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On qualitative properties of a system containing a singular parabolic functional equation

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Abstract

We consider a system consisting of a quasilinear parabolic equation and a first order ordinary differential equation containing functional dependence on the unknown functions. The existence and some properties of solutions in $(0, \infty)$ will be proved.

Introduction

In this work we consider initial-boundary value problems for the system

$$D_t u - \sum_{i=1}^{n} D_i[a_i(t, x, u(t, x), Du(t, x); u, w)]+ \tag{0.1}$$

$$a_0(t, x, u(t, x), Du(t, x); u, w) = f(t, x),$$

$$D_t w = F(t, x; u, w) \text{ in } Q_T = (0, T) \times \Omega, \quad T \in (0, \infty) \tag{0.2}$$

where the functions

$$a_i : Q_T \times \mathbb{R}^{n+1} \times L^p(0, T; V_1) \times L^2(Q_T) \to \mathbb{R}$$

(with a closed linear subspace V_1 of the Sobolev space $W^{1,p}(\Omega)$, $2 \le p < \infty$) satisfy conditions which are generalizations of the usual conditions for quasilinear parabolic differential equations, considered by using the theory of monotone type operators (see, e.g., [2], [7], [13]) but the equation (0.1) is not uniformly parabolic in a sense, analogous to the linear case. Further,

$$F : Q_T \times L^p(0, T; V_1) \times L^2(Q_T) \to \mathbb{R}$$

satisfies a Lipschitz condition.

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In [12] the existence of weak solutions in Q_T was proved. In this present paper this result will be extended to $Q_\infty = (0, \infty) \times \Omega$ and some properties (boundedness, asymptotic property as $t \to \infty$) of the solutions will be shown.

Such problems arise, e.g., when considering diffusion and transport in porous media with variable porosity, see [5], [8]. In [8] J.D. Logan, M.R. Petersen, T.S. Shores considered and numerically studied a nonlinear system, consisting of a parabolic, an elliptic and an ODE which describes reaction-mineralogy-porosity changes in porous media. System (0.1), (0.2) was motivated by that system. In [3], [4] Á. Besenyei considered a more general system of a parabolic PDE, an elliptic PDE and an ODE.

1 Existence of solutions

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain having the uniform C^1 regularity property (see [1]) and $p \geq 2$ be a real number. Denote by $W^{1,p}(\Omega)$ the usual Sobolev space of real valued functions with the norm

$$\| u \| = \left[\int_\Omega (|Du|^p + |u|^p) \right]^{1/p}.$$

Let $V_1 \subset W^{1,p}(\Omega)$ be a closed linear subspace containing $W_0^{1,p}(\Omega)$ (the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$). Denote by $L^p(0,T;V_1)$ the Banach space of the set of measurable functions $u : (0,T) \to V_1$ such that $\| u \|_{V_1}^p$ is integrable and define the norm by

$$\| u \|_{L^p(0,T;V_1)}^p = \int_0^T \| u(t) \|_{V_1}^p \, dt.$$

The dual space of $L^p(0,T;V_1)$ is $L^q(0,T;V_1^\star)$ where $1/p + 1/q = 1$ and $V_1^\star$ is the dual space of V_1 (see, e.g., [7], [13]).

On functions a_i we assume:

(A_1). The functions $a_i : Q_T \times \mathbb{R}^{n+1} \times L^p(0,T;V_1) \times L^2(\Omega) \to \mathbb{R}$ satisfy the Carathéodory conditions for arbitrary fixed $(u,w) \in L^p(0,T;V_1) \times L^2(\Omega)$ $(i = 0,1,...,n)$.

(A_2). There exist bounded (nonlinear) operators $g_1 : L^p(0,T;V_1) \times L^2(\Omega) \to \mathbb{R}^+$ and $k_1 : L^p(0,T;V_1) \times L^2(\Omega) \to L^q(\Omega)$ such that

$$|a_i(t,x,\zeta_0,\zeta;u,w)| \leq g_1(u,w)[|\zeta_0|^{p-1} + |\zeta|^{p-1}] + [k_1(u,w)](x), \quad i = 0,1,...,n$$

for a.e. $(t,x) \in Q_T$, each $(\zeta_0, \zeta) \in \mathbb{R}^{n+1}$ and $(u,w) \in L^p(0,T;V_1) \times L^2(\Omega)$.

(A_3). $\sum_{i=1}^n [a_i(t,x,\zeta_0,\zeta;u,w) - a_i(t,x,\zeta_0,\zeta^\star;u,w)](\zeta_i - \zeta_i^\star) > 0$ if $\zeta \neq \zeta^\star$.

