mu}{\sqrt{2}}\int_0^{2\pi\sqrt{\varepsilon}} (q^{\xi}+q^{\nu})m(\hat{\theta})\sin\tilde{x}ds- \\
&\quad - \int_0^{2\pi\sqrt{\varepsilon}} (1+F_J)(q(s)-\tilde{q}(s))ds, \\
p(2\pi\sqrt{\varepsilon}) &= p^0\exp(2\pi\sqrt{\varepsilon}(1+F_J)) + \frac{\mu}{\sqrt{2}}\int_0^{2\pi\sqrt{\varepsilon}} (p^{\xi}+p^{\nu})m(\hat{\theta})\sin\tilde{x}ds+ \\
&\quad + \int_0^{2\pi\sqrt{\varepsilon}} (1+F_J)(p(s)-\tilde{p}(s))ds, \qquad (2.1.16) \\
I_j(2\pi\sqrt{\varepsilon}) &= I_j^0 - \mu\int_0^{2\pi\sqrt{\varepsilon}} (\cos\tilde{x}-1)\frac{\partial m}{\partial\theta_j}(\hat{\theta})ds, \quad j=1,2, \\
\theta_1(2\pi\sqrt{\varepsilon}) &= \theta_1^0 + 2\pi, \\
\theta_2(2\pi\sqrt{\varepsilon}) &= \theta_2^0 + 2\pi I_2^0 - \mu\varepsilon^{-\frac{1}{2}}\int_0^{2\pi\sqrt{\varepsilon}}\left(\int_0^s (\cos\tilde{x}-1)\frac{\partial m}{\partial\theta_2}(\hat{\theta})d\tau\right)ds.
\end{aligned}
$$

Hence, we obtain a Poincaré map $P = \phi_{2\pi\sqrt{\varepsilon}}$ given by

$$P(q,p,I_1,I_2,\theta_1,\theta_2) = (q',p',I_1',I_2',\theta_1',\theta_2')$$

with

$$
\begin{aligned}
q' &= qa^{-1} + \frac{\mu\varepsilon^{-b(N+3)+\frac{1}{2}}}{\sqrt{2}} f_1(q,p,\theta_1,\theta_2) \\
p' &= pa - \frac{\mu\varepsilon^{-b(N+3)+\frac{1}{2}}}{\sqrt{2}} f_2(q,p,\theta_1,\theta_2) \\
I_j' &= I_j - \mu\varepsilon^{-b(N+3)+\frac{1}{2}} f_{j+2}(q,p,\theta_1,\theta_2), \quad j=1,2 \\
\theta_1' &= \theta_1 + 2\pi, \\
\theta_2' &= \theta_2 + 2\pi I_2 + \mu\varepsilon^{-b(N+3)+\frac{1}{2}} f_5(q,p,\theta_1,\theta_2), \qquad (2.1.17)
\end{aligned}
$$

where

$$a = \exp(2\pi\sqrt{\varepsilon}(1+F_J)).$$

Applying the same method as the one used to prove Lemma 1.1.6, we obtain the following result:

Lemma 2.1.1 *For every* $(q,p,\theta_1,\theta_2) \in B_\sigma^2 \times \mathcal{B}_2''$ *it follows that*

$$|f_i(q,p,\theta_1,\theta_2)| \le ctant\ (|p|+|q|),$$

for $i = 1,2,3,4,5$.

Notice that, for every θ_1^0 fixed with $|\text{Im }\theta_1^0| \le r_1 - 2\varepsilon^b$, the sets

$$\mathcal{T}_{\beta_1,\beta_2}(\theta_1^0) = \left\{(q,p,I_1,I_2,\theta_1,\theta_2) \in T_{\beta_1,\beta_2} : \theta_1 = \theta_1^0\right\}$$

are invariant under the transformation P given in (2.1.17). Furthermore, to prove the existence of non-trivial local manifolds for the ϕ-invariant tori T_{β_1,β_2} it is enough to prove the same result for each one of the P-invariant sets $\mathcal{T}_{\beta_1,\beta_2}(\theta_1^0)$.

To this end, let us fix $\theta_1 = \theta_1^0$ and write $\theta = \theta_2$. Let us consider, for every positive constant $r \le r_2 - 2\varepsilon^b$, the Banach space of holomorphic functions $\mathcal{H}(\Delta(\sigma,r))$ (used in the first chapter) formed by the analytic functions defined on $(-\sigma,\sigma) \times [0,2\pi]$ that admit an holomorphic extension to the complex domain

$$\Delta(\sigma,r) = \{(q,\theta) \in \mathbb{C}^2 : \; |q| < \sigma, |\text{Im }\theta| < r\}$$

and are continuous on the closure of $\Delta(\sigma,r)$. We will choose a subset $\overline{\mathcal{H}}$ of $\mathcal{H}(\Delta(\sigma,r))$ defined by $\overline{\mathcal{H}} = \{g \in \mathcal{H}(\Delta(\sigma,r)) : \|g\|_{\sigma,r} \le A_0\}$, where A_0 is a constant chosen in the following way: Let f_i, $i = 1,\ldots,5$ be the analytic functions defined on $B_\sigma^2 \times \mathcal{B}_2''$ for which the relations given in (2.1.17) hold. Let us set, for every $C \in \overline{\mathcal{H}}$, the functions

$$C_i : (q,\theta) \in \Delta(\sigma,r) \to C_i(q,\theta) = f_i\left(q, \mu\varepsilon^{-b(N+3)} qC(q,\theta), \theta\right), \tag{2.1.18}$$

for $i = 1,\ldots,5$. Recall that, from Lemma 2.1.1 and using that $C \in \overline{\mathcal{H}}$, we may prove that there exists a positive constant $\tilde{\mathcal{F}}$ such that

$$|C_i(q,\theta)| \le \tilde{\mathcal{F}}\,|q|\,(1 + A_0\mu\varepsilon^{-b(N+3)}) = \mathcal{F}\,|q|\,, \tag{2.1.19}$$

for every $(q,\theta) \in \Delta(\sigma,r)$.

Then, we take A_0 large enough (and μ sufficiently small) so that

$$A_0 \ge \frac{\mathcal{F}}{\sqrt{2}\pi(1 + \text{Re }F_J^*)}, \tag{2.1.20}$$

where

$$F_J^* = F_J^*(q,\theta) = F_J(q, \mu\varepsilon^{-b(N+3)} qC(q,\theta)).$$

We will prove the existence of the local stable manifold of $\mathcal{T}_{\beta_1,\beta_2}(\theta_1^0)$ (see Theorem 2.1.2) in the same way as in the proof of Theorem 1.1.8.

Theorem 2.1.2 *For any complex constants β_1 and β_2, every $\gamma \in (0,1)$ and any positive parameters ε and μ satisfying $\mu \in (0,\varepsilon^w)$, $w = w(b,N)$ large enough, there exists a unique $(A_{\varepsilon,\mu}^+, J_{\varepsilon,\mu}^{+,1}, J_{\varepsilon,\mu}^{+,2})$ in $\overline{\mathcal{H}}^3$ such that the local stable manifold, $W_{loc}^+(\mathcal{T}_{\beta_1,\beta_2}(\theta_1^0))$, of the P-invariant set $\mathcal{T}_{\beta_1,\beta_2}(\theta_1^0)$ can be written as the graph of an analytic function $R = R(q,\theta)$ defined on*

$$\{(q,\theta) \in \Delta(\sigma,r) : \; |q| < (1-\gamma)\sigma, \; |\text{Im }\theta| < r - \varepsilon^b\},$$

whose components in coordinates $(q,p,\hat{I},\theta)$ are given by

$$\begin{aligned} R_1(q,\theta) &= q, \quad R_2(q,\theta) = \mu\varepsilon^{-b(N+3)} qA_{\varepsilon,\mu}^+(q,\theta) \\ R_{2+j}(q,\theta) &= \beta_j + \mu\varepsilon^{-b(N+3)} qJ_{\varepsilon,\mu}^{+,j}(q,\theta), \; j = 1,2, \quad R_5(q,\theta) = \theta. \end{aligned}$$

Proof

The existence of an element $(A^{+}_{\varepsilon,\mu}, J^{+,1}_{\varepsilon,\mu}, J^{+,2}_{\varepsilon,\mu})$ in $\overline{\mathcal{H}}^3$ satisfying the required properties follows when we prove the existence of a fixed point for the map

$$M : (C, D, E) \in \overline{\mathcal{H}}^3 \to M(C, D, E) = (M_1(C, D, E), M_2(C, D, E), M_3(C, D, E)) \in \overline{\mathcal{H}}^3 \tag{2.1.21}$$

defined by

$$\begin{aligned} M_1(C, D, E)(q, \theta) &= \frac{\sqrt{\varepsilon}}{aq\sqrt{2}} C_2(q, \theta) + \frac{1}{a} C_1^*(q, \theta) C\left(qC_1^*(q, \theta), (C, E)_{\varepsilon,\mu}(q, \theta)\right) \\ M_2(C, D, E)(q, \theta) &= \frac{\sqrt{\varepsilon}}{q} C_3(q, \theta) + C_1^*(q, \theta) D\left(qC_1^*(q, \theta), (C, E)_{\varepsilon,\mu}(q, \theta)\right) \\ M_3(C, D, E)(q, \theta) &= \frac{\sqrt{\varepsilon}}{q} C_4(q, \theta) + C_1^*(q, \theta) E\left(qC_1^*(q, \theta), (C, E)_{\varepsilon,\mu}(q, \theta)\right) \end{aligned}$$

where the functions C_i, $i = 1, \dots, 5$ were given in (2.1.18), the function C_1^* is defined on $\Delta(\sigma, r)$ by

$$C_1^*(q, \theta) = \frac{1}{a} + \frac{\mu\varepsilon^{-b(N+3)+\frac{1}{2}}}{\sqrt{2}q} C_1(q, \theta)$$

and we have set

$$(C, E)_{\varepsilon,\mu}(q, \theta) = \theta + 2\pi\left(\beta_2 + \mu\varepsilon^{-b(N+3)} qE(q, \theta)\right) + \mu\varepsilon^{-b(N+3)+\frac{1}{2}} C_5(q, \theta). \tag{2.1.22}$$

The global strategy will be to check that M is a well defined contractive operator. Let us start by pointing out that, from (2.1.19) and (2.1.20), we have

$$\|M_j(C, D, E)\|_{\sigma,r} \leq A_0 \left(\sqrt{2\varepsilon}\pi(1 + \text{Re } F_J^*) + \frac{1}{|a|} + \frac{\mu\varepsilon^{-b(N+3)+\frac{1}{2}}\mathcal{F}}{\sqrt{2}} \right) < A_0,$$

for $j = 1, 2, 3$. Therefore, M is well-defined.

Let us remark that the main difference between the operator M introduced in (2.1.21) and the one used to prove Theorem 1.1.8 is that in the second term of the components M_j we now have $F\left(qC_1^*(q, \theta), (C, E)_{\varepsilon,\mu}(q, \theta)\right)$ instead of $F(qC_1^*(q, \theta), \theta + 2\pi\beta)$, where F means either C, D or E. Of course, this difference comes from the fact that, now, the equation in θ_2' (see (2.1.17)) is much more complicated than the respective equation in θ_2' (see (1.1.13)) appearing in the case studied in the first chapter.

To prove that M is a contractive operator, we will need to restrict the domain of definition of each of its components M_j to $\Delta(\sigma', r')$, where $\sigma' = (1 - \gamma)\sigma$ and $r' = r - \varepsilon^b \leq r_2 - 3\varepsilon^b$.

Thus, from Cauchy estimates, we deduce that

$$\left\| \frac{\partial F}{\partial q} \right\|_{\sigma',r'} \leq \frac{1}{\gamma\sigma} \|F\|_{\sigma,r} \leq \frac{A_0}{\gamma\sigma}$$

and

$$\left\| \frac{\partial F}{\partial \theta} \right\|_{\sigma',r'} \leq \varepsilon^{-b} \|F\|_{\sigma,r} \leq \varepsilon^{-b} A_0$$

for every $F \in \overline{\mathcal{H}}$. Moreover, using also Lemma 2.1.1 we get

$$\left| \frac{\partial f_i}{\partial p}(q, p, \theta_1, \theta_2) \right| < ctant, \tag{2.1.23}$$

whenever $(q, p, \theta_1, \theta_2) \in B^2_\sigma \times \mathcal{B}''_2$ with $|p| < \frac{1}{2}\sigma$.

Let (C, D, E) and (C', D', E') in $\overline{\mathcal{H}}^3$ and

$$C'_i = C'_i(q, \theta) = f_i\left(q, \mu\varepsilon^{-b(N+3)} q C'(q, \theta), \theta\right).$$

From (2.1.18) and (2.1.23) we have

$$|C_i(q, \theta) - C'_i(q, \theta)| \leq ctant\ \mu\varepsilon^{-b(N+3)} |q| \|C - C'\|_{\sigma',r'}.$$

Hence, see (2.1.22),

$$\begin{aligned} |(C, E)_{\varepsilon,\mu}(q, \theta) - (C', E')_{\varepsilon,\mu}(q, \theta)| \leq 2\pi\mu\varepsilon^{-b(N+3)} |q| \|E - E'\|_{\sigma',r'} + \\ + ctant\ \mu^2\varepsilon^{-2b(N+3)+\frac{1}{2}} |q| \|C - C'\|_{\sigma',r'}. \end{aligned}$$

Therefore, taking $\mu \in (0, \varepsilon^w)$, with $w > b(N + 3)$, it follows that

$$\begin{aligned} |F\left(qC^*_1(q, \theta), (C, E)_{\varepsilon,\mu}(q, \theta)\right) - F'\left(q(C'_1)^*(q, \theta), (C', E')_{\varepsilon,\mu}(q, \theta)\right)| \leq \\ \leq \left(1 + ctant\ \ |q|\, \mu\varepsilon^{-b(N+4)}\right) \|(C, D, E) - (C', D', E')\|_{\sigma',r'}, \end{aligned}$$

where (F, F') stands for (C, C'), (D, D') or (E, E'). Therefore, we already get a bound which is completely equivalent to the one obtained in (1.1.19) for the first case.

The rest of the arguments needed to prove the existence of a constant $c \in (0, 1)$ for which

$$\|M(C, D, E) - M(C', D', E')\|_{\sigma',r'} \leq c \|(C, D, E) - (C', D', E')\|_{\sigma',r'}$$

are the same developed in the proof of Theorem 1.1.8. Since the same considerations given in Remark 1.1.9 are valid here, the theorem is proved. □

Based on the proof of Theorem 2.1.2 (recall that we needed to restrict the range of the imaginary part of the angular variable θ to $|\text{Im}\ \theta| \leq r_2 - 3\varepsilon^b$), from now on we are going to restrict the range of the angular variables (θ_1, θ_2) to the set

$$\mathcal{B}'''_2 = \left\{(\theta_1, \theta_2) \in \mathbb{C}^2 :\ |\text{Im}\ \theta_i| \leq r_i - 3\varepsilon^b,\ i = 1, 2\right\}.$$

Let us parameterize the separatrix of the invariant tori

$$T_{\beta_1,\beta_2} = \left\{(x, y, \hat{I}, \hat{\theta}) :\ x = 0, y = 0, \hat{I} = (\beta_1, \beta_2)\right\}$$

for the unperturbed system (take $\mu = 0$ in (2.0.9))

$$\dot{x} = y, \quad \dot{y} = \sin x, \quad \dot{I}_1 = 0, \quad \dot{I}_2 = 0, \quad \dot{\theta}_1 = \frac{1}{\sqrt{\varepsilon}}, \quad \dot{\theta}_2 = \frac{I_2}{\sqrt{\varepsilon}} \tag{2.1.24}$$

in the following way

$$\begin{aligned}(\hat{\psi}, t) = (\psi_1, \psi_2, t) \in \mathcal{B}_2''' \times \mathbb{R} \to (x^0(t), y^0(t), \hat{I}^0(t), \hat{\theta}^0(\hat{\psi}, t)) = \\ = \left(4 \arctan(e^t), \frac{2}{\cosh t}, \beta_1, \beta_2, \psi_1 + \frac{t}{\sqrt{\varepsilon}}, \psi_2 + \frac{\beta_2 t}{\sqrt{\varepsilon}}\right).\end{aligned}$$

Following the ideas of the first chapter, we consider the complex subset

$$\mathcal{C}_2 = \left\{ s \in \mathbb{C} : \ |\text{Im } s| < \frac{\pi}{2} \right\}$$

and extend the above parameterization to the following one

$$\begin{aligned}(\hat{\psi}, t, s) \in \mathcal{B}_2''' \times \mathbb{R} \times \mathcal{C}_2 \to (x^0(t+s), y^0(t+s), \hat{I}^0(t+s), \hat{\theta}^0(\hat{\psi}, t)) = \\ = \left(4 \arctan(e^{t+s}), \frac{2}{\cosh(t+s)}, \beta_1, \beta_2, \psi_1 + \frac{t}{\sqrt{\varepsilon}}, \psi_2 + \frac{\beta_2 t}{\sqrt{\varepsilon}}\right).\end{aligned} \tag{2.1.25}$$

By considering the (q,p)-coordinates used to arrive at (2.0.15), we can also write two specific pieces of the complex separatrix of the unperturbed system in the following way

$$\begin{aligned}(\hat{\psi}, t, s) \in \mathcal{B}_2''' \times (-\infty, -T_0 - \text{Re } s] \times \mathcal{C}_2 \to \left(q^0(t+s), p^0(t+s), \hat{I}^0(t+s), \hat{\theta}^0(\hat{\psi}, t)\right) = \\ = \left(0, e^{t+s}, \beta_1, \beta_2, \psi_1 + \frac{t}{\sqrt{\varepsilon}}, \psi_2 + \frac{\beta_2 t}{\sqrt{\varepsilon}}\right)\end{aligned}$$

for the local unstable manifold and

$$\begin{aligned}(\hat{\psi}, t, s) \in \mathcal{B}_2''' \times [T_0 - \text{Re } s, \infty) \times \mathcal{C}_2 \to \left(q^0(t+s), p^0(t+s), \hat{I}^0(t+s), \hat{\theta}^0(\hat{\psi}, t)\right) = \\ = \left(e^{-(t+s)}, 0, \beta_1, \beta_2, \psi_1 + \frac{t}{\sqrt{\varepsilon}}, \psi_2 + \frac{\beta_2 t}{\sqrt{\varepsilon}}\right)\end{aligned} \tag{2.1.26}$$

for the local stable one, where, as in the previous chapter, T_0 is a sufficiently large positive real constant. We also remark that these last parameterizations of the local invariant manifolds satisfy, as in the first chapter, that

$$\begin{aligned}\varphi(x^0(t+s), y^0(t+s)) &= \left(0, e^{t+s}\right), \quad \text{if } t \in (-\infty, -T_0 - \text{Re } s] \\ \varphi(x^0(t+s), y^0(t+s)) &= \left(e^{-(t+s)}, 0\right), \quad \text{if } t \in [T_0 - \text{Re } s, \infty)\end{aligned}$$

where φ, as was already pointed out, denotes the change of coordinates $(q,p) = \varphi(x,y)$ defined by (2.0.13) and Lemma 1.1.4. Fortunately, all these arguments do not depend on the equation of motion in θ_2. However, this is no longer true when one looks for suitable parameterizations for the perturbed manifolds, which are furnished by the next result. Before stating this result, we observe that, from now on, we are going to assume that

$$b > \frac{1}{4}, \tag{2.1.27}$$

where b is the parameter introduced to define the complex domain $\mathcal{B}_2'$ given in (2.0.10).

Let us point out that this bound on b is not an important restriction because, as a consequence of the final arguments leading to the proof of Theorem 0.0.6, especially those related with the second Perturbing Lemma (see Lemma 2.3.9 and also (2.3.75)), we can not expect to make $b \leq 1/4$.

Lemma 2.1.3 *For every positive parameters ε and μ, with $\mu \in (0, \varepsilon^w)$, $w = w(b, N)$ big enough and for every sufficiently large $T_0 > 0$, the local perturbed stable and unstable manifolds of T_{β_1,β_2} can be parameterized in the following way*

$$(\hat\psi, t, s) \in \mathcal{B}_2''' \times [T_0 - \mathrm{Re}\ s, \infty) \times \mathcal{C}_2 \to \left(x^+(\hat\psi, t, s), y^+(\hat\psi, t, s), \hat I^+(\hat\psi, t, s), \hat\theta^+(\hat\psi, t, s)\right)$$

and

$$(\hat\psi, t, s) \in \mathcal{B}_2''' \times (-\infty, -T_0 - \mathrm{Re}\ s] \times \mathcal{C}_2 \to \left(x^-(\hat\psi, t, s), y^-(\hat\psi, t, s), \hat I^-(\hat\psi, t, s), \hat\theta^-(\hat\psi, t, s)\right)$$

in such a way that the following properties hold:

1. *If we take the notation*

$$\mathcal{U}^+ = \mathcal{B}_2''' \times [T_0 - \mathrm{Re}\ s, \infty) \times \mathcal{C}_2 \quad \text{and} \quad \mathcal{U}^- = \mathcal{B}_2''' \times (-\infty, -T_0 - \mathrm{Re}\ s] \times \mathcal{C}_2,$$

   then, it follows that

$$\left\| (x^*, y^*, \hat I^*, \hat\theta^*) - (x^0, y^0, \hat I^0, \hat\theta^0) \right\|_{\mathcal{U}^*} \leq ctant\ \mu \left| \ln \mu \right| \varepsilon^{-b(N+5)},$$

   where $(x^0, y^0, \hat I^0, \hat\theta^0)$ is the parameterization of the unperturbed manifold given in (2.1.25) and $*$ stands for $-$ or $+$.

2. *For every $(\hat\psi, s) \in \mathcal{B}_2''' \times \mathcal{C}_2$, it holds that*

$$\begin{aligned} \left| \left((x^+, y^+, \hat I^+, \hat\theta^+) - (x^0, y^0, \hat I^0, \hat\theta^0) \right) (\hat\psi, T_0 - \mathrm{Re}\ s, s) \right| &\leq ctant\ \mu \varepsilon^{-b(N+3)}, \\ \left| \left((x^-, y^-, \hat I^-, \hat\theta^-) - (x^0, y^0, \hat I^0, \hat\theta^0) \right) (\hat\psi, -T_0 - \mathrm{Re}\ s, s) \right| &\leq ctant\ \mu \varepsilon^{-b(N+3)}. \end{aligned}$$

3. *By denoting*

$$\mathcal{U}_1^+ = \mathcal{B}_2''' \times [T_0 - \mathrm{Re}\ s, 2T_0 - \mathrm{Re}\ s] \times \mathcal{C}_2, \quad \mathcal{U}_1^- = \mathcal{B}_2''' \times [-2T_0 - \mathrm{Re}\ s, -T_0 - \mathrm{Re}\ s] \times \mathcal{C}_2,$$

   we have

$$\left\| (x^*, y^*, \hat I^*, \hat\theta^*) - (x^0, y^0, \hat I^0, \hat\theta^0) \right\|_{\mathcal{U}_1^*} \leq ctant\ \mu \varepsilon^{-b(N+5)},$$

   where $*$ stands for $-$ or $+$.

4. Once $(\hat\psi, s) \in \mathcal{B}_2''' \times \mathcal{C}_2$ is fixed, the curves

$$\begin{aligned} t \in [T_0 - \text{Re } s, \infty) &\rightarrow \Big(x^+(\hat\psi,t,s), y^+(\hat\psi,t,s), \hat I^+(\hat\psi,t,s), \hat\theta^+(\hat\psi,t,s)\Big) \\ t \in (-\infty, -T_0 - \text{Re } s] &\rightarrow \Big(x^-(\hat\psi,t,s), y^-(\hat\psi,t,s), \hat I^-(\hat\psi,t,s), \hat\theta^-(\hat\psi,t,s)\Big) \end{aligned}$$

are solutions of the perturbed system (2.0.9).

Proof

The proof is essentially the same as the one of Lemma 1.1.11 in Chapter 1. However, we have to modify it slightly due to the fact that, in the present case, the relation $\theta_2^*(\hat\psi, t) = \theta_2^0(\hat\psi, t)$ (which was used to parameterize the perturbed manifolds in the first chapter) is no longer valid. In fact, the main difference between the statements of Lemma 1.1.11 and Lemma 2.1.3 is that, here, we need the parameter s (used to get (2.1.25)) to write the angular components $\hat\theta^*$ of the perturbed invariant manifolds.

Nevertheless, following the same steps as the ones given to prove Lemma 1.1.11, we can use Theorem 2.1.2 to obtain, for every T_0 large enough, a parameterization

$$(\hat\psi, t, s) \in \mathcal{B}_2''' \times [T_0 - \text{Re } s, \infty) \times \mathcal{C}_2 \rightarrow \Big(q^+(\hat\psi,t,s), p^+(\hat\psi,t,s), \hat I^+(\hat\psi,t,s), \hat\theta^+(\hat\psi,t,s)\Big)$$

of the invariant stable manifold in the $(q, p, \hat I, \hat\theta)$-coordinates, for which (see also (1.1.28))

$$\begin{aligned} \Big|q^+(\hat\psi,t,s) - q^0(t+s)\Big| &\leq ctant\ \mu \left|\ln \mu\right| \varepsilon^{-b(N+3)}, \\ \Big|p^+(\hat\psi,t,s) - p^0(t+s)\Big| &\leq ctant\ \mu\varepsilon^{-b(N+3)}, \\ \Big|I_i^+(\hat\psi,t,s) - I_i^0\Big| &\leq ctant\ \mu\varepsilon^{-b(N+3)} \Big|q^+(\hat\psi,t,s)\Big| \leq ctant\ \mu\varepsilon^{-b(N+3)}, \end{aligned} \tag{2.1.28}$$

for $i = 1, 2$ and every $(\hat\psi, t, s) \in \mathcal{B}_2''' \times [T_0 - \text{Re } s, \infty) \times \mathcal{C}_2$.

Furthermore, if $(\hat\psi, t, s) \in \mathcal{B}_2''' \times [T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}_2$, then we may improve the above bounds by applying Gronwall's estimates to get

$$\begin{aligned} \Big|q^+(\hat\psi,t,s) - q^0(t+s)\Big| &\leq ctant\ \mu\varepsilon^{-b(N+3)}, \\ \Big|p^+(\hat\psi,t,s) - p^0(t+s)\Big| &\leq ctant\ \mu\varepsilon^{-b(N+3)} \\ \Big|I_i^+(\hat\psi,t,s) - I_i^0\Big| &\leq ctant\ \mu\varepsilon^{-b(N+3)}, \quad i = 1, 2. \end{aligned} \tag{2.1.29}$$

Let us briefly recall how this parameterization of the local stable manifold can be obtained. As in the proof of Lemma 1.1.11, such parameterization is constructed by considering the solutions of the system (2.0.9) passing through the points of the frontier tori (see the proof of Lemma 1.1.11 and Remark 1.1.13). These frontier tori are obtained by using Theorem 2.1.2 and the value of the parameterization (2.1.26) at $t = T_0 - \mathrm{Re}\ s$. Hence, the frontier tori may be parameterized by $s \in \mathcal{C}_2 \to T(s) = \bigcup_{\hat\psi \in \mathcal{B}_2'''} \mathcal{P}(\hat\psi, s)$, where the components in coordinates $(q, p, \hat I, \hat\theta)$ of the function $(\hat\psi, s) \in \mathcal{B}_2''' \times \mathcal{C}_2 \to \mathcal{P}(\hat\psi, s)$ are given by

$$\begin{aligned}
\mathcal{P}_1(\hat\psi, s) &= e^{-(T_0 + \sqrt{-1}\mathrm{Im}\ s)} = h^*(s), \\
\mathcal{P}_2(\hat\psi, s) &= \mu\varepsilon^{-b(N+3)} h^*(s) \tilde A^+_{\varepsilon,\mu}\left(h^*(s), \mathcal{P}_5(\hat\psi, s), \mathcal{P}_6(\hat\psi, s)\right) \\
\mathcal{P}_{2+j}(\hat\psi, s) &= \beta_j + \mu\varepsilon^{-b(N+3)} h^*(s) \tilde J^{+;j}_{\varepsilon,\mu}\left(h^*(s), \mathcal{P}_5(\hat\psi, s), \mathcal{P}_6(\hat\psi, s)\right), \quad j = 1, 2 \\
\mathcal{P}_5(\hat\psi, s) &= \psi_1 + \frac{T_0 - \mathrm{Re}\ s}{\sqrt{\varepsilon}}, \quad \mathcal{P}_6(\hat\psi, s) = \psi_2 + \frac{\beta_2(T_0 - \mathrm{Re}\ s)}{\sqrt{\varepsilon}}.
\end{aligned}$$

The functions $\tilde A^+_{\varepsilon,\mu}$, $\tilde J^{+;j}_{\varepsilon,\mu}$, $j = 1, 2$, are defined on the set

$$\left\{(q, \hat\theta) :\ |q| < \sigma, \hat\theta = (\theta_1, \theta_2) \in \mathcal{B}_2'''\right\}$$

by

$$\tilde A^+_{\varepsilon,\mu}(q, \hat\theta) = A^+_{\varepsilon,\mu}(q, \theta_2), \quad \tilde J^{+;j}_{\varepsilon,\mu}(q, \hat\theta) = J^{+;j}_{\varepsilon,\mu}(q, \theta_2),\ j = 1, 2,$$

where $A^+_{\varepsilon,\mu}$, $J^{+;j}_{\varepsilon,\mu}$, $j = 1, 2$, are the functions (given by Theorem 2.1.2) used to express the local stable manifold of $\mathcal{T}_{\beta_1,\beta_2}(\theta_1)$. See also Remark 1.1.12 to extend the same kind of conclusions to the present case.

On the other hand, the bounds announced at (2.1.28) and (2.1.29) can be checked in the same way as the respective ones in Lemma 1.1.11 were obtained.

Moreover, since we take $t = T_0 - \mathrm{Re}\ s$ as initial time to construct the above mentioned solutions of (2.0.9) (just as in the first chapter) we have, in particular, that

$$\theta_2^+(\hat\psi, T_0 - \mathrm{Re}\ s, s) = \psi_2 + \frac{\beta_2(T_0 - \mathrm{Re}\ s)}{\sqrt{\varepsilon}} = \theta_2^0(\hat\psi, T_0 - \mathrm{Re}\ s). \tag{2.1.30}$$

Equation (2.1.30), together with (2.1.29), imply the second statement of Lemma 2.1.3.

To complete the proof of Lemma 2.1.3, let us remember that to obtain the bounds given in (2.1.28) one may use (see also (1.1.34)) that, in the present case,

$$\left|q^+(\hat\psi, t, s)\right| \le \exp\left\{(-1 + ctant\ \mu\varepsilon^{-b(N+3)})(t + \mathrm{Re}\ s)\right\},$$

for every $(\hat\psi, t, s) \in \mathcal{B}_2''' \times [T_0 - \mathrm{Re}\ s, \infty) \times \mathcal{C}_2$. Therefore, for μ small enough,

$$\int_{T_0 - \mathrm{Re}\ s}^{\infty} \left|q^+(\hat\psi, \tau, s)\right| d\tau \le ctant.$$

Hence, if we use (2.1.30) to write

$$\begin{aligned}
\left|\theta_2^+(\hat\psi, t, s) - \theta_2^0(\hat\psi, t)\right| &\le \left|\theta_2^+(\hat\psi, T_0 - \mathrm{Re}\ s, s) - \theta_2^0(\hat\psi, T_0 - \mathrm{Re}\ s)\right| + \\
&+ \frac{1}{\sqrt{\varepsilon}}\left|\int_{T_0 - \mathrm{Re}\ s}^{t} I_2^+(\hat\psi, \tau, s) d\tau - \beta_2(t - T_0 + \mathrm{Re}\ s)\right| \le \\
&\le \frac{1}{\sqrt{\varepsilon}} \int_{T_0 - \mathrm{Re}\ s}^{t} \left|I_2^+(\hat\psi, \tau, s) - \beta_2\right| d\tau,
\end{aligned}$$

then, (2.1.28) and the fact that $b > 1/4$ imply

$$\left|\theta_2^+(\hat{\psi}, t, s) - \theta_2^0(\hat{\psi}, t)\right| \leq ctant\ \mu\varepsilon^{-b(N+3)-\frac{1}{2}} \int_{T_0 - \text{Re}\ s}^{\infty} \left|q^+(\hat{\psi}, \tau, s)\right| d\tau \leq$$
$$\leq ctant\ \mu\varepsilon^{-b(N+5)}$$

for every $(\hat{\psi}, t, s) \in \mathcal{B}_2''' \times [T_0 - \text{Re}\ s, \infty) \times \mathcal{C}_2$. In this way, Lemma 2.1.3 is proven. □

Now, we will describe the behaviour of the extension of the invariant local perturbed unstable manifold of the invariant tori

$$T_{\beta_1,\beta_2} = \left\{(x, y, \hat{I}, \hat{\theta}) :\ x = y = 0, \hat{I} = (\beta_1, \beta_2)\right\}$$

by using Lemma 2.1.3. In fact, we will get useful properties of the extension to the time interval $[-T_0 - \text{Re}\ s, 2T_0 - \text{Re}\ s]$ of any solution $(x(t), y(t), \hat{I}(t), \hat{\theta}(t))$ of (2.0.9) satisfying

$$\begin{aligned} &\left|x(t_0) - x^0(t_0 + s)\right| \leq C_2\mu\varepsilon^{-b(N+3)}, \quad \left|y(t_0) - y^0(t_0 + s)\right| \leq C_2\mu\varepsilon^{-b(N+3)}, \\ &|I_i(t_0) - \beta_i| \leq C_2\mu\varepsilon^{-b(N+3)},\ i = 1, 2, \quad (\theta_1(t_0), \theta_2(t_0)) \in \mathcal{B}_2''', \end{aligned} \tag{2.1.31}$$

for $t_0 = -T_0 - \text{Re}\ s$, some positive constant C_2 and some $s \in \mathcal{C}_2'$, with

$$\mathcal{C}_2' = \left\{s \in \mathbb{C} :\ |\text{Im}\ s| \leq \frac{\pi}{2} - \varepsilon^b\right\}.$$

Thus, the second statement of Lemma 2.1.3 implies that the estimates obtained in the next theorem can be applied to the extension of the local unstable perturbed manifold of T_{β_1,β_2}.

Theorem 2.1.4 (The Extension Theorem II) *There exists a positive constant C_2' such that, if $\mu \in (0, \varepsilon^w)$ with $w > b(N+6) + \dfrac{1}{2}$, then every solution $(x(t), y(t), \hat{I}(t), \hat{\theta}(t))$ of (2.0.9) verifying (2.1.31) can be extended to the time interval*

$$[-T_0 - \text{Re}\ s, 2T_0 - \text{Re}\ s]$$

in such a way that, taking ψ_2 satisfying $\theta_2^0(\psi_2, t_0) = \theta_2(t_0)$, then

$$\begin{aligned} \left|x(t) - x^0(t + s)\right| &\leq C_2'\mu\varepsilon^{-b(N+6)}, \quad \left|y(t) - y^0(t + s)\right| \leq C_2'\mu\varepsilon^{-b(N+6)}, \\ \left|\theta_2(t) - \theta_2^0(\psi_2, t)\right| &\leq C_2'\mu\varepsilon^{-b(N+6)}, \quad |I_i(t) - \beta_i| \leq C_2'\mu\varepsilon^{-b(N+6)}, \quad i = 1, 2, \end{aligned}$$

for every $t \in [-T_0 - \text{Re}\ s, 2T_0 - \text{Re}\ s]$.

The proof of this second Extension Theorem is also given, as in the case of the Extension Theorem I, in Chapter 3.

Remark 2.1.5 *The Extension Theorem II and the fact that*

$$H_{\varepsilon,\mu}(x,y,\hat{I},\hat{\theta}) = H_{\varepsilon,\mu}(-x,y,\hat{I},-\hat{\theta})$$

imply that there exist homoclinic orbits for the perturbed system associated to the Hamiltonian (2.0.1). These homoclinic orbits are located at $x=\pi$, $(\overline{\psi}_1,\overline{\psi}_2)\in\{0,\pi\}\times\{0,\pi\}$ *(see Remark 1.1.15 for details). Furthermore, by using the parameterization of the local unperturbed manifold given by Lemma 2.1.3 and the Extension Theorem II, we may claim that*

$$\left\{(x^-(\hat{\psi},t,s),y^-(\hat{\psi},t,s),\hat{I}^-(\hat{\psi},t,s),\hat{\theta}^-(\hat{\psi},t,s)):\ \hat{\psi}=(\overline{\psi}_1,\overline{\psi}_2),\ s=0\right\}$$

are four homoclinic orbits for the perturbed system. See, once again, Remark 1.1.15 where we introduce all the needed arguments for proving such claim.

2.2 Renormalized Melnikov functions

In the present section we are going to compute the renormalized Melnikov functions

$$(s,\hat{\psi})\in\mathcal{C}_2'\times\mathcal{B}_2'''\to\mathcal{M}_j(s,\hat{\psi}),\quad j=1,2,3$$

associated to the Hamiltonian family given in (2.0.8). Those functions are going to be used to estimate the differences

$$\mathcal{Q}_j^-(\hat{\psi},t,s)-\mathcal{Q}_j^+(\hat{\psi},t,s),\quad j=1,2,3$$

of the unperturbed energies along the perturbed invariant manifolds of the invariant tori T_{β_1,β_2}. More precisely, if we denote by

$$\mathcal{Q}_1=\mathcal{Q}_1(x,y,\hat{I},\hat{\theta})=\frac{y^2}{2}+\cos x-1,\quad \mathcal{Q}_{1+j}=\mathcal{Q}_{1+j}(x,y,\hat{I},\hat{\theta})=I_j,\quad j=1,2$$

the energies (first integrals) of the system (2.1.24), then $\mathcal{Q}_j^+$ and $\mathcal{Q}_j^-$ are those functions defined on

$$\mathcal{B}_2'''\times[T_0-\mathrm{Re}\ s,2T_0-\mathrm{Re}\ s]\times\mathcal{C}_2'$$

by

$$\mathcal{Q}_j^*(\hat{\psi},t,s)=\mathcal{Q}_j(x^*(\hat{\psi},t,s),y^*(\hat{\psi},t,s),\hat{I}^*(\hat{\psi},t,s),\hat{\theta}^*(\hat{\psi},t,s))$$

where (see Lemma 2.1.3 and the Extension Theorem II)

$$(\hat{\psi},t,s)\in\mathcal{B}_2'''\times[T_0-\mathrm{Re}\ s,2T_0-\mathrm{Re}\ s]\times\mathcal{C}_2'\to(x^-,y^-,\hat{I}^-,\hat{\theta}^-)(\hat{\psi},t,s)$$

is a parameterization of a convenient piece of the (global) unstable manifold of T_{β_1,β_2} and (see Lemma 2.1.3)

$$(\hat{\psi},t,s)\in\mathcal{B}_2'''\times[T_0-\mathrm{Re}\ s,2T_0-\mathrm{Re}\ s]\times\mathcal{C}_2'\to(x^+,y^+,\hat{I}^+,\hat{\theta}^+)(\hat{\psi},t,s)$$

is a parameterization of a convenient piece of the (local) stable manifold of T_{β_1,β_2}.

As in the first chapter, we define the renormalized Melnikov functions

$$(s,\hat{\psi}) \in \mathcal{C}_2' \times \mathcal{B}_2''' \to \mathcal{M}_j(s,\hat{\psi}) = \int_{\mathbb{R}} \{\mathcal{Q}_j, H_{\varepsilon,\mu}\}(x^0, y^0, \hat{I}^0, \hat{\theta}^0)(\hat{\psi}, \gamma, \Gamma)d\gamma, \quad j = 1,2,3$$

with $\Gamma = \gamma + s$ and $(x^0, y^0, \hat{I}^0, \hat{\theta}^0)(\hat{\psi}, t, t+s)$ the parameterization of the unperturbed separatrix obtained in (2.1.25).

Then, as a first step for proving the Main Theorem II, we present the following result:

Lemma 2.2.1 *(a) For every $(\hat{\psi}, t, s) \in \mathcal{B}_2''' \times [T_0 - \mathrm{Re}\ s, 2T_0 - \mathrm{Re}\ s] \times \mathcal{C}_2'$, it follows that*

$$\left| \mathcal{Q}_j^-(\hat{\psi}, t, s) - \mathcal{Q}_j^+(\hat{\psi}, t, s) - \mathcal{M}_j(s,\hat{\psi}) \right| \leq ctant\ \mu^2 \left|\ln\mu\right| \varepsilon^{-b(2N+11)},$$

for $j = 1,2,3$.

(b) For every $(s,\hat{\psi}) \in \mathcal{C}_2' \times \mathcal{B}_2'''$, it holds that

$$\left| \mathcal{M}_j(s,\hat{\psi}) \right| \leq ctant\ \mu\varepsilon^{-b(N+4)},$$

for $j = 1,2,3$.

Proof

Let us proceed as in Subsection 1.2.1 and write

$$\mathcal{Q}_j^-(\hat{\psi}, t, s) - \mathcal{Q}_j^+(\hat{\psi}, t, s) = \mathcal{M}_j(s,\hat{\psi}) + \mathcal{R}_j^-(\hat{\psi}, t, s) + \mathcal{R}_j^+(\hat{\psi}, t, s)$$

where, for $j = 1,2,3$,

$$\mathcal{R}_j^-(\hat{\psi}, t, s) = \int_{-\infty}^{t} \mathcal{D}_j^-(\hat{\psi}, \gamma, s)d\gamma, \quad \mathcal{R}_j^+(\hat{\psi}, t, s) = \int_{t}^{\infty} \mathcal{D}_j^+(\hat{\psi}, \gamma, s)d\gamma$$

and, for $\Gamma = \gamma + s$,

$$\mathcal{D}_j^*(\hat{\psi}, \gamma, s) = \{\mathcal{Q}_j, H_{\varepsilon,\mu}\}(x^*, y^*, \hat{I}^*, \hat{\theta}^*)(\hat{\psi}, \gamma, s) - \{\mathcal{Q}_j, H_{\varepsilon,\mu}\}(x^0, y^0, \hat{I}^0, \hat{\theta}^0)(\hat{\psi}, \gamma, \Gamma).$$

In order to bound the functions $\mathcal{R}_j^*$ let us observe that from the definition of Poisson brackets and the expression for our Hamiltonians $H_{\varepsilon,\mu}$ given at (2.0.8) we have

$$\{\mathcal{Q}_1, H_{\varepsilon,\mu}\}(x, y, \hat{I}, \hat{\theta}) = \mu y m(\hat{\theta}) \sin x$$

and, for $j = 2,3$,

$$\{\mathcal{Q}_j, H_{\varepsilon,\mu}\}(x, y, \hat{I}, \hat{\theta}) = -\mu(\cos x - 1)\frac{\partial m}{\partial \theta_{j-1}}(\hat{\theta}).$$

The main difference between the present case and the one studied in the first chapter (see Lemma 1.2.1) is that, now, we must take into account that $\theta_2^*(\hat{\psi}, t, s) \neq \theta_2^0(\hat{\psi}, t)$.

Nevertheless, for any $\gamma \in \mathbb{R}$ we may write

$$\left|\mathcal{D}_j^-(\hat{\psi}, \gamma, s)\right| \leq ctant\ \mu\varepsilon^{-b(N+3)} \left|\cos(x^0(\Gamma)) - \cos(x^-(\hat{\psi}, \gamma, s))\right| + \\ +ctant\ \mu\varepsilon^{-b(N+4)} \left|\cos(x^0(\Gamma)) - 1\right| \left|\theta_2^-(\hat{\psi}, \gamma, s) - \theta_2^0(\psi_2, \gamma)\right|$$

for $j = 2, 3$, and

$$\left|\mathcal{D}_1^-(\hat{\psi}, \gamma, s)\right| \leq ctant\ \mu\varepsilon^{-b(N+2)} \left|y^-(\hat{\psi}, \gamma, s) - y^0(\Gamma)\right| \left|\sin(x^-(\hat{\psi}, \gamma, s))\right| + \\ +ctant\ \mu\varepsilon^{-b(N+2)} \left|y^0(\Gamma)\right| \left|\sin(x^-(\hat{\psi}, \gamma, s)) - \sin(x^0(\Gamma))\right| + \\ +ctant\ \mu\varepsilon^{-b(N+3)} \left|y^0(\Gamma)\right| \left|\sin(x^0(\Gamma))\right| \left|\theta_2^-(\hat{\psi}, \gamma, s) - \theta_2^0(\psi_2, \gamma)\right|.$$

Let us observe that the above two estimates are quite similar to the one obtained in (1.2.41).

Then, one may repeat the arguments used in the first chapter and use Lemma 2.1.3 and Theorem 2.1.4 to prove that

$$\left\|\mathcal{R}_j^-\right\| \leq ctant\ \mu^2 \left|\ln \mu\right| \varepsilon^{-b(2N+11)},$$

for $j = 2, 3$. The rest of the bounds announced in the first part of the lemma are obtained in the same way.

