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ASYMPTOTIC FORMULAS FOR SOLUTIONS OF HALF-LINEAR EULER-WEBER EQUATION

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Abstract. We establish improved asymptotic formulas for nonoscillatory solutions of the half-linear Euler-Weber type differential equation

$$(\Phi(x'))' + \left[\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t}\right]\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x, \quad p > 1$$

with critical coefficients

$$\gamma_p = \left(\frac{p-1}{p}\right)^p, \quad \mu_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1},$$

where this equation is viewed as a perturbation of the half-linear Euler equation.

This paper is in final form and no version of it is submitted for publication elsewhere.

1. Introduction

The aim of this paper is to present asymptotic formulas for solutions of the half-linear Euler-Weber type differential equation

$$(\Phi(x'))' + \left[\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t}\right]\Phi(x) = 0, \tag{1}$$

where $\gamma_p = \left(\frac{p-1}{p}\right)^p$, $\mu_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$. This equation is a special case of a general half-linear second order differential equation

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \tag{2}$$

where $\Phi(x) := |x|^{p-1}\operatorname{sgn} x$, $p > 1$, and r, c are continuous functions, $r(t) > 0$ (in the studied equation (1) we have $r(t) \equiv 1$). Let us recall that similarly as in the linear case, which is a special case of (2) for $p = 2$ and equation (2) then reduces to the linear Sturm-Liouville differential equation

$$(r(t)x')' + c(t)x = 0,$$

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even in the half-linear (non)oscillation theory equation (2) can be classified as *oscillatory* if every its nontrivial solution has infinitely many zeros tending to infinity and as *nonoscillatory* otherwise. The classical approach to half-linear equation (2) is to regard it as a perturbation of the one-term equation

$$(r(t)\Phi(x'))' = 0.$$

Our approach is slightly modified, we use the perturbation principle introduced in [9], [3] and applied in [2], [5], [7], [8], [14], [15]. According to this concept equation (2) can be seen as a perturbation of a general (nonoscillatory) half-linear equation

$$(r(t)\Phi(x'))' + \tilde{c}(t)\Phi(x) = 0, \tag{3}$$

i.e., (2) can be rewritten in the form

$$(r(t)\Phi(x')' + \tilde{c}(t)\Phi(x) + (c(t) - \tilde{c}(t))\Phi(x) = 0.$$

From this point of view, the studied equation (1) can be considered as a perturbation of the half-linear Euler equation

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) = 0. \tag{4}$$

This equation is nonoscillatory and γ_p is the so-called critical coefficient, critical in that sence that if it is replaced by any bigger constant, such equation becomes oscillatory, and for less constants nonoscillation is preserved.

Half-linear Euler-Weber equation (1) was studied by Elbert and Schneider in [9]. They derived the asymptotic formulas for its two linearly independent solutions in the forms

$$x_1(t) = t^{\frac{p-1}{p}} \log^{\frac{1}{p}} t\,(1 + o(1)) \quad \text{as} \quad t \to \infty$$

$$x_2(t) = t^{\frac{p-1}{p}} \log t^{\frac{1}{p}} (\log(\log t))^{\frac{2}{p}} (1 + o(1)) \quad \text{as} \quad t \to \infty.$$

Our results show that the terms $(1 + o(1))$ are special slowly varying functions.

2. Preliminaries

Let q be the conjugate number of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Let x be a solution of nonoscillatory equation (2), then the following Riccati type first order differential equation

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0 \tag{5}$$

holds, where $w(t) = r(t)\Phi(x'/x)$. It is well known from the (non)oscillation theory for half-linear equations (see e.g. [1, p. 171]), that equation (2) is nonoscillatory if and only if there exists a solution of the Riccati equation (5) on some interval of the form $[T, \infty)$.

Using the approach of perturbations, it is convenient to deal with the so-called *modified Riccati equation* (which was derived e.g. in [4]), whose solvability is again equivalent to nonoscillation of equation (2).