(A_4). There exist bounded operators $g_2 : L^p(0,T;V_1) \times L^2(\Omega) \to C[0,T]$, $k_2 : L^p(0,T;V_1) \times L^2(\Omega) \to L^1(Q_T)$ such that

$$\sum_{i=0}^n a_i(t,x,\zeta_0,\zeta;u,w)\zeta_i \geq [g_2(u,w)](t)[|\zeta_0|^p + |\zeta|^p] - [k_2(u,w)](t,x)$$

for a.e. $(t,x) \in Q_T$, all $(\zeta_0, \zeta) \in \mathbb{R}^{n+1}$, $(u,w) \in L^p(0,T;V_1) \times L^2(\Omega)$ and (with some positive constants)

$$[g_2(u,w)]t) \geq \text{const}(1+ \parallel u \parallel_{L^p(0,t;V_1)})^{-\sigma^\star}(1+ \parallel w \parallel_{L^2(Q_t)})^{-\beta^\star} \tag{1.3}$$

$$\parallel k_2(u,w) \parallel_{L^1(Q_t)} \leq \text{const}(1+ \parallel u \parallel_{L^p(0,t;V_1)})^{\sigma}(1+ \parallel w \parallel_{L^2(Q_t)})^{\beta} \tag{1.4}$$

where

$$0 < \sigma^\star < p-1, \quad 0 \leq \sigma < p - \sigma^\star$$

and $\beta, \beta^\star \geq 0$ satisfy

$$\beta^\star + \sigma^\star < p-1, \quad \beta^\star + \sigma^\star + \beta + \sigma < p.$$

(A_5). There exists $\delta > 0$ such that if $(u_k) \to u$ weakly in $L^p(0,T;V_1)$, strongly in $L^p(0,T;W^{1-\delta,p}(\Omega))$ and $(w_k) \to w$ in $L^2(\Omega)$ then for $i = 0,1,...,n$

$$a_i(t,x,u_k(t,x), Du_k(t,x); u_k, w_k) - a_i(t,x,u_k(t,x), Du_k(t,x); u, w) \to 0$$

in $L^q(Q_T)$.

Definition Assuming (A_1)-(A_5) we define operator $A : L^p(0,T;V_1) \times L^2(Q_T) \to L^q(0,T;V_1^\star)$ by

$$[A(u,w), v] = \int_0^T \langle A(u,w)(t), v(t) \rangle dt =$$

$$\int_{Q_T} \left\{ \sum_{i=1}^n a_i(t,x,u(t,x), Du(t,x); u,w) D_i v + a_0(t,x,u(t,x), Du(t,x); u,w) v \right\} dtdx,$$

$$(u,w) \in L^p(0,T;V_1) \times L^2(Q_T), \quad v \in L^p(0,T;V_1)$$

where the brackets $\langle \cdot, \cdot \rangle$, $[\cdot, \cdot]$ mean the dualities in spaces $V_1^\star, V_1$ and $L^q(0,T;V_1^\star)$, $L^p(0,T;V_1)$, respectively.

On function $F : Q_T \times L^p(0,T;V_1) \times L^2(Q_T) \to \mathbb{R}$ assume

(F_1). For each fixed $(u,w) \in L^p(0,T;V_1) \times L^2(Q_T)$, $F(\cdot; u,w) \in L^2(Q_T)$.

(F_2). F satisfies the following (global) Lipschitz condition: for each $t \in (0,T]$, $(u,w), (u^\star, w^\star) \in X$ we have

$$\parallel F(\cdot; u,w) - F(\cdot; u^\star, w^\star) \parallel^2_{L^2(Q_t)} \leq$$

$$K \left[\parallel u - u^\star \parallel^2_{L^p(0,t;W^{1-\delta,p}(\Omega))} + \parallel w - w^\star \parallel^2_{L^2(Q_t)} \right].$$

In [12] the following theorem was proved.

Theorem 1.1 *Assume (A_1) - (A_5) and (F_1), (F_2). Then for any $f \in L^q(0,T;V_1^\star)$ and $w_0 \in L^2(Q_T)$ there exists $u \in L^p(0,T;V_1)$, $w \in L^2(Q_T)$ such that $D_t u \in L^q(0,T;V_1^\star)$, $D_t w \in L^2(Q_T)$ and*

$$D_t u + A(u,w) = f, \quad u(0) = 0,. \tag{1.5}$$

$$D_t w = F(t,x;u,w) \textit{ for a.e. } (t,x) \in Q_T, \quad w(0) = w_0. \tag{1.6}$$

Now assume

(F_1') F has the form $F(t,x;u,w) = \hat{F}(t,x,w(t,x);u,w)$ and $\hat{F} : Q_T \times \mathbb{R} \times X \to R$ satisfies: for each fixed $(u,w) \in L^p(0,T;V_1) \times L^2(Q_T)$, $\xi \in \mathbb{R}$, $\hat{F}(\cdot,\xi;u,w) \in L^2(Q_T)$.