To prove the second part it is enough to take into account the definition of the renormalized Melnikov functions (it is not necessary to obtain explicit expressions for them) together with the parameterization of the unperturbed separatrix given in (2.1.25). □

Now, let us compute the renormalized Melnikov functions. To this end, let us observe that, from the expressions

$$y^0(t+s) = \frac{2}{\cosh(t+s)},\ \cos(x^0(t+s)) = 1 - \frac{2}{\cosh^2(t+s)},\ \sin(x^0(t+s)) = -\frac{2\sinh(t+s)}{\cosh^2(t+s)}$$

we have

$$\mathcal{M}_1(s, \hat{\psi}) = -4\mu \int_{\mathbb{R}} \frac{\sinh(t+s)}{\cosh^3(t+s)} m(\hat{\theta}^0(\hat{\psi}, t))dt$$

and, for $j = 1, 2$,

$$\mathcal{M}_{j+1}(s, \hat{\psi}) = 2\mu \int_{\mathbb{R}} \frac{1}{\cosh^2(t+s)} \frac{\partial m}{\partial \theta_j}(\hat{\theta}^0(\hat{\psi}, t))dt,$$

for every $(s, \hat{\psi}) \in \mathcal{C}_2' \times \mathcal{B}_2'''$.

We are not going to give all the details needed to compute those functions because the scheme is exactly the same as the one given in the first chapter. We only point out that, for $j = 1, 2, 3$, we may write

$$\mathcal{M}_j(s, \psi_1, \psi_2) = \sum_{\hat{k} \in \Lambda} M_{\hat{k}}^{(j)} \sin(k_1(\psi_1 - \frac{s}{\sqrt{\varepsilon}}) + k_2(\psi_2 - \frac{\beta_2 s}{\sqrt{\varepsilon}})). \tag{2.2.32}$$

Moreover, by denoting $\omega = (1, \beta_2)$ and keeping in mind that the functions

$$\cosh^{-3} t \cos\left(\frac{(\hat{k}\omega)t}{\sqrt{\varepsilon}}\right) \sinh t, \qquad \sin\left(\frac{(\hat{k}\omega)t}{\sqrt{\varepsilon}}\right) \cosh^{-2} t$$

are odd, one checks that

$$\begin{aligned} M_{\hat{k}}^{(1)} &= 4\mu m_{\hat{k}} \int_{\mathbb{R}} \cosh^{-3} t \sin\left(\frac{(\hat{k}\omega)t}{\sqrt{\varepsilon}}\right) \sinh t dt, \\ M_{\hat{k}}^{(j)} &= -2\mu k_{j-1} m_{\hat{k}} \int_{\mathbb{R}} \cos\left(\frac{(\hat{k}\omega)t}{\sqrt{\varepsilon}}\right) \cosh^{-2} t dt, \quad j = 2, 3 \end{aligned}$$

with $\hat{k}\omega = k_1 + \beta_2 k_2$ and $m_{\hat{k}}$ the Fourier coefficients of the function m.

Recovering the notation introduced in (1.2.47) we may write

$$M_{\hat{k}}^{(1)} = 4\mu m_{\hat{k}} T_3\left(\frac{\hat{k}\omega}{\sqrt{\varepsilon}}\right) = 2\mu m_{\hat{k}} \frac{\pi(\hat{k}\omega)^2}{\varepsilon \sinh\left(\frac{\pi(\hat{k}\omega)}{2\sqrt{\varepsilon}}\right)}$$

and, for $j = 2, 3$,

$$M_{\hat{k}}^{(j)} = -2\mu k_{j-1} m_{\hat{k}} I_2\left(\frac{\hat{k}\omega}{\sqrt{\varepsilon}}\right) = -2\mu k_{j-1} m_{\hat{k}} \frac{\pi(\hat{k}\omega)}{\sqrt{\varepsilon} \sinh\left(\frac{\pi(\hat{k}\omega)}{2\sqrt{\varepsilon}}\right)}.$$

Hence, defining

$$\frac{\hat{k}\omega}{\sinh\left(\frac{\pi(\hat{k}\omega)}{2\sqrt{\varepsilon}}\right)} = H^*(\hat{k}\omega) \left|\hat{k}\omega\right| \exp\left(-\frac{\pi\left|\hat{k}\omega\right|}{2\sqrt{\varepsilon}}\right)$$

with

$$H^*(\hat{k}\omega) = \frac{2}{1 - \exp\left(-\frac{\pi\left|\hat{k}\omega\right|}{\sqrt{\varepsilon}}\right)}, \tag{2.2.33}$$

we may write

$$M_{\hat{k}}^{(j)} = B_{\hat{k}}^{(j)} \mathcal{E}_{\hat{k}}^{*}, \tag{2.2.34}$$

where

$$\mathcal{E}_{\hat{k}}^{*} = \exp\left(-\frac{\pi\left|\hat{k}\omega\right|}{2\sqrt{\varepsilon}}\right)\exp\left(-(|k_1|\, r_1 + |k_2|\, r_2)\right) \tag{2.2.35}$$

and

$$\begin{aligned} B_{\hat{k}}^{(1)} &= 2\mu m_{\hat{k}} \frac{\pi(\hat{k}\omega)}{\varepsilon} H^{*}(\hat{k}\omega)\left|\hat{k}\omega\right| \exp(|k_1|\, r_1 + |k_2|\, r_2) \\ B_{\hat{k}}^{(j)} &= -2\mu k_{j-1} m_{\hat{k}} \frac{\pi}{\sqrt{\varepsilon}} H^{*}(\hat{k}\omega)\left|\hat{k}\omega\right| \exp(|k_1|\, r_1 + |k_2|\, r_2), \end{aligned} \tag{2.2.36}$$

for $j = 2, 3$.

Remark 2.2.2 *The renormalized Melnikov functions $\mathcal{M}_j$ are analytic on (see also (1.3.74))*

$$\mathcal{C}_2' \times \mathcal{B}_2''' = D\left(\overline{\rho}, \frac{\pi}{2} - \varepsilon^b, r_1 - 3\varepsilon^b, r_2 - 3\varepsilon^b\right),$$

being $\overline{\rho}$ any positive constant. Therefore, comparing (2.2.32) with (1.3.75) and using Lemma 2.2.1 (b), we have

$$\mathcal{M}_j \in \mathcal{A}\left(\overline{\rho}, \frac{\pi}{2} - \varepsilon^b, r_1 - 3\varepsilon^b, r_2 - 3\varepsilon^b, ctant\ \mu\varepsilon^{-b(N+4)}\right), \quad j = 1, 2, 3$$

where $\mathcal{A} = \mathcal{A}(\overline{\rho}, \rho, \rho_1, \rho_2, \rho^*)$ denotes the set of analytic functions introduced when Lemma 1.3.4 was stated.

2.3 Proof of the Main Theorem II

The objective of this section is to finish the proof of the Main Theorem II (see Theorem 0.0.4). We follow the strategy used in Section 1.3 for proving the Main Theorem I but we must pay extra attention to the differences arising from the fact that the variation of the angular variable θ_2 is not constant for the perturbed systems (2.0.15). To begin with, let us take the notation

$$Q = (q, p, \hat{I}, \hat{\theta}, s), \quad \hat{I} = (I_1, I_2),\ \hat{\theta} = (\theta_1, \theta_2)$$

and observe that, just as in the first chapter (see Remark 1.1.10), we are considering the complex parameter s used to get (2.1.25) as a new variable and we are adding $\dot{s} = 0$ to the equations given in (2.0.15) in order to define the family of dynamical systems $\dot{Q} = g_\mu(Q)$.

Let us again consider the whiskered tori

$$T_{\beta_1,\beta_2} = \left\{(q, p, \hat{I}, \hat{\theta}) : q = p = 0, I_i = \beta_i,\ i = 1, 2\right\}$$

and the pieces of their invariant manifolds

$$(\hat{\psi}, t, s) \in \mathcal{B}_2''' \times [T_0 - \mathrm{Re}\ s, 2T_0 - \mathrm{Re}\ s] \times \mathcal{C}_2' \to Q^-(\hat{\psi}, t, s)$$

and

$$(\hat{\psi}, t, s) \in \mathcal{B}_2''' \times [T_0 - \mathrm{Re}\ s, 2T_0 - \mathrm{Re}\ s] \times \mathcal{C}_2' \to Q^+(\hat{\psi}, t, s),$$

which were obtained in Section 2.1 (see Lemma 2.1.3 for the stable case and the Extension Theorem II for the unstable one). We also recall that the sets $\mathcal{B}_2'''$ and $\mathcal{C}_2'$ were already defined by

$$\mathcal{B}_2''' = \left\{(\theta_1, \theta_2) \in \mathbb{C}^2 : \ |\mathrm{Im}\ \theta_i| \leq r_i - 3\varepsilon^b, \ i = 1, 2\right\}$$

and

$$\mathcal{C}_2' = \left\{s \in \mathbb{C} : \ |\mathrm{Im}\ s| \leq \frac{\pi}{2} - \varepsilon^b\right\}$$

and point out that we use the notation

$$Q^*(\hat{\psi}, t, s) = \left(q^*(\hat{\psi}, t, s), p^*(\hat{\psi}, t, s), \hat{I}^*(\hat{\psi}, t, s), \hat{\theta}^*(\hat{\psi}, t, s), s\right). \tag{2.3.37}$$

As we said in the Introduction, the proof of the Main Theorem II is strongly based on Theorem 0.0.6. Theorem 0.0.6 gives estimates for the transversality (size of the splitting) between the pieces of invariant manifolds along one homoclinic orbit. One of the key tools used during the proof of Theorem 0.0.6 is the existence of flow-box coordinates defined in certain domain containing those pieces of invariant perturbed manifolds.

In order to prove the existence of these flow-box coordinates we will follow the same steps given in the proof of Lemma 1.3.1. In particular, as we said before, we need to consider s as a new variable in order to define an analytic complex time (see (2.3.44)), which is going to be used to prove Lemma 2.3.1.

However, in order to prove Lemma 2.3.1, additional arguments are needed due to the fact that θ_2 does not *move* in an uniform way. We recall that, in the proof of Lemma 1.3.1, we have constructed a useful family of analytic (holomorphic) conjugations h_μ between the perturbed systems ($\dot{Q} = g_\mu(Q)$) and the unperturbed one ($\dot{Q} = g_0(Q)$) which, together with the fact that the unperturbed system was easily transformed into flow-box coordinates, led us to the desired result. This is not enough in the present case, because by repeating (one by one) the above mentioned steps we are only able to obtain an analytic change of coordinates

$$\left(\mathcal{S}^0(Q), \mathcal{K}^0(Q), \mathcal{J}_1^0(Q), \mathcal{J}_2^0(Q), \hat{\theta}, s\right)$$

transforming the unperturbed system (take $\mu = 0$ in (2.0.15))

$$\begin{aligned} &\dot{q} = -q(1 + F_J), \quad \dot{p} = p(1 + F_J), \quad \dot{I}_j = 0, \quad j = 1, 2, \\ &\dot{\theta}_1 = \frac{1}{\sqrt{\varepsilon}}, \quad \dot{\theta}_2 = \frac{I_2}{\sqrt{\varepsilon}}, \end{aligned}$$

into the following one

$$\dot{\mathcal{S}}^0 = 1, \quad \dot{\mathcal{K}}^0 = 0, \quad \dot{\mathcal{J}}_j^0 = 0, \quad j = 1, 2,$$
$$\dot{\theta}_1 = \frac{1}{\sqrt{\varepsilon}}, \quad \dot{\theta}_2 = \frac{\mathcal{J}_2}{\sqrt{\varepsilon}}, \quad \dot{s} = 0.$$

The equation $\dot{\theta}_2 = \varepsilon^{-1/2}\mathcal{J}_2$ is not suitable for our purposes (see the equality (2.3.67) which strongly depends on the value of $\dot{\theta}_2$ in flow-box coordinates). We also remark that, just as in the first chapter, we need exactly the equation (2.3.67) because of (2.2.32) (see also Remark 2.2.2). In fact, from (2.2.32) one may state that the renormalized Melnikov functions can be written as transformations which only depend on the arguments

$$\psi_1 - \frac{s}{\sqrt{\varepsilon}} \quad \text{and} \quad \psi_2 - \beta_2 \frac{s}{\sqrt{\varepsilon}}.$$

Moreover, the flow-box coordinates are going to be used (in the same way as in the first chapter) to claim that the same property can be established for the error functions (defined at (2.3.62)) used to prove Theorem 0.0.6. Of course, for doing so, we must prove that the splitting functions $C_{i,u}^{\mu}$ display exactly (with the same parameter β_2) the same property, i.e., that (2.3.67) holds.

This is the reason why, in the present case, we will need to use the auxiliary integrable Hamiltonian system

$$\dot{q} = -(1 + F_J)q, \quad \dot{p} = (1 + F_J)p, \quad \dot{I}_j = 0, \quad j = 1, 2,$$
$$\dot{\theta}_1 = \frac{1}{\sqrt{\varepsilon}}, \quad \dot{\theta}_2 = \frac{\beta_2}{\sqrt{\varepsilon}}, \quad \dot{s} = 0,$$

where β_2 is the (constant) value of the action variable I_2 not only in the considered torus T_{β_1,β_2} but also along its unperturbed homoclinic separatrix (see (2.1.25)).

The use of these new arguments have a (small) price: The bound for the distance between the change of coordinates (which is henceforth denoted by V_μ) transforming our perturbed system $\dot{Q} = g_\mu(Q)$ into flow-box coordinates, and the one (denoted by V_0) transforming into flow-box coordinates the new auxiliary unperturbed system is substantially worse (see Lemma 2.3.1) than the one obtained in the previous chapter (see Lemma 1.3.1).

In the present situation, we will make also use of the piece of the unperturbed complex separatrix given in (2.1.26) by (recall that $\mathcal{C}_2' \subset \mathcal{C}_2$)

$$(\hat{\psi}, t, s) \in \mathcal{B}_2''' \times [T_0 - \text{Re } s, \infty) \times \mathcal{C}_2' \to \left(q^0(t+s), p^0(t+s), \hat{I}^0(t+s), \hat{\theta}^0(\hat{\psi}, t)\right) =$$
$$= \left(e^{-(t+s)}, 0, \beta_1, \beta_2, \psi_1 + \frac{t}{\sqrt{\varepsilon}}, \psi_2 + \frac{\beta_2 t}{\sqrt{\varepsilon}}\right)$$

and we observe that, once again, the third statement of Lemma 2.1.3 and the Extension Theorem II (recall that $(q, p) = \varphi(x, y)$ is an analytic change of coordinates) lead to

$$\begin{aligned} \left\|(q^+, p^+, \hat{I}^+, \hat{\theta}^+) - (q^0, p^0, \hat{I}^0, \hat{\theta}^0)\right\| &\leq ctant\ \mu\varepsilon^{-b(N+5)} \\ \left\|(q^-, p^-, \hat{I}^-, \hat{\theta}^-) - (q^0, p^0, \hat{I}^0, \hat{\theta}^0)\right\| &\leq ctant\ \mu\varepsilon^{-b(N+6)}, \end{aligned} \qquad (2.3.38)$$

where the norms are taken over $\mathcal{B}_2''' \times [T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}_2'$.

Furthermore, taking σ the constant given by Lemma 1.1.4, we may guarantee the existence of some constant $\sigma' \in (0, \sigma)$ such that

$$\sigma' < \left| q^*(\hat{\psi}, t, s) \right| < \sigma, \quad \text{for every } (\hat{\psi}, t, s) \in \mathcal{B}_2''' \times [T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}_2'$$

and $*$ standing for $-$ or $+$.

Finally, we also restrict the variable s to take values on the set

$$\mathcal{C}_2' = \left\{ s \in \mathbb{C} : \ |\text{Re } s| \leq \varepsilon, \ |\text{Im } s| \leq \frac{\pi}{2} - \varepsilon^b \right\}. \tag{2.3.39}$$

Now, let us go into details: In order to construct the domain in which the flow-box coordinates are going to be defined, let us observe that, by denoting $q_0(t+s) = q_1^0(t+s) + \sqrt{-1} q_2^0(t+s)$, we have

$$\frac{q_2^0(t+s)}{q_1^0(t+s)} = \tan(-\text{Im } s)$$

and therefore, due to (2.3.38), the pieces of the considered invariant perturbed manifolds are contained in the domain $\mathcal{U}' = \mathcal{U}'(\mu, \varepsilon)$ where, recalling the notation $Q = (q, p, \hat{I}, \hat{\theta}, s)$,

$$\mathcal{U}' = \left\{ Q : \sigma' < |q| < \sigma, \ \left| \arctan \frac{q_2}{q_1} + \text{Im } s \right| < ctant \ \mu \varepsilon^{-b(N+7)}, \ |p| < ctant \ \mu \varepsilon^{-b(N+7)}, \right.$$
$$\left. |I_2 - \beta_2| < ctant \ \mu \varepsilon^{-b(N+7)}, |\text{Im } \theta_i| < r_i - 3\varepsilon^b, \ i = 1, 2, \ s \in \mathcal{C}_2' \right\}, \tag{2.3.40}$$

where $q = q_1 + \sqrt{-1} q_2$ and r_i, $i = 1, 2$, are the constants for which (2.0.4) holds.

The flow-box coordinates are going to be defined in the subdomain $\mathcal{U} = \mathcal{U}(\mu, \varepsilon)$ of $\mathcal{U}'$ given by:

$$\mathcal{U} = \left\{ Q : \ \sigma' < |q| < \sigma_2, \quad \left| \arctan \frac{q_2}{q_1} + \text{Im } s \right| < \mu \varepsilon^{-b(N+7)}, |p| < \mu \varepsilon^{-b(N+7)}, \right.$$
$$\left. |I_2 - \beta_2| < \mu \varepsilon^{-b(N+7)}, \ |\text{Im } \theta_i| < r_i - 4\varepsilon^b, \ i = 1, 2, \ s \in \mathcal{C}_2' \right\}. \tag{2.3.41}$$

The constant $\sigma_2 \in (\sigma', \sigma)$ appearing in the definition of $\mathcal{U}$ is chosen in such a way that (compare with (1.3.59))

$$\left(\frac{\sigma_2}{\sigma'} \right)^3 \sigma_2 < \frac{\sigma' + \sigma}{2}.$$

We restrict, from now on, the variation of the angular variables to

$$\mathcal{B}_2'''' = \left\{ (\theta_1, \theta_2) \in \mathbb{C}^2 : \ |\text{Im } \theta_i| < r_i - 4\varepsilon^b, \ i = 1, 2 \right\}$$

to ensure that the pieces of invariant manifolds

$$(\hat{\psi}, t, s) \in \mathcal{B}_2'''' \times [T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}_2' \to Q^*(\hat{\psi}, t, s)$$

are contained in $\mathcal{U}$.

Lemma 2.3.1 (Flow-box coordinates) *For every* $\mu \in (0, \varepsilon^w)$ *with* $w > b(N+10)$, *there exists an analytic change of variables*

$$V_\mu : Q \in \mathcal{U} \to (\mathcal{S}^\mu(Q), \mathcal{K}^\mu(Q), \mathcal{J}_1^\mu(Q), \mathcal{J}_2^\mu(Q), \theta_1, \theta_2^\mu(Q), s)$$

transforming the vector field $\dot{Q} = g_\mu(Q)$ *into the flow-box system*

$$\dot{\mathcal{S}}^\mu = 1, \quad \dot{\mathcal{K}}^\mu = 0, \quad \dot{\mathcal{J}}_i^\mu = 0, \; i = 1, 2, \quad \dot{\theta}_1 = \frac{1}{\sqrt{\varepsilon}}, \quad \dot{\theta}_2^\mu = \frac{\beta_2}{\sqrt{\varepsilon}}, \quad \dot{s} = 0.$$

Moreover, the following property holds: If we define

$$V_*(Q) = \left(\mathcal{S}^*(Q), \mathcal{K}^*(Q), \mathcal{J}_1^*(Q), \mathcal{J}_2^*(Q), \hat{\theta}, s\right)$$

where

$$\mathcal{S}^*(Q) = -\frac{\ln q}{1 + F_J}, \quad \mathcal{J}_i^*(Q) = I_i, \quad i = 1, 2$$

and

$$\mathcal{K}^*(Q) = \tilde{H}(q, p) = -(qp + F(qp)),$$

the Hamiltonian given in (2.0.14), then

$$\|V_\mu - V_*\|_{\mathcal{U}} \le ctant \; \mu\varepsilon^{-b(N+9)}.$$

Proof

Let us observe that the change of variables $V^* : \mathcal{U} \to V^*(\mathcal{U})$ transforms the auxiliary vector field

$$\dot{Q} = g_*(Q) = \left(-(1 + F_J)q, (1 + F_J)p, 0, 0, \frac{1}{\sqrt{\varepsilon}}, \frac{\beta_2}{\sqrt{\varepsilon}}, 0\right) \tag{2.3.42}$$

into the flow-box system

$$\dot{\mathcal{S}}^* = 1, \quad \dot{\mathcal{K}}^* = 0, \quad \dot{\mathcal{J}}_i^* = 0, \; i = 1, 2, \quad \dot{\theta}_1 = \frac{1}{\sqrt{\varepsilon}}, \quad \dot{\theta}_2 = \frac{\beta_2}{\sqrt{\varepsilon}}, \quad \dot{s} = 0.$$

Then, as in the proof of Lemma 1.3.1, it will be enough to construct a family of analytic conjugations h_μ between the vector fields $\dot{Q} = g_\mu(Q)$ and $\dot{Q} = g_*(Q)$, satisfying

$$\|h_\mu - I\|_{\mathcal{U}} \le ctant \; \mu\varepsilon^{-b(N+9)} \tag{2.3.43}$$

and, after this, take $\mathcal{S}^\mu(Q) = \mathcal{S}^*(h_\mu(Q))$, $\mathcal{K}^\mu(Q) = \mathcal{K}^*(h_\mu(Q))$, $\mathcal{J}_i^\mu(Q) = \mathcal{J}_i^*(h_\mu(Q))$, $i = 1, 2$ and $\theta_2^\mu(Q) = \theta_2^*(h_\mu(Q)) = \theta_2(h_\mu(Q))$ the θ_2-component of $h_\mu(Q)$.

To obtain a family $\{h_\mu\}_\mu$ of holomorphic conjugations satisfying (2.3.43), let us consider the analytic function

$$T^0(Q) = -\frac{\ln(q/\sigma_2)}{1 + F_J} - s \tag{2.3.44}$$

and define a family of analytic functions $h_\mu^* : Q \in \mathcal{U} \to h_\mu^*(Q)$ by the condition

$$\phi(-T^0(Q), Q, *) = \phi(-T^0(Q), h_\mu^*(Q), \mu), \tag{2.3.45}$$

where $\phi(t, Q, *)$ denotes the flow associated to $\dot{Q} = g_*(Q)$ and $\phi(t, Q, \mu)$ the flow associated to $\dot{Q} = g_\mu(Q)$.

In the same way as in the proof of Lemma 1.3.1, the equation

$$T^0(\phi(t, Q, 0)) = T^0(Q) + t$$

implies that

$$\phi(t, h_\mu^*(Q), \mu) = h_\mu^*(\phi(t, Q, 0)),$$

so that h_μ^* is a conjugation between $\dot{Q} = g_*(Q)$ and $\dot{Q} = g_\mu(Q)$.

Now, we need to apply Gronwall's estimates in order to get (2.3.43). To this end, let us denote

$$T^0(Q) = T_1(Q) + \sqrt{-1}T_2(Q).$$

Then, for every $Q \in \mathcal{U}$ we claim that

1. $\phi(-t, Q, *) \in \mathcal{U}'$, for every $t \in [0, T_1(Q)]^*$.
2. $\Psi(-t', \phi(-T_1(Q), Q, *), *) \in \mathcal{U}'$, for every $t' \in [0, T_2(Q)]^*$.

Let us recall that by $[a, b]^*$ we denote $[a, b]$ if $a \le b$ and $[b, a]$ if $b \le a$. Moreover, $\Psi(t, Q, *)$ and $\Psi(t, Q, \mu)$ respectively denote the flows associated to the vector fields $\dot{Q} = \sqrt{-1}g_*(Q)$ and $\dot{Q} = \sqrt{-1}g_\mu(Q)$. We deem it would be useful to recall that (see the proof of Lemma 1.3.1)

$$\phi(-T^0(Q), Q, \lambda) = \Psi(-T_2(Q), \phi(-T_1(Q), Q, \lambda), \lambda)$$

where $\lambda = *$ or $\lambda = \mu$.

The first assertion of the claim directly follows, as in the first chapter, from the fact that (recall the new definition of $\mathcal{C}_2'$ given at (2.3.39) and also how (1.3.66) was deduced)

$$|T_1(Q)| < 2 \ln \frac{\sigma_2}{\sigma'},$$

where we have also used that condition (see the definition of $\mathcal{U}$ given at (2.3.41)) $|p| < \mu\varepsilon^{-b(N+7)}$ leads to $|F_J| \le ctant\ \mu\varepsilon^{-b(N+7)}$.

In order to prove the second assertion it suffices to bear in mind that, since the definition of $\mathcal{U}$ implies

$$\left| T_2(Q) + \arctan \frac{q_2}{q_1} + \mathrm{Im}\ s \right| < ctant\ \mu\varepsilon^{-b(N+7)},$$

we get (compare with (1.3.68))

$$|T_2(Q)| < ctant\ \mu\varepsilon^{-b(N+7)}.$$

Thus, proceeding as in the first chapter, if

$$\tilde{\theta}_i(-t') = \tilde{\theta}_i(-t', \phi(-T_1(Q), Q, *), *), \quad i = 1, 2$$

denotes the θ_i-component (at time $-t'$) of $\Psi(-t', \phi(-T_1(Q), Q, *), *)$, then the equations

$$\dot{\theta}_1 = \frac{\sqrt{-1}}{\sqrt{\varepsilon}}, \quad \dot{\theta}_2 = \frac{\sqrt{-1}\beta_2}{\sqrt{\varepsilon}}$$

and the fact that $b > 1/4$ allow us to write

$$\left| \mathrm{Im}\ \tilde{\theta}_i(-t') - \mathrm{Im}\ \tilde{\theta}_i(0) \right| < ctant\ \mu\varepsilon^{-b(N+7)-\frac{1}{2}} < ctant\ \mu\varepsilon^{-b(N+9)},$$

for every $t' \in [0, T_2(Q)]^*$. This last bound is completely equivalent to the respective one, see (1.3.69), obtained in the first chapter. Therefore, using the arguments introduced therein, one easily achieves, by taking $\mu \in (0, \varepsilon^w)$, with $w > b(N + 10)$,

$$\left| \mathrm{Im}\ \tilde{\theta}_i(-t') \right| < r_i - 4\varepsilon^b + ctant\ \mu\varepsilon^{-b(N+9)} < r_i - 3\varepsilon^b, \quad i = 1, 2$$

for every $t' \in [0, T_2(Q)]^*$. Hence, the second assertion of the claim is also checked.

Now, the expressions of the vector fields $\dot{Q} = g_\mu(Q)$ and $\dot{Q} = g_*(Q)$, respectively given at (2.0.15) (adding the new equation $\dot{s} = 0$) and (2.3.42) and the definition of the domain $\mathcal{U}'$ (see (2.3.40)) yield

$$\|g_* - g_\mu\|_{\mathcal{U}'} \leq ctant\ \mu\varepsilon^{-b(N+7)-\frac{1}{2}} < ctant\ \mu\varepsilon^{-b(N+9)},$$

where we have also used that $b > 1/4$ and the bounds for the functions $m = m(\hat{\theta})$ and $\dfrac{\partial m}{\partial \theta_j} = \dfrac{\partial m}{\partial \theta_j}(\hat{\theta})$ respectively given at (2.0.11) and (2.0.12).

Therefore, for each sufficiently small μ, and for every $Q \in \mathcal{U}$, we may also prove that

1. $\phi(-t, Q, \mu) \in \mathcal{U}'$, for every $t \in [0, T_1(Q)]^*$.

2. $\Psi(-t', \phi(-T_1(Q), Q, \mu), \mu) \in \mathcal{U}'$, for every $t' \in [0, T_2(Q)]^*$.

Hence, Gronwall's estimates and the definition of h^*_μ given at (2.3.45) imply

$$\left\| h^*_\mu - I \right\|_{\mathcal{U}} \leq ctant\ \mu\varepsilon^{-b(N+9)}$$

and (2.3.43) can be now easily obtained by defining $h_\mu = (h^*_\mu)^{-1}$. The rest of the arguments needed to conclude the proof of Lemma 2.3.1 are exactly those ones given for proving Lemma 1.3.1. □

Remark 2.3.2 *While we were writing this book, we became aware of a remarkable preprint of P. Lochak, J. P. Marco and D. Sauzin ([15]) containing a result which proves the existence of flow-box coordinates for a family of perturbed Hamiltonian systems similar to the one studied along this second chapter.*

The method (see also [23] for a previous version) used in [15] is completely different to the one described in the proof of Lemma 2.3.1, mainly because they do not use any dynamical argument to construct flow-box coordinates. Another interesting difference between both methods is that the one used in [15] does not provide information about the domain where the flow-box coordinates are defined.

As in the first chapter, for every μ small enough, we define on $\mathcal{C}_2' \times \mathcal{B}_2''''$ the following functions (the first three ones are called splitting functions) (see Remark 1.3.3 in order to justify why we can skip the variable t in the left-hand side of the equations)

$$\begin{aligned}
\mathcal{K}_u^\mu(s,\psi_1,\psi_2) &= \mathcal{K}^\mu(Q^-(\hat{\psi},t,s)) - \mathcal{K}^\mu(Q^+(\hat{\psi},t,s)) \\
\mathcal{J}_{i,u}^\mu(s,\psi_1,\psi_2) &= \mathcal{J}_i^\mu(Q^-(\hat{\psi},t,s)) - \mathcal{J}_i^\mu(Q^+(\hat{\psi},t,s)), \quad i=1,2 \\
\mathcal{S}_u^\mu(s,\psi_1,\psi_2) &= \mathcal{S}^\mu(Q^-(\hat{\psi},t,s)) - t \\
\theta_{2,u}^\mu(s,\psi_1,\psi_2) &= \theta_2^\mu(Q^-(\hat{\psi},t,s)) - \frac{\beta_2 t}{\sqrt{\varepsilon}}.
\end{aligned} \tag{2.3.46}$$

These functions will play, once again, a crucial role for getting asymptotic estimates for the intersection angle (transversality) between the perturbed manifolds of T_{β_1,β_2} along one homoclinic orbit for the perturbed system. The asymptotic estimates for the splitting are given by Theorem 0.0.6 whose proof also depends on two more results (see Lemma 2.3.3 and Lemma 2.3.8). These two results are naturally related to Lemma 1.3.5 and Lemma 1.3.10, respectively.

To state the first lemma, let us introduce the sets

$$\mathcal{C}_2'' = \left\{ s \in \mathbb{C} : \ |\mathrm{Re}\ s| \le \varepsilon, \ |\mathrm{Im}\ s| \le \frac{\pi}{2} - 2\varepsilon^b \right\} \tag{2.3.47}$$

and

$$\mathcal{Y}^\mathcal{V} = \left\{ \theta \in \mathbb{C} : \ |\mathrm{Im}\ \theta| < r_2 - 5\varepsilon^b \right\},$$

where r_2 is the constant given in (2.0.4).

Lemma 2.3.3 *Let $\mu \in (0,\varepsilon^w)$ with $w > b(N+10)$. Then, once $\psi_1 \in \mathbb{C}$ is fixed with $|\mathrm{Im}\ \psi_1| \le r_1 - 4\varepsilon^b$, the function*

$$U(s,\psi_2) = (U_1(s,\psi_2), U_2(s,\psi_2)) = (\mathcal{S}_u^\mu(s,\psi_1,\psi_2), \theta_{2,u}^\mu(s,\psi_1,\psi_2))$$

is invertible on $\mathcal{C}_2'' \times \mathcal{Y}^\mathcal{V}$ and its inverse, denoted by

$$(s,\psi_2) = U^{-1}(u_1,u_2) = (V_1(u_1,u_2), V_2(u_1,u_2)), \quad (u_1 = U_1(s,\psi_2), \ u_2 = U_2(s,\psi_2)),$$

satisfies

$$|V_i(u_1,u_2) - u_i| \le ctant\ \mu\varepsilon^{-b(N+9)}, \quad i=1,2.$$

Proof

Let us recall the function

$$\mathcal{S}^* = \mathcal{S}^*(Q) = -\frac{\ln q}{1+F_J}$$

introduced in the statement of Lemma 2.3.1. Then, using (2.3.46) and Lemma 2.3.1, we may write, for every $s \in \mathcal{C}_2'$ and every $\psi_2 \in \mathbb{C}$ with $|\mathrm{Im}\ \psi_2| \leq r_2 - 4\varepsilon^b$,

$$|U_1(s,\psi_2) - s| = \left|\mathcal{S}^\mu(Q^-(\hat{\psi},t,s)) - t - s\right| \leq \left|\mathcal{S}^*(Q^-(\hat{\psi},t,s)) - t - s\right| + ctant\ \mu\varepsilon^{-b(N+9)}$$

and

$$\begin{aligned} |U_2(s,\psi_2) - \psi_2| &= \left|\theta_2^\mu(Q^-(\hat{\psi},t,s)) - \frac{\beta_2 t}{\sqrt{\varepsilon}} - \psi_2\right| \leq \\ &\leq \left|\theta_2^-(\hat{\psi},t,s) - \frac{\beta_2 t}{\sqrt{\varepsilon}} - \psi_2\right| + ctant\ \mu\varepsilon^{-b(N+9)}. \end{aligned}$$

Moreover, from the Extension Theorem II and the fact that $\mathcal{S}^*$ has bounded derivatives in its domain of definition $\mathcal{U}$ given in (2.3.41), we deduce that

$$\begin{aligned} |U_1(s,\psi_2) - s| &\leq \left|\mathcal{S}^*(q^0(t+s), p^0(t+s), \hat{I}^0(t+s), \hat{\theta}^0(\hat{\psi},t), s) - t - s\right| + \\ &+ ctant\ \mu\varepsilon^{-b(N+9)} = ctant\ \mu\varepsilon^{-b(N+9)}, \end{aligned} \tag{2.3.48}$$

where we have also used that, according to (2.1.26),

$$\mathcal{S}^*(q^0(t+s), p^0(t+s), \hat{I}^0(t+s), \hat{\theta}^0(\hat{\psi},t), s) = t + s.$$

The Extension Theorem II therefore also implies (recall that $\theta_2^0(\hat{\psi},t) = \psi_2 + \dfrac{\beta_2 t}{\sqrt{\varepsilon}}$)

$$|U_2(s,\psi_2) - \psi_2| \leq ctant\ \mu\varepsilon^{-b(N+9)}. \tag{2.3.49}$$

Hence, keeping in mind that the function $U = (U_1, U_2)$ is analytic, we may apply Lemma 1.1.2 to get

$$|\det(DU) - 1| \leq ctant\ \mu\varepsilon^{-b(N+10)},$$

for every $(s,\psi_2) \in \mathcal{C}_2'' \times \mathcal{Y}^\mathcal{V}$.

Since $\mu \in (0,\varepsilon^w)$, with $w > b(N+10)$, the function U can be inverted to obtain an analytic function $(s,\psi_2) = U^{-1}(u_1,u_2) = (V_1(u_1,u_2), V_2(u_1,u_2))$. Furthermore, from equations (2.3.48) and (2.3.49), we get, for $i = 1,2$,

$$|V_i(u_1,u_2) - u_i| \leq ctant\ \mu\varepsilon^{-b(N+9)}.$$

□

In order to ensure that Lemma 2.3.3 can be used, we restrict the range of (s,ψ_1,ψ_2) to the set $\mathcal{C}_2'' \times \mathcal{B}_2^\mathcal{V}$, where $\mathcal{C}_2''$ was introduced in (2.3.47) and

$$\mathcal{B}_2^\mathcal{V} = \left\{(\theta_1,\theta_2) \in \mathbb{C}^2 :\ |\mathrm{Im}\ \theta_i| \leq r_i - 5\varepsilon^b,\ i = 1,2\right\}. \tag{2.3.50}$$

Remark 2.3.4 *If we make $s = 0$, then the functions*

$$\begin{aligned} u_1^* &= u_1^*(\psi_1,\psi_2) = U_1(0,\psi_2) = \mathcal{S}^\mu(Q^-(\hat\psi,t,0)) - t \\ u_2^* &= u_2^*(\psi_1,\psi_2) = U_2(0,\psi_2) = \theta_2^\mu(Q^-(\hat\psi,t,0)) - \frac{\beta_2 t}{\sqrt{\varepsilon}} \end{aligned}$$

are real whenever the phases ψ_i, $i = 1,2$ are real (see Remark 1.3.6 for related details). This fact will be used in the proof of Theorem 0.0.6.

To prove Theorem 0.0.6, we also need to obtain the leading order behaviour of some kind of numerical series. Let us consider a series $\hat S$ of the following form

$$\hat S = \sum_{\hat k \in \mathbb{Z}^2 \setminus \{(0,0)\}} S_{\hat k} \overline{\mathcal{E}}_{\hat k},$$

where, for $\hat k = (k_1,k_2)$, $\omega = (1,\beta_2)$ and $\hat k\omega = k_1 + \beta_2 k_2$,

$$\overline{\mathcal{E}}_{\hat k} = \overline{\mathcal{E}}_{\hat k}(v_1,v_2,d,\varepsilon,\beta_2) = \exp\left(-|k_1|\, v_1 - |k_2|\, v_2 - \frac{d}{\sqrt{\varepsilon}} \left|\hat k\omega\right| \right), \tag{2.3.51}$$

with v_1, v_2, d and ε positive parameters, ε small enough.

Definition 2.3.5 *We say that $\hat S \in \mathcal{S}_1^*(v_1,v_2,d,\varepsilon,\beta_2)$ if the coefficients $S_{\hat k}$ of $\hat S$ do not increase faster than some finite power of $\left|\hat k\right| = |k_1| + |k_2|$, i.e., there exist positive constants $\mathcal{X}_1'$ and W_1' such that, for every $\hat k \in \mathbb{Z}^2 \setminus \{(0,0)\}$,*

$$|S_{\hat k}| \le W_1' \left|\hat k\right|^{\mathcal{X}_1'}.$$

Definition 2.3.6 *We say that $\hat S \in \mathcal{S}_2^*(v_1,v_2,d,\varepsilon,\beta_2)$ if the coefficients $S_{\hat k}$ of $\hat S$ satisfy the following property: There exist positive constants $\mathcal{X}_2'$ and W_2' for which*

$$|S_{\hat k^{(j)}}| \ge W_2' \left|\hat k^{(j)}\right|^{-\mathcal{X}_2'},$$

for every $\hat k^{(j)} = (k_1^{(j)}, k_2^{(j)})$ such that $k_1^{(j)}/k_2^{(j)}$ is a best approximation to the golden mean number satisfying

$$k_2^{(j)} \in \left(\varepsilon^{-1/5}, \varepsilon^{-1/4} \left|\ln\varepsilon\right|^{1/8}\right).$$

We recall that the definition of best approximations was given in the appendix of the first chapter, see Definition 1.4.1.

Definition 2.3.7 *We say that $\hat S \in \mathcal{S}_3^*(v_1,v_2,d,\varepsilon,\beta_2)$ if the coefficients $S_{\hat k}$ of $\hat S$ satisfy $S_{\hat k} = S_{-\hat k}$, for every $\hat k \in \mathbb{Z}^2 \setminus \{(0,0)\}$.*

Once again, if $\hat{S} \in \mathcal{S}_3^*(v_1, v_2, d, \varepsilon, \beta_2)$, then we may write

$$\hat{S} = 2 \sum_{\hat{k} \in \mathbb{Z}_+^2} S_{\hat{k}} \overline{\mathcal{E}}_{\hat{k}},$$

where

$$\mathbb{Z}_+^2 = \left\{ (k_1, k_2) \in \mathbb{Z}^2 : \; k_2 > 0 \right\} \cup \left\{ (k_1, 0) \in \mathbb{Z}^2 : \; k_1 > 0 \right\}.$$

Finally, let us define

$$\mathcal{S}^*(v_1, v_2, d, \varepsilon, \beta_2) = \bigcap_{i=1}^{3} \mathcal{S}_i^*(v_1, v_2, d, \varepsilon, \beta_2)$$

in order to state the following two results whose proofs, as in the case of the Main Lemma I and the first Perturbing Lemma, are postponed to Chapter 4:

Lemma 2.3.8 (Main Lemma II) *Given any three positive constants v_1, v_2 and d there exist $\varepsilon_0 \in (0, 1)$ and a real subset $\mathcal{U}_\varepsilon^* \subset (0, \varepsilon_0]$, $\mathcal{U}_\varepsilon^* = \mathcal{U}_\varepsilon^*(v_1, v_2, d)$, with*

$$\mathcal{L}(\mathcal{U}_\varepsilon^*) = \mathrm{O}(\varepsilon^{11/6})$$

satisfying the following property: For every $\varepsilon \in \mathcal{U}_\varepsilon^*$ there exist two natural numbers n^0 and n^1, depending on ε, v_1, v_2 and d, with $|n^0 - n^1| = 1$, $k_1^{(n^0)}/k_2^{(n^0)}$, $k_1^{(n^1)}/k_2^{(n^1)}$ best approximations to the golden mean $\tilde{\beta} = \dfrac{\sqrt{5}+1}{2}$ with

$$\textit{ctant}\; \varepsilon^{-1/4} \leq k_j^{(t)} \leq \textit{ctant}\; \varepsilon^{-1/4}, \quad j = 1, 2, \quad t = n^0, n^1$$

such that, for every β_2 in a golden mean open neighbourhood $I_{\tilde{\beta}}^* = I_{\tilde{\beta}}^*(\varepsilon)$ with

$$\frac{1}{100}\varepsilon^{5/6} \leq \textit{length}(I_{\tilde{\beta}}^*) \leq \frac{1}{2}\varepsilon^{5/6}$$

and for any numerical series

$$\hat{S} = \sum_{\hat{k} \in \mathbb{Z}^2 \setminus \{(0,0)\}} S_{\hat{k}} \overline{\mathcal{E}}_{\hat{k}} \in \mathcal{S}^*(v_1, v_2, d, \varepsilon, \beta_2)$$

also satisfying

$$S_{\hat{k}(n^0)} S_{\hat{k}(n^1)} > 0,$$

one may write

$$\hat{S} = 2 \left(S_{\hat{k}(n^0)} \overline{\mathcal{E}}_{\hat{k}(n^0)} + S_{\hat{k}(n^1)} \overline{\mathcal{E}}_{\hat{k}(n^1)} \right) \left[1 + \mathrm{O}\left(\exp\left(-\frac{|\ln \varepsilon|^{-1/4}}{\varepsilon^{1/4}} \right) \right) \right].$$

Lemma 2.3.9 (Second Perturbing Lemma) *Let v_1, v_2 and d positive constants and $\mathcal{U}^*_\varepsilon = \mathcal{U}^*_\varepsilon(v_1, v_2, d)$ the set of values of ε given by the Main Lemma II. Let*

$$\hat{S} = \sum_{\hat{k}\in\mathbb{Z}^2\setminus\{(0,0)\}} S_{\hat{k}} \overline{\mathcal{E}}'_{\hat{k}}$$

be a numerical series such that

$$\hat{S} \in \mathcal{S}^*(v'_1, v'_2, d', \varepsilon, \beta_2)$$

for some $\varepsilon \in \mathcal{U}^*_\varepsilon$, $\beta_2 \in I^*_{\tilde{\beta}}(\varepsilon)$ and positive parameters v'_1, v'_2, d' verifying

$$\max\{|v_1 - v'_1|, |v_2 - v'_2|, |d - d'|\} < ctant\ \varepsilon^\alpha$$

for some constant $\alpha > 3/10$. Let n^0, n^1 be those indices depending on ε, v_1, v_2 and d furnished by the Main Lemma II. Then, one may write

$$S' = 2\left(S_{\hat{k}(n^0)}\overline{\mathcal{E}}'_{\hat{k}(n^0)} + S_{\hat{k}(n^1)}\overline{\mathcal{E}}'_{\hat{k}(n^1)}\right)\left[1 + \mathrm{O}\left(\exp\left(-\frac{|\ln\varepsilon|^{-1/4}}{\varepsilon^{1/4}}\right)\right)\right]$$

whenever condition $S_{\hat{k}(n^0)}S_{\hat{k}(n^1)} > 0$ is fulfilled.

Remark 2.3.10 *The sets $\mathcal{U}^*_\varepsilon$ and $I^*_{\tilde{\beta}}$ announced by Theorem 0.0.6 are going to coincide with those ones given by the Main Lemma II when choosing $v_1 = r_1$, $v_2 = r_2$ (r_1 and r_2 those constants for which (2.0.4) holds) and $d = \dfrac{\pi}{2}$.*

By introducing some extra burdensome notation, the Main Lemma II and the second Perturbing Lemma (and therefore Theorem 0.0.6 and the Main Theorem II) can be improved. More concretely, these four results are valid for a golden mean value neighbourhood with

$$ctant\ \varepsilon^{3/4+\alpha_1} \le length(I^*_{\tilde{\beta}}) \le ctant\ \varepsilon^{3/4+\alpha_1},$$

for any arbitrarily small positive constant α_1. Nevertheless, as in the previous chapter, we are not going to provide details.

2.3.1 Proof of Theorem 0.0.6

To prove Theorem 0.0.6 we begin, as in the proof of the Main Theorem I, by selecting the homoclinic orbit of the perturbed system (2.0.9) at which we are going to estimate the transversality. As in the first chapter, this homoclinic orbit is completely determined by choosing some values $(\overline{\psi}_1, \overline{\psi}_2) \in \{0, \pi\} \times \{0, \pi\}$ (see Remark 2.1.5) where $\overline{\psi}_i = \overline{\psi}_i(\varepsilon)$, for $i = 1.2$.