Let Φ^{-1} be the inverse function of $\Phi(x)$, $h(t)$ be a (positive) solution of (3), and $w_h(t) = r(t)\Phi(h'/h)$ be the solution of the Riccati equation associated with (3). The modified Riccati equation then reads as

$$((w - w_h)h^p)' + (C(t) - \tilde{c}(t))h^p + pr^{1-q}h^p P(\Phi^{-1}(w_h), w) = 0, \tag{6}$$

where

$$P(u, v) := \frac{|u|^p}{p} - uv + \frac{|v|^q}{q} \geq 0,$$

with the equality $P(u, v) = 0$ if and only if $v = \Phi(u)$. Observe that if $\tilde{c}(t) \equiv 0$ and $h(t) \equiv 1$, then (6) reduces to (5) and this is also the reason why we call this equation modified Riccati equation.

Let us recall that a positive measurable function $L(t)$ defined on $(0, \infty)$ is said to be a *slowly varying function* in the sence of Karamata (see e.g. [11], [12]) if it satisfies

$$\lim_{t\to\infty} \frac{L(\lambda t)}{L(t)} = 1 \quad \text{for any} \quad \lambda > 0.$$

From the representation theorem for slowly varying functions (see [10]) we know that they are in the form

$$L(t) = l(t) \exp\left\{\int_{t_0}^{t} \frac{\varepsilon(s)}{s}\, ds\right\}, \quad t \geq t_0,$$

for some $t_0 > 0$, where $l(t)$ and $\varepsilon(t)$ are measurable functions such that

$$\lim_{t\to\infty} l(t) = l \in (0, \infty) \quad \text{and} \quad \lim_{t\to\infty} \varepsilon(t) = 0.$$

If $l(t)$ is identically a positive constant, we say that $L(t)$ is a *normalized slowly varying function.*

Asymptotic formulas for some nonoscillatory solutions of the general perturbed Euler equation

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) + g(t)\Phi(x) = 0, \tag{7}$$

were studied in [13] and the following two statements were proved there.

Theorem 1. *Suppose that*

$$c(t) := \frac{\gamma_p}{t^p} + g(t) \geq 0 \quad \textit{for large } t,$$

the integral $\int^\infty g(t)t^{p-1}\,dt$ *converges, and let*

$$c := \lim_{t\to\infty} \log t \int_t^\infty g(s)s^{p-1}\,ds < \mu_p \tag{8}$$

holds. Then (7) *possesses a pair of solutions*

$$x_i(t) = t^{\frac{p-1}{p}} (\log t)^{\nu_i} L_i(t),$$

where $\lambda_i := \left(\frac{p-1}{p}\right)^{p-1} \nu_i$ *are roots of the equation*

$$\frac{\lambda^2}{4\mu_p} - \lambda + c = 0 \tag{9}$$

and $L_i(t)$ *are normalized slowly varying functions of the form* $L_i(t) = \exp\left\{\int^t \frac{\varepsilon_i(s)}{s\log s}\,ds\right\}$ *with* $\varepsilon_i(t) \to 0$ *for* $t \to \infty$, $i = 1, 2$.

Taking $g(t) := \frac{\mu_p}{t^p \log^2 t}$ we have a special perturbation of half-linear Euler equation (4) with $\lim_{t\to\infty} \log t \int_t^\infty g(s)s^{p-1}\,ds = \mu_p$. Then we have in some sence a limit case of (8) in Theorem 1 and the quadratic equation (9) has just one real zero. The derivation of the asymptotic formula for the principal solution of Euler-Weber equation (1) can be made in a similar manner as in the proof of Theorem 1 (see [13]).

Theorem 2. *Equation* (1) *has a solution satisfying the asymptotic formula*

$$x_1(t) = t^{\frac{p-1}{p}} (\log t)^{\frac{1}{p}} L_1(t), \tag{10}$$

where $L_1(t)$ *is a normalized slowly varying function in the form* $L_1(t) = \exp\left\{\int^t \frac{\varepsilon_1(s)}{s\log s}\,ds\right\}$ *and* $\varepsilon_1(t) \to 0$ *as* $t \to \infty$.

3. Main result

As the main result we introduce the asymptotic formula for the second solution of the Euler-Weber equation (1), which is linearly independent to the principal one stated in Theorem 2. This is also the answer to the open problem conjecturing such result presented in [13].