(F_2') There exist constants K, $K_1(c_1)$ such that if $|\xi|, |\xi^\star| \le c_1$ then for each $t \in (0,T]$, $(u,w), (u^\star,w^\star) \in L^p(0,T;V_1) \times L^2(Q_T$ with the property $|w|, |w^\star| \le c_1$ in Q_T

$$|\hat{F}(t,x,\xi;u,w) - \hat{F}(t,x,\xi^\star;u^\star,w^\star)|^2 \le$$

$$K \parallel u - u^\star \parallel^2_{L^p(0,t;W^{1-\delta,p}(\Omega))} + K_1(c_1)\left[\parallel w - w^\star \parallel^2_{L^2(Q_t)} + |\xi - \xi^\star|^2\right].$$

(F_3') There exists a constant $c_0 > 0$ such that for a.e. (t,x) and all u, w

$$\hat{F}(t,x,\xi;u,w)\xi \le 0 \text{ if } |\xi| \ge c_0.$$

Theorem 1.2 *Assume (A_1) - (A_5) and (F_1') - (F_3') such that operators g_2, k_2 in (A_4) satisfy the following modified inequalities instead of (1.3) and (1.4):*

$$[g_2(u,w)](t) \ge const(1+ \parallel u \parallel_{L^p(0,t;V_1)})^{-\sigma^\star}(1 + g_3(\parallel w \parallel_{L^2(Q_t)})^{-1},$$

$$\parallel k_2(u,w) \parallel_{L^1(Q_t)} \le const(1+ \parallel u \parallel_{L^p(0,t;V_1)})^{\sigma}(1+ \parallel w \parallel_{L^2(Q_t)})$$

where g_3, g_4 are monotone nondecreasing positive functions, $0 < \sigma^\star < p - 1$, $0 \le \sigma < p - \sigma^\star$.

Then for any $f \in L^q(0,T;V_1^\star)$ and $w_0 \in L^2(Q_T)$ there exists $u \in L^p(0,T;V_1)$, $w \in L^2(Q_T)$ such that $D_t u \in L^q(0,T;V_1^\star)$, $D_t w \in L^2(Q_T)$ and (1.5), (1.6) hold.

This theorem is a consequence of Theorem 1.1 (see also [12]): set

$$c_0^\star = \max\{c_0, \parallel w_0 \parallel_{L^\infty(\Omega)}\}$$

and let $\chi \in C_0^\infty(\mathbb{R})$ be such that $\chi(\xi) = \xi$ for $|\xi| \le c_0^\star$ and define functions $\tilde{F}$, $\tilde{a}_i$ by

$$\tilde{F}(t,x;u,w) = \hat{F}(t,x,\chi(w(t,x));u,\chi(w)),$$

$$\tilde{a}_i(t,x,\zeta_0,\zeta;u,w) = a_i(t,x,\zeta_0,\zeta;u,\chi(w)),$$

Then by Theorem 1.1 there exists a solution (u,w) of (1.5), (1.6) with $\tilde{F}$, $\tilde{a}_i$ (instead of F, a_i, respectively). It is not difficult to show that for a.e. $x \in \Omega$, all $t \in [0,T]$, $|w(t,x)| \le c_0^\star$ by (F_3') and so (u,w) satisfies the original problem, too.

Now we formulate existence theorems in $(0,\infty)$. Denote by $L^p_{loc}(0,\infty;V_1)$ the set of functions $v : (0,\infty) \to V_1$ such that for each fixed finite $T > 0$, $v|_{(0,T)} \in L^p(0,T;V_1)$ and let $Q_\infty = (0,\infty) \times \Omega$, $L^\alpha_{loc}(Q_\infty)$ the set of functions $v : Q_\infty \to R$ such that $v|_{Q_T} \in L^\alpha(Q_T)$ for any finite $T > 0$.

Theorem 1.3 *Assume that the functions*

$$a_i : Q_\infty \times \mathbb{R}^{n+1} \times L^p_{loc}(0,\infty;V_1) \times L^2_{loc}(Q_\infty \to \mathbb{R}$$

satisfy the assumptions (A_1) - (A_5) *for any finite* T *and that* $a_i(t,x,\zeta_0,\zeta;u,w)|_{Q_T}$ *depend only on* $u\,|_{(0,T)}$ *and* $w\,|_{Q_T}$ *(Volterra property). Further, the function*

$$F : Q_\infty \times L^p_{loc}(0,\infty;V_1) \times L^2_{loc}(Q_\infty \to \mathbb{R}$$

satisfies (F_1), (F_2) *for arbitrary fixed* T *and has the Volterra property.*

Then for each $f \in L^q_{loc}(0,\infty;V_1^\star)$, $w_0 \in L^2(\Omega)$ *there exist* $u \in L^p_{loc}(0,\infty;V_1)$, $w \in L^2_{loc}(Q_\infty)$ *which satisfy (1.5), (1.6) for any finite* T.

The idea of the proof. The Volterra property implies that if u, w are solutions in Q_T then for arbitrary $\tilde{T} < T$, their restriction to $Q_{\tilde{T}}$ are solutions in $Q_{\tilde{T}}$. Therefore, if $\lim(T_j) = +\infty$, $T_1 < T_2 < ... < T_j < ...$ and u_j, w_j are solutions in Q_{T_j} then, by using a 'diagonal process', we can select a subsequence of (u_j, w_j) which converges in Q_T for arbitrary finite T to (u, w), a solution of (1.5), (1.6) in Q_∞. (For more details see, e.g., [10].) Similarly can be proved

Theorem 1.4 *Assume that the functions*

$$a_i : Q_\infty \times \mathbb{R}^{n+1} \times L^p_{loc}(0,\infty;V_1) \times L^2_{loc}(Q_\infty \to \mathbb{R}$$

satisfy the assumptions of Theorem 1.2 for any finite T *and they have the Volterra property; the function*

$$\hat{F} : Q_\infty \times \mathbb{R} \times L^p_{loc}(0,\infty;V_1) \times L^2_{loc}(Q_\infty \to \mathbb{R}$$

satisfies (F_1') - (F_3') *for arbitrary fixed* T *and has the Volterra property.*

Then for each $f \in L^q_{loc}(0,\infty;V_1^\star)$, $w_0 \in L^2(\Omega)$ *there exist* $u \in L^p_{loc}(0,\infty;V_1)$, $w \in L^2_{loc}(Q_\infty)$ *which satisfy (1.5), (1.6) for any finite* T.