Those initial phases $\overline{\psi}_i = \overline{\psi}_i(\varepsilon)$ are going to be selected in order that the Main Lemma II can be applied to obtain two leading terms for the series

$$\frac{\partial \overline{\mathcal{M}}_i}{\partial \psi_j}(\overline{\psi}_1, \overline{\psi}_2), \quad \overline{\mathcal{M}}_i(\psi_1, \psi_2) = \mathcal{M}_i(0, \psi_1, \psi_2), \quad i = 2, 3, \ j = 1, 2,$$

where $\mathcal{M}_i = \mathcal{M}_i(s, \psi_1, \psi_2)$ are the renormalized Melnikov functions obtained in Section 2.2.

Since from (2.2.32) we have

$$\overline{\mathcal{M}}_i(\psi_1, \psi_2) = \sum_{\hat{k} \in \Lambda} M_{\hat{k}}^{(i)} \sin(k_1 \psi_1 + k_2 \psi_2), \quad i = 2, 3$$

one may use (2.2.34) and (2.2.36) to write

$$\frac{\partial \overline{\mathcal{M}}_i}{\partial \psi_j}(\overline{\psi}_1, \overline{\psi}_2) = -2\mu\pi\varepsilon^{-1/2} \sum_{\hat{k} \in \Lambda} k_j \tilde{B}_{\hat{k}}^{(i)} \mathcal{E}_{\hat{k}}^* \cos(k_1 \overline{\psi}_1 + k_2 \overline{\psi}_2)$$

with, for $i = 2, 3$,

$$\tilde{B}_{\hat{k}}^{(i)} = k_{i-1} m_{\hat{k}} H^*(\hat{k}\omega) \left|\hat{k}\omega\right| \exp\left(|k_1| r_1 + |k_2| r_2\right) \tag{2.3.52}$$

and, see (2.2.35) and (2.3.51),

$$\mathcal{E}_{\hat{k}}^* = \exp\left(-\frac{\pi \left|\hat{k}\omega\right|}{2\sqrt{\varepsilon}}\right) \exp\left(-(|k_1| r_1 + |k_2| r_2)\right) = \overline{\mathcal{E}}_{\hat{k}}\left(r_1, r_2, \frac{\pi}{2}, \varepsilon, \beta_2\right).$$

Thus, we may express

$$\frac{\partial \overline{\mathcal{M}}_i}{\partial \psi_j}(\overline{\psi}_1, \overline{\psi}_2) = -2\mu\pi\varepsilon^{-1/2} \sum_{\hat{k} \in \mathbb{Z}^2 \setminus \{(0,0)\}} S_{\hat{k}} \mathcal{E}_{\hat{k}}^* \tag{2.3.53}$$

where

$$S_{\hat{k}} = \begin{cases} k_j \tilde{B}_{\hat{k}}^{(i)} \cos(k_1 \overline{\psi}_1 + k_2 \overline{\psi}_2), & \text{if } \hat{k} \in \Lambda \\ 0, \quad \text{if } \hat{k} \in \mathbb{Z}^2 \setminus (\Lambda \cup \{(0,0)\}) \end{cases} \tag{2.3.54}$$

Since

$$\mathcal{E}_{\hat{k}}^* = \overline{\mathcal{E}}_{\hat{k}}\left(r_1, r_2, \frac{\pi}{2}, \varepsilon, \beta_2\right),$$

in order to apply the Main Lemma II to the series

$$\sum_{\hat{k} \in \mathbb{Z}^2 \setminus \{(0,0)\}} S_{\hat{k}} \mathcal{E}_{\hat{k}}^*$$

we restrict the variable ε to take values on the set

$$\mathcal{U}^*_\varepsilon\left(r_1, r_2, \frac{\pi}{2}\right).$$

This set $\mathcal{U}^*_\varepsilon\left(r_1, r_2, \frac{\pi}{2}\right)$ is furnished by the Main Lemma II for the special choice $v_1 = r_1$, $v_2 = r_2$ and $d = \frac{\pi}{2}$.

Now, once a value of $\varepsilon \in \mathcal{U}^*_\varepsilon\left(r_1, r_2, \frac{\pi}{2}\right)$ is fixed, let us consider n^0 and n^1 the indices (depending on r_1, r_2 and ε) given by the Main Lemma II when taking $v_1 = r_1$, $v_2 = r_2$ and $d = \frac{\pi}{2}$. It is clear that to apply Lemma 2.3.8 to the series

$$\hat{S} = \sum_{\hat{k}\in\mathbb{Z}^2\setminus\{(0,0)\}} S_{\hat{k}}\mathcal{E}^*_{\hat{k}}$$

we must guarantee that

$$\hat{S} \in \mathcal{S}^*\left(r_1, r_2, \frac{\pi}{2}, \varepsilon, \beta_2\right) \quad \text{and} \quad S_{\hat{k}(n^0)}S_{\hat{k}(n^1)} > 0$$

for every $\beta_2 \in I^*_{\tilde{\beta}}(\varepsilon)$.

From the fact that (see Lemma 2.3.8)

$$\text{ctant } \varepsilon^{-1/4} \le k_j^{(t)} \le \text{ctant } \varepsilon^{-1/4}, \quad j = 1, 2, \quad t = n^0, n^1$$

and the definition of the set of indices Λ given at (2.0.2), the condition $S_{\hat{k}(n^0)}S_{\hat{k}(n^1)} > 0$ is equivalent, according to (2.3.54), to

$$\text{sign}\left\{k_j^{(n^0)}\tilde{B}^{(i)}_{\hat{k}(n^0)}\cos(k_1^{(n^0)}\overline{\psi}_1 + k_2^{(n^0)}\overline{\psi}_2)\right\} = \text{sign}\left\{k_j^{(n^1)}\tilde{B}^{(i)}_{\hat{k}(n^1)}\cos(k_1^{(n^1)}\overline{\psi}_1 + k_2^{(n^1)}\overline{\psi}_2)\right\}. \tag{2.3.55}$$

Let us point out that, once three positive parameters v_1, v_2 and d are fixed, if one takes a value of ε in $\mathcal{U}^*_\varepsilon(v_1, v_2, d)$ (see the statement of the Main Lemma II) then the set $I^*_{\tilde{\beta}}(\varepsilon)$ of values of β_2 for which Lemma 2.3.8 holds satisfies the following property (see the construction process of $I^*_{\tilde{\beta}}(\varepsilon)$ given at the beginning of Section 4.2):

- If $\beta_2 \in I^*_{\tilde{\beta}}(\varepsilon)$, then the best approximations $k_1^{(j)}/k_2^{(j)}$ to β_2 satisfying $k_2^{(j)} < \varepsilon^{-1/4}\left|\ln\varepsilon\right|^{1/8}$ are exactly those ones to the golden mean $\tilde{\beta} = \frac{\sqrt{5}+1}{2}$.

Therefore, recalling the notation introduced in (1.4.124) and keeping in mind how (1.4.127) was deduced, one may also claim that, for any $\beta_2 \in I^*_{\tilde{\beta}}(\varepsilon)$,

$$A_j(\beta_2) - A_j(\tilde{\beta}) = (-1)^j(k_2^{(j)})^2(\beta_2 - \tilde{\beta}) \tag{2.3.56}$$

whenever $k_1^{(j)}/k_2^{(j)}$ is a best approximation to the golden mean (or to β_2) satisfying $k_2^{(j)} < \varepsilon^{-1/4}\left|\ln\varepsilon\right|^{1/8}$.

Hence, using that the Main Lemma II gives

$$\text{length}(I^*_{\tilde\beta}) \le ctant\ \varepsilon^{5/6}$$

it follows, for every $\beta_2 \in I^*_{\tilde\beta}$, that

$$\left| A_j(\beta_2) - A_j(\tilde\beta) \right| \le ctant\ \varepsilon^{1/3} \left|\ln \varepsilon\right|^{1/4},$$

whenever $j \in \mathbb{N}$ satisfies $k_2^{(j)} < \varepsilon^{-1/4} \left|\ln \varepsilon\right|^{1/8}$.

On the other hand, if we take a best approximation $k_1^{(j)}/k_2^{(j)}$ to $\tilde\beta$ satisfying $k_2^{(j)} \ge \varepsilon^{-1/5}$, then (1.4.125) gives

$$\left| A_j(\tilde\beta) - \frac{1}{\tilde\beta + \tilde\beta^{-1}} \right| \le \varepsilon^{2/5}.$$

Therefore, for any $\beta_2 \in I^*_{\tilde\beta}$ and any $j \in \mathbb{N}$ for which $k_1^{(j)}/k_2^{(j)}$ is a best approximation to β_2 (or to $\tilde\beta$) satisfying

$$k_2^{(j)} \in (\varepsilon^{-1/5}, \varepsilon^{-1/4} \left|\ln \varepsilon\right|^{1/8})$$

we have (see also (1.4.128))

$$\frac{1}{4(\tilde\beta + \tilde\beta^{-1})} \le |A_j(\beta_2)| \le \frac{4}{\tilde\beta + \tilde\beta^{-1}} \tag{2.3.57}$$

and, in particular, (1.4.124) implies

$$ctant\ \left| k_2^{(j)} \right|^{-1} \le \left| \hat{k}^{(j)} \omega \right| \le ctant\ \left| k_2^{(j)} \right|^{-1}. \tag{2.3.58}$$

Furthermore, since the Main Lemma II tell us that those indices n^0 and n^1 (depending on r_1, r_2 and ε) for which one expects (2.3.55) holds true, satisfy

$$\varepsilon^{-1/5} << ctant\ \varepsilon^{-1/4} \le \left| k_2^{(t)} \right| \le ctant\ \varepsilon^{-1/4} << \varepsilon^{-1/4} \left|\ln \varepsilon\right|^{1/8}, \quad t = n^0, n^1$$

we may assert that (2.3.56), (2.3.57) and (2.3.58) are satisfied when $j = n^0$ or $j = n^1$.

Now, (2.3.58) can be used to deduce that the two functions (see also (1.3.88) for related details)

$$\beta_2 \in I^*_{\tilde\beta} \to \hat{k}^{(t)}\omega \in \mathbb{R}, \quad t = n^0, n^1, \quad \omega = (1, \beta_2) \tag{2.3.59}$$

do not vanish on $I^*_{\tilde\beta}$. Hence, once again, the sign of the coefficients $\tilde{B}^{(i)}_{\hat k}$ given in (2.3.52) does not change when β_2 moves along $I^*_{\tilde\beta}$. Thus, it suffices to check (2.3.55) by assuming that β_2 coincides with the golden mean $\tilde\beta$ but, since for $t = n^0, n^1$ one easily obtain $\text{sign}(\tilde{B}^{(2)}_{\hat{k}^{(t)}}) = \text{sign}(-\tilde{B}^{(3)}_{\hat{k}^{(t)}})$ (compare with (1.3.86)), the algorithm for selecting $\overline{\psi}_i = \overline{\psi}_i(\varepsilon) \in \{0, \pi\}$, $i = 1, 2$, for which (2.3.55) holds (for $\beta_2 = \tilde\beta$) is the same as the one given in the first chapter (see the proof of Lemma 1.3.13).

Remark 2.3.11 *As in the first chapter, see Remark 1.3.14, in the case in which one of the two functions defined in (2.3.59) vanish we do not know how to get estimates on the splitting size or, in other words, our method for proving Theorem 0.0.6 and therefore the Main Theorem II, fails. In essence, this is the reason why we are only able to prove the existence of transition chains whose lengths depend on* ε *(micro-diffusion). In fact, the length of those transition chains is identified with the length of the golden mean neighbourhood* $I^*_{\tilde{\beta}}$ *furnished by the Main Lemma II when we take* $v_1 = r_1$*,* $v_2 = r_2$ *and* $d = \dfrac{\pi}{2}$.

Hence, to obtain larger transition chains, or chains whose lengths do not depend on ε *(macro-diffusion), it seems necessary to get splitting estimates for the transversality at which the perturbed manifolds of the invariant tori* T_{β_1,β_2} *intersect, for those values of* β_2 *at which one of the two functions defined in (2.3.59) vanishes.*

Before going into the details of the proof of the bounds for the transversality $\Upsilon = \Upsilon(\overline{\psi}_1, \overline{\psi}_2)$ announced by Theorem 0.0.6, we want to remark that the (good) set of parameters $\mathcal{U}^*_\varepsilon = \mathcal{U}^*_\varepsilon(v_1, v_2, d)$ given by Lemma 2.3.8 can be written as (see (4.2.27))

$$\mathcal{U}^*_\varepsilon = \bigcup_{n \geq n^*} \mathcal{U}^*_n, \quad \mathcal{U}^*_n = \mathcal{U}^*_n(v_1, v_2, d)$$

where $\{\mathcal{U}^*_n, n \geq n^*\}$ is a family of two by two disjoint real intervals satisfying the following property: If $\varepsilon \in \mathcal{U}^*_n$, then $n^0 = n$ and $n^1 = n + 1$, where n^0 and n^1 are those indices for which the Main Lemma II holds.

In particular, in the case in which $v_1 = r_1$, $v_2 = r_2$ and $d = \dfrac{\pi}{2}$, and writing the renormalized Melnikov functions coefficients obtained in (2.2.35) as

$$\mathcal{E}^*_{\hat{k}} = \overline{\mathcal{E}}_{\hat{k}}\left(r_1, r_2, \frac{\pi}{2}, \varepsilon, \beta_2\right) = \mathcal{E}^*_{\hat{k}}(\varepsilon, \beta_2)$$

then Corollary 4.2.5 yields

$$\frac{1}{4} \leq \frac{\mathcal{E}^*_{\hat{k}(n^0)}(\varepsilon, \beta_2)}{\mathcal{E}^*_{\hat{k}(n^1)}(\varepsilon, \beta_2)} \leq 4, \tag{2.3.60}$$

whenever $\varepsilon \in \mathcal{U}^*_n\left(r_1, r_2, \dfrac{\pi}{2}\right)$ and $\beta_2 \in I^*_{\tilde{\beta}}(\varepsilon)$.

Now, for every (ε, β) in $\mathcal{U}^*_\varepsilon\left(r_1, r_2, \dfrac{\pi}{2}\right) \times I^*_{\tilde{\beta}}$, we are going to obtain the conclusions of Theorem 0.0.6. This second part of the proof is divided, as was done in the second part of the proof of the Main Theorem I, in eight different steps. In fact, we are going to pay special attention in comparing, one by one, the partial results obtained in the respective steps of both proofs.

S1. We recover the notation

$$(\hat{\psi}, t, s) \in \mathcal{B}'''_2 \times [T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}'_2 \to Q^*(\hat{\psi}, t, s)$$

introduced in (2.3.37) to denote convenient pieces of the invariant manifolds of T_{β_1,β_2}. Since we are going to apply Lemma 2.3.1 and Lemma 2.3.3 several times along the next arguments, we restrict the definition domain of the above parameterizations of the invariant manifolds to the set

$$\mathcal{B}_2^{\mathcal{V}} \times [T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}_2''$$

where $\mathcal{B}_2^{\mathcal{V}}$ and $\mathcal{C}_2''$ were respectively introduced at (2.3.50) and (2.3.47).

Let us again consider the splitting functions $\mathcal{K}_u^\mu$, $\mathcal{J}_{1,u}^\mu$ and $\mathcal{J}_{2,u}^\mu$ defined in (2.3.46) and restrict their domain of definition to $\mathcal{C}_2'' \times \mathcal{B}_2^{\mathcal{V}}$.

Then, following the same steps as the ones given for proving (1.3.90) and (1.3.91), we may write, for every $(s,\psi_1,\psi_2) \in \mathcal{C}_2'' \times \mathcal{B}_2^{\mathcal{V}}$ and any $t \in [T_0 - \text{Re } s, 2T_0 - \text{Re } s]$,

$$\left|\mathcal{K}_u^\mu(s,\psi_1,\psi_2) - H_1(x^-,y^-,\hat{I}^-,\hat{\theta}^-)(\hat{\psi},t,s) + H_1(x^+,y^+,\hat{I}^+,\hat{\theta}^+)(\hat{\psi},t,s)\right| \leq \\ \leq ctant\ \mu^2\varepsilon^{-2b(N+8)}$$

with

$$H_1 = H_1(x,y,\hat{I},\hat{\theta}) = \frac{y^2}{2} + \cos x - 1$$

and, for $i = 1,2$,

$$\left|\mathcal{J}_{i,u}^\mu(s,\psi_1,\psi_2) - I_i(x^-,y^-,\hat{I}^-,\hat{\theta}^-)(\hat{\psi},t,s) + I_i(x^+,y^+,\hat{I}^+,\hat{\theta}^+)(\hat{\psi},t,s)\right| \leq \\ \leq ctant\ \mu^2\varepsilon^{-2b(N+8)}. \tag{2.3.61}$$

To obtain the above bounds one may apply (2.3.38) and use the family of conjugations h_μ between the vector fields $\dot{Q} = g_\mu(Q)$ and $\dot{Q} = g_*(Q)$ used to prove Lemma 2.3.1, which satisfy (see (2.3.43)),

$$\|h_\mu - I\| \leq ctant\ \mu\varepsilon^{-b(N+9)}$$

and

$$\|Dh_\mu - I\| \leq ctant\ \mu\varepsilon^{-b(N+10)}.$$

Therefore, if we take the notation

$$C_{1,u}^\mu = \mathcal{K}_u^\mu, \qquad C_{i,u}^\mu = \mathcal{J}_{i-1,u}^\mu, \quad i = 2,3,$$

and introduce the error functions

$$(s,\psi_1,\psi_2) \in \mathcal{C}_2'' \times \mathcal{B}_2^{\mathcal{V}} \to E_i^u(s,\psi_1,\psi_2) = C_{i,u}^\mu(s,\psi_1,\psi_2) - \mathcal{M}_i(s,\psi_1,\psi_2), \tag{2.3.62}$$

then Lemma 2.2.1 (a) and (2.3.61) yield

$$|E_i^u(s,\psi_1,\psi_2)| \leq ctant\ \mu^2\,|\ln\mu|\,\varepsilon^{-2b(N+8)}, \tag{2.3.63}$$

for every $(s,\psi_1,\psi_2)\in\mathcal{C}_2''\times\mathcal{B}_2^{\mathcal{V}}$. Hence, Lemma 2.2.1 (b) leads to

$$\left|C_{i,u}^{\mu}(s,\psi_1,\psi_2)\right|\le ctant\ \mu\varepsilon^{-b(N+4)}, \tag{2.3.64}$$

whenever $\mu\in(0,\varepsilon^w)$, with $w>\dfrac{5}{4}b(N+12)$ and $\mu^{1/5}\left|\ln\mu\right|<1$ (the factor 5/4 could be replaced by any λ bigger than one).

S2. Now, once a value of ψ_1 with $|\mathrm{Im}\ \psi_1|\le r_1-5\varepsilon^b$ is fixed, we apply Lemma 2.3.3 to write, for any $s\in\mathcal{C}_2''$ and any ψ_2 satisfying $|\mathrm{Im}\ \psi_2|<r_2-5\varepsilon^b$, (see also (2.3.46))

$$\begin{aligned} u_1 &= U_1(s,\psi_2)=\mathcal{S}^{\mu}(Q^-(\hat{\psi},t,s))-t\\ u_2 &= U_2(s,\psi_2)=\theta_2^{\mu}(Q^-(\hat{\psi},t,s))-\frac{\beta_2 t}{\sqrt{\varepsilon}}\end{aligned}$$

where the right-hand side terms do not depend on t.

Furthermore, using that Lemma 2.3.3 also gives

$$\|(s,\psi_2)-(u_1,u_2)\|\le ctant\ \mu\varepsilon^{-b(N+9)},$$

we may apply Lemma 1.1.2 and (2.3.64) to deduce

$$\left|C_{i,u}^{\mu}(s,\psi_1,\psi_2)-C_{i,u}^{\mu}(u_1,\psi_1,u_2)\right|\le ctant\ \mu^2\varepsilon^{-2b(N+7)}, \tag{2.3.65}$$

for every $(s,\psi_1,\psi_2)\in\mathcal{C}_2'''\times\mathcal{B}_2^{\mathcal{V}I}$ with

$$\mathcal{C}_2'''=\left\{s\in\mathbb{C}:\ |\mathrm{Re}\ s|\le\varepsilon,\ |\mathrm{Im}\ s|\le\frac{\pi}{2}-3\varepsilon^b\right\}$$

and

$$\mathcal{B}_2^{\mathcal{V}I}=\left\{(\theta_1,\theta_2)\in\mathbb{C}^2:\ |\mathrm{Im}\ \theta_i|\le r_i-6\varepsilon^b,\ i=1,2\right\}.$$

Moreover, in the same way (using Lemma 2.2.1 (b) instead of (2.3.64)) we also achieve

$$|\mathcal{M}_i(s,\psi_1,\psi_2)-\mathcal{M}_i(u_1,\psi_1,u_2)|\le ctant\ \mu^2\varepsilon^{-2b(N+7)}$$

for $i=1,2,3$ and for every $(s,\psi_1,\psi_2)\in\mathcal{C}_2'''\times\mathcal{B}_2^{\mathcal{V}I}$.

Therefore, (2.3.62) and (2.3.63) imply

$$|E_i^{\mu}(u_1,\psi_1,u_2)|\le ctant\ \mu^2\left|\ln\mu\right|\varepsilon^{-2b(N+8)}. \tag{2.3.66}$$

Now, following the respective arguments given in the second step (entitled "Setting suitable coordinates") of the second part of the proof of the Main Theorem I (Subsection 1.3.2) let us fix a value of ψ_1 with $|\mathrm{Im}\ \psi_1|\le r_1-6\varepsilon^b$. Once ψ_1 is fixed, we take, once more, the functions $U_1=U_1(s,\psi_2)$ and $U_2=U_2(s,\psi_2)$ given by Lemma 2.3.3, satisfying, for every $t\in[T_0-\mathrm{Re}\ s,2T_0-\mathrm{Re}\ s]$, that

$$U_1(s,\psi_2)+t=u_1+t=\mathcal{S}^{\mu}(Q^-(\hat{\psi},t,s))$$

and

$$U_1(s,\psi_2) + \frac{\beta_2 t}{\sqrt{\varepsilon}} = u_2 + \frac{\beta_2 t}{\sqrt{\varepsilon}} = \theta_2^\mu(Q^-(\hat{\psi},t,s)).$$

We recall that $\mathcal{S}^\mu$ and θ^μ are, respectively, the first and the sixth components of the analytic change of coordinates given by Lemma 2.3.1 transforming our initial system $\dot{Q} = g_\mu(Q)$ into the flow box system:

$$\dot{\mathcal{S}}^\mu = 1, \quad \dot{\mathcal{K}}^\mu = 0, \quad \dot{\mathcal{J}}_i^u = 0, \; i = 1,2, \quad \dot{\theta}_1 = \frac{1}{\sqrt{\varepsilon}}, \quad \dot{\theta}_2^\mu = \frac{\beta_2}{\sqrt{\varepsilon}}, \quad \dot{s} = 0.$$

Let us take the notation:

$$\begin{aligned} \mathcal{S}^\mu &= \mathcal{S}^\mu(s,t) = \mathcal{S}^\mu(Q^-(\hat{\psi},t,s)) = u_1 + t \\ \theta_2^\mu &= \theta_2^\mu(s,t) = \theta_2^\mu(Q^-(\hat{\psi},t,s)) = u_2 + \frac{\beta_2 t}{\sqrt{\varepsilon}} \end{aligned}$$

Then, in the time-dependent coordinates $(\mathcal{S}^\mu, \theta_1, \theta_2^\mu)$, with $\theta_1 = \psi_1 + \frac{t}{\sqrt{\varepsilon}}$, our splitting functions satisfy

$$C_{i,u}^\mu(\mathcal{S}^\mu,\theta_1,\theta_2^\mu) = C_{i,u}^\mu(\mathcal{S}^\mu,\psi_1 + \frac{t}{\sqrt{\varepsilon}}, u_2 + \frac{\beta_2 t}{\sqrt{\varepsilon}}) = C_{i,u}^\mu(\mathcal{S}^\mu - t,\psi_1,u_2) = C_{i,u}^\mu(u_1,\psi_1,u_2).$$

Hence, from the same arguments as the ones used to reach (1.3.99) we now obtain

$$C_{i,u}^\mu(\mathcal{S}^\mu,\theta_1,\theta_2^\mu) = C_{i,u}^\mu\left(\tilde{\mathcal{S}}^\mu,\tilde{\theta}_1,\tilde{\theta}_2^\mu\right),$$

whenever

$$\theta_1 - \frac{\mathcal{S}^\mu}{\sqrt{\varepsilon}} = \tilde{\theta}_1 - \frac{\tilde{\mathcal{S}}^\mu}{\sqrt{\varepsilon}}, \quad \theta_2^\mu - \frac{\beta_2\mathcal{S}^\mu}{\sqrt{\varepsilon}} = \tilde{\theta}_2^\mu - \frac{\beta_2\tilde{\mathcal{S}}^\mu}{\sqrt{\varepsilon}}.$$

Thus, there exist analytic functions $(\tilde{F}_i^\mu)^*$, $i = 1,2,3$, for which

$$C_{i,u}^\mu(u_1,\psi_1,u_2) = (\tilde{F}_i^\mu)^*\left(\psi_1 - \frac{u_1}{\sqrt{\varepsilon}}, u_2 - \frac{\beta_2 u_1}{\sqrt{\varepsilon}}\right). \tag{2.3.67}$$

On the other hand, equations (2.3.64) and (2.3.65) imply

$$\left|C_{i,u}^\mu(u_1,\psi_1,u_2)\right| \le ctant\ \mu\varepsilon^{-b(N+4)},$$

whenever $\mu \in (0,\varepsilon^w)$ with $w > b(N+10)$. Therefore, since from (1.3.74), we have

$$\mathcal{C}_2''' \times \mathcal{B}^{\mathcal{V}I} = D\left(\varepsilon, \frac{\pi}{2} - 3\varepsilon^b, r_1 - 6\varepsilon^b, r_2 - 6\varepsilon^b\right)$$

we may use (2.3.67) to write

$$C_{i,u}^\mu \in \mathcal{A}\left(\varepsilon, \frac{\pi}{2} - 3\varepsilon^b, r_1 - 6\varepsilon^b, r_2 - 6\varepsilon^b, ctant\ \mu\varepsilon^{-b(N+4)}\right).$$

Now, Remark 2.2.2 together with (2.3.66) give

$$E_i^\mu \in \mathcal{A}\left(\varepsilon, \frac{\pi}{2} - 3\varepsilon^b, r_1 - 6\varepsilon^b, r_2 - 6\varepsilon^b, ctant\ \mu^2 \left|\ln \mu\right| \varepsilon^{-2b(N+8)}\right).$$

Therefore, we may write

$$E_i^\mu(u_1, \psi_1, u_2) = G_i^\mu\left(\psi_1 - \frac{u_1}{\sqrt{\varepsilon}}, u_2 - \frac{\beta_2 u_1}{\sqrt{\varepsilon}}\right). \tag{2.3.68}$$

Hence, we may apply Lemma 1.3.4 to obtain, for any $\hat{k} = (k_1, k_2) \in \mathbb{Z}^2 \backslash \{(0,0)\}$,

$$\left|G^\mu_{i,\hat{k}}\right| \leq ctant\ \mu^2 \left|\ln \mu\right| \varepsilon^{-2b(N+8)} \overline{\Lambda}_{\hat{k}}(r_1, r_2, \varepsilon), \tag{2.3.69}$$

where $G^\mu_{i,\hat{k}}$ are the Fourier coefficients of the functions G_i^μ, $i = 1, 2, 3$ and (see also (2.3.51))

$$\overline{\Lambda}_{\hat{k}}(r_1, r_2, \varepsilon) = \overline{\mathcal{E}}_{\hat{k}}\left(r_1 - 6\varepsilon^b, r_2 - 6\varepsilon^b, \frac{\pi}{2} - 3\varepsilon^b, \varepsilon, \beta_2\right). \tag{2.3.70}$$

S3. We select the homoclinic orbit (see Remark 2.1.5)

$$\left\{(x^-(\hat{\psi}, t, s), y^-(\hat{\psi}, t, s), \hat{I}^-(\hat{\psi}, t, s), \hat{\theta}^-(\hat{\psi}, t, s)) :\ \hat{\psi} = (\overline{\psi}_1, \overline{\psi}_2),\ s = 0\right\}$$

of the perturbed system in order that equation (2.3.55) holds and define the splitting size (transversality) of the perturbed manifolds along this orbit by

$$\Upsilon = \Upsilon(\overline{\psi}_1, \overline{\psi}_2) = \det \begin{pmatrix} \dfrac{\partial \overline{\mathcal{J}}^\mu_{1,u}}{\partial \psi_1}(\overline{\psi}_1, \overline{\psi}_2) & \dfrac{\partial \overline{\mathcal{J}}^\mu_{1,u}}{\partial \psi_2}(\overline{\psi}_1, \overline{\psi}_2) \\ \dfrac{\partial \overline{\mathcal{J}}^\mu_{2,u}}{\partial \psi_1}(\overline{\psi}_1, \overline{\psi}_2) & \dfrac{\partial \overline{\mathcal{J}}^\mu_{2,u}}{\partial \psi_2}(\overline{\psi}_1, \overline{\psi}_2) \end{pmatrix}, \tag{2.3.71}$$

where we have introduced, for $i = 2, 3$, the functions

$$\overline{\mathcal{J}}^\mu_{i-1,u}(\psi_1, \psi_2) = \mathcal{J}^\mu_{i-1,u}(0, \psi_1, \psi_2).$$

As in the first chapter, we also define, for $i = 2, 3$,

$$\overline{\mathcal{M}}_i(\psi_1, \psi_2) = \mathcal{M}_i(0, \psi_1, \psi_2), \quad \overline{E}_i^\mu(\psi_1, \psi_2) = E_i^\mu(0, \psi_1, \psi_2),$$

and write

$$\frac{\partial \overline{\mathcal{J}}_{i-1,u}}{\partial \psi_j}(\overline{\psi}_1, \overline{\psi}_2) = m^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2) + e^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2)$$

where, for $j = 1, 2$ and $i = 2, 3$,

$$m^{i-1,j}(\psi_1, \psi_2) = \frac{\partial \overline{\mathcal{M}}_i}{\partial \psi_j}(\psi_1, \psi_2) \quad \text{and} \quad e^{i-1,j}(\psi_1, \psi_2) = \frac{\partial \overline{E}_i^\mu}{\partial \psi_j}(\psi_1, \psi_2).$$

S4. Now, let us bound $\left|e^{i-1,j}(\overline{\psi}_1,\overline{\psi}_2)\right|$, for $i=2,3$ and $j=1,2$.

Let U_1 and U_2 be the functions introduced in the statement of Lemma 2.3.3 and let us choose (see also Remark 2.3.4) the real functions

$$\begin{array}{rcl}(\psi_1,\psi_2)\in\mathbb{R}^2 & \to & u_1^*(\psi_1,\psi_2)=U_1(0,\psi_2)\\ (\psi_1,\psi_2)\in\mathbb{R}^2 & \to & u_2^*(\psi_1,\psi_2)=U_2(0,\psi_2).\end{array}$$

Then, using (2.3.68), we may write, for $\hat{k}=(k_1,k_2)$,

$$\begin{aligned}\overline{E}_i^{\mu}(\psi_1,\psi_2) &= E_i^{\mu}(0,\psi_1,\psi_2)=E_i^{\mu}(u_1^*,\psi_1,u_2^*)=G_i^{\mu}\left(\psi_1-\frac{u_1^*}{\sqrt{\varepsilon}},u_2^*-\frac{\beta u_1^*}{\sqrt{\varepsilon}}\right)=\\ &= \sum_{\hat{k}\in\mathbb{Z}^2}G^{\mu}_{i,\hat{k}}\exp\left(\sqrt{-1}\left(k_1\left(\psi_1-\frac{u_1^*}{\sqrt{\varepsilon}}\right)+k_2\left(u_2^*-\frac{\beta_2u_1^*}{\sqrt{\varepsilon}}\right)\right)\right).\end{aligned}$$

Therefore,

$$e^{i-1,j}(\psi_1,\psi_2)=\sum_{\hat{k}\in\mathbb{Z}^2}G^{j,\mu}_{i,\hat{k}}\exp\left(\sqrt{-1}\left(k_1\left(\psi_1-\frac{u_1^*}{\sqrt{\varepsilon}}\right)+k_2\left(u_2^*-\frac{\beta_2u_1^*}{\sqrt{\varepsilon}}\right)\right)\right)$$

where

$$G^{1,\mu}_{i,\hat{k}}=\sqrt{-1}\left(k_1+k_2\frac{\partial u_2^*}{\partial\psi_1}-\frac{k_1+k_2\beta_2}{\sqrt{\varepsilon}}\frac{\partial u_1^*}{\partial\psi_1}\right)G^{\mu}_{i,\hat{k}} \tag{2.3.72}$$

and

$$G^{2,\mu}_{i,\hat{k}}=\sqrt{-1}\left(k_2\frac{\partial u_2^*}{\partial\psi_2}-\frac{k_1+k_2\beta_2}{\sqrt{\varepsilon}}\frac{\partial u_1^*}{\partial\psi_2}\right)G^{\mu}_{i,\hat{k}}. \tag{2.3.73}$$

Hence, we easily obtain that $G^{j,\mu}_{i,0,0}=0$, for $j=1,2$ and $i=2,3$, and moreover

$$\left|e^{i-1,j}(\overline{\psi}_1,\overline{\psi}_2)\right|\le\sum_{\hat{k}\in\mathbb{Z}^2\setminus\{(0,0)\}}\left|G^{j,\mu}_{i,\hat{k}}\right|, \tag{2.3.74}$$

whenever $(\overline{\psi}_1,\overline{\psi}_2)\in\mathbb{R}^2$. Now, Lemma 2.3.3 implies

$$\|u_1^*\|_{\mathcal{B}_2^{\mathcal{V}}}\le ctant\ \mu\varepsilon^{-b(N+9)},\quad \|u_2^*-P_2\|_{\mathcal{B}_2^{\mathcal{V}}}\le ctant\ \mu\varepsilon^{-b(N+9)},$$

where $P_2(\psi_1,\psi_2)=\psi_2$.

Furthermore, since the functions u_i^*, $i=1,2$, are analytic in $\mathcal{B}_2^{\mathcal{V}}$, we may apply Cauchy estimates in order to guarantee that

$$\left|\frac{\partial u_l^*}{\partial\psi_\lambda}(\psi_1,\psi_2)\right|<2$$

for $l,\lambda=1,2$ and any $(\psi_1,\psi_2)\in\mathcal{B}_2^{\mathcal{V}I}$, whenever $\mu\in(0,\varepsilon^w)$, with $w>b(N+10)$.

Then, from (2.3.72) and (2.3.73) we get, for $j = 1, 2$, that

$$\left|G^{j,\mu}_{i,\hat{k}}\right| \leq ctant \ \left|\hat{k}\right| \left|G^{\mu}_{i,\hat{k}}\right|$$

and therefore (2.3.74) leads to

$$\left|e^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2)\right| \leq ctant \ \varepsilon^{-1/2} \sum_{\hat{k}\in\mathbb{Z}^2\setminus\{(0,0)\}} \left|\hat{k}\right| \left|G^{\mu}_{i,\hat{k}}\right|,$$

where, as usual, $\left|\hat{k}\right| = |k_1| + |k_2|$.

Hence, from (2.3.69), we get

$$\left|e^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2)\right| \leq ctant \ \mu^2 \left|\ln \mu\right| \varepsilon^{-2b(N+8)-1/2} \sum_{\hat{k}\in\mathbb{Z}^2\setminus\{(0,0)\}} \left|\hat{k}\right| \overline{\Lambda}_{\hat{k}}(r_1, r_2, \varepsilon).$$

Therefore, taking into account that (see (2.3.70))

$$\overline{\Lambda}_{\hat{k}}(r_1, r_2, \varepsilon) = \overline{\mathcal{E}}_{\hat{k}} \left(r_1 - 6\varepsilon^b, r_2 - 6\varepsilon^b, \frac{\pi}{2} - 3\varepsilon^b, \varepsilon, \beta_2\right)$$

it is easy to see that the series

$$\sum_{\hat{k}\in\mathbb{Z}^2\setminus\{(0,0)\}} \left|\hat{k}\right| \overline{\Lambda}_{\hat{k}}(r_1, r_2, \varepsilon)$$

belongs to $\mathcal{S}^*_i \left(r_1 - 6\varepsilon^b, r_2 - 6\varepsilon^b, \frac{\pi}{2} - 3\varepsilon^b, \varepsilon, \beta_2\right)$ (see Definition 2.3.5, Definition 2.3.6 and Definition 2.3.7) for $i = 1, 2, 3$. Hence, assuming

$$b > \frac{3}{10} \ \left(> \frac{1}{4}\right) \tag{2.3.75}$$

we may apply the second Perturbing Lemma (see Lemma 2.3.9), in order to ensure that, for every $\varepsilon \in \mathcal{U}^*_\varepsilon \left(r_1, r_2, \frac{\pi}{2}\right)$ and any $\beta_2 \in I^*_{\tilde{\beta}}(\varepsilon)$, one may write

$$\left|e^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2)\right| \leq ctant \ \mu^2 \left|\ln \mu\right| \varepsilon^{-2b(N+8)-1/2} \left(\left|\hat{k}^{(n^0)}\right| \overline{\Lambda}_{\hat{k}^{(n^0)}} + \left|\hat{k}^{(n^1)}\right| \overline{\Lambda}_{\hat{k}^{(n^1)}}\right), \tag{2.3.76}$$

for $j = 1, 2$, $i = 2, 3$, where $k_1^{(n^0)}/k_2^{(n^0)}$, $k_1^{(n^1)}/k_2^{(n^1)}$ are the best approximations to the golden mean given by the Main Lemma II (when taking $v_1 = r_1$, $v_2 = r_2$, $d = \frac{\pi}{2}$) satisfying $|n^0 - n^1| = 1$ and

$$\text{ctant } \varepsilon^{-1/4} \leq k_r^{(t)} \leq \text{ctant } \varepsilon^{-1/4} \tag{2.3.77}$$

for $r = 1, 2$ and $t = n^0, n^1$.

Now, (2.3.58) and (2.3.77) allow us to deduce

$$\left|\hat{k}^{(t)}\omega\right| = \left|k_1^{(t)} + \beta_2 k_2^{(t)}\right| \leq ctant\ \left|k_2^{(t)}\right|^{-1} \leq ctant\ \varepsilon^{1/4}, \quad t = n^0, n^1.$$

Hence, using also that $b > 3/10$, it is easy to check that, for $t = n^0, n^1$,

$$\overline{\Lambda}_{\hat{k}^{(t)}} \leq ctant\ \mathcal{E}^*_{\hat{k}^{(t)}}$$

where

$$\mathcal{E}^*_{\hat{k}} = \overline{\mathcal{E}}_{\hat{k}}\left(r_1, r_2, \frac{\pi}{2}, \varepsilon, \beta_2\right)$$

are the coefficients taking part in the expressions for renormalized Melnikov functions computed in Section 2.2 (see, in particular, (2.2.35)).

Hence, from (2.3.76), we get

$$\left|e^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2)\right| \leq ctant\ \mu^2 \left|\ln \mu\right| \varepsilon^{-2b(N+8)-1/2} \left(\left|\hat{k}^{(n^0)}\right| \mathcal{E}^*_{\hat{k}^{(n^0)}} + \left|\hat{k}^{(n^1)}\right| \mathcal{E}^*_{\hat{k}^{(n^1)}}\right), \quad (2.3.78)$$

whenever $\mu \in (0, \varepsilon^w)$, $w > b(N + 10)$ and $b > 3/10$.

S5. Let us observe that, as was already obtained in (2.3.53), we have, for $j = 1, 2$ and $i = 2, 3$, that

$$m^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2) = \frac{\partial \overline{\mathcal{M}}_i}{\partial \psi_j}(\overline{\psi}_1, \overline{\psi}_2) = -2\mu\pi\varepsilon^{-1/2} \sum_{\hat{k} \in \mathbb{Z}^2 \setminus \{(0,0)\}} S_{\hat{k}} \mathcal{E}^*_{\hat{k}} \quad (2.3.79)$$

where the coefficients

$$S_{\hat{k}} = \begin{cases} k_j \tilde{B}^{(i)}_{\hat{k}} \cos(k_1 \overline{\psi}_1 + k_2 \overline{\psi}_2), & \text{if } \hat{k} \in \Lambda \\ 0, \quad \text{if } \hat{k} \in \mathbb{Z}^2 \setminus (\Lambda \cup \{(0,0)\}) \end{cases}$$

were given at (2.3.54).

Let us denote again

$$\hat{S} = \sum_{\hat{k} \in \mathbb{Z}^2 \setminus \{(0,0)\}} S_{\hat{k}} \mathcal{E}^*_{\hat{k}}$$

and prove that (recall Definition 2.3.5, Definition 2.3.6 and Definition 2.3.7)

$$\hat{S} \in \mathcal{S}^*\left(r_1, r_2, \frac{\pi}{2}, \varepsilon, \beta_2\right) = \bigcap_{i=1}^{3} \mathcal{S}^*_i\left(r_1, r_2, \frac{\pi}{2}, \varepsilon, \beta_2\right)$$

for every $\varepsilon \in \mathcal{U}^*_\varepsilon\left(r_1, r_2, \frac{\pi}{2}\right)$ and any $\beta \in I^*_{\tilde{\beta}}(\varepsilon)$.

We begin by writing the coefficients (see (2.3.52))

$$\tilde{B}^{(i)}_{\hat{k}} = k_{i-1} m_{\hat{k}} H^*(\hat{k}\omega) \left|\hat{k}\omega\right| \exp\left(\left|k_1\right| r_1 + \left|k_2\right| r_2\right)$$

appearing in the definition of $S_{\hat{k}}$. From equations (2.0.3), (2.0.4) and (2.0.5), one easily gets, for every $\hat{k} \in \mathbb{Z}^2 \backslash \{(0,0)\}$, that

$$|m_{\hat{k}}| \exp\left(|k_1| r_1 + |k_2| r_2\right) \leq ctant \left|\hat{k}\right|^N.$$

Moreover, from the definition of the function $H^* = H^*(\hat{k}\omega)$ given in (2.2.33), it follows that

$$\left|H^*(\hat{k}\omega)\right| \left|\hat{k}\omega\right| < ctant \left|\hat{k}\right|.$$

Hence,

$$\left|\tilde{B}^{(i)}_{\hat{k}}\right| < ctant \left|\hat{k}\right|^{N+2}$$

and therefore, for every $\hat{k} \in \mathbb{Z}^2 \backslash \{(0,0)\}$, it holds that

$$|S_{\hat{k}}| < ctant \left|\hat{k}\right|^{N+3}.$$

Thus,

$$\hat{S} \in \mathcal{S}_1^* \left(r_1, r_2, \frac{\pi}{2}, \varepsilon, \beta_2\right).$$

Now, to prove that $\hat{S} \in \mathcal{S}_2^* \left(r_1, r_2, \frac{\pi}{2}, \varepsilon, \beta_2\right)$, let us take $k_1^{(h)}/k_2^{(h)}$ a best approximation to the golden mean satisfying

$$k_2^{(h)} \in \left(\varepsilon^{-1/5}, \varepsilon^{-1/4} \left|\ln \varepsilon\right|^{1/8}\right).$$

Due to the assumption imposed to the set of indices Λ (see (2.0.2)) and, more concretely, once $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}_3$ are fixed, we restrict our study to those values of ε for which, if $k_2^{(h)} \geq \varepsilon^{-1/5}$, then $(k_1^{(h)}, k_2^{(h)}) \in \Lambda$. Moreover, see the property given in (2.0.7) on the coefficients $g(k_1, k_2)$, once the value of n_0 is fixed we also take sufficiently small values of ε in order that, if $k_2^{(h)} \geq \varepsilon^{-1/5}$ and $k_1^{(h)}/k_2^{(h)}$ is a best approximation to the golden mean, then $h \geq n_0$. In this way, (2.0.7) leads to

$$\left|g(k_1^{(h)}, k_2^{(h)})\right| \geq ctant\ \exp\left(-\left|k_1^{(h)}\right| r_1 - \left|k_2^{(h)}\right| r_2\right)$$

and therefore, from (2.0.3) and (2.0.5), we have

$$|m_{\hat{k}^{(h)}}| \exp\left(\left|k_1^{(h)}\right| r_1 + \left|k_2^{(h)}\right| r_2\right) > ctant \left|\hat{k}^{(h)}\right|^{-N}.$$

On the other hand, (see (2.3.58)), for those best approximations $k_1^{(h)}/k_2^{(h)}$ to the golden mean number such that $k_2^{(h)} \in (\varepsilon^{-1/5}, \varepsilon^{-1/4} \left|\ln \varepsilon\right|^{1/8})$, it follows that $\left|\hat{k}^{(h)}\omega\right| \geq ctant \left|\hat{k}^{(h)}\right|^{-1}$.

Hence, since it also follows that $\left|H^*(\hat{k}^{(h)}\omega)\right| > 1$, we deduce

$$\left|\tilde{B}^{(i)}_{\hat{k}^{(h)}}\right| \geq ctant \ \left|\hat{k}^{(h)}\right|^{-(N+1)}$$

and, since $(\overline{\psi}_1, \overline{\psi}_2) \in \{0, \pi\} \times \{0, \pi\}$,

$$|S_{\hat{k}^{(h)}}| \geq ctant \ \left|\hat{k}^{(h)}\right|^{-(N+1)}$$

in such a way that, according to Definition 2.3.6,

$$\hat{S} \in \mathcal{S}^*_2\left(r_1, r_2, \frac{\pi}{2}, \varepsilon, \beta_2\right).$$

Finally, since (2.0.6) implies $\tilde{B}^{(i)}_{\hat{k}} = -\tilde{B}^{(i)}_{-\hat{k}}$, one easily gets $S_{\hat{k}} = S_{-\hat{k}}$, thus that

$$\hat{S} \in \mathcal{S}^*_3\left(r_1, r_2, \frac{\pi}{2}, \varepsilon, \beta_2\right). \tag{2.3.80}$$

Therefore, we may apply the Main Lemma II to the series

$$\hat{S} = \sum_{\hat{k}\in\mathbb{Z}^2\setminus\{(0,0)\}} S_{\hat{k}}\mathcal{E}^*_{\hat{k}} \in \mathcal{S}^*\left(r_1, r_2, \frac{\pi}{2}, \varepsilon, \beta_2\right)$$

to write

$$\hat{S} = 2\left(S_{\hat{k}^{(n^0)}}\mathcal{E}^*_{\hat{k}^{(n^0)}} + S_{\hat{k}^{(n^1)}}\mathcal{E}^*_{\hat{k}^{(n^1)}}\right)\left[1 + \mathrm{O}\left(\exp\left(-\frac{|\ln\varepsilon|^{-1/4}}{\varepsilon^{1/4}}\right)\right)\right],$$

for every $\varepsilon \in \mathcal{U}^*_\varepsilon\left(r_1, r_2, \frac{\pi}{2}\right)$, any $\beta_2 \in I^*_{\tilde{\beta}}$ and n^0, n^1 those indices for which (2.3.55) and (2.3.78) hold.

Now, if we again consider the coefficients (see (2.2.36))

$$B^{(i)}_{\hat{k}} = -2\mu k_{i-1} m_{\hat{k}} \frac{\pi}{\sqrt{\varepsilon}} H^*(\hat{k}\omega)\left|\hat{k}\omega\right| \exp\left(|k_1|\, r_1 + |k_2|\, r_2\right), \quad i = 2, 3 \tag{2.3.81}$$

and use (2.3.79) and (2.3.54), then we may definitively write

$$m^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2) = 2A^*_{i,j}\left[1 + \mathrm{O}\left(\exp\left(-\frac{|\ln\varepsilon|^{-1/4}}{\varepsilon^{1/4}}\right)\right)\right],$$

where, for $i = 2, 3$, $j = 1, 2$, we have denoted

$$\begin{aligned} A^*_{i,j} &= A^*_{i,j}(n^0, n^1) = P^*_0 k^{(n^0)}_j B^{(i)}_{\hat{k}^{(n^0)}}\mathcal{E}^*_{\hat{k}^{(n^0)}} + P^*_1 k^{(n^1)}_j B^{(i)}_{\hat{k}^{(n^1)}}\mathcal{E}^*_{\hat{k}^{(n^1)}} \\ P^*_\nu &= \cos(k^{(n^\nu)}_1\overline{\psi}_1 + k^{(n^\nu)}_2\overline{\psi}_2), \quad \nu = 0, 1. \end{aligned} \tag{2.3.82}$$

S6. We now may write (as was done in (1.3.115) and (1.3.116))

$$\Upsilon = 4(1 + h_1(\varepsilon))^2 \Upsilon_1^* + 2(1 + h_1(\varepsilon))\tilde{\Upsilon}_1 + \overline{\Upsilon}_1$$

with

$$\begin{aligned}
h_1(\varepsilon) &= \mathrm{O}\left(\exp\left(-\frac{|\ln\varepsilon|^{-1/4}}{\varepsilon^{1/4}}\right)\right) \\
\Upsilon_1^* &= A_{2,1}^* A_{3,2}^* - A_{2,2}^* A_{3,1}^* \\
\tilde{\Upsilon}_1 &= A_{2,1}^* e^{2,2} + A_{3,2}^* e^{1,1} - A_{2,2}^* e^{2,1} - A_{3,1}^* e^{1,2} \\
\overline{\Upsilon}_1 &= e^{1,1} e^{2,2} - e^{1,2} e^{2,1}
\end{aligned}$$

where, for $j = 1, 2$, $i = 2, 3$, we have denoted $e^{i-1,j} = e^{i-1,j}(\overline{\psi}_1, \overline{\psi}_2)$.

S7. From the definition of $A_{i,j}^*$ given in (2.3.82) we have

$$|\Upsilon_1^*| = \mathcal{E}_{\hat{k}^{(n^0)}}^* \mathcal{E}_{\hat{k}^{(n^1)}}^* \left| B_{\hat{k}^{(n^0)}}^{(2)} B_{\hat{k}^{(n^1)}}^{(3)} - B_{\hat{k}^{(n^1)}}^{(2)} B_{\hat{k}^{(n^0)}}^{(3)} \right|,$$

where we have also used Proposition 1.4.2. Moreover, recalling (2.3.81), we obtain

$$\left| B_{\hat{k}^{(n^0)}}^{(2)} B_{\hat{k}^{(n^1)}}^{(3)} - B_{\hat{k}^{(n^1)}}^{(2)} B_{\hat{k}^{(n^0)}}^{(3)} \right| = \mathcal{L}_0' \mathcal{L}_1'$$

with, for $\nu = 0, 1$,

$$\mathcal{L}_\nu' = 2\mu\pi\varepsilon^{-1/2} \left| \hat{k}^{(n^\nu)}\omega \right| H^*(\hat{k}^{(n^\nu)}\omega)\, |m_{\hat{k}^{(n^\nu)}}| \exp\left(\left| k_1^{(n^\nu)} \right| r_1 + \left| k_2^{(n^\nu)} \right| r_2 \right).$$

Then, using (2.0.4), (2.0.5), (2.0.7), (2.3.77) and the inequalities (see also (2.3.58))

$$\left| \hat{k}^{(n^\nu)}\omega \right| H^*(\hat{k}^{(n^\nu)}\omega) < ctant, \quad H^*(\hat{k}^{(n^\nu)}\omega) > 1, \quad \left| \hat{k}^{(n^\nu)}\omega \right| > ctant \left| \hat{k}^{(n^\nu)} \right|^{-1},$$

we get

$$ctant\ \mu\varepsilon^{\frac{N-1}{4}} \le \mathcal{L}_\nu' \le ctant\ \mu\varepsilon^{-\frac{N+2}{4}}, \quad \nu = 0, 1,$$

from which it follows that

$$ctant\ \mu^2\varepsilon^{\frac{N-1}{2}} \mathcal{E}_{\hat{k}^{(n^0)}}^* \mathcal{E}_{\hat{k}^{(n^1)}}^* \le |\Upsilon_1^*| \le ctant\ \mu^2\varepsilon^{-\frac{N+2}{2}} \mathcal{E}_{\hat{k}^{(n^0)}}^* \mathcal{E}_{\hat{k}^{(n^1)}}^*.$$

Hence, from (2.3.60),

$$ctant\ \mu^2\varepsilon^{\frac{N-1}{2}} (\mathcal{E}_{\hat{k}^{(n^0)}}^*)^2 \le |\Upsilon_1^*| \le ctant\ \mu^2\varepsilon^{-\frac{N+2}{2}} (\mathcal{E}_{\hat{k}^{(n^0)}}^*)^2.$$

S8. From (2.3.81) and (2.3.77) we easily check that

$$\left|k_j^{(n^\nu)} B^{(i)}_{\hat{k}^{(n^\nu)}}\right| \le ctant\ \mu\varepsilon^{-\frac{N+4}{4}}.$$

Thus, (2.3.60) and (2.3.82) leads to

$$\left|A^*_{i,j}\right| \le ctant\ \mu\varepsilon^{-\frac{N+4}{4}}\mathcal{E}^*_{\hat{k}^{(n^0)}}$$

and, moreover, from (2.3.78),

$$\left|e^{i-1,j}\right| \le ctant\ \mu^2 \left|\ln\mu\right| \varepsilon^{-2b(N+8)-3/4}\mathcal{E}^*_{\hat{k}^{(n^0)}}.$$

Then,

$$\left|\tilde{\Upsilon}_1\right| \le ctant\ \mu^3 \left|\ln\mu\right| \varepsilon^{-2b(N+8)-\frac{N+7}{4}}(\mathcal{E}^*_{\hat{k}^{(n^0)}})^2$$

and

$$\left|\overline{\Upsilon}_1\right| \le ctant\ \mu^4 \left|\ln\mu\right|^2 \varepsilon^{-4b(N+8)-3/2}(\mathcal{E}^*_{\hat{k}^{(n^0)}})^2.$$

Hence, taking

$$\mu \in (0,\varepsilon^w), \quad \text{with } w > \frac{5}{4}\left(2b(N+8) + \frac{3N+5}{4}\right), \quad \mu^{1/5}\left|\ln\mu\right| < 1,$$

we finally obtain

$$ctant\ \mu^2\varepsilon^{\frac{N-1}{2}}(\mathcal{E}^*_{\hat{k}^{(n^0)}})^2 \le |\Upsilon| \le ctant\ \mu^2\varepsilon^{-\frac{N+2}{2}}(\mathcal{E}^*_{\hat{k}^{(n^0)}})^2.$$

Now, since

$$\mathcal{E}^*_{\hat{k}^{(n^0)}} = \exp\left(-\left|k_1^{(n^0)}\right| r_1 - \left|k_2^{(n^0)}\right| r_2 - \frac{\pi\left|\hat{k}^{(n^0)}\omega\right|}{2\sqrt{\varepsilon}}\right)$$

the relations

$$\text{ctant } \varepsilon^{-1/4} \le k_r^{(n^0)} \le \text{ctant } \varepsilon^{-1/4},$$

and

$$ctant < \frac{\left|\hat{k}^{(n^0)}\omega\right|}{k_2^{(n^0)}} \le ctant$$

yield

$$ctant\ \exp\left(-\frac{ctant}{\varepsilon^{1/4}}\right) \le \mathcal{E}^*_{\hat{k}^{(n^0)}} \le ctant\ \exp\left(-\frac{ctant}{\varepsilon^{1/4}}\right),$$

where, as usual, *ctant* denotes several different constants, all of them not depending neither on μ nor on ε.

Therefore, there exist two positive constants b_1' and b_2' depending on N, r_1 and r_2, but uniform on ε and on μ, for which

$$\mu^2 \exp\left(-\frac{b_1'}{\varepsilon^{1/4}}\right) \le |\Upsilon| \le \mu^2 \exp\left(-\frac{b_2'}{\varepsilon^{1/4}}\right), \qquad (2.3.83)$$

whenever $\varepsilon \in \mathcal{U}^*_\varepsilon\left(r_1, r_2, \frac{\pi}{2}\right)$ and $\mu \in (0, \varepsilon^w)$, with

$$w > \frac{5}{4}\left(2b(N+8) + \frac{3N+5}{4}\right).$$

Once again, the factor 5/4 can be replaced by any constant bigger than one and, moreover, $b - \dfrac{3}{10}$ can be assumed to be as small as we want. Therefore, the upper and lower bounds for the transversality given in (2.3.83) are valid whenever

$$\mu \in (0, \varepsilon^w), \quad \text{for some } w > \frac{3N}{2} + 7 .$$

Therefore, Theorem 0.0.6 is proved. Now, let us show how Theorem 0.0.6 implies the Main Theorem II:

2.3.2 Inclination Lemma

The dynamics of discrete systems in a vicinity of the invariant manifolds of a hyperbolic fixed point is well-described by the so-called Inclination Lemma, see [19]. In a few words, this result ensures that any sufficiently smooth manifold transversally intersecting the stable manifold of the fixed point contains a submanifold whose iterates converge to the unstable one.

In order to obtain diffusion chains in the phase space associated to our Hamiltonian (2.0.1) we need an adapted version of the standard Inclination Lemma to a not completely hyperbolic scenario. More concretely, we will use a similar result but, instead of fixed hyperbolic points, we must deal with partially hyperbolic two-dimensional tori having three-dimensional invariant manifolds in a space of dimension six. In this setting, we will use the results given in [8] (see also [16]).

To begin with, let us consider a μ-parametric family of analytic maps

$$f_\mu(q, p, \hat{I}, \hat{\theta}) = (q', p', \hat{I}', \hat{\theta}'),$$

$\hat{I} = (I_1, I_2)$, $\hat{\theta} = (\theta_1, \theta_2)$, defined on the subset of $\mathbb{R}^4 \times \mathbb{T}^2$ given by

$$U_\delta = \left\{(q, p, \hat{I}, \hat{\theta}) : |p| < \delta, |q| < \delta, \left\|\hat{I} - \hat{\beta}\right\| < \delta\right\},$$

where $\hat{\beta} = (\beta_1, \beta_2)$, β_1 and β_2 arbitrary positive real numbers.

Moreover, we will assume that

$$\begin{aligned}
q' &= \Lambda_- q + F_q(q, p, \hat{I}, \hat{\theta}, \mu) \\
p' &= \Lambda_+ p + F_p(q, p, \hat{I}, \hat{\theta}, \mu) \\
\hat{I}' &= \hat{I} + F_{\hat{I}}(q, p, \hat{I}, \hat{\theta}, \mu) \\
\hat{\theta}' &= \hat{\theta} + F_{\hat{\theta}}(q, p, \hat{I}, \hat{\theta}, \mu)
\end{aligned}$$

and that there exists $\mu_0 > 0$ such that, for any $\mu \in [0, \mu_0]$, the following properties hold:

H1.- The transformation f_μ depends analytically on μ.

H2.- $0 < |\Lambda_-| < 1$, $0 < \left|\Lambda_+^{-1}\right| < 1$.

H3.- The functions F_q, F_p, $F_{\hat{I}}$ and $F_{\hat{\theta}}$ are 2π-periodic in θ_1 and θ_2. Moreover, for $X = q$, $X = p$ and $X = \hat{I}$, $F_X(0, 0, \hat{\beta}, \hat{\theta}, \mu) = 0$, for every $\hat{\theta} \in \mathbb{T}^2$.

H4.- There exists a constant K such that

$$\max\left\{\left\|\frac{\partial F_X}{\partial q}\right\|_{U_\delta}, \left\|\frac{\partial F_X}{\partial p}\right\|_{U_\delta}, \left\|\frac{\partial F_X}{\partial \hat{I}}\right\|_{U_\delta}\right\} \leq K\left(\delta + \mu\right)$$

$$\left\|\frac{\partial F_X}{\partial \hat{\theta}}\right\|_{U_\delta} \leq K\left(\delta + \mu\right)\left(|p| + |q| + \left\|\hat{I} - \hat{\beta}\right\|\right), \quad \left\|\frac{\partial F_{\hat{\theta}}}{\partial X}\right\|_{U_\delta} \leq K\left(\delta + \mu\right),$$

for $X = q$, $X = p$ and $X = \hat{I}$. Finally

$$\left\|\frac{\partial F_{\hat{\theta}}}{\partial \hat{\theta}}\right\|_{U_\delta} \leq K.$$

Under these hypotheses, the sets

$$T_{\beta_1,\beta_2} = \left\{(q, p, \hat{I}, \hat{\theta}) : p = q = 0, \hat{I} = \hat{\beta}\right\}$$

are two-dimensional invariant tori and (see [8]) they have three-dimensional stable and unstable manifolds $W^+(T_{\beta_1,\beta_2})$ and $W^-(T_{\beta_1,\beta_2})$.

We assume that there exists a five-dimensional analytic manifold M in such a way that $W^+(T_{\beta_1,\beta_2}) \cup W^-(T_{\beta_1,\beta_2}) \subset M$ and, moreover, for any $\mu \in [0, \mu_0]$, f_μ restricted to T_{β_1,β_2} is a non-resonant rotation.

Under these assumptions, we have the following result:

Lemma 2.3.12 (Inclination Lemma) *Let $\Gamma \subset M$ be a three-dimensional analytic manifold transversally intersecting $W^+(T_{\beta_1,\beta_2})$ in M, i.e., the tangent spaces of Γ and $W^+(T_{\beta_1,\beta_2})$ at their intersection span the tangent space of M. Then, for every $\mu \in [0, \mu_0]$,*

$$W^-(T_{\beta_1,\beta_2}) \subset \overline{\bigcup_{n\geq 0} f_\mu^n(\Gamma)}.$$

Let us remark that, instead of transversal intersection in the whole space, assumptions H1-H4 allow us to impose a weaker hypothesis in such a way that it suffices to assume that a transversal intersection in a codimension one manifold takes place. See [8] and [16] for details.

Let us now explain how to apply Lemma 2.3.12 to prove the Main Theorem II. Let us again consider our real family of perturbed Hamiltonian systems

$$H_{\varepsilon,\mu}(x, y, \hat{I}, \hat{\theta}) = \frac{I_1}{\sqrt{\varepsilon}} + \frac{I_2^2}{2\sqrt{\varepsilon}} + \frac{y^2}{2} + (\cos x - 1)(1 + \mu m(\hat{\theta}))$$

introduced in (2.0.8). Let us recall that, by means of analytic changes of variables (see (2.0.13) and Lemma 1.1.4), the associated Hamilton equations in $(q,p,\hat{I},\hat{\theta})$-coordinates can be written as in (2.0.15), in such a way that, by denoting $Q=(q,p,\hat{I},\hat{\theta})$ and once a real positive value of time τ is fixed, we may construct, for any small enough value of ε, the following μ-parametric family of maps

$$f_\mu(Q)=\phi\left(\tau,Q,\mu\right), \tag{2.3.84}$$

where, as usual, $\phi\left(\tau,Q,\mu\right)$ denotes the flow associated to the Hamiltonian $H_{\varepsilon,\mu}$.

In the same way as equations (2.1.16) were obtained, we can write

$$f_\mu(q,p,\hat{I},\hat{\theta})=(q',p',\hat{I}',\hat{\theta}'),$$

with

$$\begin{aligned}
q' &= qa^{-1}+F_q(q,p,\hat{I},\hat{\theta},\mu)\\
p' &= pa+F_p(q,p,\hat{I},\hat{\theta},\mu)\\
\hat{I}' &= \hat{I}+F_{\hat{I}}(q,p,\hat{I},\hat{\theta},\mu)\\
\hat{\theta}' &= \hat{\theta}+F_{\hat{\theta}}(q,p,\hat{I},\hat{\theta},\mu)
\end{aligned}$$

where $a=\exp(2\pi\sqrt{\varepsilon}\,(1+F_J))$,

$$F_q(q,p,\hat{I},\hat{\theta},\mu)=\frac{\mu}{\sqrt{2}}\int_0^\tau (q^\xi+q^\nu)m(\hat{\theta})\sin(\tilde{x}(q,p))ds-\int_0^\tau (1+F_J)\left(q\left(s\right)-\tilde{q}\left(s\right)\right)ds$$

$$F_p(q,p,\hat{I},\hat{\theta},\mu)=\frac{\mu}{\sqrt{2}}\int_0^\tau (p^\xi+p^\nu)m(\hat{\theta})\sin(\tilde{x}(q,p))ds+\int_0^\tau (1+F_J)\left(p\left(s\right)-\tilde{p}\left(s\right)\right)ds$$

and, if we denote by $F_{\hat{I}}=(F_{I_1},F_{I_2})$, $F_{\hat{\theta}}=(F_{\theta_1},F_{\theta_2})$, then, for $j=1,2$

$$\begin{aligned}
F_{I_j}(q,p,\hat{I},\hat{\theta},\mu) &= -\mu\int_0^\tau \left(\cos\left(\tilde{x}\left(q,p\right)\right)-1\right)\frac{\partial m}{\partial\theta_j}(\hat{\theta})ds,\\
F_{\theta_1}\left(q,p,\hat{I},\hat{\theta},\mu\right) &= \frac{\tau}{\sqrt{\varepsilon}}
\end{aligned}$$

and

$$F_{\theta_2}\left(q,p,\hat{I},\hat{\theta},\mu\right)=\frac{\tau I_2}{\sqrt{\varepsilon}}-\mu\varepsilon^{-1/2}\int_0^\tau\left(\int_0^s \left(\cos\left(\tilde{x}\left(q,p\right)\right)-1\right)\frac{\partial m}{\partial\theta_2}(\hat{\theta})d\rho\right)ds.$$

Let us recall that $(\tilde{q},\tilde{p})$ is the solution of

$$\begin{cases}\dot{q}=-q\left(1+F_J\right), & \dot{p}=p\left(1+F_J\right)\\ \tilde{q}\left(0\right)=q, & \tilde{p}\left(0\right)=p\end{cases}$$

Let us take $\delta=\sigma$ (see Lemma 1.1.4) and observe that, from (2.0.4) and (2.0.5), we have

$$|m(\theta_1,\theta_2)|\le\sum_{\hat{k}\in\Lambda}|m_{\hat{k}}|<ctant\sum_{\hat{k}\in\Lambda}\left|\hat{k}\right|^N\exp\left(-r_1\left|k_1\right|-r_2\left|k_2\right|\right)<K,$$

whenever $(\theta_1, \theta_2) \in \mathbb{T}^2$. In the same way, we may get $\max\{\|Dm\|, \|D^2m\|\} \leq K$. Therefore, see also Lemma 2.1.1, if we choose $\tau = \tau(\varepsilon)$ small enough, we may check that f_μ satisfies hypotheses H1-H4 and, therefore, as it was already obtained in Section 2.1, the sets

$$T_{\beta_1,\beta_2} = \left\{(q,p,\hat{I},\hat{\theta}) : p = q = 0, \hat{I} = \hat{\beta}\right\}$$

are f_μ-invariant and, moreover, since

$$F_{\hat{\theta}}(0,0,\hat{\beta},\hat{\theta},\mu) = \left(\frac{\tau}{\sqrt{\varepsilon}}, \frac{\beta_2\tau}{\sqrt{\varepsilon}}\right),$$

by taking an irrational value of β_2 we guarantee that f_μ restricted to T_{β_1,β_2} is a non-resonant rotation.

Let us consider the set $\mathcal{U}^*_\varepsilon = \mathcal{U}^*_\varepsilon(r_1, r_2, \frac{\pi}{2})$ constructed in the proof of Theorem 0.0.6 and, once $\varepsilon \in \mathcal{U}^*_\varepsilon$ is fixed, let us take an arbitrary real number β_2^0 in $I^*_{\tilde{\beta}}$, where $I^*_{\tilde{\beta}}$ is the neighbourhood of the golden mean number furnished by the Main Lemma II.

Let us take an arbitrary real number β_1^0 and consider the invariant torus $T_{\beta_1^0,\beta_2^0}$. Let us denote by M the energy level $H_{\varepsilon,\mu} =$*ctant* where $T_{\beta_1^0,\beta_2^0}$ is contained, and consider its three-dimensional stable and unstable manifolds $W^+(T_{\beta_1^0,\beta_2^0})$ and $W^-(T_{\beta_1^0,\beta_2^0})$. We remark that

$$W^+(T_{\beta_1^0,\beta_2^0}) \cup W^-(T_{\beta_1^0,\beta_2^0}) \subset M$$

and, moreover, using the splitting functions introduced in (2.3.46) it is clear that on homoclinic orbits $(\overline{\psi}_1, \overline{\psi}_2)$ (see also Remark 2.1.5) we have

$$\overline{\mathcal{J}}^\mu_{1,u}(\overline{\psi}_1, \overline{\psi}_2) = 0, \quad \overline{\mathcal{J}}^\mu_{2,u}(\overline{\psi}_1, \overline{\psi}_2) = 0.$$

Let us recall that, since ε is fixed, then $\overline{\psi}_i = \overline{\psi}_i(\varepsilon) \in \{0, \pi\}$ are completely determinated according to the algorithm described in the proof of Theorem 0.0.6.

Now, from the definition of $\Upsilon = \Upsilon(\overline{\psi}_1, \overline{\psi}_2)$ (see (2.3.71)) and due to the fact that Theorem 0.0.6 implies $\Upsilon \neq 0$, the Implicit Function Theorem allows us to deduce that, not only the intersection between $W^+(T_{\beta_1^0,\beta_2^0})$ and $W^-(T_{\beta_1^0,\beta_2^0})$ restricted to a neighbourhood of the homoclinic orbit $(\overline{\psi}_1, \overline{\psi}_2)$ reduces to such homoclinic orbit, but also that $W^+(T_{\beta_1^0,\beta_2^0})$ and $W^-(T_{\beta_1^0,\beta_2^0})$ transversally intersects along that homoclinic orbit in the energy level M.

Furthermore, from (2.3.83), if

$$\mu \in (0, \varepsilon^w), \quad w > \frac{3N}{2} + 7$$

the same holds between $W^+(T_{\beta_1^0,\beta_2^0})$ and $W^-(T_{\beta_1^*,\beta_2^*})$, for every torus $T_{\beta_1^*,\beta_2^*}$ contained in M with (β_1^*, β_2^*) close enough to (β_1^0, β_2^0).

In fact, let us observe that there exists a dense set of irrational values of β_2^* for which such transversal intersections take place.

Hence, the Inclination Lemma yields

$$W^-(T_{\beta_1^0,\beta_2^0}) \subset \overline{\bigcup_{n\geq 0} f_\mu^n \left(W^-(T_{\beta_1^*,\beta_2^*})\right)},$$

where f_μ are the τ-flow transformations defined in (2.3.84).

Therefore,

$$W^-(T_{\beta_1^0,\beta_2^0}) \subset \overline{W^-(T_{\beta_1^*,\beta_2^*})}.$$

Of course, we may repeat the arguments for a finite sequence of irrational values of β_2, $\{\beta_2^1, \beta_2^2, ..., \beta_2^n\}$, contained in $I^*_{\tilde{\beta}}$ (just where our estimates on the transversality are valid) with $\beta_2^1 = \beta_2^0$ and (see the Main Lemma II)

$$\left|\beta_2^n - \beta_2^0\right| > \frac{1}{2}\text{length}(I^*_{\tilde{\beta}}) > ctant\ \varepsilon^{5/6}$$

in such a way that

$$W^-\left(T_{\beta_1^0,\beta_2^0}\right) \subset \overline{W^-\left(T_{\beta_1^n,\beta_2^n}\right)}.$$

Hence, a transition chain is constructed and, consequently, the Main Theorem II is proven.

Chapter 3

Proofs of the Extension Theorems

This chapter is devoted to prove the Extension Theorem I (see Theorem 1.1.14) and the Extension Theorem II (see Theorem 2.1.4). For proving the Extension Theorem I we will follow the scheme used in [6] or [7] and, later, we will adapt those arguments to the non-quasiperiodic case in order to prove the Extension Theorem II.

3.1 Proof of the Extension Theorem I

Let us again consider the invariant tori

$$T_{\alpha_1,\alpha_2} = \{(x, y, I_1, I_2, \theta_1, \theta_2) : \; x = y = 0, \; I_1 = \alpha_1, \; I_2 = \alpha_2\}$$

for the perturbed dynamical systems given in (1.1.6). Let us recall that, in Lemma 1.1.11, we proved the existence of suitable parameterizations of the local invariant manifolds of those tori. Those parameterizations were given by

$$(\hat{\psi}, t, s) \in \mathcal{U}^* \to \Big(x^*(\hat{\psi}, t, s), y^*(\hat{\psi}, t, s), \hat{I}^*(\hat{\psi}, t, s), \hat{\theta}^0(\hat{\psi}, t)\Big)$$

where

$$\begin{aligned} \mathcal{U}^+ &= \mathcal{B}_1'' \times [T_0 - \text{Re } s, \infty) \times \mathcal{C}_1, \quad \mathcal{U}^- = \mathcal{B}_1'' \times (-\infty, -T_0 - \text{Re } s] \times \mathcal{C}_1 \\ \mathcal{B}_1'' &= \big\{(\psi_1, \psi_2) \in \mathbb{C}^2 : \; |\text{Im } \psi_i| \le -\ln(a\varepsilon^p) - 2\varepsilon^b, \; i = 1, 2\big\} \end{aligned}$$

and

$$\mathcal{C}_1 = \left\{ s \in \mathbb{C} : \; |\text{Im } s| < \frac{\pi}{2\sqrt{A}} \right\}.$$

One of the most remarkable properties of those parameterizations (see the second statement of Lemma 1.1.11) is that

$$\left\| (x^*, y^*, \hat{I}^*, \hat{\theta}^0) - (x^0, y^0, \hat{I}^0, \hat{\theta}^0) \right\|_{\mathcal{U}_1^*} \le ctant \; \mu\varepsilon^{-b(N+3)},$$

where

$$\mathcal{U}_1^+ = \mathcal{B}_1'' \times [T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}_1, \quad \mathcal{U}_1^- = \mathcal{B}_1'' \times [-2T_0 - \text{Re } s, -T_0 - \text{Re } s] \times \mathcal{C}_1$$

and $(x^0, y^0, \hat{I}^0, \hat{\theta}^0)$ is the parameterization of the unperturbed separatrix introduced in (1.1.22). This is the key which allows us to assert that

$$\left\| (x^-, y^-, \hat{I}^-, \hat{\theta}^0) - (x^0, y^0, \hat{I}^0, \hat{\theta}^0) \right\|_{\mathcal{U}^{--}} \leq ctant\ \mu\varepsilon^{-b(N+5)},$$

where

$$\mathcal{U}^{--} = \mathcal{B}_1'' \times [-T_0 - \text{Re } s, 2T_0 - \text{Re } s] \times \mathcal{C}_1'$$

and

$$\mathcal{C}_1' = \left\{ s \in \mathbb{C} : \ |\text{Im } s| \leq \frac{\pi}{2\sqrt{A}} - \varepsilon^b \right\}.$$

We have often used this last bound during the proof of the Main Theorem I and we want to point out that it directly follows from the statement of the Extension Theorem I.

Along this section we will always assume that $\mu \in (0, \varepsilon^m)$ with $m > b(N+6)$ and, to begin with the proof of the first Extension Theorem, let us assume that we have a solution $(x(t), y(t), I_1(t), I_2(t), \theta_1(t), \theta_2(t))$ of (1.1.6) satisfying the conditions given in (1.1.35), i.e.,

$$\begin{aligned} \left|x(t_0) - x^0(t_0+s)\right| &\leq C_1\mu\varepsilon^{-b(N+3)}, \quad \left|y(t_0) - y^0(t_0+s)\right| \leq C_1\mu\varepsilon^{-b(N+3)}, \\ |I_i(t_0) - \alpha_i| &\leq C_1\mu\varepsilon^{-b(N+3)}, \ i = 1, 2, \quad (\theta_1(t_0), \theta_2(t_0)) \in \mathcal{B}_1'', \end{aligned}$$

for $t_0 = -T_0 - \text{Re } s$, some positive constant C_1 and some $s \in \mathcal{C}_1'$. Under these assumptions we have to prove that, for every t in $[-T_0 - \text{Re } s, 2T_0 - \text{Re } s]$, the following bounds hold:

$$\begin{aligned} \left|x(t) - x^0(t+s)\right| &\leq C_1'\mu\varepsilon^{-b(N+5)}, \quad \left|y(t) - y^0(t+s)\right| \leq C_1'\mu\varepsilon^{-b(N+5)}, \\ |I_i(t) - \alpha_i| &\leq C_1'\mu\varepsilon^{-b(N+5)}, \ i = 1, 2, \end{aligned}$$

where C_1' will be a positive constant depending on K_1 (see the statement of Lemma 3.1.3), C_1, A and T_0.

To this end, let us remark that, once a value of $s \in \mathcal{C}_1'$ is fixed, we may construct the following vector field

$$\begin{aligned} \dot{\lambda} &= \kappa + \mu M_1(\hat{\theta}) \sin\left(x^0(t+s) + \lambda\right) \\ \dot{\kappa} &= A\left[\sin(x^0(t+s) + \lambda) - \sin(x^0(t+s))\right] - \\ &\quad -\mu\left(\kappa + y^0(t+s)\right) M_1(\hat{\theta}) \cos\left(x^0(t+s) + \lambda\right), \\ \dot{\theta}_1 &= \frac{1}{\varepsilon}, \quad \dot{\theta}_2 = \frac{\beta}{\varepsilon} \end{aligned} \tag{3.1.1}$$

in such a way that, if $(\lambda(t), \kappa(t))$ satisfies the first two equations of (3.1.1), then

$$x(t) = \lambda(t) + x^0(t+s), \qquad y(t) = \kappa(t) + y^0(t+s)$$

are the solutions of the first two ones of (1.1.6) satisfying the initial conditions

$$x(t_0) = \lambda(t_0) + x^0(t_0+s), \qquad y(t_0) = \kappa(t_0) + y^0(t_0+s).$$

Therefore, the statement of the Extension Theorem I may be replaced by the following: If, for $t_0 = -T_0 - \text{Re}\ s$, we have

$$|\lambda(t_0)| \leq C_1 \mu \varepsilon^{-b(N+3)}, \quad |\kappa(t_0)| \leq C_1 \mu \varepsilon^{-b(N+3)}, \tag{3.1.2}$$

then, for $t \in [-T_0 - \text{Re}\ s, 2T_0 - \text{Re}\ s]$, it follows that

$$|\lambda(t)| \leq C_1' \mu \varepsilon^{-b(N+5)}, \quad |\kappa(t)| \leq C_1' \mu \varepsilon^{-b(N+5)}. \tag{3.1.3}$$

Let us comment that the bounds for $|I_i(t) - \alpha_i|$ announced by the Extension Theorem I are going to be easily checked from the bounds obtained for $|\lambda(t)|$ and $|\kappa(t)|$, see Remark 3.1.7 for details.

It will be very useful to write the system (3.1.1) in the following way

$$\dot{z} = B(t+s)z + \mu G(x^0(t+s), y^0(t+s), \hat{\theta}) + F(x^0(t+s), y^0(t+s), \lambda(t), \kappa(t), \hat{\theta}), \tag{3.1.4}$$

where

$$z = \begin{pmatrix} \lambda \\ \kappa \end{pmatrix}, \quad B(t+s) = \begin{pmatrix} 0 & 1 \\ A\cos(x^0(t+s)) & 0 \end{pmatrix}, \quad G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

and

$$\begin{aligned} g_1(x,y,\hat{\theta}) &= \sin x M_1(\hat{\theta}), \qquad g_2(x,y,\hat{\theta}) = -y\cos x M_1(\hat{\theta}) \\ f_1(x,y,\lambda,\kappa,\hat{\theta}) &= \mu\left[g_1(x+\lambda, y+\kappa, \hat{\theta}) - g_1(x,y,\hat{\theta})\right] \\ f_2(x,y,\lambda,\kappa,\hat{\theta}) &= A\left[\sin(x+\lambda) - \sin x - \lambda\cos x\right] + \\ &\quad +\mu\left[g_2(x+\lambda, y+\kappa, \hat{\theta}) - g_2(x,y,\hat{\theta})\right]. \end{aligned} \tag{3.1.5}$$

Let us consider the respective homogeneous linear system

$$\frac{\mathrm{d}z}{\mathrm{d}t} = B(t+s)z \tag{3.1.6}$$

associated to (3.1.4). According to (1.1.20), a solution of (3.1.6) is given by

$$\lambda(t) = \dot{x}^0(t+s), \quad \kappa(t) = \dot{y}^0(t+s).$$

Another independent solution can be obtained in the following way: For an arbitrary complex number b_0, let us define

$$W(t+s) = \int_{b_0}^{t+s} \frac{d\sigma}{(y^0(\sigma+s))^2} \tag{3.1.7}$$

and take

$$\lambda(t) = y^0(t+s)W(t+s) \quad \text{and} \quad \kappa(t) = \dot{\lambda}(t).$$

Hence, letting

$$\Psi(t+s) = \dot{x}^0(t+s), \quad \Phi(t+s) = y^0(t+s)W(t+s) = \Psi(t+s)W(t+s) \tag{3.1.8}$$

and using that $\Psi(t+s)\dot{\Phi}(t+s) - \Phi(t+s)\dot{\Psi}(t+s) = 1$, we obtain

$$M(t+s) = \begin{pmatrix} \Psi(t+s) & \Phi(t+s) \\ \dot{\Psi}(t+s) & \dot{\Phi}(t+s) \end{pmatrix} \tag{3.1.9}$$

as a fundamental matrix associated to the system (3.1.6). Therefore, the solution $\varphi(t,v)$ of (3.1.6) satisfying the initial condition $\varphi(t_0) = v$ is given by

$$\varphi(t,v) = M(t+s)M^{-1}(t_0+s)v,$$

where

$$M^{-1}(t_0+s) = \begin{pmatrix} \dot{\Phi}(t_0+s) & -\Phi(t_0+s) \\ -\dot{\Psi}(t_0+s) & \Psi(t_0+s) \end{pmatrix}.$$

Finally, the solution $z(t)$ of (3.1.4) satisfying the initial condition $z(t_0)$ is given by

$$z(t) = z^1(t) + M(t+s) \int_{t_0}^{t} M^{-1}(\sigma+s)F(x^0(\sigma+s), y^0(\sigma+s), \lambda(\sigma), \kappa(\sigma), \hat{\theta})\, \mathrm{d}\sigma \tag{3.1.10}$$

with

$$z^1(t) = M(t+s)\left[M^{-1}(t_0+s)z(t_0) + \mu \int_{t_0}^{t} M^{-1}(\sigma+s)G(x^0(\sigma+s), y^0(\sigma+s), \hat{\theta})\, \mathrm{d}\sigma\right]. \tag{3.1.11}$$

Thus, (3.1.10) yields an iterative process which will be used to state fine properties of the solution of (3.1.1) leading to (3.1.3).

We are going to develop the proof of the Extension Theorem (or the conclusion given in (3.1.3)) for the case in which $s \in \mathcal{C}_1'(+)$, where

$$\mathcal{C}_1'(+) = \left\{ s \in \mathbb{C} : \; 0 \le \text{Im } s \le \frac{\pi}{2\sqrt{A}} - \varepsilon^b \right\}.$$

For the values of s not belonging to $\mathcal{C}_1'(+)$, i.e., those s such that

$$-\frac{\pi}{2\sqrt{A}} + \varepsilon^b < \text{Im } s < 0$$

the result is obtained with the same method : The unique difference between both cases arises in Lemma 3.1.3. In fact, it is easy to see that Lemma 3.1.3 also holds by taking $b_0 = -\frac{\sqrt{-1}\pi}{2\sqrt{A}}$ and this is enough to conclude the proof of the Extension Theorem in the second case, because the rest of the arguments are those of the first case.

The Extension Theorem will directly follow from Proposition 3.1.1 and Proposition 3.1.2 stated below.

In order to establish these results, we need to introduce some definitions: In Proposition 3.1.1 we will extend the solutions of (3.1.1) to the domain $[-T_0 - \text{Re } s, t_1(s)]$ where $t_1(s)$ is the *separation time* defined by

$$t_1(s) + \text{Re } s = \begin{cases} \varepsilon^{\frac{2}{3}b}, \text{ if } \frac{\pi}{2\sqrt{A}} - \varepsilon^{\frac{2}{3}b} \leq \text{Im } s \leq \frac{\pi}{2\sqrt{A}} - \varepsilon^b \\ 0, \text{ if } 0 \leq \text{Im } s < \frac{\pi}{2\sqrt{A}} - \varepsilon^{\frac{2}{3}b}. \end{cases} \tag{3.1.12}$$

Next, in Proposition 3.1.2, $t_1(s)$ will be chosen as the initial time in order to extend the respective solutions of (3.1.1) to the domain $[t_1(s), 2T_0 - \text{Re } s]$. We refer the reader to Remark 3.1.6, where the advantages to work with this separation time are described.

Let us recall that s is fixed and recover the definition of the function τ given in (1.2.38) by

$$t \in [-T_0 - \text{Re } s, 2T_0 - \text{Re } s] \to \tau(t) = \left| t + s - \frac{\sqrt{-1}\pi}{2\sqrt{A}} \right|.$$

We observe that, since $s \in \mathcal{C}_1'$, then

$$\varepsilon^b \leq \tau(t) \leq \frac{\pi}{2\sqrt{A}} + 2T_0.$$

We also define, for every positive real constant γ, the following norm:

$$z = (z_1, z_2) \in \mathbb{C}^2 \to |z|_\gamma = |z_1| + \gamma\, |z_2|$$

and note that, for every $z = (z_1, z_2) \in \mathbb{C}^2$, we have

$$|z_1| \leq |z|_\gamma \quad \text{and} \quad |z_2| \leq \gamma^{-1} |z|_\gamma \, .$$

In order to make clear the relation between the different constants taking part in the proof of the Extension Theorem, during the rest of this section, we will denote by K_1 the constant given by the first statement of Lemma 3.1.3, by $\overline{K}_1 = \overline{K}_1(T_0, K_1, A)$ the constant given by Lemma 3.1.4 and by

$$\tilde{K}_1 = \max_{\lambda \in \{0,2,3\}} \left\{ \tilde{K}_1(T_0, \lambda) \right\},$$

where $\tilde{K}_1(T_0, \lambda)$ is the constant given by Lemma 3.1.5.

Proposition 3.1.1 *Let s be a fixed complex number in $\mathcal{C}_1'(+)$. If $z(t) = (\lambda(t), \kappa(t))$ is a solution of (3.1.4) satisfying (3.1.2) for $t_0 = -T_0 - \mathrm{Re}\ s$, then $z(t)$ can be extended to $[-T_0 - \mathrm{Re}\ s, t_1(s)]$, with $t_1(s)$ given by (3.1.12), in such a way that*

$$|z(t)|_{\tau(t)} \leq 4K_1^* \tau^{-1}(t)\mu\varepsilon^{-b(N+3)},$$

for every $t \in [-T_0 - \mathrm{Re}\ s, t_1(s)]$, where $K_1^* = \max\{18K_1\overline{K}_1\tilde{K}_1, \hat{K}_1\}$ and

$$\hat{K}_1 = 8C_1K_1^2 \left(1 + T_0 + \frac{\pi}{2\sqrt{A}}\right)\left(T_0 + \frac{\pi}{2\sqrt{A}}\right),$$

with C_1 the constant introduced in (3.1.2).

Proposition 3.1.2 *Let s be a fixed complex number in $\mathcal{C}_1'(+)$. If $z(t) = (\lambda(t), \kappa(t))$ is a solution of (3.1.4) satisfying*

$$|z(t_1)|_{\tau(t_1)} \leq 4K_1^* \tau^{-1}(t_1)\mu\varepsilon^{-b(N+3)}$$

with $t_1 = t_1(s)$ given by (3.1.12), then $z(t)$ can be extended to $[t_1(s), 2T_0 - \mathrm{Re}\ s]$ in such a way that

$$|z(t)|_{\tau(t)} \leq 40K_1^2K_1^* \tau^2(t)\mu\varepsilon^{-b(N+5)},$$

for every $t \in [t_1(s), 2T_0 - \mathrm{Re}\ s]$.

To prove both propositions we need three auxiliary lemmas. The first one gives standard estimates on the functions Φ and Ψ introduced in (3.1.8).

Lemma 3.1.3 *(a) There exists a positive constant K_1 such that, if we take $b_0 = \dfrac{\sqrt{-1}\pi}{2\sqrt{A}}$ in (3.1.7), then for every $s \in \mathcal{C}_1'(+)$ and every $t \in [-T_0 - \mathrm{Re}\ s, 2T_0 - \mathrm{Re}\ s]$ the following bounds hold*

$$\left|\frac{\partial^k \Psi}{\partial t^k}(t+s)\right| \leq \frac{K_1}{\tau^{1+k}(t)}, \quad \left|\frac{\partial^k \Phi}{\partial t^k}(t+s)\right| \leq K_1\tau^{2-k}(t), \quad k = 0, 1, 2.$$

(b) For each $v = (v_1, v_2) \in \mathbb{C}^2$ and every $t, \bar{t} \in [-T_0 - \mathrm{Re}\ s, 2T_0 - \mathrm{Re}\ s]$ it follows that

$$|M(t+s)v|_{\tau(t)} \leq 2K_1\left(\frac{|v_1|}{\tau(t)} + \tau^2(t)\,|v_2|\right)$$

and

$$\left|M(t+s)M^{-1}(\bar{t}+s)v\right|_{\tau(t)} \leq 2K_1^2\,|v|_{\tau(\bar{t})}\left(\frac{\tau(\bar{t})}{\tau(t)} + \frac{\tau^2(t)}{\tau^2(\bar{t})}\right).$$

Proof

The function

$$\Psi(t+s) = \dot{x}^0(t+s) = y^0(t+s) = \frac{2\sqrt{A}}{\cosh(\sqrt{A}(t+s))}$$

has a simple pole at $b_0 = \dfrac{\sqrt{-1}\pi}{2\sqrt{A}}$. Hence, the function $W = W(t+s)$ introduced in (3.1.7) has a triple zero at this point and thus the function $\Phi(t+s) = \Psi(t+s)W(t+s)$ has a double zero at b_0.