2 Boundedness and stabilization

Theorem 2.1 *Assume that the functions* $a_i, \hat{F}$ *satisfy the conditions of Theorem 1.4 such that for all* $u \in L^p_{loc}(0,\infty;V_1)$, $w \in L^\infty_{loc}(Q_\infty)$, $t \in (0,\infty)$ *the operators* g_2, k_2 *in* (A_4) *satisfy*

$$[g_2(u,w)](t) \geq const \left[1 + \sup_{\tau\in[0,T]} y(\tau)\right]^{-\sigma^\star/2} \cdot \left[1 + g_3(\sup_{\tau\in[0,T]} z(\tau))\right]^{-1} \tag{2.7}$$

$$\int_\Omega [k_2(u,w)](t,x)dx \leq \tag{2.8}$$

$$const \left[1 + \sup_{\tau\in[0,T]} y(\tau)^{\sigma/2} + \varphi(t) \sup_{\tau\in[0,T]} y(\tau)^{(p-\sigma^\star)/2}\right] \cdot \left[1 + g_4(\sup_{\tau\in[0,T]} z(\tau))\right]$$

where $0 < \sigma^\star < p-1$, $0 < \sigma < p - \sigma^\star$, $\lim_\infty \varphi = 0$, g_3, g_4 *are monotone nondecreasing positive functions,*

$$y(\tau) = \int_\Omega u(\tau,x)^2 dx, \quad z(\tau) = \| w(\tau,\cdot) \|_{L^\infty(\Omega)} .$$

Further, the constant c_0 *in* (F_3') *is independent of* T, $\| f(t) \|_{V_1^\star}$ *is bounded for* $t \in (0,\infty)$.

Then for a solution $u \in L^p_{loc}(0,\infty;V_1)$, $w \in L^\infty_{loc}(Q_\infty)$ *of (1.5), (1.6) with* $w_0 \in L^\infty(\Omega)$ *and arbitrary initial condition on* u, y *and* z *are bounded in* $(0,\infty)$.

Proof Since the constant c_0 in (F_3') is independent of T, it is easy to show that

$$|w(t,x)| \le \max\left\{c_0, \| w_0 \|_{L^\infty(\Omega)}\right\}$$

for a.e. $x \in \Omega$, all $t > 0$ (see the idea of the proof of Theorem 1.2, i.e. z is bounded.

Further, applying (1.5) to $u(t) \in V_1$, by (A_4) we obtain

$$\frac{1}{2}y'(t) + [g_2(u,w)](t) \| u(t) \|^p_{V_1} - \int_\Omega [k_2(u,w)](t,x)dx \le \tag{2.9}$$

$$\langle f(t), u(t)\rangle \le \| f(t) \|_{V_1^\star} \| u(t) \|_{V_1} \le \text{const} \| u(t) \|_{V_1}$$

since $\| f(t) \|_{V_1^\star}$ is bounded. Young's inequality implies

$$\| u(t) \|_{V_1} \le \varepsilon [g_2(u,w)](t)^{1/p} \| u(t) \|_{V_1} \cdot \frac{1}{\varepsilon [g_2(u,w)](t)^{1/p}} \le \tag{2.10}$$

$$\frac{\varepsilon^p}{p}[g_2(u,w)](t) \| u(t) \|^p_{V_1} + \frac{1}{q\varepsilon^q [g_2(u,w)](t)^{q/p}}.$$

Choosing sufficiently small $\varepsilon < 0$, one obtains from (2.9), (2.10)

$$\frac{1}{2}y'(t) + \frac{1}{2}[g_2(u,w)](t) \| u(t) \|^p_{V_1} \le \int_\Omega [k_2(u,w)](t,x)dx + \text{const}[g_2(u,w)](t)^{-q/p} \tag{2.11}$$

Since by Hölder's inequality , $p \ge 2$,

$$\| u(t) \|^p_{V_1} \ge \text{const} y(t)^{p/2},$$

(2.7), (2.8), (2.11) and the boundedness of z imply (with some positive constant $c^\star$)

$$y'(t) + c^\star y(t)^{p/2}\left[1 + \sup_{\tau\in[0,T]} y(\tau)\right]^{-\sigma^\star/2} \le \tag{2.12}$$

$$\text{const}\left[1 + \sup_{\tau\in[0,T]} y(\tau)^{\sigma/2} + \varphi(t) \sup_{\tau\in[0,T]} y(\tau)^{(p-\sigma^\star)/2} + \sup_{\tau\in[0,T]} y(\tau)^{(q/p)(\sigma^\star/2)}\right].$$

Since $0 \le \sigma < p - \sigma^\star < p$, $(q/p)\sigma^\star < p - \sigma^\star$, $\lim_\infty \varphi = 0$, it is not difficult to show that (2.12) implies the boundedness of $y(t)$ (see [11]).