Now, the statement (a) of the lemma follows by expanding the functions Ψ and Φ to obtain

$$\Psi(t+s) = \frac{C_\Psi}{t+s-\dfrac{\sqrt{-1}\pi}{2\sqrt{A}}}\left(1+\mathrm{O}(\tau^2(t))\right)$$

and

$$\Phi(t+s) = C_\Phi\left(t+s-\frac{\sqrt{-1}\pi}{2\sqrt{A}}\right)^2\left(1+\mathrm{O}(\tau^2(t))\right),$$

where C_Ψ and C_Φ are constants.

Doing so, the statement (b) directly follows from the first one by using the expressions of the fundamental matrix $M = M(t+s)$ given in (3.1.9). □

The next result gives properties of the functions g_1, g_2, f_1 and f_2 introduced in (3.1.5):

Lemma 3.1.4 *There exists a positive constant $\overline{K}_1 = \overline{K}_1(T_0, K_1, A)$ such that*

(a) For every $(t, s, \hat{\theta}) \in [-T_0 - \mathrm{Re}\ s, 2T_0 - \mathrm{Re}\ s] \times \mathcal{C}_1'(+) \times \mathcal{B}_1''$ it follows that

$$\left|g_i(x^0(t+s), y^0(t+s), \hat{\theta})\right| \le \frac{\overline{K}_1\varepsilon^{-b(N+2)}}{\tau^{1+i}(t)}, \quad i = 1, 2.$$

(b) For any $\Lambda_1 = (\lambda_1, \kappa_1)$, $\Lambda_2 = (\lambda_2, \kappa_2) \in \mathbb{C}^2$ satisfying

$$\max_{i=1,2}\{|\lambda_i|, |\kappa_i|\} < \mu\varepsilon^{-b(N+6)}$$

and every $(t, s, \hat{\theta}) \in [-T_0 - \mathrm{Re}\ s, 2T_0 - \mathrm{Re}\ s] \times \mathcal{C}_1'(+) \times \mathcal{B}_1''$, we have

$$\left|f_1(x^0(t+s), y^0(t+s), \Lambda_1, \hat{\theta}) - f_1(x^0(t+s), y^0(t+s), \Lambda_2, \hat{\theta})\right| \le \frac{\overline{K}_1\mu\varepsilon^{-b(N+2)}}{\tau^2(t)}|\lambda_1 - \lambda_2| \tag{3.1.13}$$

and

$$\left|f_2(x^0(t+s), y^0(t+s), \Lambda_1, \hat{\theta}) - f_2(x^0(t+s), y^0(t+s), \Lambda_2, \hat{\theta})\right| \le$$
$$\le \frac{A\overline{K}_1}{\tau^2(t)}(|\lambda_1| + |\lambda_2|)\,|\lambda_1 - \lambda_2| + \frac{\overline{K}_1\mu\varepsilon^{-b(N+2)}}{\tau^3(t)}\,|\Lambda_1 - \Lambda_2|_{\tau(t)}. \tag{3.1.14}$$

Proof

The equalities $\sin(x^0(t+s)) = \dot{y}^0(t+s)$ and $\ddot{y}^0(t+s) = A\cos(x^0(t+s))y^0(t+s)$ imply that the functions $\sin(x^0(t+s))$ and $\cos(x^0(t+s))$ have double poles at $b_0 = \dfrac{\sqrt{-1}\pi}{2\sqrt{A}}$. This fact, together with Lemma 1.1.1 and Lemma 3.1.3, implies the statement (a) of Lemma 3.1.4.

To prove (3.1.13) let us observe that

$$\begin{aligned} &\left| f_1(x^0(t+s), y^0(t+s), \Lambda_1, \hat{\theta}) - f_1(x^0(t+s), y^0(t+s), \Lambda_2, \hat{\theta}) \right| = \\ &= \mu \left| M_1(\hat{\theta}) \left(\sin(x^0(t+s) + \lambda_1) - \sin(x^0(t+s) + \lambda_2) \right) \right| \leq \\ &\leq \mu \varepsilon^{-b(N+2)} \left| \cos(x^0(t+s) + \tilde{\lambda}) \right| \left| \lambda_1 - \lambda_2 \right|, \end{aligned}$$

with $\left|\tilde{\lambda}\right| < \max\left\{ |\lambda_1|, |\lambda_2| \right\} < \mu\varepsilon^{-b(N+6)}$ and where we have used the Mean Value Theorem and Lemma 1.1.1.

Hence, (3.1.13) follows by recalling that $\mu \in (0, \varepsilon^m)$, with $m > b(N+6)$, and therefore $\left|\cos(x^0(t+s) + \tilde{\lambda})\right| \leq \overline{K}_1 \tau^{-2}(t)$, with $\overline{K}_1$ some constant depending on K_1, T_0 and A.

Now, let us bound

$$\left| f_2(x^0(t+s), y^0(t+s), \Lambda_1, \hat{\theta}) - f_2(x^0(t+s), y^0(t+s), \Lambda_2, \hat{\theta}) \right|$$

in order to get (3.1.14).

Using a Taylor expansion to write

$$\sin(x+\lambda) = \sin x + \lambda \cos x - \frac{\lambda^2}{2} \sin(x + h(\lambda))$$

it is easy to see that (we take the notation $x^0 = x^0(t+s)$, $y^0 = y^0(t+s)$)

$$\begin{aligned} &A\left[\sin(x^0 + \lambda_1) - \sin x^0 - \lambda_1 \cos x^0\right] - A\left[\sin(x^0 + \lambda_2) - \sin x^0 - \lambda_2 \cos x^0\right] = \\ &= \frac{A}{2}\left[\sin(x^0 + h(\lambda_1))(\lambda_1^2 - \lambda_2^2) + \lambda_2^2(\sin(x^0 + h(\lambda_1)) - \sin(x^0 + h(\lambda_2))\right]. \end{aligned}$$

Now, since $|h(\lambda)| < |\lambda|$, we may write

$$\left|\sin(x^0 + h(\lambda_1)) - \sin(x^0 + h(\lambda_2))\right| \leq \left|\cos(x^0 + h(\tilde{\lambda}))\right| |\lambda_1 - \lambda_2|,$$

for some complex number $\tilde{\lambda}$ satisfying

$$\left|\tilde{\lambda}\right| < \max\{|\lambda_1|, |\lambda_2|\} < \mu\varepsilon^{-b(N+6)}.$$

Hence,

$$\left|A\left[\sin(x^0+\lambda_1)-\sin x^0-\lambda_1\cos x^0\right]-A\left[\sin(x^0+\lambda_2)-\sin x^0-\lambda_2\cos x^0\right]\right|\le$$
$$\le\frac{A\overline{K}_1}{\tau^2(t)}(|\lambda_1|+|\lambda_2|)\,|\lambda_1-\lambda_2|\,. \tag{3.1.15}$$

On the other hand, since

$$\begin{aligned}&g_2(x^0+\lambda_1,y^0+\kappa_1,\hat\theta)-g_2(x^0+\lambda_2,y^0+\kappa_2,\hat\theta)=\\&\quad=M_1(\hat\theta)\left[(y^0+\kappa_2)\cos(x^0+\lambda_2)-(y^0+\kappa_1)\cos(x^0+\lambda_1)\right]=\\&\qquad=M_1(\hat\theta)\left[(y_0+\kappa_2)\sin(x_0+\lambda^*)(\lambda_1-\lambda_2)-\cos(x_0+\lambda_1)(\kappa_1-\kappa_2)\right],\end{aligned}$$

with

$$|\lambda^*|<\max\{|\lambda_1|\,,|\lambda_2|\}<\mu\varepsilon^{-b(N+6)},$$

one checks easily that

$$\left|g_2(x^0+\lambda_1,y^0+\kappa_1,\hat\theta)-g_2(x^0+\lambda_2,y^0+\kappa_2,\hat\theta)\right|\le\frac{\overline{K}_1\mu\varepsilon^{-b(N+2)}}{\tau^3(t)}\left|\Lambda_1-\Lambda_2\right|_{\tau(t)}.$$

Therefore, using also (3.1.5) and (3.1.15) we deduce that (3.1.14) is proved. □

To state the last technical lemma we recall that, in Chapter 1 (see, for instance, (1.2.39)), we already defined, for a fixed value of s and any $\lambda\in\mathbb{R}$, the function

$$\rho_{[t_0,t]}(\lambda):=\begin{cases}\sup\limits_{\sigma\in[t_0,t]}\dfrac{1}{\tau^\lambda(\sigma)}, & \text{if } \lambda\ne 0\\ \sup\limits_{\sigma\in[t_0,t]}|\ln\tau(\sigma)|\,, & \text{if } \lambda=0.\end{cases} \tag{3.1.16}$$

Lemma 3.1.5 *There exists a positive constant $\tilde K_1=\tilde K_1(T_0,\lambda)$ such that if*

$$[t_0,t]\subset[-T_0-\text{Re } s,2T_0-\text{Re } s]$$

and $v:\sigma\in[t_0,t]\to v(\sigma)\in\mathbb{C}^2$ is a function satisfying

$$|v(\sigma)|_{\tau(\sigma)}\le\frac{C^*}{\tau^\lambda(\sigma)},\quad \textit{for every } \sigma\in[t_0,t],$$

then

$$\left|M(t+s)\int_{t_0}^t M^{-1}(\sigma+s)v(\sigma)d\sigma\right|_{\tau(t)}\le C^*K_1\tilde K_1\left(\frac{\rho_{[t_0,t]}(\lambda-2)}{\tau(t)}+\tau^2(t)\rho_{[t_0,t]}(\lambda+1)\right).$$

Proof

The proof easily follows from Lemma 3.1.3 (a), together with the fact that (see also (1.2.39)), since $[t_0, t] \subset [-T_0 - \text{Re } s, 2T_0 - \text{Re } s]$, there exists a positive constant $\tilde{K}_1$ for which

$$\int_{t_0}^{t} \frac{d\sigma}{\tau^{\lambda}(\sigma)} \leq \tilde{K}_1 \rho_{[t_0,t]}(\lambda - 1).$$

$\square$

Proof of Proposition 3.1.1

Let us start by recalling that, from (3.1.10), the solution $z(t) = (\lambda(t), \kappa(t))$ of (3.1.4) passing through $z(t_0)$ is implicitly given by

$$z(t) = z^1(t) + M(t+s) \int_{t_0}^{t} M^{-1}(\sigma + s) F(x^0(\sigma + s), y^0(\sigma + s), z(\sigma), \hat{\theta})\, \mathrm{d}\sigma$$

with (see (3.1.11))

$$z^1(t) = M(t+s) \left[M^{-1}(t_0 + s) z(t_0) + \mu \int_{t_0}^{t} M^{-1}(\sigma + s) G(x^0(\sigma + s), y^0(\sigma + s), \hat{\theta}) d\sigma \right].$$

For proving Proposition 3.1.1 we are going to express $z(t)$ as the limit of the following iterative process: Take $z_0(t) = 0$ and define

$$z_{n+1}(t) = z^1(t) + M(t+s) \int_{t_0}^{t} M^{-1}(\sigma + s) F(x^0(\sigma + s), y^0(\sigma + s), z_n(\sigma), \hat{\theta})\, \mathrm{d}\sigma. \quad (3.1.17)$$

Then, it is clear that $z_1 \equiv z^1$.

We will inductively prove that, for every $n \in \mathbb{N}$,

$$\|z_n\|_1 \leq 4K_1^* \mu \varepsilon^{-b(N+3)}, \quad (3.1.18)$$

where, see the statement of Proposition 3.1.1, $K_1^* = \max\{18K_1\overline{K}_1\tilde{K}_1, \hat{K}_1\}$ and, by definition,

$$\|z\|_1 = \sup_{t \in [t_0, t_1(s)]} \left\{ |z(t)|_{\tau(t)} \, \tau(t) \right\} \quad (3.1.19)$$

with $t_1(s)$ the separation time defined in (3.1.12).

To begin with, let us prove (3.1.18) for $n = 1$. To this end, we use the condition given in (3.1.2) together with Lemma 3.1.3 (b) to write

$$\left| M(t+s) M^{-1}(t_0 + s) z(t_0) \right|_{\tau(t)} \leq 2K_1^2 \left(|\lambda(t_0)| + \tau(t) \, |\kappa(t_0)| \right) \left(\frac{\tau(t_0)}{\tau(t)} + \frac{\tau^2(t)}{\tau^2(t_0)} \right) \leq$$

$$\leq 8K_1^2 C_1 \mu \varepsilon^{-b(N+3)} \left(1 + T_0 + \frac{\pi}{2\sqrt{A}} \right) \left(T_0 + \frac{\pi}{2\sqrt{A}} \right) \frac{1}{\tau(t)} = \frac{\hat{K}_1}{\tau(t)} \mu \varepsilon^{-b(N+3)}, \quad (3.1.20)$$

where we have used that $\tau(t) \leq \tau(t_0) = \tau(-T_0 - \text{Re } s) \leq T_0 + \dfrac{\pi}{2\sqrt{A}}$.

On the other hand, for every $\sigma \in [t_0, t_1(s)]$, Lemma 3.1.4 (a) yields

$$\left| G(x^0(\sigma + s), y^0(\sigma + s), \hat{\theta}) \right|_{\tau(\sigma)} \leq \frac{2\overline{K}_1 \varepsilon^{-b(N+2)}}{\tau^2(\sigma)}.$$

Hence, Lemma 3.1.5 implies, for every $t \in [t_0, t_1(s)]$, that

$$\left| M(t+s) \int_{t_0}^{t} M^{-1}(\sigma + s) G(x^0(\sigma + s), y^0(\sigma + s), \hat{\theta}) d\sigma \right|_{\tau(t)} \leq$$
$$\leq 2K_1 \overline{K}_1 \tilde{K}_1 \varepsilon^{-b(N+2)} \left(\frac{\rho(0)}{\tau(t)} + \tau^2(t)\rho(3) \right), \tag{3.1.21}$$

where ρ means $\rho_{[t_0,t]}$. In order to bound $\rho(0)$ and $\rho(3)$ let us fix $t \in [t_0, t_1(s)]$ and note that, if $t + \text{Re } s < 0$, then $\tau(\sigma) \geq \tau(t)$ for every $\sigma \in [t_0, t]$ and thus

$$\rho(\lambda) \leq \begin{cases} \tau^{-\lambda}(t), & \text{if } \lambda > 0 \\ \left|\ln \tau(t)\right|, & \text{if } \lambda = 0 \end{cases} \tag{3.1.22}$$

On the contrary, if $t + \text{Re } s > 0$, then $\tau(\sigma) \geq \tau(0) > \varepsilon^b$ for every $\sigma \in [t_0, t]$. Thus,

$$\rho(\lambda) \leq \begin{cases} \varepsilon^{-\lambda b}, & \text{if } \lambda > 0 \\ b\left|\ln \varepsilon\right|, & \text{if } \lambda = 0. \end{cases} \tag{3.1.23}$$

Therefore, we have that

$$\tau^2(t)\rho(3) \leq \frac{8\varepsilon^{-b}}{\tau(t)} \quad \text{and} \quad \rho(0) < b\left|\ln \varepsilon\right|. \tag{3.1.24}$$

Indeed, the second bound is clear and the first one easily follows when $t + \text{Re } s < 0$. Otherwise, bearing in mind the definition of the separation time given in (3.1.12), we obtain that

$$\tau(t) \leq 2\varepsilon^{\frac{2}{3}b} \tag{3.1.25}$$

and then (3.1.23) implies (3.1.24).

Therefore, using (3.1.20) and (3.1.21) we deduce that

$$\left|z^1(t)\right|_{\tau(t)} \leq \frac{\hat{K}_1}{\tau(t)} \mu \varepsilon^{-b(N+3)} + 2K_1 \overline{K}_1 \tilde{K}_1 \mu \varepsilon^{-b(N+2)} \left(\frac{b\left|\ln \varepsilon\right|}{\tau(t)} + \frac{8\varepsilon^{-b}}{\tau(t)} \right)$$

from which we finally obtain

$$\left\|z^1\right\|_1 \leq 2K_1^* \mu \varepsilon^{-b(N+3)},$$

whenever ε is sufficiently small.

Remark 3.1.6 *Disregarding the separation time defined at (3.1.12), we only would get*

$$\tau^3(t)\rho(3) \leq ctant\ \varepsilon^{-3b},$$

which is a substantially worse bound than the one obtained in (3.1.24). Moreover, the separation time is exclusively used to obtain the previous bound for $\|z^1\|_1$ *(more concretely, for using (3.1.25)) together with the bound given in (3.1.29). Both bounds explain why the definition of the separation time given in (3.1.12) is optimum.*

Now, let us assume that (3.1.18) holds for $k = 1, \ldots, n$ and let $z_{n-1} = (\lambda_{n-1}, \kappa_{n-1})$, $z_n = (\lambda_n, \kappa_n)$. Since, by assumption

$$\max\{|\lambda_{n-1}|, |\lambda_n|, |\kappa_{n-1}|, |\kappa_n|\} < 4K_1^*\mu\varepsilon^{-b(N+3)} < \mu\varepsilon^{-b(N+6)},$$

we may apply equations (3.1.13) and (3.1.14) given by Lemma 3.1.4 (b) to get

$$\begin{aligned}
&\left|F(x^0(\sigma+s), y^0(\sigma+s), z_n(\sigma), \hat{\theta}) - F(x^0(\sigma+s), y^0(\sigma+s), z_{n-1}(\sigma), \hat{\theta})\right|_{\tau(\sigma)} \leq \\
&\leq \frac{A\overline{K}_1}{\tau(\sigma)}(|\lambda_n(\sigma)| + |\lambda_{n-1}(\sigma)|)\,|\lambda_n(\sigma) - \lambda_{n-1}(\sigma)| + \\
&\quad + \frac{2\overline{K}_1\mu\varepsilon^{-b(N+2)}}{\tau^2(\sigma)}\,|z_n(\sigma) - z_{n-1}(\sigma)|_{\tau(\sigma)}\,,
\end{aligned} \tag{3.1.26}$$

where we recall that F denotes $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ with f_1 and f_2 the functions defined at (3.1.5).

Using the inductive hypothesis it follows that

$$\begin{aligned}
&(|\lambda_n(\sigma)| + |\lambda_{n-1}(\sigma)|)\,|\lambda_n(\sigma) - \lambda_{n-1}(\sigma)| \leq \\
&\leq \left(|z_n(\sigma)|_{\tau(\sigma)} + |z_{n-1}(\sigma)|_{\tau(\sigma)}\right)|z_n(\sigma) - z_{n-1}(\sigma)|_{\tau(\sigma)} \leq \\
&\leq \frac{8K_1^*\mu\varepsilon^{-b(N+3)}}{\tau(\sigma)}\,|z_n(\sigma) - z_{n-1}(\sigma)|_{\tau(\sigma)}\,.
\end{aligned} \tag{3.1.27}$$

Hence,

$$\begin{aligned}
&\left|F(x^0(\sigma+s), y^0(\sigma+s), z_n(\sigma), \hat{\theta}) - F(x^0(\sigma+s), y^0(\sigma+s), z_{n-1}(\sigma), \hat{\theta})\right|_{\tau(\sigma)} \leq \\
&\leq \left(\frac{2\overline{K}_1\mu\varepsilon^{-b(N+2)}}{\tau^2(\sigma)} + \frac{8A\overline{K}_1K_1^*\mu\varepsilon^{-b(N+3)}}{\tau^2(\sigma)}\right)|z_n(\sigma) - z_{n-1}(\sigma)|_{\tau(\sigma)} \leq \\
&\leq \left(\frac{2\overline{K}_1\mu\varepsilon^{-b(N+2)}}{\tau^3(\sigma)} + \frac{8A\overline{K}_1K_1^*\mu\varepsilon^{-b(N+3)}}{\tau^3(\sigma)}\right)\|z_n - z_{n-1}\|_1\,.
\end{aligned}$$

Therefore, we may apply Lemma 3.1.5 to get (see also the definition of the iterative process given in (3.1.17))

$$\begin{aligned}
&|z_{n+1}(t) - z_n(t)|_{\tau(t)} \leq \\
&\leq \left(2K_1\overline{K}_1\tilde{K}_1\mu\varepsilon^{-b(N+2)} + 8AK_1\overline{K}_1\tilde{K}_1K_1^*\mu\varepsilon^{-b(N+3)}\right)\left(\frac{\rho(1)}{\tau(t)} + \tau^2(t)\rho(4)\right)\|z_n - z_{n-1}\|_1\,.
\end{aligned}$$

Now, (3.1.22), (3.1.23) and the fact that $\tau > \varepsilon^b$ give $\rho(l) \leq \varepsilon^{-bl}$, for every $l > 0$ and, in the same way as the first bound in (3.1.24) was deduced, we get $\tau^3(t)\rho(4) \leq 8\varepsilon^{-2b}$. Therefore,

$$\begin{aligned} &\|z_{n+1} - z_n\|_1 \leq \\ &\left(2K_1\overline{K}_1\tilde{K}_1\mu\varepsilon^{-b(N+2)} + 8AK_1\overline{K}_1\tilde{K}_1K_1^*\mu\varepsilon^{-b(N+3)}\right)\left(\rho(1) + \tau^3(t)\rho(4)\right)\|z_n - z_{n-1}\|_1 \leq \\ &\leq 90AK_1\overline{K}_1\tilde{K}_1K_1^*\mu\varepsilon^{-b(N+5)}\|z_n - z_{n-1}\|_1 . \end{aligned}$$

Thus, since $\mu \in (0, \varepsilon^m)$ with $m > b(N+6)$, we have

$$\|z_{n+1} - z_n\|_1 < \frac{1}{2}\|z_n - z_{n-1}\|_1 .$$

Then,

$$\|z_{n+1}\|_1 \leq \sum_{j=1}^{n+1}\|z_j - z_{j-1}\|_1 < 2\|z_1\|_1 < 4\hat{K}_1\mu\varepsilon^{-b(N+3)}$$

and, not only (3.1.18) is proved, but also we may claim that $\{z_n\}_{n\in\mathbb{N}}$ converges uniformly. Hence, it necessarily holds that $\{z_n\}_{n\in\mathbb{N}}$ converges to the solution z and therefore, now, (3.1.18) implies the proposition. □

Proof of Proposition 3.1.2

In order to prove Proposition 3.1.2 we essentially follow the same scheme used to prove Proposition 3.1.1 by looking for the solution of (3.1.4) passing through $z(t_1)$ at time $t_1 = t_1(s)$ as the limit of the following iterative process:

$$z_{n+1}(t) = z^1(t) + M(t+s)\int_{t_1}^{t} M^{-1}(\sigma+s)F(x^0(\sigma+s), y^0(\sigma+s), z_n(\sigma), \hat{\theta})\,\mathrm{d}\sigma$$

with

$$z^1(t) = M(t+s)\left[M^{-1}(t_1+s)z(t_1) + \mu\int_{t_1}^{t} M^{-1}(\sigma+s)G(x^0(\sigma+s), y^0(\sigma+s), \hat{\theta})\,\mathrm{d}\sigma\right]$$

where, by assumption,

$$|z(t_1)|_{\tau(t_1)} \leq 4K_1^*\tau^{-1}(t_1)\mu\varepsilon^{-b(N+3)}.$$

We will prove that

$$\|z_n\|_{-2} \leq 40K_1^2K_1^*\mu\varepsilon^{-b(N+5)}, \tag{3.1.28}$$

for every $n \in \mathbb{N}$, where, by definition

$$\|z\|_{-2} = \sup_{t\in[t_1(s), 2T_0 - \mathrm{Re}\ s]}\left\{|z(t)|_{\tau(t)}\,\tau^{-2}(t)\right\}$$

from which, as in the previous proposition, the result follows.

Notice that, from Lemma 3.1.3, we have

$$\begin{aligned} \left|M(t+s)M^{-1}(t_1+s)z(t_1)\right|_{\tau(t)} &\leq 2K_1^2 \left|z(t_1)\right|_{\tau(t_1)} \left(\frac{\tau(t_1)}{\tau(t)} + \frac{\tau^2(t)}{\tau^2(t_1)}\right) \leq \\ &\leq 8K_1^2 K_1^* \mu \varepsilon^{-b(N+3)} \left(\frac{1}{\tau(t)} + \frac{\tau^2(t)}{\tau^3(t_1)}\right) \end{aligned}$$

and, as in (3.1.21), we now deduce that

$$\left|z^1(t)\right|_{\tau(t)} \leq$$

$$\leq 8K_1^2 K_1^* \mu \varepsilon^{-b(N+3)} \left(\frac{1}{\tau(t)} + \frac{\tau^2(t)}{\tau^3(t_1)}\right) + 2K_1 \overline{K}_1 \tilde{K}_1 \mu \varepsilon^{-b(N+2)} \left(\frac{\rho'(0)}{\tau(t)} + \tau^2(t)\rho'(3)\right),$$

where ρ' means $\rho_{[t_1,t]}$. Now, since τ is an increasing function in $[t_1, t]$, we obtain

$$\rho'(\lambda) \leq \begin{cases} \tau^{-\lambda}(t), & \text{if } \lambda < 0 \\ \dfrac{2b}{3}\left|\ln\varepsilon\right|, & \text{if } \lambda = 0 \text{ (and } \varepsilon \text{ small enough)} \\ \tau^{-\lambda}(t_1), & \text{if } \lambda > 0 \end{cases}$$

where we have also used that $\tau(t_1) > \varepsilon^{\frac{2}{3}b}$.

Therefore, we get

$$\begin{aligned} \frac{\left|z^1(t)\right|_{\tau(t)}}{\tau^2(t)} &\leq \frac{16K_1^2 K_1^* \mu \varepsilon^{-b(N+3)}}{\tau^3(t_1)} + \frac{8bK_1 \overline{K}_1 K_1^* \mu \left|\ln\varepsilon\right| \varepsilon^{-b(N+2)}}{3\tau^3(t_1)} < \\ &< 20K_1^2 K_1^* \mu \varepsilon^{-b(N+5)}. \end{aligned} \tag{3.1.29}$$

To get (3.1.28) by an inductive method, it suffices to repeat the respective estimates of the proof of Proposition 3.1.1 (see how (3.1.26) and (3.1.27) were obtained) to deduce, for ε small enough, that

$$\left|F(x^0(\sigma+s), y^0(\sigma+s), z_n(\sigma), \hat{\theta}) - F(x^0(\sigma+s), y^0(\sigma+s), z_{n-1}(\sigma), \hat{\theta})\right|_{\tau(\sigma)} \leq$$

$$\leq \left(\frac{2\overline{K}_1 \mu \varepsilon^{-b(N+2)}}{\tau^2(\sigma)} + \frac{80AK_1^2 \overline{K}_1 K_1^* \mu \varepsilon^{-b(N+5)}}{\tau^2(\sigma)}\right) \left|z_n(\sigma) - z_{n-1}(\sigma)\right|_{\tau(\sigma)} \leq$$

$$\leq \left(2\overline{K}_1 \mu \varepsilon^{-b(N+2)} + 80AK_1^2 \overline{K}_1 K_1^* \mu \varepsilon^{-b(N+5)}\right) \left\|z_n - z_{n-1}\right\|_{-2},$$

whenever (3.1.28) is assumed to be true for n and $n-1$ (note that, therefore, the hypotheses needed for applying Lemma 3.1.4 (b), are fulfilled).

Hence, Lemma 3.1.5 implies

$$\left|z_{n+1}(t) - z_n(t)\right|_{\tau(t)} \leq 200K_1 \overline{K}_1 \tilde{K}_1 \mu \varepsilon^{-b(N+2)} \left(\frac{\rho'(-2)}{\tau(t)} + \tau^2(t)\rho'(1)\right) \left\|z_n - z_{n-1}\right\|_{-2} +$$

$$+80AK_1^3 \overline{K}_1 \tilde{K}_1 K_1^* \mu \varepsilon^{-b(N+5)} \left(\frac{\rho'(-2)}{\tau(t)} + \tau^2(t)\rho'(1)\right) \left\|z_n - z_{n-1}\right\|_{-2}$$

and thus, using that $\mu \in (0, \varepsilon^m)$ with $m > b(N+6)$, we conclude that

$$\|z_{n+1} - z_n\|_{-2} < 200AK_1^3\overline{K}_1\tilde{K}_1K_1^*\mu\varepsilon^{-b(N+6)} \|z_n - z_{n-1}\|_{-2} < \frac{1}{2} \|z_n - z_{n-1}\|_{-2} ,$$

provided that μ is small enough. $\square$

Proof of the Extension Theorem I

Since $\tau(t) \geq \varepsilon^b$, Proposition 3.1.1 implies that, for every $t \in [-T_0 - \text{Re } s, t_1(s)]$,

$$|\lambda(t)| \leq |z(t)|_{\tau(t)} \leq 4K_1^*\tau^{-1}(t)\mu\varepsilon^{-b(N+3)} < 4K_1^*\mu\varepsilon^{-b(N+4)}$$

and

$$|\kappa(t)| \leq \tau^{-1}(t)\, |z(t)|_{\tau(t)} \leq 4K_1^*\mu\varepsilon^{-b(N+5)}.$$

Moreover, if $t \in [t_1(s), 2T_0 - \text{Re } s]$, Proposition 3.1.2 yields

$$|\lambda(t)| \leq |z(t)|_{\tau(t)} \leq 40K_1^2K_1^*\tau^2(t)\mu\varepsilon^{-b(N+5)} \leq C_1'\mu\varepsilon^{-b(N+5)}$$

and

$$|\kappa(t)| \leq \tau^{-1}(t)\, |z(t)|_{\tau(t)} \leq C_1'\mu\varepsilon^{-b(N+5)},$$

where C_1' is a sufficiently large constant depending on K_1, C_1, A and T_0. $\square$

Remark 3.1.7 *To obtain the required bounds for $|I_i(t) - \alpha_i|$, $i = 1, 2$, for any t in $[-T_0 - \text{Re } s, 2T_0 - \text{Re } s]$, we proceed as follows: From the corresponding equation in (1.1.6) and the bound provided by (1.1.8), it holds that*

$$\begin{aligned} |I_i(t) - \alpha_i| &\leq |I_i(t_0) - \alpha_i| + \mu \left\| \frac{\partial M_1}{\partial \theta_i} \right\|_{\mathcal{B}_1''} \int_{-T_0 - \text{Re } s}^{t} |y(r)|\, |\sin x(r)|\, dr \leq \\ &\leq C_1\mu\varepsilon^{-b(N+3)} + ctant\ \mu\varepsilon^{-b(N+3)} \int_{-T_0 - \text{Re } s}^{t} |y(r)|\, |\sin x(r)|\, dr. \qquad (3.1.30) \end{aligned}$$

By taking into account that $x(r) = x^0(r+s) + \lambda(r)$ and $y(r) = y^0(r+s) + \kappa(r)$ with

$$\max\{|\lambda(r)|, |\kappa(r)|\} \leq C_1'\mu\varepsilon^{-b(N+5)},$$

we may conclude, for μ sufficiently small, that $|y(r)|\, |\sin x(r)| \leq ctant\ \tau^{-3}(r)$. Hence, from (1.2.40) we finally deduce

$$|I_i(t) - \alpha_i| \leq C_1'\ \mu\varepsilon^{-b(N+5)},$$

where we still denote by C_1' a (new) sufficiently large constant only depending on K_1, C_1, A and T_0.

3.2 Proof of the Extension Theorem II

To prove Theorem 2.1.4 we will follow the same strategy used to prove the Extension Theorem I. However, due to the fact that we have to deal with a non-quasiperiodic case, we must carry out a more *global* analysis in the phase space.

Let us consider a solution $(x(t), y(t), \hat{I}(t), \hat{\theta}(t))$ of (2.0.9) satisfying (2.1.31), i.e.,

$$\begin{aligned} \left|x(t_0) - x^0(t+s)\right| &\leq C_2 \mu \varepsilon^{-b(N+3)}, \quad \left|y(t_0) - y^0(t+s)\right| \leq C_2 \mu \varepsilon^{-b(N+3)}, \\ \left|I_i(t_0) - \beta_i\right| &\leq C_2 \mu \varepsilon^{-b(N+3)}, \ i = 1, 2, \quad (\theta_1(t_0), \theta_2(t_0)) \in \mathcal{B}_2''' \end{aligned}$$

for some positive constant C_2, $t_0 = -T_0 - \text{Re } s$, T_0 a sufficiently large positive constant (see Lemma 2.1.3), s a complex number in

$$\mathcal{C}_2' = \left\{ s \in \mathbb{C} : \ |\text{Im } s| \leq \frac{\pi}{2} - \varepsilon^b \right\}$$

and

$$\mathcal{B}_2''' = \left\{ (\theta_1, \theta_2) \in \mathbb{C}^2 : \ |\text{Im } \theta_i| \leq r_i - 3\varepsilon^b, \ i = 1, 2 \right\}.$$

Instead of directly looking for suitable properties of the extensions to the time interval $[-T_0 - \text{Re } s, 2T_0 - \text{Re } s]$ of such solutions $(x(t), y(t), \hat{I}(t), \hat{\theta}(t))$, we will get information about the solutions of the following system of differential equations

$$\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= \sin(x^0(t+s) + X) - \sin(x^0(t+s)) + \mu \sin(x^0(t+s) + X) m(\theta_1^0, \theta_2^0 + \nu) \\ \dot{\nu} &= \frac{\alpha}{\sqrt{\varepsilon}} \\ \dot{\alpha} &= -\mu \left(\cos(x^0(t+s) + X) - 1\right) \frac{\partial m}{\partial \theta_2}(\theta_1^0, \theta_2^0 + \nu), \\ \dot{\theta}_1^0 &= \frac{1}{\sqrt{\varepsilon}}, \quad \dot{\theta}_2^0 = \frac{\beta_2}{\sqrt{\varepsilon}}, \end{aligned} \tag{3.2.31}$$

where $(x^0, y^0, \hat{I}^0, \hat{\theta}^0)$ is the parameterization of the complex separatrix given in (2.1.25).

Therefore, if $(X(t), Y(t), \nu(t), \alpha(t), \theta_1^0(t), \theta_2^0(t))$ is a solution of (3.2.31), then

$$\begin{aligned} x(t) = X(t) + x^0(t+s), &\qquad y(t) = Y(t) + y^0(t+s), \\ I_2(t) = \alpha(t) + \beta_2, &\qquad \theta_2(t) = \nu(t) + \theta_2^0(t) \end{aligned}$$

satisfy the respective equations (the first two ones, the fourth and the sixth) given in (2.0.9).

Let us also recall that the Main Theorem II is proved whenever we prove the following claim: If, for $t_0 = -T_0 - \text{Re } s$ and some positive constant C_2,

$$\max\left\{ |X(t_0)|, |Y(t_0)|, |\nu(t_0)|, |\alpha(t_0)| \right\} \leq C_2 \mu \varepsilon^{-b(N+3)} \tag{3.2.32}$$

then, for every $t \in [-T_0 - \text{Re } s, 2T_0 - \text{Re } s]$,

$$\max\{|X(t)|, |Y(t)|, |\nu(t)|, |\alpha(t)|\} \le C_2' \mu \varepsilon^{-b(N+6)},$$

where C_2' is some positive constant depending on C_2, T_0 and K_2 (K_2 some positive constant which, as we will see along the arguments, essentially coincide with the constant K_1 given by Lemma 3.1.3).

Proceeding as in the previous section, let us start by pointing out that, along the present section, we will always assume that $\mu \in (0, \varepsilon^w)$, with $w > b(N+6) + \dfrac{1}{2}$.

Moreover, let us write the dynamical system (3.2.31) in an equivalent form: By denoting

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \nu \\ \alpha \end{pmatrix},$$

then (3.2.31) can be written as

$$\begin{aligned} \dot{Z} &= R(t+s)Z + \mu G_1(x^0(t+s), \hat{\theta}^0) + F_1(x^0(t+s), X, \nu, \hat{\theta}^0) \\ \dot{\Lambda} &= B\Lambda + \mu G_2(x^0(t+s), \hat{\theta}^0) + F_2(x^0(t+s), X, \nu, \hat{\theta}^0), \end{aligned} \tag{3.2.33}$$

where

$$R(t+s) = \begin{pmatrix} 0 & 1 \\ \cos(x^0(t+s)) & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \dfrac{1}{\sqrt{\varepsilon}} \\ 0 & 0 \end{pmatrix}$$

and, for $i = 1, 2$,

$$G_i = \begin{pmatrix} 0 \\ g_i \end{pmatrix}, \qquad F_i = \begin{pmatrix} 0 \\ f_i \end{pmatrix}$$

with

$$\begin{aligned} g_1(x, \hat{\theta}) &= m(\hat{\theta}) \sin x \\ g_2(x, \hat{\theta}) &= -(\cos x - 1) \frac{\partial m}{\partial \theta_2}(\hat{\theta}) \\ f_1(x, X, \nu, \hat{\theta}) &= \sin(x + X) - \sin x - X \cos x + \mu\left[g_1(x + X, \theta_1, \theta_2 + \nu) - g_1(x, \theta_1, \theta_2)\right] \\ f_2(x, X, \nu, \hat{\theta}) &= \mu\left[g_2(x + X, \theta_1, \theta_2 + \nu) - g_2(x, \hat{\theta})\right]. \end{aligned}$$

In the same way as was done in (3.1.10) and (3.1.11) the Z-component of the solution $(Z(t), \Lambda(t))$ of (3.2.33) satisfying the initial condition $(Z(t_0), \Lambda(t_0))$ is given by

$$Z(t) = Z^1(t) + M(t+s) \int_{t_0}^{t} M^{-1}(\sigma + s) F_1(x^0(\sigma + s), X(\sigma), \nu(\sigma), \hat{\theta}^0) d\sigma,$$

where

$$Z^1(t) = M(t+s)\left[M^{-1}(t_0 + s)Z(t_0) + \mu \int_{t_0}^{t} M^{-1}(\sigma + s) G_1(x^0(\sigma + s), \hat{\theta}^0)\, \mathrm{d}\sigma\right] \tag{3.2.34}$$

and M is the fundamental matrix computed in (3.1.9). In order to be precise the (new) fundamental matrix does not coincide with the one given in (3.1.9). Nevertheless, their components only differ in a constant (depending on the factor A which takes part in the definition of the Hamiltonians studied in the first chapter) which does not play an essential role in the arguments. Hence, let us denote both matrices by M and observe that Lemma 3.1.3 still holds in this new scenario by taking $b_0 = \dfrac{\sqrt{-1}\pi}{2}$ instead of $b_0 = \dfrac{\sqrt{-1}\pi}{2\sqrt{A}}$. Of course, the (new) function τ is now defined, for a fixed value $s \in \mathcal{C}_2'$, by

$$\tau(t) = \left| t + s - \frac{\sqrt{-1}\pi}{2} \right|.$$

We replace the constant K_1 in the statement of Lemma 3.1.3 by a new constant K_2 (valid for this second setting) although K_1 and K_2 would be essentially the same.

In a similar way, the Λ-component of the solution $(Z(t), \Lambda(t))$ of (3.2.33) satisfying the initial condition $(Z(t_0), \Lambda(t_0))$ can be written as

$$\Lambda(t) = \Lambda^1(t) + N(t+s) \int_{t_0}^{t} N^{-1}(\sigma + s) F_2(x^0(\sigma + s), X(\sigma), \nu(\sigma), \hat{\theta}^0)\, \mathrm{d}\sigma,$$

with

$$\Lambda^1(t) = N(t+s) \left[N^{-1}(t_0 + s)\Lambda(t_0) + \mu \int_{t_0}^{t} N^{-1}(\sigma + s) G_2(x^0(\sigma + s), \hat{\theta}^0)\, \mathrm{d}\sigma \right] \quad (3.2.35)$$

and

$$N(t+s) = \begin{pmatrix} 1 & \dfrac{t+s}{\sqrt{\varepsilon}} \\ 0 & 1 \end{pmatrix}$$

the fundamental matrix associated to the homogeneous system $\dot{\Lambda} = B\Lambda$.

We are going to develop the proof of the Extension Theorem II for the case in which $s \in \mathcal{C}_2'(+)$, with

$$\mathcal{C}_2'(+) = \left\{ s \in \mathbb{C} : \ 0 \le \mathrm{Im}\ s < \frac{\pi}{2} - \varepsilon^b \right\}.$$

Let us begin by introducing two auxiliary results, whose proofs can be obtained following the same methods applied to prove Lemma 3.1.4 and Lemma 3.1.5, respectively.

Lemma 3.2.1 *There exists a positive constant $\overline{K}_2 = \overline{K}_2(K_2, T_0)$ such that:*
(a) For every $(t, s, \hat{\theta}) \in [-T_0 - \mathrm{Re}\ s, 2T_0 - \mathrm{Re}\ s] \times \mathcal{C}_2'(+) \times \mathcal{B}_2'''$ we have

$$\left| g_j(x^0(t+s), \hat{\theta}) \right| \le \frac{\overline{K}_2 \varepsilon^{-b(N+j+1)}}{\tau^2(t)}, \quad j = 1, 2.$$

(b) For any complex numbers X_1, X_2, ν_1, ν_2 with

$$\max\{|X_1|, |X_2|, |\nu_1|, |\nu_2|\} < \mu\varepsilon^{-b(N+5)-1/2}$$

and every $(t, s, \hat{\theta}) \in [-T_0 - \mathrm{Re}\ s, 2T_0 - \mathrm{Re}\ s] \times \mathcal{C}_2'(+) \times \mathcal{B}_2'''$ it follows that

$$\left| f_1(x^0(t+s), X_1, \nu_1, \hat{\theta}^0) - f_1(x^0(t+s), X_2, \nu_2, \hat{\theta}^0) \right| \leq$$
$$\leq \frac{\overline{K}_2}{\tau^2(t)} (|X_1| + |X_2|) |X_1 - X_2| + \frac{\overline{K}_2 \mu\varepsilon^{-b(N+3)}}{\tau^2(t)} \left(\varepsilon^b |X_1 - X_2| + |\nu_1 - \nu_2| \right)$$

and

$$\left| f_2(x^0(t+s), X_1, \nu_1, \hat{\theta}^0) - f_2(x^0(t+s), X_2, \nu_2, \hat{\theta}^0) \right| \leq$$
$$\leq \frac{\overline{K}_2 \mu\varepsilon^{-b(N+4)}}{\tau^2(t)} \left(\varepsilon^b |X_1 - X_2| + |\nu_1 - \nu_2| \right).$$

Proof

With respect to the proof of Lemma 3.2.1 we only underline that conditions

$$\mu \in (0, \varepsilon^w), \quad w > b(N+6) + \frac{1}{2}$$

and

$$|\nu_i| < \mu\varepsilon^{-b(N+5)-1/2}, \quad i = 1, 2$$

imply that $(\theta_1, \theta_2 + \nu) \in \mathcal{B}_2''$ whenever $\hat{\theta} = (\theta_1, \theta_2) \in \mathcal{B}_2'''$. Then, the arguments used during the proof of Lemma 3.1.4 are enough to prove Lemma 3.2.1. □

To state the second auxiliary lemma, we recall the definition of the function $\rho = \rho_{[t_0,t]}(\lambda)$ given in (3.1.16). We will also make use of the following norm on $\mathbb{C}^2$: For any positive real constant Υ let

$$z = (z_1, z_2) \in \mathbb{C}^2 \to |z|^{\Upsilon} = \Upsilon |z_1| + |z_2|$$

and let us remark that, for every $z = (z_1, z_2) \in \mathbb{C}^2$,

$$|z_1| \leq \Upsilon^{-1} |z|^{\Upsilon} \quad \text{and} \quad |z_2| \leq |z|^{\Upsilon}.$$

We also recall that in the last section we have also introduced the norm

$$z = (z_1, z_2) \in \mathbb{C}^2 \to |z|_{\gamma} = |z_1| + \gamma |z_2|$$

for any positive real constant γ.

Lemma 3.2.2 *There exists a positive constant* $K_2^* = K_2^*(T_0)$ *such that for every* $v \in \mathbb{C}^2$ *and* $t, \overline{t} \in [-T_0 - \mathrm{Re}\ s, 2T_0 - \mathrm{Re}\ s]$,

$$\left|N(t+s)N^{-1}(\overline{t}+s)v\right|^{\sqrt{\varepsilon}} \le K_2^* \left|v\right|^{\sqrt{\varepsilon}}.$$

Furthermore, if $[t_0, t] \subset [-T_0 - \mathrm{Re}\ s, 2T_0 - \mathrm{Re}\ s]$ *and*

$$w : \sigma \in [t_0, t] \to w(\sigma) = (0, w_2(\sigma)) \in \mathbb{C}^2$$

is a function satisfying

$$|w_2(\sigma)| \le \frac{C_2^*}{\tau^{\lambda}(\sigma)}$$

for some positive constant C_2^* *and any* $\sigma \in [t_0, t]$, *then*

$$\left|N(t+s)\int_{t_0}^{t} N^{-1}(\sigma+s)w(\sigma)d\sigma\right|^{\sqrt{\varepsilon}} \le \tilde{K}_1 K_2^* C_2^* \rho_{[t_0,t]}(\lambda - 1),$$

where $\tilde{K}_1 = \tilde{K}_1(T_0, \lambda)$ *is the constant for which (1.2.39) or Lemma 3.1.5 holds.*

Now, let us provide the final details to prove the second Extension Theorem: Let

$$S(t) = (Z(t), \Lambda(t)) = (X(t), Y(t), \nu(t), \alpha(t))$$

be the solution of (3.2.33) passing through $S(t_0) = (X(t_0), Y(t_0), \nu(t_0), \alpha(t_0))$.