Now we formulate an attractivity result.

Theorem 2.2 *Assume that the functions $a_i, \hat{F}$ satisfy the conditions of Theorem 2.1 such that for all $u \in L^p_{loc}(0, \infty; V_1)$, $w \in L^\infty_{loc}(Q_\infty)$, $t \in (0, \infty)$*

$$\int_\Omega [k_2(u,w)](t,x)dx \le \varphi(t) \left[\sup_{\tau \in [0,T]} y(\tau)^{(p-\sigma^\star)/2} \right] \cdot \left[1 + g_4(\sup_{\tau \in [0,T]} z(\tau)) \right]. \tag{2.13}$$

Further,

$$\lim_{t \to \infty} \| f(t) \|_{V_1^\star} = 0, \tag{2.14}$$

$$\xi \hat{F}(t, x, \xi; u, w) \le -g(\xi)\xi \tag{2.15}$$

with a strictly monotone increasing continuous function g satisfying $g(0) = 0$.

Then for a solution $u \in L^p_{loc}(0, \infty; V_1)$, $w \in L^\infty_{loc}(Q_\infty)$ of (1.5), (1.6) with $w_0 \in L^\infty(\Omega)$ and arbitrary initial condition on u, for the functions defined in Theorem 2.1 we have

$$\lim_\infty y = 0, \tag{2.16}$$

$$\lim_\infty z = 0. \tag{2.17}$$

Proof By (1.6) and (2.15) for a.e. $x \in \Omega$, $t \mapsto w(t,x)$ is continuous and monotone decreasing and for a.e. (t,x)

$$D_t w(t,x) \le -g(w(t,x)) \text{ if } w(t,x) > 0$$

thus for a.e. $x \in \Omega$ satisfying $w_0(x) > 0$,

$$w(t,x) \le w_0(x) - tg(w(t,x)) \text{ for a.e. } x \in \Omega \text{ until } w(t,x) > 0$$

(g is monotone increasing, $t \mapsto w(t,x)$ is monotone decreasing). Consequently,

$$tg(w(t,x)) \le w_0(x), \text{ thus } w(t,x) \le g^{-1}\left(\frac{\| w_0 \|_{L^\infty(\Omega)}}{t} \right)$$

for a.e. $x \in \Omega$ with $w_0(x) > 0$ until $w(t,x) > 0$. In the case $w_0(x) < 0$ we obtain

$$w(t,x) \ge -g^{-1}\left(\frac{\| w_0 \|_{L^\infty(\Omega)}}{t} \right)$$

for a.e. $x \in \Omega$ until $w(t,x) < 0$. If for some t_1, $w(t_1, x) = 0$ then $w(t,x) = 0$ for $t > t_1$. Hence we obtain (2.17).

In order to prove (2.16), we use (2.13) and so we obtain (similarly to (2.12))

$$y'(t) + c^\star y(t)^{p/2} \left[1 + \sup_{\tau \in [0,T]} y(\tau) \right]^{-\sigma^\star/2} \le \tag{2.18}$$

$$\text{const } \varphi(t) \left[1 + \sup_{\tau \in [0,T]} y(\tau)^{(p-\sigma^\star)/2} \right] + \text{const } \| f(t) \|_{V_1^\star} \sup_{\tau \in [0,T]} y(\tau)^{(q/p)(\sigma^\star/2)}.$$

Since y is bounded and $\lim \varphi_\infty = 0$, by using (2.14) one can derive from (2.18) the equality (2.16) (see, e.g., [9]).

Remark In the case $g(\xi) = -\alpha_1 \xi$ (where α_1 is a positive constant)

$$|w(t,x)| \leq |w_0(x)| \exp(-\alpha_1 t) \text{ for a.e. } (t,x)$$

and the inequality

$$-\alpha_2 \xi^2 \leq \xi \hat{F}(t,x,\xi;u,w) \leq 0 \quad (\alpha_2 > 0)$$

implies

$$|w(t,x)| \geq |w_0(x)| \exp(-\alpha_2 t) \text{ for a.e. } (t,x).$$

Now we formulate a stabilization result.

Theorem 2.3 *Assume that conditions of Theorem 2.1 are satisfied such that (A_2), (A_4) hold for all $T > 0$ with operators*

$$g_1, g_2 : L^p_{loc}(0,\infty;V_1) \times L^2_{loc}(Q_\infty) \to \mathbb{R}^+, \tag{2.19}$$

$$k_1 : L^p_{loc}(0,\infty;V_1) \times L^2_{loc}(Q_\infty) \to L^q(\Omega); \tag{2.20}$$

for arbitrary fixed $u \in L^p_{loc}(0,\infty;V_1)$, $w \in L^2_{loc}(Q_\infty)$ such that

$$\int_\Omega u(t,x)^2 dx, \quad \| w(t,\cdot) \|_{L^\infty}, \quad t \in (0,\infty) \text{ are bounded} \tag{2.21}$$

and for every $(\zeta_0, \zeta) \in \mathbb{R}$, a.a. $x \in \Omega$

$$\lim_{t\to\infty} a_i(t,x,\zeta_0,\zeta;u,w) = a_{i,\infty}(x,\zeta_0,\zeta), \quad i = 0,1,...,n \tag{2.22}$$

exist and are finite where $a_{i,\infty}$ satisfy the Carathéodory conditions.

Further, for every fixed $u \in L^p_{loc}(0,\infty;V_1)$, $w \in L^2_{loc}(Q_\infty)$

$$\sum_{i=0}^n [a_i(t,x,\zeta_0,\zeta;u,w) - a_i(t,x,\zeta_0^\star,\zeta^\star;u,w)](\zeta_i - \zeta_i^\star) \geq \tag{2.23}$$

$$[g_2(u,w)](t)[|\zeta_0 - \zeta_0^\star|^p + |\zeta - \zeta^\star|^p] - [k_3(u,w)](t,x)$$

with some operator

$$k_3 : L^p_{loc}(0,\infty;V_1) \times L^2_{loc}(Q_\infty) \to L^1(Q_\infty) \tag{2.24}$$

satisfying

$$\lim_{t\to\infty} \int_\Omega [k_3(u,w)](t,x) dx = 0 \tag{2.25}$$

for all fixed u, w satisfying (2.21).

On $\hat{F}$ assume

$$\hat{F}(t,x,\xi;u,w)[\xi - w_\infty(x)] \leq -g(\xi - w_\infty(x))[\xi - w_\infty(x)] \tag{2.26}$$

with some $w_\infty \in L^\infty(\Omega)$ *where* g *is a strictly monotone increasing function with* $g(0) = 0$.

Finally, there exists $f_\infty \in V_1^\star$ *such that*

$$\lim_{t\to\infty} \| f(t) - f_\infty \|_{V_1^\star} = 0. \tag{2.27}$$

Then for a solution $u \in L^p_{loc}(0,\infty;V_1)$, $w \in L^\infty_{loc}(Q_\infty)$ *of (1.5), (1.6) in* $(0,\infty)$ *with* $w_0 \in L^\infty(\Omega)$, *any initial condition on* u *we have*

$$\lim_{t\to\infty} \| u(t) - u_\infty \|_{L^2(\Omega)} = 0, \tag{2.28}$$

$$\lim_{T\to\infty} \int_{T-a}^{T+a} \| u(t) - u_\infty \|^p_{V_1} \, dt = 0 \text{ for arbitrary fixed } a > 0, \tag{2.29}$$

$$\lim_{t\to\infty} \| w(t,\cdot) - w_\infty \|_{L^\infty(\Omega)} = 0, \tag{2.30}$$

where $u_\infty \in V_1$ *is the unique solution to*

$$A_\infty(u_\infty) = f_\infty \tag{2.31}$$

and the operator $A_\infty : V_1 \to V_1^\star$ *is defined by*

$$\langle A_\infty(z), v\rangle = \sum_{i=1}^n \int_\Omega a_{i,\infty}(x, z(x), Dz(x)) D_i v(x) dx +$$
$$\int_\Omega a_{0,\infty}(x, z(x), Dz(x)) v(x) dx, \quad z, v \in V_1.$$

Proof Equality (2.30) follows from (2.26) similarly as it was proved in Theorem 2.2. By Theorem 2.1 (2.21) holds. Applying (A_2) (by using (2.19), (2.20)) to $u(t) = \tilde{u}$, $w(t) = \tilde{w}$ where $\tilde{u} \in V_1$, $\tilde{w} \in L^2(\Omega)$, we obtain from (2.22)

$$|a_{i,\infty}(x, \zeta_0, \zeta)| \le c_1(|\zeta_0|^{p-1} + |\zeta|^{p-1}) + \tilde{k}_1(x)$$

with some constant c_1 and $\tilde{k}_1 \in L^q(\Omega)$. Similarly, Vitali's theorem, (2.19), (2.20), (2.22) - (2.25) imply

$$\sum_{i=1}^n \int_\Omega [a_{i,\infty}(x, z(x), Dz(x)) - a_{i,\infty}(x, z^\star(x), Dz^\star(x))][D_i z(x) - D_i z^\star(x)] dx +$$
$$\int_\Omega [a_{0,\infty}(x, z(x), Dz(x)) - a_{0,\infty}(x, z^\star(x), Dz^\star(x))][z(x) - z^\star(x)] dx \ge$$
$$c_2 \int_\Omega [|z(x) - z^\star(x)|^p + |Dz(x) - Dz^\star(x)|^p] dx \text{ for any } z, z^\star \in V_1.$$

Consequently, $A_\infty : V_1 \to V_1^\star$ is bounded, hemicontinuous, strictly monotone and coercive which implies the existence of a unique solution of (2.31) (see, e.g., [13]).