Once again, it will be useful to express S as the limit function of an iterative process: Let us take $Z_0(t) = \Lambda_0(t) = 0$ and define

$$\begin{aligned} Z_{n+1}(t) &= Z^1(t) + M(t+s)\int_{t_0}^{t} M^{-1}(\sigma+s)F_1(x^0(\sigma+s), X_n(\sigma), \nu_n(\sigma), \hat{\theta}^0)d\sigma \\ \Lambda_{n+1}(t) &= \Lambda^1(t) + N(t+s)\int_{t_0}^{t} N^{-1}(\sigma+s)F_2(x^0(\sigma+s), X_n(\sigma), \nu_n(\sigma), \hat{\theta}^0)d\sigma \end{aligned} \tag{3.2.36}$$

with Z^1 and Λ^1 the functions respectively defined in (3.2.34) and (3.2.35).

If we write

$$S_n(t) = (Z_n(t), \Lambda_n(t)) = (X_n(t), Y_n(t), \nu_n(t), \alpha_n(t)),$$

we will inductively prove that, for every $n \in \mathbb{N}$,

$$\|S_n\|_3 \le 4K_2'\mu\varepsilon^{-b(N+4)} \tag{3.2.37}$$

where K_2' is a constant depending on K_2, C_2, T_0 and also on the constants $\tilde{K}_1$ (see Lemma 3.1.5), $\overline{K}_2$ (see Lemma 3.2.1) and K_2^* (see Lemma 3.2.2). Moreover, by definition, for $S = S(t) = (Z, \Lambda) = (Z(t), \Lambda(t))$, let

$$\|S\|_3 = \|Z\|' + \|\Lambda\|^*$$

with

$$\|Z\|' = \sup_{t\in[-T_0-\text{Re } s, 2T_0-\text{Re } s]} \left\{ |Z(t)|_{\tau(t)} \right\}, \qquad \|\Lambda\|^* = \sup_{t\in[-T_0-\text{Re } s, 2T_0-\text{Re } s]} \left\{ |\Lambda(t)|^{\sqrt{\varepsilon}} \right\}.$$

Let us prove (3.2.37) for $n = 1$ observing that $Z_1 = Z^1$ and $\Lambda_1 = \Lambda^1$.

From Lemma 3.1.3 we have

$$\left|M(t+s)M^{-1}(t_0+s)Z(t_0)\right|_{\tau(t)} \leq 2K_2^2 \left(|X(t_0)| + \tau(t_0)\,|Y(t_0)|\right) \left(\frac{\tau(t_0)}{\tau(t)} + \frac{\tau^2(t)}{\tau^2(t_0)} \right).$$

Therefore, using (3.2.32) and, in the same way as (3.1.20) was obtained, we may conclude

$$\left|M(t+s)M^{-1}(t_0+s)Z(t_0)\right|_{\tau(t)} \leq \frac{\hat{K}_2}{\tau(t)} \mu\varepsilon^{-b(N+3)} < \hat{K}_2 \mu\varepsilon^{-b(N+4)},$$

where $\hat{K}_2 = \hat{K}_2(K_2, C_2, T_0)$ is some positive constant and we have used that $\tau(t) > \varepsilon^b$, for every $t \in [-T_0 - \text{Re } s, 2T_0 - \text{Re } s]$.

Moreover, since Lemma 3.2.1 (a) yields

$$\left|g_j(x^0(\sigma+s), \hat{\theta}^0)\right| \leq \frac{\overline{K}_2 \varepsilon^{-b(N+j+1)}}{\tau^2(\sigma)} \tag{3.2.38}$$

for every $\sigma \in [-T_0 - \text{Re } s, 2T_0 - \text{Re } s]$, and $G_1 = (0, g_1)$, we conclude

$$\left|G_1(x^0(\sigma+s), \hat{\theta}^0)\right|_{\tau(\sigma)} \leq \frac{\overline{K}_2 \varepsilon^{-b(N+2)}}{\tau(\sigma)}$$

and therefore Lemma 3.1.5 leads to

$$\begin{aligned} &\left| M(t+s) \int_{t_0}^{t} M^{-1}(\sigma+s) G_1(x^0(\sigma+s), \hat{\theta}^0) d\sigma \right|_{\tau(t)} \leq \\ &\qquad \leq K_2 \overline{K}_2 \tilde{K}_1 \varepsilon^{-b(N+2)} \left(\tau^{-1}(t)\rho(-1) + \tau^2(t)\rho(2) \right), \end{aligned}$$

where ρ means $\rho_{[t_0,t]}$.

Hence, using that

$$\rho(\lambda) \leq \begin{cases} \varepsilon^{-\lambda b}, & \lambda > 0, \\ b\,|\ln\varepsilon|\,, & \lambda = 0, \\ C_3 = C_3(\lambda, T_0), & \lambda < 0 \end{cases} \tag{3.2.39}$$

we finally deduce

$$\left|Z^1(t)\right|_{\tau(t)} \leq \hat{K}_2 \mu\varepsilon^{-b(N+4)} + 2\left(4T_0^2 + \frac{\pi^2}{4}\right) K_2 \overline{K}_2 \tilde{K}_1 \mu\varepsilon^{-b(N+4)} < K_2' \mu\varepsilon^{-b(N+4)}, \tag{3.2.40}$$

whenever ε is small enough.

Furthermore, from Lemma 3.2.2, we obtain

$$\left|N(t+s)N^{-1}(t_0+s)\Lambda(t_0)\right|^{\sqrt{\varepsilon}} \leq K_2^* \left(\sqrt{\varepsilon}\,|\nu(t_0)| + |\alpha(t_0)|\right) \leq 2C_2 K_2^* \mu\varepsilon^{-b(N+3)}.$$

Moreover, using (3.2.38) for $j=2$, we may apply the second statement of Lemma 3.2.2 to deduce, from the inequality $\rho(1) \leq \varepsilon^{-b}$, that

$$\left|N(t+s)\int_{t_0}^{t} N^{-1}(\sigma+s)G_2(x^0(\sigma+s),\hat{\theta}^0)d\sigma\right|^{\sqrt{\varepsilon}} \leq \tilde{K}_1 K_2^* \overline{K}_2 \varepsilon^{-b(N+4)}.$$

Therefore,

$$\left|\Lambda^1(t)\right|^{\sqrt{\varepsilon}} \leq K_2' \mu\varepsilon^{-b(N+4)}$$

and this bound, together with (3.2.40), implies that (3.2.37) holds for $n=1$.

Remark 3.2.3 *As in the last section, we could have used a separation time (or even an auxiliary convenient norm $\|\cdot\|_1$ on $\mathbb{C}^2$ like the one introduced in (3.1.19)) in order to try to amend the above bound for S_1. Nevertheless, in the present setting, this procedure is unfruitful due to the fact that only the bounds for the Z^1-component could be improved, but not those ones for the Λ^1-component.*

Let us continue the proof of the Extension Theorem II by assuming that (3.2.37) holds for $k=1,\ldots,n$. Let $S_k=(Z_k,\Lambda_k)$ with $Z_k=(X_k,Y_k)$ and $\Lambda_k=(\nu_k,\alpha_k)$. Then, since

$$\max\{|X_n|,|\nu_n|,|X_{n-1}|,|\nu_{n-1}|\} < 4K_2'\mu\varepsilon^{-b(N+4)-1/2} < \mu\varepsilon^{-b(N+5)-1/2},$$

we may apply Lemma 3.2.1 (b) to get

$$\begin{aligned}&\left|f_1(x^0(\sigma+s),X_n(\sigma),\nu_n(\sigma),\hat{\theta}^0) - f_1(x^0(\sigma+s),X_{n-1}(\sigma),\nu_{n-1}(\sigma),\hat{\theta}^0)\right| \leq \\ &\quad\leq \frac{\overline{K}_2}{\tau^2(\sigma)}\mu\varepsilon^{-b(N+3)}\left[\varepsilon^b\,|X_n(\sigma)-X_{n-1}(\sigma)| + |\nu_n(\sigma)-\nu_{n-1}(\sigma)|\right] + \\ &\quad\quad + \frac{\overline{K}_2}{\tau^2(\sigma)}(|X_n(\sigma)|+|X_{n-1}(\sigma)|)\,|X_n(\sigma)-X_{n-1}(\sigma)|\end{aligned}$$

with, due to the inductive assumption

$$\begin{aligned}&(|X_n(\sigma)|+|X_{n-1}(\sigma)|)\,|X_n(\sigma)-X_{n-1}(\sigma)| \leq 8K_2'\mu\varepsilon^{-b(N+4)}\,|Z_n-Z_{n-1}|_{\tau(\sigma)} \leq \\ &\quad\leq 8K_2'\mu\varepsilon^{-b(N+4)}\,\|S_n-S_{n-1}\|_3\,.\end{aligned}$$

Moreover, since

$$\begin{aligned}&\varepsilon^b\,|X_n(\sigma)-X_{n-1}(\sigma)| + |\nu_n(\sigma)-\nu_{n-1}(\sigma)| \leq \\ &\quad\leq |Z_n-Z_{n-1}|_{\tau(t)} + \varepsilon^{-1/2}\,|\Lambda_n-\Lambda_{n-1}|^{\sqrt{\varepsilon}} < \varepsilon^{-1/2}\,\|S_n-S_{n-1}\|_3\,,\end{aligned}$$

we deduce that (recall that K_2' depends on $\overline{K}_2$)

$$\left| F_1(x^0(\sigma+s), X_n(\sigma), \nu_n(\sigma), \hat{\theta}^0) - F_1(x^0(\sigma+s), X_{n-1}(\sigma), \nu_{n-1}(\sigma), \hat{\theta}^0) \right|_{\tau(\sigma)} \leq$$
$$\leq \frac{10}{\tau(\sigma)} K_2' \mu \varepsilon^{-b(N+4)} (1 + \varepsilon^{b-\frac{1}{2}}) \left\| S_n - S_{n-1} \right\|_3 .$$

Therefore, using the expression (3.2.36) together with Lemma 3.1.5, we obtain

$$|Z_{n+1}(t) - Z_n(t)|_{\tau(t)} \leq$$
$$\leq 10 K_2 K_2' \tilde{K}_1 \mu \varepsilon^{-b(N+4)} (1 + \varepsilon^{b-\frac{1}{2}}) \left(\tau^{-1}(t)\rho(-1) + \tau^2(t)\rho(2) \right) \left\| S_n - S_{n-1} \right\|_3 \leq$$
$$\leq 20 K_2 K_2' \tilde{K}_1 \left(4T_0^2 + \frac{\pi^2}{4} \right) \mu \varepsilon^{-b(N+6)} (1 + \varepsilon^{b-\frac{1}{2}}) \left\| S_n - S_{n-1} \right\|_3 ,$$

where we have also used (3.2.39).

On the other hand, Lemma 3.2.1 (b) also gives

$$\left| f_2(x^0(\sigma+s), X_n(\sigma), \nu_n(\sigma), \hat{\theta}^0) - f_2(x^0(\sigma+s), X_{n-1}(\sigma), \nu_{n-1}(\sigma), \hat{\theta}^0) \right| \leq$$
$$\leq \frac{\overline{K}_2}{\tau^2(\sigma)} \mu \varepsilon^{-b(N+4)} \left(\varepsilon^b \left| X_n(\sigma) - X_{n-1}(\sigma) \right| + \left| \nu_n(\sigma) - \nu_{n-1}(\sigma) \right| \right) \leq$$
$$\leq \frac{\overline{K}_2 \mu \varepsilon^{-b(N+4)-1/2}}{\tau^2(\sigma)} \left\| S_n - S_{n-1} \right\|_3 .$$

Then, Lemma 3.2.2 leads to

$$\begin{aligned} |\Lambda_{n+1}(t) - \Lambda_n(t)|^{\sqrt{\varepsilon}} &\leq \tilde{K}_1 \overline{K}_2 K_2^* \mu \varepsilon^{-b(N+4)-1/2} \rho(1) \left\| S_n - S_{n-1} \right\|_3 < \\ &< \tilde{K}_1 \overline{K}_2 K_2^* \mu \varepsilon^{-b(N+5)-1/2} \left\| S_n - S_{n-1} \right\|_3 . \end{aligned}$$

Hence, since $\mu \in (0, \varepsilon^w)$, with

$$w > \max\{b(N+5) + 1/2, b(N+6)\},$$

we finally get

$$\left\| S_{n+1} - S_n \right\|_3 < \frac{1}{2} \left\| S_n - S_{n-1} \right\|_3 .$$

Then,

$$\left\| S_{n+1} \right\|_3 < 2 \left\| S_1 \right\|_3 < 4 K_2' \mu \varepsilon^{-b(N+4)}.$$

Consequently, we have proved that the solution S of (3.2.33) passing through $S(t_0) = (X(t_0), Y(t_0), \nu(t_0), \alpha(t_0))$ satisfies

$$\left\| S \right\|_3 \leq 4 K_2' \mu \varepsilon^{-b(N+4)}.$$

Therefore, using that $\tau(t) \geq \varepsilon^b$ for every $t \in [-T_0 - \text{Re } s, 2T_0 - \text{Re } s]$, we conclude that

$$\begin{aligned}
|X(t)| &\leq |Z(t)|_{\tau(t)} \leq \|S\|_3 \leq C_2' \mu \varepsilon^{-b(N+4)}, \\
|Y(t)| &\leq \tau^{-1}(t) \, |Z(t)|_{\tau(t)} \leq C_2' \mu \varepsilon^{-b(N+5)}, \\
|\nu(t)| &\leq \varepsilon^{-1/2} \, |\Lambda(t)|^{\sqrt{\varepsilon}} \leq \varepsilon^{-1/2} \, \|S\|_3 \leq C_2' \mu \varepsilon^{-b(N+4)-1/2}
\end{aligned}$$

and

$$|\alpha(t)| \leq |\Lambda(t)|^{\sqrt{\varepsilon}} \leq C_2' \mu \varepsilon^{-b(N+4)}.$$

On the other hand, to obtain the required bound for the norm of $I_1(t) - \beta_1$, we proceed as in the above section (see (3.1.30)) writing

$$|I_1(t) - \beta_1| \leq |I_1(t_0) - \beta_1| + ctant \ \mu \varepsilon^{-b(N+3)} \int_{t_0}^{t} |\cos(x(r)) - 1| \, dr.$$

Thus, using that $x(r) = x^0(r+s) + X(r)$ with $|X(r)| \leq C_2' \mu \varepsilon^{-b(N+4)}$, we have, for μ sufficiently small,

$$|\cos(x(r)) - 1| < ctant \ \tau^{-2}(r).$$

Then, (1.2.40) leads to

$$|I_1(t) - \beta_1| \leq C_2 \mu \varepsilon^{-b(N+3)} + ctant \ \mu \varepsilon^{-b(N+4)} < C_2' \mu \varepsilon^{-b(N+4)}$$

and, assuming $b > 1/4$ (recall that this assumption was already made in (2.1.27)), we finally get

$$\max \{ |X(t)| \, , |Y(t)| \, , |\nu(t)| \, , |\alpha(t)| \, , |I_1(t) - \beta_1| \} \leq C_2' \mu \varepsilon^{-b(N+6)},$$

whenever C_2' is a sufficiently large positive constant depending on C_2, K_2 and T_0. Therefore, the Extension Theorem II is proven. □

Chapter 4

Proofs of the Main Lemmas

The aim of this chapter is to prove the Main Lemma I (see Lemma 1.3.10) and the Main Lemma II (see Lemma 2.3.8), as well as the respective perturbing lemmas, i.e., Lemma 1.3.11 and Lemma 2.3.9.

4.1 Proof of the Main Lemma I

To begin with the proof of the Main Lemma I, let us recall that in the first chapter (see Definition 1.3.7, Definition 1.3.8 and Definition 1.3.9) we have introduced, for positive parameters c, l, d, ε and β, the set

$$\mathcal{S} = \mathcal{S}(c, l, d, \varepsilon, \beta) = \bigcap_{i=1}^{3} \mathcal{S}_i(c, l, d, \varepsilon, \beta)$$

whose elements are series of the type

$$S = \sum_{\hat{k} \in \mathbb{Z}^2 \setminus \{(0,0)\}} S_{\hat{k}} \hat{\mathcal{E}}_{\hat{k}}.$$

According to (1.3.78), for every $\hat{k} = (k_1, k_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, we have

$$\hat{\mathcal{E}}_{\hat{k}} = \hat{\mathcal{E}}_{\hat{k}}(c, l, d, \varepsilon, \beta) = \exp\left(- \left|\hat{k}\right| \left(c \left|\ln \varepsilon\right| + l\right) - \frac{d \left|\hat{k}\omega\right|}{\varepsilon} \right),$$

with

$$\left|\hat{k}\right| = |k_1| + |k_2|, \quad \omega = (1, \beta), \quad \left|\hat{k}\omega\right| = |k_1 + \beta k_2|.$$

From the fact that

$$S \in \mathcal{S}_3(c, l, d, \varepsilon, \beta)$$

we may write, see Definition 1.3.9,

$$S = 2 \sum_{\hat{k} \in \mathbb{Z}_+^2} S_{\hat{k}} \hat{\mathcal{E}}_{\hat{k}}$$

with

$$\mathbb{Z}_+^2 = \{(k_1, k_2) \in \mathbb{Z}^2 : \; k_2 > 0\} \cup \{(k_1, 0) \in \mathbb{Z}^2 : \; k_1 > 0\}.$$

Therefore, in order to look for the dominant terms of the series S it suffices to restrict our study for those indices $\hat{k} = (k_1, k_2)$ for which $k_2 > 0$.

As was sketched in the Appendix of Chapter 1, where we have introduced the definition of best approximations in the same manner as some results related to Continued Fraction Theory, the behaviour of the coefficients $\hat{\mathcal{E}}_{\hat{k}}$ depends on the arithmetic properties of the frequency β. Hence, we deem it wise to start the proof of the Main Lemma I by studying such behaviour and, more concretely, by pointing out that, in order to find the dominant terms of the initial series S, we must pay special attention to those indices $\hat{k} = (k_1, k_2)$ for which $\hat{k} = \hat{k}^{(j)} = (k_1^{(j)}, k_2^{(j)})$ and $k_1^{(j)}/k_2^{(j)}$ is a best approximation to β.

For such indices we have

$$\hat{\mathcal{E}}_{\hat{k}^{(j)}} = \exp\left(- \left(\left| k_1^{(j)} \right| + \left| k_2^{(j)} \right| \right) (c \left| \ln \varepsilon \right| + l) - \frac{d \left| \hat{k}^{(j)} \omega \right|}{\varepsilon} \right)$$

with, see (1.4.124),

$$\left| \hat{k}^{(j)} \omega \right| = \frac{1}{k_2^{(j)} (z_j + x_j)} = \frac{A_j(\beta)}{k_2^{(j)}}$$

and, from (1.4.123) (recall also that $k_1^{(j)} < 0$ and $k_2^{(j)} > 0$),

$$\left| k_1^{(j)} \right| + \left| k_2^{(j)} \right| = -k_1^{(j)} + k_2^{(j)} = k_2^{(j)} (1 + \beta) - (-1)^j \frac{A_j(\beta)}{k_2^{(j)}}.$$

Therefore, if we define the family of functions

$$x \in \mathbb{R}^+ \to \Phi_{\varepsilon,\beta}(x) = - \left(x(1 + \beta) - \frac{B(x)}{x} \right) (c \left| \ln \varepsilon \right| + l) - \frac{d A(x)}{\varepsilon x}, \tag{4.1.1}$$

where

$$\begin{array}{rcl} A(x) & = & A_j(\beta), \quad \text{if } x \in [k_2^{(j)}, k_2^{(j+1)}) \\ B(x) & = & (-1)^j A_j(\beta), \quad \text{if } x \in [k_2^{(j)}, k_2^{(j+1)}), \end{array}$$

then it follows that

$$\hat{\mathcal{E}}_{\hat{k}^{(j)}} = \exp\left(\Phi_{\varepsilon,\beta}(k_2^{(j)}) \right). \tag{4.1.2}$$

Thus, it seems clear that we will obtain a lot of information about the dominant terms of S by maximizing the function $\Phi_{\varepsilon,\beta}$. In fact, as we will see along the arguments, the two dominant terms of S, $S_{\hat{k}(n^0)}\hat{\mathcal{E}}_{\hat{k}(n^0)}$ and $S_{\hat{k}(n^1)}\hat{\mathcal{E}}_{\hat{k}(n^1)}$, are located among those ones whose indices $\hat{k} = (k_1, k_2)$ satisfy

$$k_2 \in \mathcal{I}_\varepsilon = \left(\varepsilon^{-3/8}, \varepsilon^{-1/2} \left|\ln \varepsilon\right|^{-1/4}\right).$$

Hence, it will be enough to restrict the study of the function $\Phi_{\varepsilon,\beta}$ to the interval $\mathcal{I}_\varepsilon$ and, moreover, the global strategy for proving the Main Lemma I is to look for a whole neighbourhood $I_{\tilde{\beta}}$ (as large as possible) of the golden mean value satisfying the following properties:

- If $\beta \in I_{\tilde{\beta}}$, then those best approximations $k_1^{(j)}/k_2^{(j)}$ to β satisfying

$$k_2^{(j)} \leq \varepsilon^{-1/2} \left|\ln \varepsilon\right|^{-1/4}$$

coincide with the best approximations to the golden mean $\tilde{\beta} = \dfrac{\sqrt{5}+1}{2}$. Therefore, $k_2^{(j)}$ coincide with the respective members of the sequence $\{k_2^{(j)}\}_{j\in\mathbb{N}}$ of Fibonacci numbers given by (see also (1.4.120))

$$k_2^{(1)} = 1, \quad k_2^{(2)} = 2, \quad k_2^{(n)} = k_2^{(n-2)} + k_2^{(n-1)}, \quad n > 2.$$

- If $\beta \in I_{\tilde{\beta}}$, then the "behaviour" of $\Phi_{\varepsilon,\beta}$ on $\mathcal{I}_\varepsilon$ essentially coincide with the respective one of $\Phi_{\varepsilon,\tilde{\beta}}$.

It is obvious that the first property will imply that the length of $I_{\tilde{\beta}}$ depends on ε and, moreover, this length goes to zero as ε tends to zero. But, on the other hand, the second property implies that, by moving β along $I_{\tilde{\beta}}$, we only slightly perturb the function $\Phi_{\varepsilon,\beta}$ on $\mathcal{I}_\varepsilon$ and, in particular, these facts will allow us to get some uniformity in the obtained results (note that the statement of the Main Lemma I guarantees that the dominant terms of S are localized at best approximations to the golden mean, independently of the value of $\beta \in I_{\tilde{\beta}}$).

Therefore, let us describe how the neighbourhood $I_{\tilde{\beta}}$ (depending on ε) of the golden mean is constructed. We are going to do that for any small enough value of ε although in forthcoming arguments we must restrict the range of ε to some open real subset $\mathcal{U}_\varepsilon$. We also point out that the construction of those neighbourhoods $I_{\tilde{\beta}}$ does not depend on the value of the parameters c, l and d.

Once a value of ε is fixed, let us take $k_2^{(m)}$ the minimum natural number satisfying the following properties:

- There exists $k_1 = k_1^{(m)} \in \mathbb{Z}$ such that $k_1^{(m)}/k_2^{(m)}$ is a best approximation to the golden mean, i.e., $k_2^{(m)}$ is a Fibonacci number.
- $k_2^{(m)} \geq \varepsilon^{-1/2} \left|\ln \varepsilon\right|^{-1/4}$.

Let us observe that, for the golden mean $\tilde{\beta}$, we have (see (1.4.121))

$$\tilde{\beta} = [a_0, a_1, \ldots, a_m, z_m(\tilde{\beta})]$$

with $a_i = 1$, $0 \le i \le m$ and $z_m(\tilde{\beta}) = \tilde{\beta}$.

Then, we consider all the real numbers

$$\beta = [a_0, a_1, \ldots, a_m, z_m(\beta)]$$

such that

$$a_i = 1, \ 0 \le i \le m \quad \text{and} \quad \left| z_m(\beta) - \tilde{\beta} \right| \le \varepsilon^{2/3}.$$

If we denote by $I_{\tilde{\beta}} = I_{\tilde{\beta}}(\varepsilon)$ the neighbourhood of $\tilde{\beta}$ constructed in this way, we may prove the following results:

Lemma 4.1.1 *It follows that*

$$\frac{1}{100}\varepsilon^{5/3} \left|\ln \varepsilon\right|^{1/2} \le \mathit{length}(I_{\tilde{\beta}}) \le \frac{1}{2}\varepsilon^{5/3} \left|\ln \varepsilon\right|^{1/2}.$$

Proof

Note that, if β belongs to $I_{\tilde{\beta}}$ then, from (1.4.120), $k_1^{(j)}$ and $k_2^{(j)}$ do not depend on β for $1 \le j \le m$. Then, see (1.4.126) for details, for any number

$$\beta_1 = [a_0, a_1, \ldots, a_m, z_m(\beta_1)]$$

in $I_{\tilde{\beta}}$, we have

$$\left|\beta_1 - \tilde{\beta}\right| = \frac{\left|z_m(\beta_1) - \tilde{\beta}\right|}{\left|(k_2^{(m)} z_m(\beta_1) + k_2^{(m-1)})(k_2^{(m)}\tilde{\beta} + k_2^{(m-1)})\right|}.$$

On the other hand, since $k_1^{(j)}/k_2^{(j)}$, $1 \le j \le m$, are best approximations to the golden mean, it follows that $k_2^{(m)} = -k_1^{(m-1)}$ and therefore, for ε small enough, $k_2^{(m)} < 2\tilde{\beta}k_2^{(m-1)}$ while, by definition of m, $k_2^{(m)} > \varepsilon^{-1/2}\left|\ln \varepsilon\right|^{-1/4}$. Hence,

$$\left|\beta_1 - \tilde{\beta}\right| \le \frac{\left|z_m(\beta_1) - \tilde{\beta}\right|}{(2 + \tilde{\beta}^{-1})(k_2^{(m)})^2} \le \frac{1}{2}\varepsilon^{5/3}\left|\ln \varepsilon\right|^{1/2},$$

for every $\beta_1 \in I_{\tilde{\beta}}$.

Moreover, since $k_2^{(m)} \le 2\tilde{\beta}k_2^{(m-1)}$ and $k_2^{(m-1)} < \varepsilon^{-1/2}\left|\ln \varepsilon\right|^{-1/4}$, we also have

$$\left|\beta_1 - \tilde{\beta}\right| \ge \frac{\varepsilon\left|z_m(\beta_1) - \tilde{\beta}\right|\left|\ln \varepsilon\right|^{1/2}}{(1 + 4\tilde{\beta})^2}.$$

Then, taking β_1 with $\left|z_m(\beta_1) - \tilde{\beta}\right| = \varepsilon^{2/3}$ it follows that

$$\text{length}(I_{\tilde{\beta}}) \ge \frac{1}{100}\varepsilon^{5/3}\left|\ln \varepsilon\right|^{1/2}.$$

□

Remark 4.1.2 *Let us observe that, using (1.4.126) in a direct way, we can slightly improve the above estimates on the length of $I_{\tilde\beta}$ by proving that, for $I^r_{\tilde\beta} = \left\{\beta \in I_{\tilde\beta} : \ \beta > \tilde\beta\right\}$ and $I^l_{\tilde\beta} = I_{\tilde\beta} \backslash I^r_{\tilde\beta}$ one has*

$$\frac{1}{100}\varepsilon^{5/3} \left|\ln \varepsilon\right|^{1/2} \le length(I^\delta_{\tilde\beta}) \le \frac{1}{2}\varepsilon^{5/3} \left|\ln \varepsilon\right|^{1/2},$$

for $\delta = r$ and $\delta = l$.

Lemma 4.1.3 *For every $\beta, \beta' \in I_{\tilde\beta}$ and every best approximation $k_1^{(j)}/k_2^{(j)}$ to the golden mean satisfying $k_2^{(j)} \le \varepsilon^{-1/2} \left|\ln \varepsilon\right|^{-1/4}$, it follows that*

$$\left|\Phi_{\varepsilon,\beta}(k_2^{(j)}) - \Phi_{\varepsilon,\beta'}(k_2^{(j)})\right| \le \frac{1}{2} d\varepsilon^{1/6} \left|\ln \varepsilon\right|^{1/4},$$

where $\Phi_{\varepsilon,\beta}$ are the transformations defined at (4.1.1).

Proof

Let us observe that, since $k_1^{(j)}/k_2^{(j)}$ is a best approximation to β' and β, we may write

$$\Phi_{\varepsilon,\beta}(k_2^{(j)}) - \Phi_{\varepsilon,\beta'}(k_2^{(j)}) = \frac{d}{\varepsilon}\left(\left|\hat k^{(j)}\omega'\right| - \left|\hat k^{(j)}\omega\right|\right)$$

where $\omega' = (1, \beta')$ and $\omega = (1, \beta)$. Hence

$$\Phi_{\varepsilon,\beta}(k_2^{(j)}) - \Phi_{\varepsilon,\beta'}(k_2^{(j)}) = \frac{d}{\varepsilon}\left(\frac{A_j(\beta')}{k_2^{(j)}} - \frac{A_j(\beta)}{k_2^{(j)}}\right) = \frac{(-1)^j d}{\varepsilon} k_2^{(j)}(\beta - \beta')$$

where we have used that (see (1.4.127))

$$A_j(\beta) - A_j(\beta') = (-1)^j (k_2^{(j)})^2 (\beta - \beta'). \tag{4.1.3}$$

Then, the result follows from the condition $k_2^{(j)} < \varepsilon^{-1/2} \left|\ln \varepsilon\right|^{-1/4}$ together with Lemma 4.1.1. □

As we said before, in order to prove the Main Lemma I, it will be very important to maximize the functions $\Phi_{\varepsilon,\beta}$. We start by pointing out that, since we pretend to localize the maximum of $\Phi_{\varepsilon,\beta}$ by solving the equation $\dot\Phi_{\varepsilon,\beta} \equiv 0$, the presence of points of discontinuity implies that a direct approach to the problem would not be fruitful.

Hence, let us define the family of continuous functions

$$x \in \mathbb{R}^+ \to \Phi_{1,\varepsilon,\beta}(x) = -x(1+\beta)(c\left|\ln \varepsilon\right| + l) - \frac{d\tilde A}{\varepsilon x}, \tag{4.1.4}$$

where

$$\tilde A = \frac{1}{\tilde\beta + \tilde\beta^{-1}}.$$

Lemma 4.1.4 *For any positive* ε *and every* $\beta \in I_{\tilde{\beta}}(\varepsilon)$*, it follows that*

$$|\Phi_{\varepsilon,\beta}(x) - \Phi_{1,\varepsilon,\beta}(x)| \leq ctant\ \varepsilon^{1/24},$$

whenever $x \in \mathcal{I}_\varepsilon = \left(\varepsilon^{-3/8}, \varepsilon^{-1/2}\,|\ln\varepsilon|^{-1/4}\right)$.

Proof

Let us observe that

$$\Phi_{\varepsilon,\beta}(x) - \Phi_{1,\varepsilon,\beta}(x) = \frac{B(x)}{x}\,(c\,|\ln\varepsilon| + l) + \frac{d}{\varepsilon x}(\tilde{A} - A(x)),$$

where, by definition, $A(x) = A_j(\beta)$, $B(x) = (-1)^j A_j(\beta)$, if $x \in [k_2^{(j)}, k_2^{(j+1)})$.

Now, (1.4.125) implies that, for every $x \in \mathcal{I}_\varepsilon$, $|B(x)| < ctant$, and, furthermore, using also (4.1.3) and Lemma 4.1.1, we get

$$\left|\tilde{A} - A(x)\right| \leq \left|\tilde{A} - A_j(\tilde{\beta})\right| + \left|A_j(\tilde{\beta}) - A_j(\beta)\right| \leq \varepsilon^{2/3},$$

for every $\beta \in I_{\tilde{\beta}}(\varepsilon)$ and any $x \in \mathcal{I}_\varepsilon$.

Therefore, for every $x \in \mathcal{I}_\varepsilon$,

$$|\Phi_{\varepsilon,\beta}(x) - \Phi_{1,\varepsilon,\beta}(x)| \leq ctant\ \varepsilon^{3/8}\,|\ln\varepsilon| + ctant\ \varepsilon^{1/24} \leq ctant\ \varepsilon^{1/24}.$$

□

In this way, for ε sufficiently small, the behaviour of $\Phi_{\varepsilon,\beta}$ over

$$\mathcal{I}_\varepsilon = \left(\varepsilon^{-3/8}, \varepsilon^{-1/2}\,|\ln\varepsilon|^{-1/4}\right)$$

can be approximated by the respective one of $\Phi_{1,\varepsilon,\beta}$. In particular, in order to choose our two candidates for giving the leading order behaviour of the whole initial series S, we compute the absolute maximum of $\Phi_{1,\varepsilon,\tilde{\beta}}$, which is easily localized at

$$x_\varepsilon = x_\varepsilon(d,c,l) = \left(\frac{d\tilde{A}}{(1+\tilde{\beta})(c\,|\ln\varepsilon| + l)\varepsilon}\right)^{1/2}, \quad \tilde{A} = \frac{1}{\tilde{\beta} + \tilde{\beta}^{-1}}. \tag{4.1.5}$$

It is clear that, once a sufficiently small value of ε is fixed, not only $x_\varepsilon \in \mathcal{I}_\varepsilon$, but also there exist two consecutive Fibonacci numbers $k_2^{(n)}$, $k_2^{(n+1)}$ such that

$$x_\varepsilon \in [k_2^{(n)}, k_2^{(n+1)}) \subset \mathcal{I}_\varepsilon. \tag{4.1.6}$$

Moreover, let us observe that $\lim_{\varepsilon\to 0} x_\varepsilon = \infty$ and therefore the continuity of $\Phi_{1,\varepsilon,\tilde{\beta}}$ implies that, if we choose the complete sequence $\{k_2^{(n)}\}_{n\in\mathbb{N}}$ of Fibonacci numbers, then we find a sequence $\{\tilde{\varepsilon}_n\}_{n\in\mathbb{N}}$ for which

$$\Phi_{1,\tilde{\varepsilon}_n,\tilde{\beta}}(k_2^{(n)}) = \Phi_{1,\tilde{\varepsilon}_n,\tilde{\beta}}(k_2^{(n+1)}) \tag{4.1.7}$$

and $\lim_{n\to\infty}\tilde{\varepsilon}_n = 0$. Let us choose n^* a sufficiently large natural number in such a way that any argument (used for proving the Main Theorem I) related to the smallness of ε is fulfilled whenever

$$\varepsilon < \tilde{\varepsilon}_{n^*}. \tag{4.1.8}$$

Now we are going to construct the open real subset $\mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon(c, l, d)$ announced by the Main Lemma I.

We begin by choosing the decreasing sequence $\{\overline{\varepsilon}_n\}_{n\in\mathbb{N}}$ of values of ε for which (see (4.1.5))

$$x_{\overline{\varepsilon}_n} = \left(\frac{d\tilde{A}}{(1+\tilde{\beta})(c\,|\ln\overline{\varepsilon}_n| + l)\overline{\varepsilon}_n}\right)^{1/2} = k_2^{(n)} \tag{4.1.9}$$

and by observing that, for every $n \geq n^*$, $\overline{\varepsilon}_{n+1} < \tilde{\varepsilon}_n < \overline{\varepsilon}_n$. Then, the subset $\mathcal{U}_\varepsilon$ is going to be constructed as

$$\mathcal{U}_\varepsilon = \bigcup_{n\geq n^*} \mathcal{U}_n, \tag{4.1.10}$$

where $\mathcal{U}_n$ are certain open real intervals to be constructed later on with $\tilde{\varepsilon}_n \in \mathcal{U}_n \subset (\overline{\varepsilon}_{n+1}, \overline{\varepsilon}_n)$. In fact, if $\varepsilon \in \mathcal{U}_\varepsilon$, then we are going to prove that the indices n^0 and n^1, for which the Main Lemma I holds, are given by $n^0 = n$, $n^1 = n+1$, where n is the unique natural number satisfying $\varepsilon \in \mathcal{U}_n$.

Remark 4.1.5 *The indices n and $n+1$ (and therefore the indices n^0 and n^1 announced in the statement of the Main Lemma I) are determined by the condition (see (4.1.6)) $x_\varepsilon \in [k_2^{(n)}, k_2^{(n+1)})$. Hence, the indices n^0 and n^1 only depend, according to (4.1.5), on the parameters c, l, d and ε.*

Now, let us explain how to construct each one of the real subsets $\mathcal{U}_n$: Let us define

$$\Pi_n = \left\{(\varepsilon, \beta) \in \mathbb{R}^2 : \ \varepsilon \in [\overline{\varepsilon}_{n+1}, \overline{\varepsilon}_n]\right\}$$

and consider the family of maps (recall that $\{k_2^{(n)}\}_{n\in\mathbb{N}}$ denotes the sequence of Fibonacci numbers)

$$G_n : (\varepsilon, \beta) \in \Pi_n \to G_n(\varepsilon, \beta) = \Phi_{1,\varepsilon,\beta}(k_2^{(n)}) - \Phi_{1,\varepsilon,\beta}(k_2^{(n+1)}).$$

Then, (4.1.7) implies that $G_n(\tilde{\varepsilon}_n, \tilde{\beta}) = 0$, and moreover, if we define

$$\Lambda_n : \varepsilon \in [\overline{\varepsilon}_{n+1}, \overline{\varepsilon}_n] \to \Lambda_n(\varepsilon) = \frac{d\tilde{A}}{(c\,|\ln\varepsilon| + l)\varepsilon k_2^{(n)} k_2^{(n+1)}} - 1, \tag{4.1.11}$$

then (see (4.1.4)), $G_n(\varepsilon, \Lambda_n(\varepsilon)) = 0$, for every $\varepsilon \in [\overline{\varepsilon}_{n+1}, \overline{\varepsilon}_n]$ and every $n \geq n^*$.

On the other hand, for every $\varepsilon \in (\overline{\varepsilon}_{n+1}, \overline{\varepsilon}_n)$, we have

$$\Lambda_n'(\varepsilon) = -\frac{d\tilde{A}(c\,|\ln\varepsilon| + l - c)}{(c\,|\ln\varepsilon| + l)^2 \varepsilon^2 k_2^{(n)} k_2^{(n+1)}}$$

and thus, since for any $\varepsilon \in [\overline{\varepsilon}_{n+1}, \overline{\varepsilon}_n]$ and any n large enough it holds that

$$\frac{1}{2} k_2^{(n+1)} \leq k_2^{(n)} \leq x_\varepsilon \leq k_2^{(n+1)} \leq 2k_2^{(n)},$$

where x_ε was defined in (4.1.5), it is easy to see that there exists some constant $A > 1$ for which

$$\frac{1}{A\varepsilon} \leq |\Lambda_n'(\varepsilon)| \leq \frac{A}{\varepsilon}, \tag{4.1.12}$$

for any $\varepsilon \in (\overline{\varepsilon}_{n+1}, \overline{\varepsilon}_n)$.

Therefore, Λ_n is a strongly decreasing function satisfying $\Lambda_n(\tilde{\varepsilon}_n) = \tilde{\beta}$, and (see (4.1.9))

$$\begin{aligned} 3 < \Lambda_n(\overline{\varepsilon}_{n+1}) &= \frac{k_2^{(n+1)}}{k_2^{(n)}}(1+\tilde{\beta}) - 1 < 4, \\ \frac{1}{2} < \Lambda_n(\overline{\varepsilon}_n) &= \frac{k_2^{(n)}}{k_2^{(n+1)}}(1+\tilde{\beta}) - 1 < \frac{3}{4}. \end{aligned} \tag{4.1.13}$$

Then, we deduce the existence of a neighbourhood $\mathcal{U}_n$ of $\tilde{\varepsilon}_n$, $\mathcal{U}_n \subset (\overline{\varepsilon}_{n+1}, \overline{\varepsilon}_n)$, which is characterized by the condition: $\varepsilon \in \mathcal{U}_n$ if and only if $\Lambda_n(\varepsilon) \in I_{\tilde{\beta}}(\varepsilon)$, where $I_{\tilde{\beta}}(\varepsilon)$ is the already constructed neighbourhood of the golden mean.

Remark 4.1.6 *Once a value of n is fixed, the function Λ_n, defined at (4.1.11), depends on the parameters c, l and d. Therefore, the same holds for the set $\mathcal{U}_n$:*

$$\mathcal{U}_n = \mathcal{U}_n(c, l, d).$$

Now, let

$$\mathcal{U}_\varepsilon = \bigcup_{n \geq n^*} \mathcal{U}_n \subset (0, \overline{\varepsilon}_{n^*}],$$

with n^* given by condition (4.1.8), let us denote $\mathcal{L}$ the Lebesgue measure on $\mathbb{R}$ and define

$$F : \varepsilon \in \mathbb{R}^+ \to F(\varepsilon) = \varepsilon\,|\ln \varepsilon| \in \mathbb{R}^+.$$

Proposition 4.1.7 *Let $\varepsilon_0 = \overline{\varepsilon}_{n^*}$ with $\overline{\varepsilon}_{n^*}$ given in (4.1.9). Then, it holds that*

$$ctant\ \varepsilon_0^{8/3} |\ln \varepsilon_0|^{3/2} < \mathcal{L}(F(\mathcal{U}_\varepsilon)) < ctant\ \varepsilon_0^{8/3} |\ln \varepsilon_0|^{8/3}.$$

Proof

For any $n \geq n^*$ let $(a_n, b_n) = \mathcal{U}_n$ and let us observe that the definition of $\mathcal{U}_n$ and Remark 4.1.2 imply

$$\begin{aligned} \frac{1}{100}\left(a_n^{5/3}|\ln a_n|^{1/2} + b_n^{5/3}|\ln b_n|^{1/2}\right) \leq \Lambda_n(a_n) - \Lambda_n(b_n) \leq \\ \leq \frac{1}{2}\left(a_n^{5/3}|\ln a_n|^{1/2} + b_n^{5/3}|\ln b_n|^{1/2}\right). \end{aligned}$$

Let us take a positive constant $\alpha = \alpha(\varepsilon_0)$ satisfying $\varepsilon_0^{\alpha} \left|\ln \varepsilon_0\right|^{\frac{7}{6}+\alpha} = 1$. Note that, since $\alpha > 0$, then for every $n \geq n^*$, we may write (recall that $\overline{\varepsilon}_{n+1} < a_n < b_n < \overline{\varepsilon}_n$)

$$\frac{1}{50}(\overline{\varepsilon}_{n+1})^{\frac{5}{3}+\alpha} \left|\ln \overline{\varepsilon}_{n+1}\right|^{\frac{5}{3}+\alpha} \leq \Lambda_n(a_n) - \Lambda_n(b_n) \leq (\overline{\varepsilon}_n)^{5/3} \left|\ln \overline{\varepsilon}_n\right|^{5/3}.$$

Now, let $t = F(\varepsilon) = \varepsilon \left|\ln \varepsilon\right|$ and denote, for any $n \geq n^*$,

$$\overline{t}_n = F(\overline{\varepsilon}_n), \quad \overline{a}_n = F(a_n), \quad \overline{b}_n = F(b_n).$$

Then, if we define

$$\overline{\Lambda}_n : t \in [\overline{t}_{n+1}, \overline{t}_n] \to \overline{\Lambda}_n(t) = \Lambda_n(F^{-1}(t)),$$

we know that

$$\frac{1}{50}(\overline{t}_{n+1})^{\frac{5}{3}+\alpha} \leq \overline{\Lambda}_n(\overline{a}_n) - \overline{\Lambda}_n(\overline{b}_n) \leq (\overline{t}_n)^{5/3}$$

and, moreover, from (4.1.13),

$$2 \leq \overline{\Lambda}_n(\overline{t}_{n+1}) - \overline{\Lambda}_n(\overline{t}_n) \leq 4.$$

On the other hand, since

$$\overline{\Lambda}'_n(t) = \Lambda'_n(F^{-1}(t)) \frac{1}{\left|\ln F^{-1}(t)\right| - 1}$$

we have that (4.1.12) gives

$$\frac{1}{A(t - F^{-1}(t))} \leq \left|\overline{\Lambda}'_n(t)\right| \leq \frac{A}{t - F^{-1}(t)},$$

for every $t \in (\overline{t}_{n+1}, \overline{t}_n)$.