If u, w are solutions of (1.5), (1.6) in $(0, \infty)$ then by (2.31) we obtain

$$\langle D_t[u(t) - u_\infty], u(t) - u_\infty\rangle + \langle A(u,w)(t) - A_\infty(u_\infty), u(t) - u_\infty\rangle = \tag{2.32}$$
$$\langle f(t) - f_\infty, u(t) - u_\infty\rangle.$$

It is well known (see [13]) that

$$y(t) = \langle u(t) - u_\infty, u(t) - u_\infty\rangle = \int_\Omega [u(t) - u_\infty]^2 dx$$

is absolutely continuous and the first term in (2.32) equals to $1/2y'(t)$ for a.e. t. Further, for the second term in (2.32) we have by (2.23) and Young's inequality

$$\langle [A(u,w)](t) - A_\infty(u_\infty), u(t) - u_\infty\rangle = \tag{2.33}$$
$$\langle [A(u,w)](t) - [A_{u,w}(u_\infty)](t), u(t) - u_\infty\rangle +$$
$$\langle [A_{u,w}(u_\infty)](t) - A_\infty(u_\infty), u(t) - u_\infty\rangle \geq$$
$$g_2(u,w) \parallel u(t) - u_\infty \parallel_{V_1}^p - \int_\Omega [k_3(u,w)](t,x)dx -$$
$$\frac{\varepsilon^p}{p} \parallel u(t) - u_\infty \parallel_{V_1}^p - \frac{1}{q\varepsilon^q} \parallel [A_{u,w}(u_\infty)](t) - A_\infty(u_\infty) \parallel_{V_1^\star}^q$$

with arbitrary $\varepsilon > 0$ where we used the notation

$$\langle [A_{u,w}(u_\infty)](t), z\rangle =$$
$$\int_\Omega \left\{ \sum_{i=1}^n a_i(t,x,u_\infty(x), Du_\infty(x); u,w) D_i z + a_0(t,x,u_\infty(x), Du_\infty(x); u,w) z \right\}.$$

By Vitali's theorem we obtain from (A_2), (2.19), (2.20), (2.22)

$$\lim_{t\to\infty} \parallel [A_{u,w}(u_\infty)](t) - A_\infty(u_\infty) \parallel_{V_1^\star} = 0. \tag{2.34}$$

Finally, by Young's inequality, for the right hand side of (2.32) we have

$$|\langle f(t) - f_\infty, u(t) - u_\infty\rangle| \leq \frac{\varepsilon^p}{p} \parallel u(t) - u_\infty \parallel_{V_1}^p + \frac{1}{q\varepsilon^q} \parallel f(t) - f_\infty \parallel_{V_1^\star}^q . \tag{2.35}$$

Thus, choosing sufficiently small $\varepsilon > 0$, (2.21), (2.25), (2.27), (2.32) - (2.35) yield

$$y'(t) + c^\star \parallel u(t) - u_\infty \parallel_{V_1}^p \leq \psi(t), \tag{2.36}$$

thus by Hölder's inequality

$$y'(t) + c^{\star\star} y(t)^{p/2} \leq \psi(t), \tag{2.37}$$

where $\lim_{t\to\infty} \psi(t) = 0$ and $c^\star, c^{\star\star}$ are positive constants. Similarly to (2.16), one obtains (2.28) from (2.37). Combining (2.28) and (2.36) one obtains (2.29).

Examples Now we consider examples satisfying the conditions of Theorems 1.4 - 2.3. (Examples for Theorems 1.1, 1.3 see in [12].) Let $a_i(t,x,\zeta_0,\zeta;u,w)$ have the form

$$a_i(t,x,\zeta_0,\zeta;u,w)=b_1([H_1(u)])b_2([H_2(w)])\alpha_i(t,x,\zeta_0,\zeta),\quad i=1,...,n,\tag{2.38}$$

$$\begin{gathered}a_0(t,x,\zeta_0,\zeta;u,w)=b_1([H_1(u)])b_2([H_2(w)])\alpha_0(t,x,\zeta_0,\zeta)+\\ \hat{b}_0([F_0(u)](t,x))\tilde{b}_0(G_0(w))\hat{\alpha}_0(t,x,\zeta_0,\zeta)\end{gathered}\tag{2.39}$$

where α_i satisfy the usual conditions: they are Carathéodory functions;

$$|\alpha_i(t,x,\zeta_0,\zeta)|\le c_1(|\zeta_0|^{p-1}+|\zeta|^{p-1})+k_1(x)$$

with some constant c_1, $k_1\in L^q(\Omega)$, $i=0,1,...,n$;

$$\sum_{i=1}^n[\alpha_i(t,x,\zeta_0,\zeta)-\alpha_i(t,x,\zeta_0,\zeta^\star)](\zeta_i-\zeta_i^\star)>0 \text{ if } \zeta\ne\zeta^\star;$$

$$\sum_{i=0}^n\alpha_i(t,x,\zeta_0,\zeta)\zeta_i\ge c_2(|\zeta_0|^p+|\zeta|^p)$$

with some constant $c_2>0$. E.g. functions

$$\alpha_i=\zeta_i|\zeta|^{p-2},\quad i=1,...,n,\quad \alpha_0=\zeta_0|\zeta_0|^{p-2}$$

satisfy the above conditions. The function $\hat{\alpha}_0$ satisfies the Carathéodory condition and

$$|\hat{\alpha}_0(t,x,\zeta_0,\zeta)|\le c_1(|\zeta_0|^{\hat{\rho}}+|\zeta|^{\hat{\rho}}),\quad 0\le\hat{\rho}<p-1\tag{2.40}$$

Further, $b_1,b_2,\hat{b}_0,\tilde{b}_0$ are continuous functions, satisfying (with some positive constants)

$$b_1(\Theta)\ge\frac{\text{const}}{1+|\Theta|^{\sigma^\star}},\quad |\hat{b}_0(\Theta)|\le\text{const}|\Theta|^{p-1-\rho^\star}$$

with $0<\sigma^\star<p-1$, $\sigma^\star+\hat{\rho}<\rho^\star<p-1$,

$$b_2(\Theta)\ge\frac{\text{const}}{1+g_3(\Theta)},\quad |\tilde{b}_0(\Theta)|\le\text{const}[1+g_4(\Theta)].$$

(g_3,g_4 are monotone nondecreasing positive functions.)

Finally,

$$H_1:L^p_{loc}(0,\infty;W^{1-\delta,p}(\Omega)))\to C(\overline{Q_\infty}),\quad H_2:L^2_{loc}(Q_\infty)\to C(\overline{Q_\infty}),$$

$$F_0:L^p_{loc}(0,\infty;W^{1-\delta,p}(\Omega)))\to L^p_{loc}(Q_\infty),\quad G_0:L^2_{loc}(Q_\infty)\to L^2_{loc}(Q_\infty)$$

are linear operators of Volterra type such that for any fixed finite $T>0$ their restrictions

$$H_1:L^p(0,T;W^{1-\delta,p}(\Omega)))\to C(\overline{Q_T}),\quad H_2:L^2(Q_T)\to C(\overline{Q_T}),$$

$$F_0 : L^p(0,T;W^{1-\delta,p}(\Omega))) \to L^p(Q_T), \quad G_0 : L^2(Q_T) \to L^2(Q_T)$$

are uniformly bounded with respect to $T \in (0,\infty)$. $[H_1(u)](t,x)$ may have e.g. one of the forms:

$$\int_{Q_t} d(t,x,\tau,\xi)u(\tau,\xi)d\tau d\xi \text{ where } \sup_{(t,x)\in Q_T} \int_{Q_T} |d(t,x,\tau,\xi)|^q d\tau d\xi < \infty,$$

$$\int_{\Gamma_t} d(t,x,\tau,\xi)v(\tau,\xi)d\tau d\sigma_\xi \text{ where } \sup_{(t,x)\in Q_T} \int_{\Gamma_T} |d(t,x,\tau,\xi)|^q d\tau d\sigma_\xi < \infty.$$

Examples for F_0, G_0 see in [11].

By using Young's inequality, one can prove that the assumptions on a_i in Theorem 1.4 are fulfilled for the above example (see [11]). The assumptions on a_i in Theorem 2.1 are satisfied for the above example if

$$\| H_1(u) \|_{C(\overline{Q_T})} \le \text{const} \sup_{\tau\in[0,T]} \left\{ \int_\Omega u(\tau,x)^2 dx \right\}^{1/2}, \tag{2.41}$$

$$\| F_0(u) \|_{L^p(Q_T)} \le \text{const} \sup_{\tau\in[0,T]} \left\{ \int_\Omega u(\tau,x)^2 dx \right\}^{1/2} \tag{2.42}$$

with constants not depending on T. (2.41) is satisfied if e.g.

$$[H_1(u)](t,x) = \int_{Q_t} d(t,x,\tau,\xi)u(\tau,\xi)d\tau d\xi \text{ where}$$

$$\sup_{(t,x)\in Q_\infty} \int_0^\infty \left[\int_\Omega |d(t,x,\tau,\xi)|^2 d\xi \right]^{1/2} d\tau < \infty.$$

The assumptions on a_i in Theorem 2.2 are satisfied if (2.41), (2.42) hold and (instead of (2.40))

$$|\hat{\alpha}_0(t,x,\zeta_0,\zeta)| \le \varphi_1(t)(|\zeta_0|^{\hat{\rho}} + |\zeta|^{\hat{\rho}}), \quad 0 \le \hat{\rho} < p-1 \tag{2.43}$$

where $\lim_\infty \varphi_1 = 0$.

Finally, the following modification of functions (2.38), (2.39) satisfy the conditions of Theorem 2.3: for simplicity e.g.

$$\alpha_i = \zeta_i |\zeta|^{p-2} \text{ for } i = 1,...,n, \quad \alpha_0 = \zeta_0|\zeta_0|^{p-2},$$

(2.43) is valid and instead of $b_1(H_1(u)), b_2(H_2(w))$ we have $b_1(t,H_1(u)), b_2(t,H_2(w))$, respectively, where (with some positive constants)

$$b_1(t,\Theta) = \frac{\text{const}}{1+\psi_1(t)|\Theta|^{\sigma^\star}}, \quad b_2(t,\Theta) = \frac{\text{const}}{1+\psi_2(t)g_3(\Theta)}, \quad \lim_\infty \psi_j = 0.$$

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