Hence, since there exist $c_n \in (\overline{a}_n, \overline{b}_n)$, $d_n \in (\overline{t}_{n+1}, \overline{t}_n)$ such that

$$\frac{\overline{\Lambda}_n(\overline{a}_n) - \overline{\Lambda}_n(\overline{b}_n)}{\overline{\Lambda}_n(\overline{t}_{n+1}) - \overline{\Lambda}_n(\overline{t}_n)} = \frac{\left|\overline{\Lambda}'_n(c_n)\right|}{\left|\overline{\Lambda}'_n(d_n)\right|} \frac{\overline{b}_n - \overline{a}_n}{\overline{t}_n - \overline{t}_{n+1}}, \tag{4.1.14}$$

we get

$$\begin{aligned} \frac{1}{200}(\overline{t}_{n+1})^{\frac{5}{3}+\alpha} &\leq \frac{A^2(d_n - F^{-1}(d_n))}{c_n - F^{-1}(c_n)} \frac{\overline{b}_n - \overline{a}_n}{\overline{t}_n - \overline{t}_{n+1}} \leq 2A^2 \frac{d_n}{c_n} \frac{\overline{b}_n - \overline{a}_n}{\overline{t}_n - \overline{t}_{n+1}} < \\ &< 2A^2 \frac{\overline{t}_n}{\overline{t}_{n+1}} \frac{\overline{b}_n - \overline{a}_n}{\overline{t}_n - \overline{t}_{n+1}}, \end{aligned}$$

where we have used that, for every $t \in (0, \overline{t}_{n^*}]$, $t - F^{-1}(t) > \frac{1}{2}t$. Now, since for every $n \geq n^*$, $\overline{t}_n = \overline{\varepsilon}_n \left|\ln \overline{\varepsilon}_n\right|$, from (4.1.9) it follows that

$$2 < \frac{\overline{t}_n}{\overline{t}_{n+1}} < 4.$$

Let us remark that the above uniform bounds for $\dfrac{\overline{t}_n}{\overline{t}_{n+1}}$ are not satisfied by replacing $\overline{t}_j$ by ε_j and this is the reason why we must introduce the transformation $F(\varepsilon) = \varepsilon\,|\ln \varepsilon|$.

Therefore, for any $n \geq n^*$,

$$\overline{b}_n - \overline{a}_n > ctant\ (\overline{t}_{n+1})^{\frac{5}{3}+\alpha}(\overline{t}_n - \overline{t}_{n+1}) > ctant\ (\overline{t}_n)^{\frac{8}{3}+\alpha}$$

and thus

$$\begin{aligned}\mathcal{L}(F(\mathcal{U}_\varepsilon)) &> \mathcal{L}(F(\mathcal{U}_{n^*})) = \overline{b}_{n^*} - \overline{a}_{n^*} > ctant(\overline{t}_{n^*})^{\frac{8}{3}+\alpha} = \\ &= ctant\ \varepsilon_0^{\frac{8}{3}+\alpha}\,|\ln \varepsilon_0|^{\frac{8}{3}+\alpha} = ctant\ \varepsilon_0^{8/3}\,|\ln \varepsilon|^{3/2}.\end{aligned}$$

From (4.1.14) we also deduce

$$\begin{aligned}\frac{1}{2}(\overline{t}_n)^{5/3} &> \frac{d_n - F^{-1}(d_n)}{A^2(c_n - F^{-1}(c_n))}\frac{\overline{b}_n - \overline{a}_n}{\overline{t}_n - \overline{t}_{n+1}} > \frac{1}{2A^2}\frac{d_n}{c_n}\frac{\overline{b}_n - \overline{a}_n}{\overline{t}_n - \overline{t}_{n+1}} > \\ &> \frac{1}{2A^2}\frac{\overline{t}_{n+1}}{\overline{t}_n}\frac{\overline{b}_n - \overline{a}_n}{\overline{t}_n - \overline{t}_{n+1}} > \frac{1}{8A^2}\frac{\overline{b}_n - \overline{a}_n}{\overline{t}_n - \overline{t}_{n+1}}.\end{aligned}$$

Doing so, for any $n \geq n^*$, we derive

$$\overline{b}_n - \overline{a}_n < ctant\ (\overline{t}_n)^{5/3}(\overline{t}_n - \overline{t}_{n+1}) < ctant\ (\overline{t}_n)^{8/3}.$$

Hence, for any $j \geq 0$,

$$\overline{b}_{n^*+j} - \overline{a}_{n^*+j} < ctant\ (\overline{t}_{n^*+j})^{8/3} < ctant\ \left(\frac{1}{4^{8/3}}\right)^j (\overline{t}_{n^*})^{8/3},$$

from which we finally arrive at

$$\mathcal{L}(F(\mathcal{U}_\varepsilon)) = \sum_{j=0}^{\infty}(\overline{b}_{n^*+j} - \overline{a}_{n^*+j}) \leq ctant\ \varepsilon_0^{8/3}\,|\ln \varepsilon_0|^{8/3}$$

and therefore the proposition is proved. □

Once the set $\mathcal{U}_\varepsilon$ was constructed we present the following corollary which is used to prove the Main Theorem I. Before stating it, we introduce the following notation: Let us fix the parameters c, l and d and write, for any $\varepsilon \in \mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon(c, l, d)$ and any $\beta \in I_{\tilde{\beta}}(\varepsilon)$,

$$\hat{\mathcal{E}}_{\hat{k}} = \hat{\mathcal{E}}_{\hat{k}}(c, l, d, \varepsilon, \beta) = \exp\left(-\left|\hat{k}\right|(c\,|\ln \varepsilon| + l) - \frac{d\left|\hat{k}\omega\right|}{\varepsilon}\right) = \hat{\mathcal{E}}_{\hat{k}}(\varepsilon, \beta).$$

Moreover, let us recall that, if $n \geq n^*$ is the (unique) natural number for which $\varepsilon \in \mathcal{U}_n$, then

$$x_\varepsilon \in [k_2^{(n)}, k_2^{(n+1)}) \subset \mathcal{I}_\varepsilon = \left(\varepsilon^{-3/8}, \varepsilon^{-1/2}\,|\ln \varepsilon|^{-1/4}\right),$$

where x_ε was defined in (4.1.5) and $\{k_2^{(j)}\}_{j\in\mathbb{N}}$ is the sequence of Fibonacci numbers. Finally, we also recall that, for any $\varepsilon \in \mathcal{U}_\varepsilon$ and every $\beta \in I_{\tilde{\beta}}(\varepsilon)$, the best approximations $k_1^{(j)}/k_2^{(j)}$ to β satisfying $k_2^{(j)} \in \mathcal{I}_\varepsilon$ are exactly those ones to the golden mean.

Corollary 4.1.8 *For every* $\varepsilon \in \mathcal{U}_\varepsilon(c,l,d)$ *and any* $\beta \in I_{\tilde{\beta}}(\varepsilon)$*, it holds that*

$$\frac{1}{4} \le \frac{\hat{\mathcal{E}}_{\hat{k}^{(n)}}(\varepsilon,\beta)}{\hat{\mathcal{E}}_{\hat{k}^{(n+1)}}(\varepsilon,\beta)} \le 4,$$

where $n \ge n^*$ *is the natural number for which* $\varepsilon \in \mathcal{U}_n$*.*

Proof

Let us observe that, by definition of $\mathcal{U}_n$, we have $\Lambda_n(\varepsilon) \in I_{\tilde{\beta}}(\varepsilon)$ and

$$\Phi_{1,\varepsilon,\Lambda_n(\varepsilon)}(k_2^{(n)}) = \Phi_{1,\varepsilon,\Lambda_n(\varepsilon)}(k_2^{(n+1)}).$$

Hence, Lemma 4.1.3 and Lemma 4.1.4 imply, for every $\beta \in I_{\tilde{\beta}}(\varepsilon)$, that

$$\left|\Phi_{\varepsilon,\beta}(k_2^{(n)}) - \Phi_{\varepsilon,\beta}(k_2^{(n+1)})\right| < ctant\ \varepsilon^{1/24},$$

where, from (4.1.2),

$$\exp\left(\Phi_{\varepsilon,\beta}(k_2^{(j)})\right) = \hat{\mathcal{E}}_{\hat{k}^{(j)}}(\varepsilon,\beta),$$

for $j = n,\ n+1$. Hence, the corollary follows easily. □

Lemma 4.1.9 *For every* $\varepsilon \in \mathcal{U}_\varepsilon$ *and any* $\beta \in I_{\tilde{\beta}}(\varepsilon)$ *it follows that*

$$\begin{aligned} \Phi_{\varepsilon,\beta}(k_2^{(n)}) - \Phi_{\varepsilon,\beta}(k_2^{(j)}) &> ctant\ \varepsilon^{-1/2}\,|\ln\varepsilon|^{1/2} \\ \Phi_{\varepsilon,\beta}(k_2^{(n+1)}) - \Phi_{\varepsilon,\beta}(k_2^{(j)}) &> ctant\ \varepsilon^{-1/2}\,|\ln\varepsilon|^{1/2} \end{aligned}$$

for every Fibonacci number $k_2^{(j)} \in \mathcal{I}_\varepsilon = (\varepsilon^{-3/8}, \varepsilon^{-1/2}\,|\ln\varepsilon|^{-1/4})$*, with* $j \ne n$*,* $j \ne n+1$ *and* n *the natural number for which* $\varepsilon \in \mathcal{U}_n$*.*

Proof

Once again, we use the definition of $\mathcal{U}_n$ to assert that $\Lambda_n(\varepsilon) \in I_{\tilde{\beta}}(\varepsilon)$. Let us begin the proof by replacing the function $\Phi_{\varepsilon,\beta}$ by $\Phi_{1,\varepsilon,\Lambda_n(\varepsilon)}$. Observe that the condition

$$\Phi_{1,\varepsilon,\Lambda_n(\varepsilon)}(k_2^{(n)}) = \Phi_{1,\varepsilon,\Lambda_n(\varepsilon)}(k_2^{(n+1)}) \tag{4.1.15}$$

implies that the critical point c_ε of $\Phi_{1,\varepsilon,\Lambda_n(\varepsilon)}$ belongs to $(k_2^{(n)}, k_2^{(n+1)})$. Then, a Taylor expansion gives

$$\Phi_{1,\varepsilon,\Lambda_n(\varepsilon)}(k_2^{(n)}) = \Phi_{1,\varepsilon,\Lambda_n(\varepsilon)}(c_\varepsilon) + \frac{1}{2}\ddot{\Phi}_{1,\varepsilon,\Lambda_n(\varepsilon)}(y_n)(k_2^{(n)} - c_\varepsilon)^2 \tag{4.1.16}$$

and

$$\Phi_{1,\varepsilon,\Lambda_n(\varepsilon)}(k_2^{(n+1)}) = \Phi_{1,\varepsilon,\Lambda_n(\varepsilon)}(c_\varepsilon) + \frac{1}{2}\ddot{\Phi}_{1,\varepsilon,\Lambda_n(\varepsilon)}(z_n)(k_2^{(n+1)} - c_\varepsilon)^2, \tag{4.1.17}$$

where $k_2^{(n)} \le y_n \le c_\varepsilon \le z_n \le k_2^{(n+1)}$.

Now, since the definition of $\Phi_{1,\varepsilon,\Lambda_n(\varepsilon)}$ (see (4.1.4)) implies

$$\ddot{\Phi}_{1,\varepsilon,\Lambda_n(\varepsilon)}(x) = -\frac{2d\tilde{A}}{\varepsilon x^3},$$

we obtain that there exist two constants c_1 and c_2 for which

$$c_1\varepsilon^{1/2}\left|\ln\varepsilon\right|^{3/2} \le \left|\ddot{\Phi}_{1,\varepsilon,\Lambda_n(\varepsilon)}(x)\right| \le c_2\varepsilon^{1/2}\left|\ln\varepsilon\right|^{3/2},$$

for any $x \in [k_2^{(n)}, k_2^{(n+1)}]$. Let us observe that those bounds follow by bearing in mind that, since

$$\frac{1}{2}k_2^{(n+1)} \le k_2^{(n)} \le x_\varepsilon \le k_2^{(n+1)} \le 2k_2^{(n)},$$

where x_ε was given in (4.1.5), then, for $t = n, n+1$, we have

$$\text{ctant } \varepsilon^{-1/2}\left|\ln\varepsilon\right|^{-1/2} \le k_2^{(t)} \le \text{ctant } \varepsilon^{-1/2}\left|\ln\varepsilon\right|^{-1/2}. \tag{4.1.18}$$

Then, using (4.1.15), (4.1.16) and (4.1.17), we get the existence of some constant $\tilde{A} > 1$ such that

$$\frac{1}{\tilde{A}} < \frac{k_2^{(n+1)} - c_\varepsilon}{c_\varepsilon - k_2^{(n)}} < \tilde{A}.$$

Hence, we deduce that $k_2^{(n+1)} - k_2^{(n)} < (1+\tilde{A})(c_\varepsilon - k_2^{(n)})$. Now, it is easy to see that

$$\begin{aligned}
&\Phi_{1,\varepsilon,\Lambda_n(\varepsilon)}(k_2^{(n)}) - \Phi_{1,\varepsilon,\Lambda_n(\varepsilon)}(k_2^{(n-1)}) > \\
&\quad > \dot{\Phi}_{1,\varepsilon,\Lambda_n(\varepsilon)}(k_2^{(n)})(k_2^{(n)} - k_2^{(n-1)}) = \left|\ddot{\Phi}_{1,\varepsilon,\Lambda_n(\varepsilon)}(\tilde{y}_n)\right|(c_\varepsilon - k_2^{(n)})(k_2^{(n)} - k_2^{(n-1)}) > \\
&\quad\quad > ctant\ \varepsilon^{1/2}\left|\ln\varepsilon\right|^{3/2}(k_2^{(n+1)} - k_2^{(n)})(k_2^{(n)} - k_2^{(n-1)}) > \\
&\quad\quad\quad > ctant\ \varepsilon^{1/2}\left|\ln\varepsilon\right|^{3/2}(k_2^{(n-1)})^2 > ctant\ \varepsilon^{-1/2}\left|\ln\varepsilon\right|^{1/2}.
\end{aligned}$$

Of course, we may also check that

$$\Phi_{1,\varepsilon,\Lambda_n(\varepsilon)}(k_2^{(n+1)}) - \Phi_{1,\varepsilon,\Lambda_n(\varepsilon)}(k_2^{(n+2)}) > ctant\ \varepsilon^{-1/2}\left|\ln\varepsilon\right|^{1/2}.$$

Therefore, for any $j \ne n$, $j \ne n+1$, we have

$$\Phi_{1,\varepsilon,\Lambda_n(\varepsilon)}(k_2^{(*)}) - \Phi_{1,\varepsilon,\Lambda_n(\varepsilon)}(k_2^{(j)}) > ctant\ \varepsilon^{-1/2}\left|\ln\varepsilon\right|^{1/2},$$

where $*$ means n or $n+1$. Now, since Lemma 4.1.4 implies

$$\left|\Phi_{\varepsilon,\Lambda_n(\varepsilon)}(k_2^{(h)}) - \Phi_{1,\varepsilon,\Lambda_n(\varepsilon)}(k_2^{(h)})\right| < ctant\ \varepsilon^{1/24},$$

for every $k_2^{(h)} \in \mathcal{I}_\varepsilon = (\varepsilon^{-3/8}, \varepsilon^{-1/2}\left|\ln\varepsilon\right|^{-1/4})$, and Lemma 4.1.3 leads to

$$\left|\Phi_{\varepsilon,\beta}(k_2^{(h)}) - \Phi_{\varepsilon,\Lambda_n(\varepsilon)}(k_2^{(h)})\right| < \frac{1}{2}d\varepsilon^{1/6}\left|\ln\varepsilon\right|^{1/4},$$

for any $\beta \in I_{\tilde{\beta}}$ and every $k_2^{(h)} \in \mathcal{I}_\varepsilon$, Lemma 4.1.9 follows easily. □

Let us remark that Lemma 4.1.9 will be strongly used to prove the Main Lemma I. Therefore, since we must also prove the first Perturbing Lemma (see Lemma 1.3.11) let us state and prove a perturbing version of Lemma 4.1.9.

Before that, we consider again the family of functions given in (4.1.1) and take the new notation

$$\Phi_{\varepsilon,\beta}(x) = -\left(x(1+\beta) - \frac{B(x)}{x}\right)(c\,|\ln\varepsilon| + l) - \frac{dA(x)}{\varepsilon x} = \Psi_{\varepsilon,\beta}(x,c,d,l). \qquad (4.1.19)$$

Lemma 4.1.10 *For every $\varepsilon \in \mathcal{U}_\varepsilon = \mathcal{U}_\varepsilon(c,l,d)$ and any $\beta \in I_{\tilde{\beta}}(\varepsilon)$, it follows that*

$$\begin{aligned} \Psi_{\varepsilon,\beta}(k_2^{(n)}, c, d', l') - \Psi_{\varepsilon,\beta}(k_2^{(j)}, c, d', l') &> ctant\ \varepsilon^{-1/2}\,|\ln\varepsilon|^{1/2} \\ \Psi_{\varepsilon,\beta}(k_2^{(n+1)}, c, d', l') - \Psi_{\varepsilon,\beta}(k_2^{(j)}, c, d', l') &> ctant\ \varepsilon^{-1/2}\,|\ln\varepsilon|^{1/2} \end{aligned}$$

for every Fibonacci number $k_2^{(j)} \in \mathcal{I}_\varepsilon$, with $j \neq n$, $j \neq n+1$, n the natural number for which $\varepsilon \in \mathcal{U}_n$ and every positive parameters d', l' satisfying

$$\max\{|d-d'|\,, |l-l'|\} < ctant\ \varepsilon^{\alpha},$$

for some $\alpha > 5/8$.

Proof

Let us take the notation (see also (4.1.4))

$$\Phi_{1,\varepsilon,\beta}(x) = -x(1+\beta)(c\,|\ln\varepsilon| + l) - \frac{d\tilde{A}}{\varepsilon x} = \Psi_{1,\varepsilon,\beta}(x,c,d,l)$$

and recall that Lemma 4.1.4 implies

$$|\Psi_{1,\varepsilon,\beta}(x,c,d,l) - \Psi_{\varepsilon,\beta}(x,c,d,l)| \leq ctant\ \varepsilon^{1/24},$$

for every $x \in \mathcal{I}_\varepsilon$. Therefore, Lemma 4.1.10 easily follows by taking into account that, from Lemma 4.1.9, we have

$$\Psi_{\varepsilon,\beta}(k_2^{(*)}, c, d, l) - \Psi_{\varepsilon,\beta}(k_2^{(j)}, c, d, l) > ctant\ \varepsilon^{-1/2}\,|\ln\varepsilon|^{1/2},$$

where $*$ stands for n and $n+1$, and, on the other hand, since

$$\mathcal{I}_\varepsilon = \left(\varepsilon^{-3/8}, \varepsilon^{-1/2}\,|\ln\varepsilon|^{-1/4}\right),$$

then

$$\max_{x\in\mathcal{I}_\varepsilon}\left\{\left|\frac{\partial\Psi_{1,\varepsilon,\beta}}{\partial l}(x,c,d,l)\right|, \left|\frac{\partial\Psi_{1,\varepsilon,\beta}}{\partial d}(x,c,d,l)\right|\right\} < ctant\ \varepsilon^{-5/8}.$$

□

Now, we have all the ingredients needed for proving the principal auxiliary result, see Lemma 4.1.11, leading to the proof of the Main Lemma I and to the proof of the first Perturbing Lemma.

Let us pick three arbitrary positive parameters c, l and d and fix $\varepsilon \in \mathcal{U}_\varepsilon(c,l,d)$ and $\beta \in I_{\tilde\beta}(\varepsilon)$.

Let us recall that, once c, l, d and $\varepsilon \in \mathcal{U}_\varepsilon(c,l,d)$ are fixed, there exists a unique natural number n (depending on c, l, d and ε) for which $\varepsilon \in \mathcal{U}_n(c,l,d)$. In fact, see also Remark 4.1.5 and (4.1.5), n is given by the condition

$$x_\varepsilon = \left(\frac{d\tilde{A}}{(1+\tilde\beta)(c\,|\ln\varepsilon| + l)\varepsilon}\right)^{1/2} \in [k_2^{(n)}, k_2^{(n+1)}),$$

with $k_2^{(n)}$ and $k_2^{(n+1)}$ two consecutive Fibonacci numbers.

Now, see the statements of Lemma 1.3.10 and Lemma 1.3.11, we need to obtain the two leading terms of any series

$$S = 2\sum_{\hat k\in\mathbb{Z}^2_+} S_{\hat k}\hat{\mathcal{E}}_{\hat k} \in \mathcal{S}(c,l',d',\varepsilon,\beta), \quad \hat{\mathcal{E}}_{\hat k} = \hat{\mathcal{E}}_{\hat k}(c,l',d',\varepsilon,\beta)$$

whenever

$$\max\{|l-l'|\,, |d-d'|\} < ctant\ \varepsilon^\alpha,$$

for some constant $\alpha > 5/8$.

It will be necessary to divide the set of indices

$$\mathbb{Z}^2_+ = \{(k_1,k_2)\in\mathbb{Z}^2 :\ k_2>0\} \cup \{(k_1,0)\in\mathbb{Z}^2 :\ k_1>0\}$$

in a suitable way in order to study the different contribution which each of the terms $S_{\hat k}\hat{\mathcal{E}}_{\hat k}$ brings to the whole series S. A similar kind of decomposition was also elaborated in [22] for reaching results on the leading order behaviour like the ones that we present here, but only for the case in which β is a sufficiently good irrational number.

Now, let us describe the partition of $\mathbb{Z}^2_+$ which we are going to use. Since the value of β is fixed, there exists a unique way of writing $\mathbb{Z}^2_+$ as

$$\mathbb{Z}^2_+ = \mathcal{A}_0 \cup \mathcal{A},$$

where

$$\mathcal{A}_0 = \{\hat k\in\mathbb{Z}^2_+ :\ \left|\hat k\omega\right| \geq 1/2\}, \quad \mathcal{A} = \mathbb{Z}^2_+\backslash\mathcal{A}_0. \tag{4.1.20}$$

In order to divide $\mathcal{A}$ in a convenient way, let us denote by $\hat k^{(j)} = (k_1^{(j)}, k_2^{(j)})$, $k_1^{(j)}/k_2^{(j)}$ the best approximations to β. We decompose

$$\mathcal{A} = \bigcup_{i=1}^{7}\mathcal{A}_i \tag{4.1.21}$$

where

$$
\begin{aligned}
\mathcal{A}_1 &= \{\hat{k} : \ k_2 > \varepsilon^{-1/2} \left|\ln \varepsilon\right|^{-1/4}\} \\
\mathcal{A}_2 &= \{\hat{k} : \ k_2 < k_2^{(100)}\} \\
\mathcal{A}_3 &= \{\hat{k} : \ \hat{k} = \hat{k}^{(j)}, \ j \geq 100, \ k_2^{(j)} < \varepsilon^{-3/8}\} \\
\mathcal{A}_4 &= \{\hat{k} : \ \hat{k} = \hat{k}^{(j)}, \ k_2^{(j)} \in \mathcal{I}_\varepsilon, \ j \neq n, \ j \neq n+1\} \\
\mathcal{A}_5 &= \{\hat{k} : \ k_2 \in (k_2^{(j)}, k_2^{(j+1)}), \ k_2^{(j)} < \varepsilon^{-3/8}, \ j \geq 100\} \\
\mathcal{A}_6 &= \{\hat{k} : \ k_2 \in (k_2^{(j)}, k_2^{(j+1)}), \ \text{for some } (k_2^{(j)}, k_2^{(j+1)}) \subset \mathcal{I}_\varepsilon\} \\
\mathcal{A}_7 &= \{\hat{k}^{(n)}, \hat{k}^{(n+1)}\}.
\end{aligned}
$$

We point out that, since $\beta \in I_{\tilde{\beta}}(\varepsilon)$, we know that the best approximations $k_1^{(j)}/k_2^{(j)}$ to β satisfying $k_2^{(j)} < \varepsilon^{-1/2} \left|\ln \varepsilon\right|^{-1/4}$ are those ones to the golden mean. Therefore, the partition of $\mathcal{A}$ is uniform in β whenever $\beta \in I_{\tilde{\beta}}(\varepsilon)$.

On the other hand, since n only depends on c, l, d and ε, the same holds for the partition of $\mathbb{Z}_+^2$ defined in (4.1.20) and (4.1.21) (in particular, such partition does not depend on l', d' whenever $|l - l'| < \varepsilon^\alpha$ and $|d - d'| < \varepsilon^\alpha$).

Now, let us write, for $i = 0, 1, \ldots, 7$,

$$
N_i = N_i(c, l', d', \varepsilon, \beta) = \sum_{\hat{k} \in \mathcal{A}_i} S_{\hat{k}} \hat{\mathcal{E}}_{\hat{k}}, \quad \hat{\mathcal{E}}_{\hat{k}} = \hat{\mathcal{E}}_{\hat{k}}(c, l', d', \varepsilon, \beta).
$$

Thus, since

$$
\{\mathcal{A}_j\}_{j=0}^7
$$

is a family of two by two disjoint sets, we have

$$
S = 2 \sum_{i=0}^{7} N_i.
$$

Lemma 4.1.11 *If* $S_{\hat{k}^{(n)}} S_{\hat{k}^{(n+1)}} > 0$, *then, for* $i = 0, 1, \ldots, 6$, *it follows that*

$$
\frac{|N_i(c, l', d', \varepsilon, \beta)|}{|N_7(c, l', d', \varepsilon, \beta)|} < \exp\left(-\frac{\left|\ln \varepsilon\right|^{1/5}}{\varepsilon^{1/2}} \right),
$$

whenever $\max\{|d - d'|, |l - l'|\} < ctant\, \varepsilon^\alpha$, *for some* $\alpha > 5/8$.

Proof

During the proof we keep the notation (see also (4.1.19))

$$
N_i = N_i(c, l', d', \varepsilon, \beta), \quad \hat{\mathcal{E}}_{\hat{k}} = \hat{\mathcal{E}}_{\hat{k}}(c, l', d', \varepsilon, \beta), \quad \Phi_{\varepsilon,\beta} = \Psi_{\varepsilon,\beta}(\cdot, c, d', l')
$$

by putting special emphasis in those places in which the arguments depend on the value of the parameters l' and d'.

Let us start by giving a lower bound for

$$|N_7| = \left| S_{\hat{k}^{(n)}} \hat{\mathcal{E}}_{\hat{k}^{(n)}} + S_{\hat{k}^{(n+1)}} \hat{\mathcal{E}}_{\hat{k}^{(n+1)}} \right| .$$

To this end, let us recall that

$$\hat{\mathcal{E}}_{\hat{k}^{(j)}} = \exp \left(- \left| \hat{k}^{(j)} \right| (c \left| \ln \varepsilon \right| + l') - \frac{d' \left| \hat{k}^{(j)} \omega \right|}{\varepsilon} \right) ,$$

for every $j \in \mathbb{N}$.

Moreover, since $k_1^{(t)}/k_2^{(t)}$, $t = n$ or $t = n+1$, are those best approximations to the golden mean for which (4.1.6) holds, we easily get (see also (4.1.18))

$$\left| \hat{k}^{(t)} \right| = \left| k_1^{(t)} \right| + \left| k_2^{(t)} \right| < 4 \left| k_2^{(t)} \right| < \mathit{ctant}\ \varepsilon^{-1/2} \left| \ln \varepsilon \right|^{-1/2} .$$

Then, (4.1.3) and Lemma 4.1.1 lead to $\left| A_t(\beta) - A_t(\tilde{\beta}) \right| < \mathit{ctant}\ \varepsilon^{2/3}$, from which (1.4.125) yields $|A_t(\beta)| < \mathit{ctant}$ and then (1.4.124) yields

$$\left| \hat{k}^{(t)} \omega \right| = \frac{A_t(\beta)}{k_2^{(t)}} < \mathit{ctant}\ \varepsilon^{1/2} \left| \ln \varepsilon \right|^{1/2} .$$

Therefore, for $t = n$ or $t = n+1$, it follows that

$$\hat{\mathcal{E}}_{\hat{k}^{(t)}} > \exp \left(- \frac{\mathit{ctant}}{\varepsilon^{1/2} \left| \ln \varepsilon \right|^{1/2}} (c \left| \ln \varepsilon \right| + l') - \frac{\mathit{ctant}\ \left| \ln \varepsilon \right|^{1/2}}{\varepsilon^{1/2}} \right) > \exp \left(-\mathit{ctant}\ \frac{\left| \ln \varepsilon \right|^{1/2}}{\varepsilon^{1/2}} \right) .$$

Furthermore, since

$$S \in \mathcal{S}_2(c, l', d', \varepsilon, \beta)$$

we have, according to Definition 1.3.8, that

$$|S_{\hat{k}^{(t)}}| \geq W_2 \left| \hat{k}^{(t)} \right|^{-\mathcal{X}_2} \geq \mathit{ctant}\ \varepsilon^{\mathcal{X}_2/2},$$

for $t = n$, $n+1$. In this way, using the assumption $S_{\hat{k}^{(n)}} S_{\hat{k}^{(n+1)}} > 0$, we have

$$|N_7| > \mathit{ctant}\ \varepsilon^{\mathcal{X}_2/2} \left(\hat{\mathcal{E}}_{\hat{k}^{(n)}} + \hat{\mathcal{E}}_{\hat{k}^{(n+1)}} \right) > \exp \left(-\mathit{ctant}\ \frac{\left| \ln \varepsilon \right|^{1/2}}{\varepsilon^{1/2}} \right) . \qquad (4.1.22)$$

Let us also recall that, as usual, we denote by *ctant* several different constants not depending neither on ε nor on μ.

Now, let us observe that, if $\hat{k} \in \mathcal{A}_0$, then

$$\hat{\mathcal{E}}_{\hat{k}} = \exp \left(- \left| \hat{k} \right| (c \left| \ln \varepsilon \right| + l') - \frac{d' \left| \hat{k} \omega \right|}{\varepsilon} \right) < \exp \left(- \frac{d'}{2\varepsilon} \right) \exp \left(- \left| \hat{k} \right| (c \left| \ln \varepsilon \right| + l') \right) .$$

Hence, using that $S \in \mathcal{S}_1(c, l', d', \varepsilon, \beta)$, we get

$$|N_0| \leq W_1 \exp\left(-\frac{d'}{2\varepsilon}\right) \sum_{\hat{k}\in\mathbb{Z}^2} \left|\hat{k}\right|^{\mathcal{X}_1} \exp\left(-\left|\hat{k}\right|(c\,|\ln\varepsilon| + l')\right) \leq ctant\ \exp\left(-\frac{d'}{2\varepsilon}\right).$$

On the other hand, the relation

$$\left|\hat{k}\omega\right| < \frac{1}{2}$$

implies that the series N_i, $i = 1, \ldots, 6$ have only one index. In other words, once a value of k_2 is fixed there exists, at most, one value of k_1 (satisfying $|k_1| < 3k_2$) such that $\hat{k} = (k_1, k_2) \in \mathcal{A} = \mathbb{Z}^2_+ \backslash \mathcal{A}_0$. In particular, using also that $S \in \mathcal{S}_1(c, l', d', \varepsilon, \beta)$, we deduce

$$\begin{aligned} |N_1| &\leq W_1 \sum_{k_2 > \varepsilon^{-1/2}|\ln\varepsilon|^{-1/4}} (4k_2)^{\mathcal{X}_1} \exp\left(-k_2(c\,|\ln\varepsilon| + l')\right) \leq \\ &\leq \sum_{k_2 > \varepsilon^{-1/2}|\ln\varepsilon|^{-1/4}} \exp\left(-\frac{k_2(c\,|\ln\varepsilon| + l')}{2}\right) \leq ctant\ \exp\left(-\frac{ctant\ |\ln\varepsilon|^{3/4}}{2\varepsilon^{1/2}}\right). \end{aligned}$$

Now, for giving an upper bound for $|N_2|$ let us observe that the cardinal of $\mathcal{A}_2$ is finite and it does not depend on ε. Moreover, there exists a constant c^* (which does not depend on β, whenever $\beta \in I_{\tilde{\beta}}$), for which $\left|\hat{k}\omega\right| > c^*$, for every $\hat{k} \in \mathcal{A}_2$. Thus, if $\hat{k} \in \mathcal{A}_2$, we may write

$$\hat{\mathcal{E}}_{\hat{k}} < \exp\left(-\frac{d'\left|\hat{k}\omega\right|}{\varepsilon}\right) < \exp\left(-\frac{ctant}{\varepsilon}\right)$$

and therefore

$$|N_2| < ctant\ \exp\left(-\frac{ctant}{\varepsilon}\right).$$

Let us assume $\hat{k} \in \mathcal{A}_3$. This means that $\hat{k} = (k_1^{(j)}, k_2^{(j)})$ with $k_2^{(j)} < \varepsilon^{-3/8}$, $j \geq 100$ and $k_1^{(j)}/k_2^{(j)}$ a best approximation to β (and also a best approximation to $\hat{\beta}$, whenever $\hat{\beta}$ belongs to $I_{\tilde{\beta}}$). Since $j \geq 100$, (1.4.125) and (4.1.3) imply $A_j(\beta) > 1/6$. Therefore, condition $k_2^{(j)} < \varepsilon^{-3/8}$ leads to

$$\left|\hat{k}^{(j)}\omega\right| = \frac{A_j(\beta)}{k_2^{(j)}} > \frac{1}{6}\varepsilon^{3/8}.$$

Then,

$$\hat{\mathcal{E}}_{\hat{k}^{(j)}} < \exp\left(-\frac{d'}{6\varepsilon^{5/8}}\right).$$

On the other hand, since

$$|S_{\hat{k}^{(j)}}| < W_1 \left|\hat{k}^{(j)}\right|^{\mathcal{X}_1} < W_1(4k_2^{(j)})^{\mathcal{X}_1} < ctant\ \varepsilon^{-\frac{3}{8}\mathcal{X}_1}$$

we obtain, for ε sufficiently small,

$$|N_3| \leq ctant \sum_{\hat{k}\in\mathcal{A}_3} \varepsilon^{-\frac{3}{8}\mathcal{X}_1}\hat{\mathcal{E}}_{\hat{k}^{(j)}} < ctant\ \varepsilon^{-\frac{3}{8}(1+\mathcal{X}_1)} \exp\left(-\frac{d'}{6\varepsilon^{5/8}}\right) < \exp\left(-\frac{d'}{12\varepsilon^{5/8}}\right).$$

Once the above upper bounds were obtained for $|N_i|$, $i = 0, 1, 2, 3$, it is easy to see that (4.1.22) gives

$$\frac{|N_i|}{|N_7|} < \exp\left(-\frac{|\ln\varepsilon|^{1/5}}{\varepsilon^{1/2}}\right)$$

for $i = 0, 1, 2, 3$.

Let us continue the proof of the lemma by searching an upper bound for

$$\frac{\left|S_{\hat{k}^{(j)}}\hat{\mathcal{E}}_{\hat{k}^{(j)}}\right|}{|N_7|},$$

when $\hat{k}^{(j)} = (k_1^{(j)}, k_2^{(j)})$, $k_2^{(j)} \in \mathcal{I}_\varepsilon = \left(\varepsilon^{-3/8}, \varepsilon^{-1/2}|\ln\varepsilon|^{-1/4}\right)$, $k_1^{(j)}/k_2^{(j)}$ are best approximations to β, $j \neq n$, $j \neq n+1$.

Of course, the fact that $S \in \mathcal{S}_1(c, l', d', \varepsilon, \beta)$ gives $|S_{\hat{k}^{(j)}}| < W_1\left|\hat{k}^{(j)}\right|^{\mathcal{X}_1}$ and therefore, since

$$\left|\hat{k}^{(j)}\right| < 4k_2^{(j)} < 4\varepsilon^{-1/2}|\ln\varepsilon|^{-1/4} < 4\varepsilon^{-1/2},$$

we achieve

$$|S_{\hat{k}^{(j)}}| < ctant\ \varepsilon^{-\mathcal{X}_1/2}.$$

Hence, using (4.1.22), we get

$$\frac{\left|S_{\hat{k}^{(j)}}\hat{\mathcal{E}}_{\hat{k}^{(j)}}\right|}{|N_7|} < ctant\ \varepsilon^{-\frac{\mathcal{X}_1+\mathcal{X}_2}{2}} \frac{\hat{\mathcal{E}}_{\hat{k}^{(j)}}}{\hat{\mathcal{E}}_{\hat{k}^{(n)}} + \hat{\mathcal{E}}_{\hat{k}^{(n+1)}}}.$$

Now, we have to deduce a lower bound for

$$\frac{\hat{\mathcal{E}}_{\hat{k}^{(n)}} + \hat{\mathcal{E}}_{\hat{k}^{(n+1)}}}{\hat{\mathcal{E}}_{\hat{k}^{(j)}}}.$$

Here is where we use Lemma 4.1.9 and Lemma 4.1.10 to ensure the existence of a positive constant $\tilde{c}$ satisfying (see also (4.1.2))

$$\frac{\hat{\mathcal{E}}_{\hat{k}^{(t)}}}{\hat{\mathcal{E}}_{\hat{k}^{(j)}}} = \exp\left(\Phi_{\varepsilon,\beta}(k_2^{(t)}) - \Phi_{\varepsilon,\beta}(k_2^{(j)})\right) > \exp\left(\frac{\tilde{c}|\ln\varepsilon|^{1/2}}{\varepsilon^{1/2}}\right)$$

for $t = n$ and $t = n+1$. Namely, Lemma 4.1.9 is used to reach the announced bound for the case in which $l = l'$ and $d = d'$, while Lemma 4.1.10 is applied to extend that bound for every l', d' satisfying $\max\{|l - l'|, |d - d'|\} < ctant\ \varepsilon^\alpha$, for some $\alpha > 5/8$.

Hence, for ε small enough, we finally deduce

$$\frac{\left|S_{\hat{k}^{(j)}}\hat{\mathcal{E}}_{\hat{k}^{(j)}}\right|}{|N_7|} < \exp\left(-\frac{\tilde{c}\,|\ln\varepsilon|^{1/2}}{2\varepsilon^{1/2}}\right). \tag{4.1.23}$$

In particular, this last bound allows us to write

$$\frac{|N_4|}{|N_7|} < \sum_{\hat{k}\in\mathcal{A}_4}\exp\left(-\frac{\tilde{c}\,|\ln\varepsilon|^{1/2}}{2\varepsilon^{1/2}}\right) <$$

$$< \ \varepsilon^{-1/2}\,|\ln\varepsilon|^{-1/4}\exp\left(-\frac{\tilde{c}\,|\ln\varepsilon|^{1/2}}{2\varepsilon^{1/2}}\right) < \exp\left(-\frac{\tilde{c}\,|\ln\varepsilon|^{1/2}}{4\varepsilon^{1/2}}\right) < \exp\left(-\frac{|\ln\varepsilon|^{1/5}}{\varepsilon^{1/2}}\right).$$

Now, in order to prove Lemma 4.1.11 for the case in which $i = 5$, we are going to use the next result whose proof can be found in [22]:

Proposition 4.1.12 *For every $\beta > 0$ let $\{k_1^{(j)}\}_{j\in\mathbb{N}}$ and $\{k_2^{(j)}\}_{j\in\mathbb{N}}$ be the sequences defined in (1.4.120). Then, for every $\hat{k} = (k_1, k_2) \in \mathbb{Z}_+^2$ satisfying $k_2 \in (k_2^{(j)}, k_2^{(j+1)})$, $j \geq 1$, it follows that*

$$|k_1 + \beta k_2| > \left|k_1^{(j)} + \beta k_2^{(j)}\right| + \left|k_1^{(j+1)} + \beta k_2^{(j+1)}\right|.$$

Now, let $\hat{k} = (k_1, k_2)$ belonging to $\mathcal{A}_5$. This means that $k_2 \in (k_2^{(j)}, k_2^{(j+1)})$ for some $j \geq 100$ and $k_2^{(j)} < \varepsilon^{-3/8}$. Let $\tilde{n}$ be the smallest natural number for which

$$k_2^{(\tilde{n})} > \varepsilon^{-5/12}$$

and write

$$\frac{|N_5|}{|N_7|} = \frac{|N_5|}{\left|S_{\hat{k}^{(\tilde{n})}}\hat{\mathcal{E}}_{\hat{k}^{(\tilde{n})}}\right|}\frac{\left|S_{\hat{k}^{(\tilde{n})}}\hat{\mathcal{E}}_{\hat{k}^{(\tilde{n})}}\right|}{|N_7|}. \tag{4.1.24}$$

By applying Proposition 4.1.12, we have that condition $\hat{k} \in (k_2^{(j)}, k_2^{(j+1)})$ implies

$$\left|\hat{k}\omega\right| > \frac{A_j(\beta)}{k_2^{(j)}} > \frac{ctant}{k_2^{(j)}} > ctant\ \varepsilon^{3/8}.$$

Therefore,

$$\hat{\mathcal{E}}_{\hat{k}} < \exp\left(-\frac{d'\left|\hat{k}\omega\right|}{\varepsilon}\right) < \exp\left(-\frac{ctant}{\varepsilon^{5/8}}\right).$$

On the other hand, since $k_2^{(\tilde{n})} < ctant\ k_2^{(\tilde{n}-1)} < ctant\ \varepsilon^{-5/12}$, we get

$$\left|\hat{k}^{(\tilde{n})}\right|(c\,|\ln\varepsilon| + l') < ctant\ \varepsilon^{-5/12}\,|\ln\varepsilon|.$$

Furthermore, since we also have

$$\frac{\left|\hat{k}^{(\tilde{n})}\omega\right|}{\varepsilon} = \frac{A_{\tilde{n}}(\beta)}{\varepsilon k_2^{(\tilde{n})}} < ctant\ \varepsilon^{-7/12},$$

we obtain

$$\hat{\mathcal{E}}_{\hat{k}^{(\tilde{n})}} > \exp\left(-ctant\ \varepsilon^{-7/12}\right).$$

Hence, since $k_2^{(\tilde{n})} \in \mathcal{I}_\varepsilon$ and $S \in \mathcal{S}_1(c, l', d', \varepsilon, \beta) \cap \mathcal{S}_2(c, l', d', \varepsilon, \beta)$, we deduce

$$\frac{\left|S_{\hat{k}}\hat{\mathcal{E}}_{\hat{k}}\right|}{\left|S_{\hat{k}^{(\tilde{n})}}\hat{\mathcal{E}}_{\hat{k}^{(\tilde{n})}}\right|} < ctant\left|\hat{k}\right|^{\mathcal{X}_1}\left|\hat{k}^{(\tilde{n})}\right|^{\mathcal{X}_2}\frac{\hat{\mathcal{E}}_{\hat{k}}}{\hat{\mathcal{E}}_{\hat{k}^{(\tilde{n})}}} <$$
$$< ctant\ \varepsilon^{-\frac{3}{8}\mathcal{X}_1 - \frac{5}{12}\mathcal{X}_2}\exp\left(-ctant\ \varepsilon^{-7/12}\right) < \exp\left(-\frac{ctant}{\varepsilon^{7/12}}\right).$$

Therefore,

$$\frac{|N_5|}{\left|S_{\hat{k}^{(\tilde{n})}}\hat{\mathcal{E}}_{\hat{k}^{(\tilde{n})}}\right|} < \sum_{\hat{k}\in\mathcal{A}_5}\exp\left(-\frac{ctant}{\varepsilon^{7/12}}\right) < \varepsilon^{-3/8}\exp\left(-\frac{ctant}{\varepsilon^{7/12}}\right) < \exp\left(-\frac{ctant}{\varepsilon^{1/2}}\right).$$

On the other hand, since $k_2^{(\tilde{n})} \in \mathcal{I}_\varepsilon$ and $\tilde{n} \neq n$, $\tilde{n} \neq n+1$, we may apply (4.1.23) to deduce from (4.1.24) that

$$\frac{|N_5|}{|N_7|} < \exp\left(-\frac{ctant}{\varepsilon^{1/2}}\right)\exp\left(-\frac{\tilde{c}\left|\ln\varepsilon\right|^{1/2}}{2\varepsilon^{1/2}}\right) < \exp\left(-\frac{\left|\ln\varepsilon\right|^{1/5}}{\varepsilon^{1/2}}\right).$$

Finally, let us consider the smallest natural number $\hat{n}$ for which $k_2^{(\hat{n})} > \varepsilon^{-3/8}$ and take $\hat{m}$ the greatest natural number satisfying $k_2^{(\hat{m})} < \varepsilon^{-1/2}\left|\ln\varepsilon\right|^{-1/4}$. Then, we may write

$$\mathcal{A}_6 = \bigcup_{j=\hat{n}}^{\hat{m}-1}\mathcal{A}_6^j,$$

where, for $j = \hat{n}, \ldots, \hat{m}-1$,

$$\mathcal{A}_6^j = \left\{\hat{k} \in \mathcal{A}_6 :\ k_2 \in (k_2^{(j)}, k_2^{(j+1)})\right\}.$$

Now, by denoting

$$N_6^j = \sum_{\hat{k}\in\mathcal{A}_6^j} S_{\hat{k}}\hat{\mathcal{E}}_{\hat{k}},$$

we obtain

$$\frac{|N_6|}{|N_7|} \leq \sum_{j=\hat{n}}^{\hat{m}-1}\frac{\left|N_6^j\right|}{|N_7|} = \sum_{j=\hat{n}}^{\hat{m}-1}\frac{\left|N_6^j\right|}{\left|S_{\hat{k}^{(j)}}\hat{\mathcal{E}}_{\hat{k}^{(j)}}\right|}\frac{\left|S_{\hat{k}^{(j)}}\hat{\mathcal{E}}_{\hat{k}^{(j)}}\right|}{|N_7|}. \qquad (4.1.25)$$

Let us take $\hat{k} \in \mathcal{A}_6^j$ and observe that

$$\frac{\left|S_{\hat{k}}\hat{\mathcal{E}}_{\hat{k}}\right|}{\left|S_{\hat{k}^{(j)}}\hat{\mathcal{E}}_{\hat{k}^{(j)}}\right|} < ctant\ \varepsilon^{-\frac{\varkappa_1+\varkappa_2}{2}} \frac{\hat{\mathcal{E}}_{\hat{k}}}{\hat{\mathcal{E}}_{\hat{k}^{(j)}}}.$$

Now, we have

$$\frac{\hat{\mathcal{E}}_{\hat{k}}}{\hat{\mathcal{E}}_{\hat{k}^{(j)}}} = \exp\left(\left(\left|\hat{k}^{(j)}\right| - \left|\hat{k}\right|\right)(c\left|\ln\varepsilon\right| + l') + \frac{d'}{\varepsilon}\left(\left|\hat{k}^{(j)}\omega\right| - \left|\hat{k}\omega\right|\right)\right),$$

with (note that, if $\hat{k} = (k_1, k_2) \notin \mathcal{A}_0$, then $k_2 > 0$ implies $k_1 < 0$)

$$\left|\hat{k}^{(j)}\right| - \left|\hat{k}\right| = -k_1^{(j)} + k_2^{(j)} + k_1 - k_2 < k_1 - k_1^{(j)}.$$

Then, the relations (see Proposition 4.1.12)

$$\left|k_1^{(j)} + \beta k_2^{(j)}\right| < |k_1 + k_2\beta| < \frac{1}{2}, \qquad k_2^{(j)} < k_2$$

imply that $k_1 < k_1^{(j)}$ and thus $\left|\hat{k}^{(j)}\right| < \left|\hat{k}\right|$. Hence,

$$\frac{\hat{\mathcal{E}}_{\hat{k}}}{\hat{\mathcal{E}}_{\hat{k}^{(j)}}} < \exp\left(\frac{d'}{\varepsilon}\left(\left|\hat{k}^{(j)}\omega\right| - \left|\hat{k}\omega\right|\right)\right) < \exp\left(-\frac{d'\left|\hat{k}^{(j+1)}\omega\right|}{\varepsilon}\right),$$

where we have again used Proposition 4.1.12.

Therefore, since

$$\left|\hat{k}^{(j+1)}\omega\right| = \frac{A_{j+1}(\beta)}{k_2^{(j+1)}} > ctant\ \varepsilon^{1/2}\left|\ln\varepsilon\right|^{1/4},$$

we deduce

$$\frac{\left|S_{\hat{k}}\hat{\mathcal{E}}_{\hat{k}}\right|}{\left|S_{\hat{k}^{(j)}}\hat{\mathcal{E}}_{\hat{k}^{(j)}}\right|} < \exp\left(-ctant\ \frac{\left|\ln\varepsilon\right|^{1/4}}{\varepsilon^{1/2}}\right).$$

Consequently, we get

$$\begin{aligned}\frac{\left|N_6^j\right|}{\left|S_{\hat{k}^{(j)}}\hat{\mathcal{E}}_{\hat{k}^{(j)}}\right|} &< \sum_{\hat{k}\in\mathcal{A}_6^j} \exp\left(-ctant\ \frac{\left|\ln\varepsilon\right|^{1/4}}{\varepsilon^{1/2}}\right) < \\ &< \varepsilon^{-1/2}\left|\ln\varepsilon\right|^{-1/4}\exp\left(-ctant\ \frac{\left|\ln\varepsilon\right|^{1/4}}{\varepsilon^{1/2}}\right) < \exp\left(-\frac{ctant\ \left|\ln\varepsilon\right|^{1/4}}{\varepsilon^{1/2}}\right).\end{aligned}$$

Hence, at sight of (4.1.25), in order to conclude the proof of the lemma it suffices to bound

$$\frac{\left|S_{\hat{k}(j)}\hat{\mathcal{E}}_{\hat{k}(j)}\right|}{|N_7|}.$$

To this end, let us observe that, if $j \neq n$ and $j \neq n+1$, then (4.1.23) directly gives

$$\frac{\left|S_{\hat{k}(j)}\hat{\mathcal{E}}_{\hat{k}(j)}\right|}{|N_7|} \leq \exp\left(-\frac{\tilde{c}\,|\ln\varepsilon|^{1/2}}{2\varepsilon^{1/2}}\right).$$

Nevertheless, if $j = n$ or $j = n+1$, we may only use that $S \in \mathcal{S}_1(c, l', d', \varepsilon, \beta)$ and (4.1.22) to obtain

$$\frac{\left|S_{\hat{k}(j)}\hat{\mathcal{E}}_{\hat{k}(j)}\right|}{|N_7|} \leq ctant\ \varepsilon^{-\frac{\varkappa_1+\varkappa_2}{2}} \frac{\hat{\mathcal{E}}_{\hat{k}(j)}}{\hat{\mathcal{E}}_{\hat{k}(n)} + \hat{\mathcal{E}}_{\hat{k}(n+1)}} < ctant\ \varepsilon^{-\frac{\varkappa_1+\varkappa_2}{2}}.$$

Hence, (4.1.25) gives

$$\frac{|N_6|}{|N_7|} \leq ctant\ \varepsilon^{-\frac{\varkappa_1+\varkappa_2}{2}} \sum_{j=\hat{n}}^{\hat{m}-1} \exp\left(-\frac{ctant\ \ |\ln\varepsilon|^{1/4}}{\varepsilon^{1/2}}\right) <$$

$$< ctant\ \varepsilon^{-1/2}\,|\ln\varepsilon|^{-1/4}\,\varepsilon^{-\frac{\varkappa_1+\varkappa_2}{2}} \exp\left(-\frac{ctant\ \ |\ln\varepsilon|^{1/4}}{\varepsilon^{1/2}}\right) < \exp\left(-\frac{|\ln\varepsilon|^{1/5}}{\varepsilon^{1/2}}\right).$$

Therefore, the lemma is proved. □

Now, keeping in mind that, for $\hat{\mathcal{E}}_{\hat{k}} = \hat{\mathcal{E}}_{\hat{k}}(c, l', d', \varepsilon, \beta)$ and $N_i = N_i(c, l', d', \varepsilon, \beta)$, we may write

$$S = 2\sum_{\hat{k}\in\mathbb{Z}_+^2} S_{\hat{k}}\hat{\mathcal{E}}_{\hat{k}} = 2\sum_{i=0}^{7} N_i$$

and that Lemma 4.1.11 gives

$$\left|\frac{S}{2N_7} - 1\right| \leq 6\exp\left(-\frac{|\ln\varepsilon|^{1/5}}{\varepsilon^{1/2}}\right),$$

whenever $\varepsilon \in \mathcal{U}_\varepsilon$ (more concretely, $\varepsilon \in \mathcal{U}_n$), $\beta \in I_{\tilde{\beta}}(\varepsilon)$ and $\max\{|l-l'|, |d-d'|\} < \varepsilon^\alpha$, $\alpha > 5/8$, we may definitively write

$$S = 2\left(S_{\hat{k}(n)}\hat{\mathcal{E}}_{\hat{k}(n)} + S_{\hat{k}(n+1)}\hat{\mathcal{E}}_{\hat{k}(n+1)}\right)\left[1 + \mathrm{O}\left(\exp\left(-\frac{|\ln\varepsilon|^{1/5}}{\varepsilon^{1/2}}\right)\right)\right].$$

Therefore the Main Lemma I and the first Perturbing Lemma are proved by taking $n^0 = n$ and $n^1 = n+1$.

4.2 Proof of the Main Lemma II

The course of the proof of the second main lemma is essentially the same as for the Main Lemma I. Hence, we are going to outline the needed arguments for getting the same kind of partial results in the new scenario.

Let us begin by making some considerations on a numerical series (see Definition 2.3.5, Definition 2.3.6 and Definition 2.3.7)

$$\hat{S} = \sum_{\hat{k} \in \mathbb{Z}^2 \setminus \{(0,0)\}} S_{\hat{k}} \overline{\mathcal{E}}_{\hat{k}} = 2 \sum_{\hat{k} \in \mathbb{Z}^2_+} S_{\hat{k}} \overline{\mathcal{E}}_{\hat{k}} \in \mathcal{S}^*(v_1, v_2, d, \varepsilon, \beta_2) = \bigcap_{i=1}^{3} \mathcal{S}^*_i(v_1, v_2, d, \varepsilon, \beta_2)$$

where (see (2.3.51)), for $\omega = (1, \beta_2)$, $\hat{k} = (k_1, k_2)$ and $\hat{k}\omega = k_1 + \beta_2 k_2$,

$$\overline{\mathcal{E}}_{\hat{k}} = \overline{\mathcal{E}}_{\hat{k}}(v_1, v_2, d, \varepsilon, \beta_2) = \exp\left(- |k_1| v_1 - |k_2| v_2 - \frac{d \left| \hat{k}\omega \right|}{\sqrt{\varepsilon}} \right),$$

v_1, v_2, d and ε are positive parameters, ε sufficiently small.

Let us observe that if we define the function

$$x \in \mathbb{R}^+ \to \Phi^*_{\varepsilon,\beta_2}(x) = -\left(x(v_2 + v_1\beta_2) - \frac{B^*(x)}{x} \right) - \frac{dA(x)}{\sqrt{\varepsilon} x},$$

where

$$\begin{aligned} A(x) &= A_j(\beta_2), \quad \text{if } x \in [k_2^{(j)}, k_2^{(j+1)}) \\ B^*(x) &= (-1)^j v_1 A_j(\beta_2), \quad \text{if } x \in [k_2^{(j)}, k_2^{(j+1)}), \end{aligned}$$

then (1.4.123) and (1.4.124) lead us to write

$$\overline{\mathcal{E}}_{\hat{k}^{(j)}} = \exp\left(\Phi^*_{\varepsilon,\beta_2}(k_2^{(j)}) \right),$$

whenever $\hat{k}^{(j)} = (k_1^{(j)}, k_2^{(j)})$, with $k_1^{(j)}/k_2^{(j)}$ a best approximation to β_2.

Construction of $I^*_{\tilde{\beta}}$.- Let us fix a sufficiently small value of ε and take the minimum Fibonacci number $k_2^{(m^*)}$ satisfying $k_2^{(m^*)} \geq \varepsilon^{-1/4} |\ln \varepsilon|^{1/8}$.

Let us take $I^*_{\tilde{\beta}}$ the set of all real numbers

$$\beta_2 = [a_0, a_1, \ldots, a_{m^*}, z_{m^*}(\beta_2)]$$

satisfying (recall that $\tilde{\beta}$ always denotes the golden mean)

$$a_i = 1, \quad 0 \leq i \leq m^* \quad \text{and} \quad \left| z_{m^*}(\beta_2) - \tilde{\beta} \right| \leq \varepsilon^{1/3} |\ln \varepsilon|^{1/4}.$$

Lemma 4.2.1 *It holds that*

$$\frac{1}{100}\varepsilon^{5/6} \leq length(I^*_{\tilde\beta}) \leq \frac{1}{2}\varepsilon^{5/6}.$$

Proof

Since for every $\beta_2' \in I^*_{\tilde\beta}$ we have (see the proof of Lemma 4.1.1)

$$\frac{\varepsilon^{1/2}\left|\ln\varepsilon\right|^{-1/4}\left|z_{m^*}(\beta_2')-\tilde\beta\right|}{(1+4\tilde\beta)^2} \leq \left|\beta_2'-\tilde\beta\right| \leq \frac{\left|z_{m^*}(\beta_2')-\tilde\beta\right|}{(2+\tilde\beta^{-1})(k_2^{(m^*)})^2},$$

the result follows easily. □

Lemma 4.2.2 *For every best approximation $k_1^{(j)}/k_2^{(j)}$ to $\tilde\beta$ satisfying*

$$k_2^{(j)} \leq \varepsilon^{-1/4}\left|\ln\varepsilon\right|^{1/8}$$

and any $\beta_2, \beta_2' \in I^*_{\tilde\beta}$, it holds that

$$\left|\Phi^*_{\varepsilon,\beta_2}(k_2^{(j)}) - \Phi^*_{\varepsilon,\beta_2'}(k_2^{(j)})\right| \leq \frac{1}{2}d\varepsilon^{1/12}\left|\ln\varepsilon\right|^{1/8}.$$

Proof

The definition of $I^*_{\tilde\beta}$ implies that, for any $\beta_2 \in I^*_{\tilde\beta}$, the best approximations $k_1^{(j)}/k_2^{(j)}$ to β_2 satisfying $k_2^{(j)} \leq \varepsilon^{-1/4}\left|\ln\varepsilon\right|^{1/8}$ are those ones to the golden mean number $\tilde\beta$. Therefore, see also (1.4.127)

$$\begin{aligned}\left|\Phi^*_{\varepsilon,\beta_2}(k_2^{(j)}) - \Phi^*_{\varepsilon,\beta_2'}(k_2^{(j)})\right| &= \frac{d}{\sqrt{\varepsilon}k_2^{(j)}}\left|A_j(\beta_2)-A_j(\beta_2')\right| = \\ &= \frac{d}{\sqrt{\varepsilon}}k_2^{(j)}\left|\beta_2-\beta_2'\right| \leq \frac{1}{2}d\varepsilon^{1/12}\left|\ln\varepsilon\right|^{1/8},\end{aligned}$$

where we have also used Lemma 4.2.1. □

For proving the Main Lemma II we are going to make use of the following family of continuous functions:

$$x \in \mathbb{R}^+ \to \Phi^*_{1,\varepsilon,\beta_2}(x) = -x(v_2+v_1\beta_2) - \frac{d\tilde A}{\sqrt{\varepsilon}x}, \quad \tilde A = \frac{1}{\tilde\beta+\tilde\beta^{-1}}.$$

Lemma 4.2.3 *For any positive ε and any $\beta_2 \in I^*_{\tilde\beta}(\varepsilon)$, it follows that*

$$\left|\Phi^*_{\varepsilon,\beta_2}(x) - \Phi^*_{1,\varepsilon,\beta_2}(x)\right| \leq ctant\ \varepsilon^{1/30}\left|\ln\varepsilon\right|^{1/4},$$

for every $x \in \left(\varepsilon^{-1/5}, \varepsilon^{-1/4}\left|\ln\varepsilon\right|^{1/8}\right)$.

Proof

The proof follows from Lemma 4.2.1 together with (1.4.125) and (1.4.127). □

Now, as in the first case, we look for the absolute maximum of $\Phi^*_{1,\varepsilon,\tilde{\beta}}$ which now is attained at

$$x^*_\varepsilon = x^*_\varepsilon(v_1, v_2, d) = \left(\frac{d\tilde{A}}{\sqrt{\varepsilon}(v_2 + v_1\tilde{\beta})}\right)^{1/2} \in \mathcal{I}^*_\varepsilon$$

in order to select the two consecutive Fibonacci numbers $k_2^{(n)}$, $k_2^{(n+1)}$ satisfying

$$x^*_\varepsilon \in [k_2^{(n)}, k_2^{(n+1)}).$$

Moreover, we define the sequence of values of ε, $\{\tilde{\varepsilon}^*_n\}_{n\in\mathbb{N}}$, for which

$$\Phi^*_{1,\tilde{\varepsilon}^*_n,\tilde{\beta}}(k_2^{(n)}) = \Phi^*_{1,\tilde{\varepsilon}^*_n,\tilde{\beta}}(k_2^{(n+1)})$$

and also those $\{\overline{\varepsilon}^*_n\}_{n\in\mathbb{N}}$ such that

$$x_{\overline{\varepsilon}^*_n} = \left(\frac{d\tilde{A}}{\sqrt{\overline{\varepsilon}^*_n}(v_2 + v_1\tilde{\beta})}\right)^{1/2} = k_2^{(n)}, \tag{4.2.26}$$

for every $n \in \mathbb{N}$.

The good set of parameters $\mathcal{U}^*_\varepsilon$ is going to be defined as

$$\mathcal{U}^*_\varepsilon = \bigcup_{n\geq n^*} \mathcal{U}^*_n \tag{4.2.27}$$

with $\{\mathcal{U}^*_n\}_{n\geq n^*}$ a family of two by two disjoint real intervals satisfying $\tilde{\varepsilon}^*_n \subset \mathcal{U}^*_n \subset (\overline{\varepsilon}^*_{n+1}, \overline{\varepsilon}^*_n)$. Moreover, if $\varepsilon \in \mathcal{U}^*_n$, then those indices for which the conclusions of the Main Lemma II hold are going to coincide with n and $n+1$.

Construction of $\mathcal{U}^*_n$.- For any $n \geq n^*$, let us set

$$\Lambda^*_n : \varepsilon \in [\overline{\varepsilon}^*_{n+1}, \overline{\varepsilon}^*_n] \to \Lambda^*_n(\varepsilon) = \frac{d\tilde{A}}{v_1\sqrt{\varepsilon}k_2^{(n)}k_2^{(n+1)}} - \frac{v_2}{v_1}$$

in such a way that, for every $\varepsilon \in [\overline{\varepsilon}^*_{n+1}, \overline{\varepsilon}^*_n]$, one gets

$$\Phi^*_{1,\varepsilon,\Lambda^*_n(\varepsilon)}(k_2^{(n)}) = \Phi^*_{1,\varepsilon,\Lambda^*_n(\varepsilon)}(k_2^{(n+1)}).$$

It is easy to see that, for every $\varepsilon \in [\overline{\varepsilon}^*_{n+1}, \overline{\varepsilon}^*_n]$,

$$\text{ctant}\ \varepsilon^{-1/4} \leq k_2^{(l)} \leq \text{ctant}\ \varepsilon^{-1/4}, \quad l = n, n+1,$$

and thus there exists some constant A^* with

$$\frac{1}{A^*\varepsilon} \leq |(\Lambda_n^*)'(\varepsilon)| \leq \frac{A^*}{\varepsilon}. \tag{4.2.28}$$

On the other hand, since for n large enough,

$$\frac{v_2}{v_1} + 1 < \Lambda_n^*(\overline{\varepsilon}_{n+1}) - \Lambda_n^*(\overline{\varepsilon}_n) \leq \frac{v_2}{v_1} + 2, \tag{4.2.29}$$

it is easy to check the existence of some subset $\mathcal{U}_n^*$ of $(\overline{\varepsilon}_{n+1}^*, \overline{\varepsilon}_n^*)$ characterized by the following condition: $\varepsilon \in \mathcal{U}_n^*$ if and only if $\Lambda_n^*(\varepsilon) \in I_{\tilde{\beta}}^*(\varepsilon)$. Now, we provide estimates on the Lebesgue measure of the set $\mathcal{U}_\varepsilon^* = \mathcal{U}_\varepsilon^*(v_1, v_2, d)$.

Proposition 4.2.4 *Let $\varepsilon_0 = \overline{\varepsilon}_{n^*}$. Then,*

$$ctant\ \varepsilon_0^{11/6} \leq \mathcal{L}(\mathcal{U}_\varepsilon^*) \leq ctant\ \varepsilon_0^{11/6}.$$

Proof

Let us write $(a_n^*, b_n^*) = \mathcal{U}_n^*$. We have

$$\frac{\Lambda_n^*(a_n^*) - \Lambda_n^*(b_n^*)}{\Lambda_n^*(\overline{\varepsilon}_{n+1}^*) - \Lambda_n^*(\overline{\varepsilon}_n^*)} = \frac{|(\Lambda_n^*)'(c_n^*)|}{|(\Lambda_n^*)'(d_n^*)|} \frac{b_n^* - a_n^*}{\overline{\varepsilon}_n^* - \overline{\varepsilon}_{n+1}^*},$$

for some $c_n^* \in \mathcal{U}_n^*$ and some $d_n^* \in (\overline{\varepsilon}_{n+1}^*, \overline{\varepsilon}_n^*)$. From Lemma 4.2.1 and (4.2.29) it follows that

$$\frac{v_1}{50(v_2 + 2v_1)} (\overline{\varepsilon}_{n+1}^*)^{5/6} \leq \frac{\Lambda_n^*(a_n^*) - \Lambda_n^*(b_n^*)}{\Lambda_n^*(\overline{\varepsilon}_{n+1}^*) - \Lambda_n^*(\overline{\varepsilon}_n^*)} \leq \frac{v_1}{v_2 + v_1} (\overline{\varepsilon}_n^*)^{5/6}.$$

Moreover, from (4.2.26) and (4.2.28)

$$\frac{1}{16(A^*)^2} \leq \frac{\overline{\varepsilon}_{n+1}^*}{(A^*)^2 \overline{\varepsilon}_n^*} \leq \frac{d_n^*}{(A^*)^2 c_n^*} \leq \frac{|(\Lambda_n^*)'(c_n^*)|}{|(\Lambda_n^*)'(d_n^*)|} \leq \frac{(A^*)^2 d_n^*}{c_n^*} \leq \frac{(A^*)^2 \overline{\varepsilon}_n^*}{\overline{\varepsilon}_{n+1}^*} \leq 16(A^*)^2.$$

Let us observe that it is in the above inequalities where the greatest differences with respect to the first case arise. In fact, here we may bound (from above and below) the quotients

$$\frac{\overline{\varepsilon}_n^*}{\overline{\varepsilon}_{n+1}^*} = \left(\frac{k_2^{(n+1)}}{k_2^{(n)}} \right)^4$$

and this fact allows us to get (direct) estimates on the Lebesgue measure of $\mathcal{U}_\varepsilon^*$.

Furthermore, the rest of the arguments needed for proving the present result are quite standard, because we already deduce that

$$ctant\ (\overline{\varepsilon}_n^*)^{11/6} \leq b_n^* - a_n^* \leq ctant\ (\overline{\varepsilon}_n^*)^{11/6}.$$

□

In the same way as Corollary 4.1.8 was proven, one may obtain the following:

Corollary 4.2.5 *For every* $\varepsilon \in \mathcal{U}_n^*$ *and any* $\beta_2 \in I_{\tilde{\beta}}^*(\varepsilon)$, *it holds that*

$$\frac{1}{4} \leq \frac{\overline{\mathcal{E}}_{\hat{k}^{(n)}}(\varepsilon, \beta_2)}{\overline{\mathcal{E}}_{\hat{k}^{(n+1)}}(\varepsilon, \beta_2)} \leq 4,$$

where, for v_1, v_2 *and* d *fixed,*

$$\overline{\mathcal{E}}_{\hat{k}}(\varepsilon, \beta_2) = \overline{\mathcal{E}}_{\hat{k}}(v_1, v_2, d, \varepsilon, \beta_2).$$

Once the set $\mathcal{U}_\varepsilon^* = \mathcal{U}_\varepsilon^*(v_1, v_2, d)$ was constructed, let us introduce the notation

$$\begin{aligned}
\Psi^*_{\varepsilon,\beta_2}(x, v_1, v_2, d) &= \Phi^*_{\varepsilon,\beta_2}(x) = -\left(x(v_2 + v_1\beta_2) - \frac{B^*(x)}{x}\right) - \frac{dA(x)}{\sqrt{\varepsilon x}} \\
\Psi^*_{1,\varepsilon,\beta_2}(x, v_1, v_2, d) &= \Phi^*_{1,\varepsilon,\beta_2}(x) = -x(v_2 + v_1\beta_2) - \frac{d\tilde{A}}{\sqrt{\varepsilon x}}
\end{aligned}$$

in order to state and prove the next result:

Lemma 4.2.6 *For every* $\varepsilon \in \mathcal{U}_\varepsilon^*(v_1, v_2, d)$ *and any* $\beta_2 \in I_{\tilde{\beta}}^*(\varepsilon)$, *it follows that*

$$\begin{aligned}
\Psi^*_{\varepsilon,\beta_2}(k_2^{(n)}, v_1', v_2', d') - \Psi^*_{\varepsilon,\beta_2}(k_2^{(j)}, v_1', v_2', d') &\geq \mathit{ctant}\ \varepsilon^{-1/4} \\
\Psi^*_{\varepsilon,\beta_2}(k_2^{(n+1)}, v_1', v_2', d') - \Psi^*_{\varepsilon,\beta_2}(k_2^{(j)}, v_1', v_2', d') &\geq \mathit{ctant}\ \varepsilon^{-1/4}
\end{aligned}$$

for every Fibonacci number $k_2^{(j)} \in \mathcal{I}_\varepsilon^* = (\varepsilon^{-1/5}, \varepsilon^{-1/4} |\ln \varepsilon|^{1/8})$, $j \neq n$, $j \neq n+1$, n *the natural number for which* $\varepsilon \in \mathcal{U}_n^*$ *and every positive parameters* v_1', v_2' *and* d' *satisfying*

$$\max\{|v_1 - v_1'|, |v_2 - v_2'|, |d - d'|\} < \mathit{ctant}\ \varepsilon^\alpha,$$

for some constant $\alpha > 3/10$.

Proof

By definition of $\mathcal{U}_n^*$ it follows that $\Lambda_n^*(\varepsilon) \in I_{\tilde{\beta}}^*(\varepsilon)$. Therefore, Lemma 4.2.2 implies

$$\left|\Psi^*_{\varepsilon,\Lambda_n^*(\varepsilon)}(k_2^{(t)}, v_1', v_2', d') - \Psi^*_{\varepsilon,\beta_2}(k_2^{(t)}, v_1', v_2', d')\right| \leq \frac{1}{2} d' \varepsilon^{1/12} |\ln \varepsilon|^{1/8},$$

for every $\beta_2 \in I_{\tilde{\beta}}^*(\varepsilon)$ and any Fibonacci number $k_2^{(t)} \in \mathcal{I}_\varepsilon^*$. Now, since Lemma 4.2.3 implies, for every $x \in \mathcal{I}_\varepsilon^*$, that

$$\left|\Psi^*_{1,\varepsilon,\Lambda_n^*(\varepsilon)}(x, v_1', v_2', d') - \Psi^*_{\varepsilon,\Lambda_n^*(\varepsilon)}(x, v_1', v_2', d')\right| \leq \mathit{ctant}\ \varepsilon^{1/30} |\ln \varepsilon|^{1/4},$$

where the constant may depend on v_1', v_2' and d' but not on ε, we conclude that it is enough to prove Lemma 4.2.6 by replacing $\Psi^*_{\varepsilon,\beta_2}$ by $\Psi^*_{1,\varepsilon,\Lambda_n^*(\varepsilon)}$.

Furthermore, under the condition $x \in \mathcal{I}^*_\varepsilon$, we have

$$\max\left\{\left\|\frac{\partial \Psi^*_{1,\varepsilon,\beta_2}}{\partial v_1}\right\|, \left\|\frac{\partial \Psi^*_{1,\varepsilon,\beta_2}}{\partial v_2}\right\|, \left\|\frac{\partial \Psi^*_{1,\varepsilon,\beta_2}}{\partial d}\right\|\right\} \le ctant\ \varepsilon^{-3/10}$$

in such a way that it suffices to prove

$$\Psi^*_{1,\varepsilon,\Lambda^*_n(\varepsilon)}(k_2^{(l)}, v_1, v_2, d) - \Psi^*_{1,\varepsilon,\Lambda^*_n(\varepsilon)}(k_2^{(j)}, v_1, v_2, d) > ctant\ \varepsilon^{-1/4}$$

for $l = n$ and $l = n+1$. In this case, the fact that ctant $\varepsilon^{-1/4} \le k_2^{(l)} \le$ ctant $\varepsilon^{-1/4}$, $l = n, n+1$, implies the existence of two positive constants c_1^* and c_2^* for which

$$c_1^* \varepsilon^{1/4} \le \left|\frac{\partial^2}{\partial x^2}\Psi^*_{1,\varepsilon,\Lambda^*_n(\varepsilon)}(x, v_1, v_2, d)\right| \le c_2^* \varepsilon^{1/4},$$

for every $x \in [k_2^{(n)}, k_2^{(n+1)}]$.

Therefore, since we also have

$$\Psi^*_{1,\varepsilon,\Lambda^*_n(\varepsilon)}(k_2^{(n)}, v_1, v_2, d) = \Psi^*_{1,\varepsilon,\Lambda^*_n(\varepsilon)}(k_2^{(n+1)}, v_1, v_2, d),$$

it is easy to show the existence of some constant $\tilde{A}^* > 1$ satisfying

$$\frac{1}{\tilde{A}^*} \le \frac{k_2^{(n+1)} - c^*_\varepsilon}{c^*_\varepsilon - k_2^{(n)}} \le \tilde{A}^*,$$

where c^*_ε is the critical point of $\Psi^*_{1,\varepsilon,\Lambda^*_n(\varepsilon)}(\cdot, v_1, v_2, d)$. The rest of the arguments are exactly those leading to the proof of Lemma 4.1.9. □

Proceeding as in the previous section, let us fix three positive constants v_1, v_2 and d. For any $\varepsilon \in \mathcal{U}^*_\varepsilon(v_1, v_2, d)$ and any $\beta_2 \in I^*_{\tilde{\beta}}(\varepsilon)$, we define (recall that $\omega = (1, \beta_2)$)

$$\mathcal{A}^*_0 = \left\{\hat{k} \in \mathbb{Z}^2_+ :\ \left|\hat{k}\omega\right| \ge \frac{1}{2}\right\}, \qquad \mathcal{A}^* = \mathbb{Z}^2_+ \backslash \mathcal{A}^*_0 = \bigcup_{i=1}^{7} \mathcal{A}^*_i,$$

where, if $n \ge n^*$ is the unique natural number with $\varepsilon \in \mathcal{U}^*_n$, then

$$\begin{aligned}
\mathcal{A}^*_1 &= \{\hat{k} :\ k_2 > \varepsilon^{-1/4} |\ln \varepsilon|^{1/8}\},\\
\mathcal{A}^*_2 &= \{\hat{k} :\ k_2 < k_2^{(100)}\},\\
\mathcal{A}^*_3 &= \{\hat{k} :\ \hat{k} = \hat{k}^{(j)},\ j \ge 100,\ k_2^{(j)} < \varepsilon^{-1/5}\},\\
\mathcal{A}^*_4 &= \{\hat{k} :\ \hat{k} = \hat{k}^{(j)},\ k_2^{(j)} \in \mathcal{I}^*_\varepsilon,\ j \ne n,\ j \ne n+1\},\\
\mathcal{A}^*_5 &= \{\hat{k} :\ k_2 \in (k_2^{(j)}, k_2^{(j+1)}),\ k_2^{(j)} < \varepsilon^{-1/5},\ j \ge 100\},\\
\mathcal{A}^*_6 &= \{\hat{k} :\ k_2 \in (k_2^{(j)}, k_2^{(j+1)}),\ \text{for some } (k_2^{(j)}, k_2^{(j+1)}) \subset \mathcal{I}^*_\varepsilon\},\\
\mathcal{A}^*_7 &= \{\hat{k}^{(n)}, \hat{k}^{(n+1)}\},
\end{aligned}$$

with $k_1^{(j)}/k_2^{(j)}$ denoting the best approximations to β_2.

Now, let us take a series

$$\hat{S} = 2 \sum_{\hat{k}\in\mathbb{Z}_+^2} S_{\hat{k}} \overline{\mathcal{E}}_{\hat{k}} \in \mathcal{S}^*(v_1', v_2', d', \varepsilon, \beta_2), \quad \overline{\mathcal{E}}_{\hat{k}} = \overline{\mathcal{E}}_{\hat{k}}(v_1', v_2', d', \varepsilon, \beta_2)$$

with

$$\max\{|v_1 - v_1'|, |v_2 - v_2'|, |d - d'|\} < ctant\ \varepsilon^{\alpha},$$

for some constant $\alpha > 3/10$.

We define

$$N_i^* = N_i^*(v_1', v_2', d', \varepsilon, \beta_2) = \sum_{\hat{k}\in\mathcal{A}_i^*} S_{\hat{k}} \overline{\mathcal{E}}_{\hat{k}}, \quad \overline{\mathcal{E}}_{\hat{k}} = \overline{\mathcal{E}}_{\hat{k}}(v_1', v_2', d', \varepsilon, \beta_2),$$

for $i = 0, \ldots, 7$. As in the previous section, it is easy to conclude that the Main Lemma II and the second Perturbing Lemma are direct consequences (by taking $n^0 = n$ and $n^1 = n+1$) of the following result:

Lemma 4.2.7 *If $S_{\hat{k}(n)} S_{\hat{k}(n+1)} > 0$, then, for $i = 0, 1, \ldots, 6$, it holds that*

$$\frac{|N_i^*(v_1', v_2', d', \varepsilon, \beta_2)|}{|N_7^*(v_1', v_2', d', \varepsilon, \beta_2)|} \leq \exp\left(-\frac{|\ln \varepsilon|^{-1/4}}{\varepsilon^{1/4}}\right),$$

whenever v_1', v_2', d' are positive parameters satisfying

$$\max\{|v_1 - v_1'|, |v_2 - v_2'|, |d - d'|\} < ctant\ \varepsilon^{\alpha},$$

for some constant $\alpha > 3/10$.

Proof

Bearing in mind that, for $j = 1, 2$ and $t = n, n+1$, ctant $\varepsilon^{-1/4} \leq k_j^{(t)} \leq$ ctant $\varepsilon^{-1/4}$, we obtain (see (1.4.127) and Lemma 4.2.1)

$$|A_t(\beta_2)| \leq \left|A_t(\tilde{\beta})\right| + ctant\ \varepsilon^{1/3} \leq ctant.$$

Therefore, one may use that $\hat{S} \in \mathcal{S}_2^*(v_1', v_2', d', \varepsilon, \beta_2)$ and the assumption $S_{\hat{k}(n)} S_{\hat{k}(n+1)} > 0$ to check that

$$|N_7^*| > ctant\ \varepsilon^{\mathcal{X}_2'/4} \left(\overline{\mathcal{E}}_{\hat{k}(n)} + \overline{\mathcal{E}}_{\hat{k}(n+1)}\right) > \exp\left(-\frac{ctant}{\varepsilon^{1/4}}\right). \tag{4.2.30}$$

On the other hand, since

$$|k_1|\, v_1' + |k_2|\, v_2' > \min\{v_1', v_2'\} \left|\hat{k}\right|,$$

from the fact that $\hat{S} \in \mathcal{S}_1^*(v_1', v_2', d', \varepsilon, \beta_2)$ it is easy to see that

$$|N_0^*| \leq W_1' \exp\left(-\frac{d'}{2\sqrt{\varepsilon}}\right) \sum_{\hat{k}\in\mathbb{Z}^2} \left|\hat{k}\right|^{\mathcal{X}_1'} \exp\left(-ctant\ \left|\hat{k}\right|\right) \leq ctant\ \exp\left(-\frac{d'}{2\sqrt{\varepsilon}}\right).$$

Moreover,

$$|N_1^*| \leq W_1' \sum_{k_2>\varepsilon^{-1/4}|\ln\varepsilon|^{1/8}} (4k_2)^{\mathcal{X}_1'} \exp(-k_2 v_2') \leq ctant\ \exp\left(-\frac{ctant\ |\ln\varepsilon|^{1/8}}{\varepsilon^{1/4}}\right).$$

Now, if $\hat{k} \in \mathcal{A}_2^*$, we have

$$\overline{\mathcal{E}}_{\hat{k}} < \exp\left(-\frac{d'\left|\hat{k}\omega\right|}{\sqrt{\varepsilon}}\right) < \exp\left(-\frac{ctant}{\sqrt{\varepsilon}}\right)$$

in such a way that

$$|N_2^*| < ctant\ \exp\left(-\frac{ctant}{\sqrt{\varepsilon}}\right).$$

For any $\hat{k} = \hat{k}^{(j)} \in \mathcal{A}_3^*$ it is easy to see that $6\left|\hat{k}^{(j)}\omega\right| > \varepsilon^{1/5}$, from which

$$\overline{\mathcal{E}}_{\hat{k}^{(j)}} < \exp\left(-\frac{d'}{6\varepsilon^{3/10}}\right).$$

Hence, since $|S_{\hat{k}^{(j)}}| < W_1'(4k_2^{(j)})^{\mathcal{X}_1'} < ctant\ \varepsilon^{-\mathcal{X}_1'/5}$, we get, for ε small enough,

$$|N_3^*| \leq ctant\ \varepsilon^{-\frac{1}{5}(1+\mathcal{X}_1')} \exp\left(-\frac{d'}{6\varepsilon^{3/10}}\right) < \exp\left(-\frac{d'}{12\varepsilon^{3/10}}\right).$$

Therefore, using (4.2.30), we deduce, for $i = 0, 1, 2, 3$, that

$$\frac{|N_i^*|}{|N_7^*|} \leq \exp\left(-\frac{|\ln\varepsilon|^{-1/4}}{\varepsilon^{1/4}}\right).$$

Let $\hat{k} = \hat{k}^{(j)}$, with $k_2^{(j)} \in \mathcal{I}_\varepsilon^* = (\varepsilon^{-1/5}, \varepsilon^{-1/4}\,|\ln\varepsilon|^{1/8})$ and $j \neq n$, $j \neq n+1$. Since

$$|S_{\hat{k}^{(j)}}| < W_1' \left|\hat{k}^{(j)}\right|^{\mathcal{X}_1'} < ctant\ \varepsilon^{-\mathcal{X}_1'/4}\,|\ln\varepsilon|^{\mathcal{X}_1'/8},$$

we may use (4.2.30) to write

$$\frac{\left|S_{\hat{k}^{(j)}}\overline{\mathcal{E}}_{\hat{k}^{(j)}}\right|}{|N_7^*|} < ctant\ \varepsilon^{-\frac{\mathcal{X}_1'+\mathcal{X}_2'}{4}}\,|\ln\varepsilon|^{\mathcal{X}_1'/8}\,\frac{\overline{\mathcal{E}}_{\hat{k}^{(j)}}}{\overline{\mathcal{E}}_{\hat{k}^{(n)}} + \overline{\mathcal{E}}_{\hat{k}^{(n+1)}}}.$$

Therefore, Lemma 4.2.6 yields

$$\frac{\left|S_{\hat{k}^{(j)}}\overline{\mathcal{E}}_{\hat{k}^{(j)}}\right|}{|N_7^*|} < \exp\left(-\frac{c^*}{2\varepsilon^{1/4}}\right) \tag{4.2.31}$$

for some positive constant c^* which may depend on v_1', v_2' and d' but not on ε. Hence,

$$\frac{|N_4^*|}{|N_7^*|} \leq (\operatorname{card} \mathcal{A}_4^*) \exp\left(-\frac{c^*}{2\varepsilon^{1/4}}\right) \leq \exp\left(-\frac{|\ln\varepsilon|^{-1/4}}{\varepsilon^{1/4}}\right).$$

Now, let $\hat{k} = (k_1, k_2) \in \mathcal{A}_5^*$ ($k_2 \in (k_2^{(j)}, k_2^{(j+1)})$ for some $j \geq 100$ and $k_2^{(j)} < \varepsilon^{-1/5}$). Then, since Proposition 4.1.12 gives

$$\left|\hat{k}\omega\right| > \frac{ctant}{k_2^{(j)}} > ctant\ \varepsilon^{1/5},$$

we have

$$\overline{\mathcal{E}}_{\hat{k}} < \exp\left(-\frac{ctant}{\varepsilon^{3/10}}\right).$$

Let us choose $\tilde{n}^*$ the first natural number for which

$$k_2^{(\tilde{n}^*)} > \varepsilon^{-9/40}.$$

Then, the inequalities

$$\max\left\{\left|k_1^{(\tilde{n}^*)}\right|, k_2^{(\tilde{n}^*)}\right\} < ctant\ \varepsilon^{-9/40}, \qquad \frac{\left|\hat{k}^{(\tilde{n}^*)}\omega\right|}{\sqrt{\varepsilon}} < ctant\ \varepsilon^{-11/40}$$

allow us to write

$$\overline{\mathcal{E}}_{\hat{k}^{(\tilde{n}^*)}} > \exp\left(-ctant\ \varepsilon^{-11/40}\right).$$

Then, the assumption $\hat{S} \in \mathcal{S}_1^*(v_1', v_2', d', \varepsilon, \beta_2) \cap \mathcal{S}_2^*(v_1', v_2', d', \varepsilon, \beta_2)$ leads to

$$\frac{\left|S_{\hat{k}}\overline{\mathcal{E}}_{\hat{k}}\right|}{\left|S_{\hat{k}^{(\tilde{n}^*)}}\overline{\mathcal{E}}_{\hat{k}^{(\tilde{n}^*)}}\right|} \leq ctant\ \varepsilon^{-\frac{1}{5}\mathcal{X}_1' - \frac{9}{40}\mathcal{X}_2'} \exp\left(-\frac{ctant}{\varepsilon^{11/40}}\right)$$

and therefore, using also (4.2.31) (note that $\tilde{n}^* \neq n$, $\tilde{n}^* \neq n+1$), we obtain

$$\begin{aligned} \frac{|N_5^*|}{|N_7^*|} \ &\leq \ \frac{\left|S_{\hat{k}^{(\tilde{n}^*)}}\overline{\mathcal{E}}_{\hat{k}^{(\tilde{n}^*)}}\right|}{|N_7^*|} \sum_{k \in \mathcal{A}_5^*} \frac{\left|S_{\hat{k}}\overline{\mathcal{E}}_{\hat{k}}\right|}{\left|S_{\hat{k}^{(\tilde{n}^*)}}\overline{\mathcal{E}}_{\hat{k}^{(\tilde{n}^*)}}\right|} \leq \\ &\leq \ \exp\left(-\frac{c^*}{2\varepsilon^{1/4}}\right) \exp\left(-\frac{ctant}{\varepsilon^{11/40}}\right) < \exp\left(-\frac{|\ln\varepsilon|^{-1/4}}{\varepsilon^{1/4}}\right). \end{aligned}$$

Finally, we write

$$\mathcal{A}_6^* = \bigcup_{j=\hat{n}^*}^{\hat{m}^*-1} \mathcal{A}_6^{j,*}$$

where $\hat{n}^*$ is the smallest natural number with $k_2^{(\hat{n}^*)} > \varepsilon^{-1/5}$, $\hat{m}^*$ is the largest one with $k_2^{(\hat{m}^*)} < \varepsilon^{-1/4} |\ln \varepsilon|^{1/8}$ and

$$\mathcal{A}_6^{j,*} = \left\{ \hat{k} \in \mathcal{A}_6^* : \ k_2 \in (k_2^{(j)}, k_2^{(j+1)}) \right\}, \quad j = \hat{n}^*, \ldots, \hat{m}^* - 1.$$

Let us write

$$N_6^* = \sum_{j=\hat{n}^*}^{\hat{m}^*-1} N_6^{j,*}, \qquad N_6^{j,*} = \sum_{\hat{k} \in \mathcal{A}_6^{j,*}} S_{\hat{k}} \overline{\mathcal{E}}_{\hat{k}}.$$

For every $\hat{k} \in \mathcal{A}_6^{j,*}$ let us observe that

$$\frac{\left|S_{\hat{k}} \overline{\mathcal{E}}_{\hat{k}}\right|}{\left|S_{\hat{k}^{(j)}} \overline{\mathcal{E}}_{\hat{k}^{(j)}}\right|} \leq ctant\ \varepsilon^{-\frac{\varkappa_1' + \varkappa_2'}{4}} |\ln \varepsilon|^{\frac{\varkappa_1' + \varkappa_2'}{8}} \frac{\overline{\mathcal{E}}_{\hat{k}}}{\overline{\mathcal{E}}_{\hat{k}^{(j)}}},$$

with, using Proposition 4.1.12,

$$\frac{\overline{\mathcal{E}}_{\hat{k}}}{\overline{\mathcal{E}}_{\hat{k}^{(j)}}} < \exp\left(\frac{d'}{\sqrt{\varepsilon}} \left(\left|\hat{k}^{(j)}\omega\right| - \left|\hat{k}\omega\right|\right)\right) < \exp\left(-\frac{d'\left|\hat{k}^{(j+1)}\omega\right|}{\sqrt{\varepsilon}}\right) < \exp\left(-ctant\ \frac{|\ln \varepsilon|^{-1/8}}{\varepsilon^{1/4}}\right).$$

Therefore, since from (4.2.30) and (4.2.31), we get

$$\frac{\left|S_{\hat{k}^{(j)}} \overline{\mathcal{E}}_{\hat{k}^{(j)}}\right|}{|N_7^*|} \leq ctant\ \varepsilon^{-\frac{\varkappa_1' + \varkappa_2'}{4}}, \quad j = \hat{n}^*, \ldots, \hat{m}^* - 1,$$

we finally obtain

$$\begin{aligned} \frac{|N_6^*|}{|N_7^*|} \ &\leq \ \sum_{j=\hat{n}^*}^{\hat{m}^*-1} \frac{\left|N_6^{j,*}\right|}{\left|S_{\hat{k}^{(j)}} \overline{\mathcal{E}}_{\hat{k}^{(j)}}\right|} \frac{\left|S_{\hat{k}^{(j)}} \overline{\mathcal{E}}_{\hat{k}^{(j)}}\right|}{|N_7^*|} \leq ctant\ |\ln \varepsilon|^{1/8} \varepsilon^{-\frac{\varkappa_1' + \varkappa_2' + 1}{4}} \sum_{\hat{k} \in \mathcal{A}_6^{j,*}} \frac{\left|S_{\hat{k}} \overline{\mathcal{E}}_{\hat{k}}\right|}{\left|S_{\hat{k}^{(j)}} \overline{\mathcal{E}}_{\hat{k}^{(j)}}\right|} \leq \\ &\leq \ ctant\ \left(\varepsilon^{-1} |\ln \varepsilon|^{1/4}\right)^{\frac{\varkappa_1' + \varkappa_2' + 1}{2}} \exp\left(-ctant\ \frac{|\ln \varepsilon|^{-1/8}}{\varepsilon^{1/4}}\right) \leq \exp\left(-\frac{|\ln \varepsilon|^{-1/4}}{\varepsilon^{1/4}}\right). \end{aligned}$$

Now, Lemma 4.2.7 is proved. □

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