Theorem 3. *Equation* (1) *has a solution satisfying the asymptotic formula*

$$x_2(t) = t^{\frac{p-1}{p}} \log t^{\frac{1}{p}} (\log(\log t))^{\frac{2}{p}} L_2(t),$$

where $L_2(t)$ *is a normalized slowly varying function in the form*

$$L_2(t) = \exp\left\{\int^t \frac{\varepsilon_2(s)}{s\log s \log(\log s)}\,ds\right\}$$

and $\varepsilon_2(t) \to 0$ *as* $t \to \infty$.

Proof. First we formulate the modified Riccati equation associated with (1). Let w be a solution of the Riccati equation

$$w' + \frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t} + (p-1)|w|^q = 0. \tag{11}$$

Since $\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t} \geq 0$ for large t, from [6, Cor. 4.2.1] we have $w(t) \geq 0$ for large t. Let

$$w_h(t) = \Phi\left(\frac{h'}{h}\right) = \left(\frac{p-1}{p}\right)^{p-1} t^{1-p}$$

be the solution of Riccati equation associated with (4) generated by the solution $h(t) = t^{\frac{p-1}{p}}$, and denote

$$v(t) = (w(t) - w_h(t))h^p(t) = t^{p-1}\left(w - \left(\frac{p-1}{p}\right)^{p-1} t^{1-p}\right). \tag{12}$$

Modified Riccati equation (6), where $\tilde{c}(t) = \frac{\gamma_p}{t^p}, C(t) = \frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t}$, has then the form

$$v' + \frac{\mu_p}{t \log^2 t} + pt^{p-1} P\left(\left(\frac{p-1}{p}\right)\frac{1}{t}, w\right) = 0,$$

which, by an easy calculation, arrives at

$$v' + \frac{\mu_p}{t \log^2 t} + \frac{p-1}{t} G(v) = 0, \tag{13}$$

where

$$G(v) = \left| v + \left(\frac{p-1}{p}\right)^{p-1} \right|^q - v - \left(\frac{p-1}{p}\right)^p,$$

with the equality $G(v) = 0$ if and only if $v = 0$.

Now we show that $v(t) \to 0$ for $t \to \infty$. Integrating (13) from T to t, $T \leq t$, we have

$$\left[\left(\frac{p-1}{p}\right)^{p-1} - t^{p-1} w\right]_T^t = \int_T^t \frac{\mu_p}{s \log^2 s}\, ds + (p-1) \int_T^t \frac{G(v)}{s}\, ds$$

Letting $t \to \infty$ and taking into account that $w(t) \geq 0$ for large t,

$$\left[\left(\frac{p-1}{p}\right)^{p-1} - t^{p-1} w\right]_T^\infty \leq T^{p-1} w(T).$$

Hence

$$\int_T^\infty \frac{\mu_p}{s \log^2 s}\, ds + (p-1) \int_T^\infty \frac{G(v)}{s}\, ds \leq T^{p-1} w(T)$$

and since the integral $\int_T^\infty \frac{\mu_p}{s\log^2 s}\,ds$ converges, the integral $\int_T^\infty \frac{G(v)}{s}\,ds$ converges too. This means that $\lim_{t\to\infty} v(t)$ exists and $v(t) \to 0$, since if $v(t) \to v_0 \neq 0$, then $G(v(t)) \to G(v_0) > 0$ which contradicts the convergence of $\int_T^\infty \frac{G(v)}{s}\,ds$.

Let us investigate the behavior of the function $G(v)$. By L'Hospital's rule (used twice) we have

$$\lim_{v\to 0}\frac{G(v)}{v^2} = \frac{q-1}{2}\left(\frac{p}{p-1}\right)^{p-1} = \frac{q-1}{4\mu_p}.$$

Hence, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left(\frac{q-1}{4\mu_p} - \varepsilon\right)v^2 \le G(v) \le \left(\frac{q-1}{4\mu_p} + \varepsilon\right)v^2. \tag{14}$$

for v satisfying $|v| < \delta$. Similarly for $\frac{\partial G}{\partial v}$, as

$$\lim_{v\to 0}\frac{\frac{\partial G}{\partial v}}{v} = (q-1)\left(\frac{p}{p-1}\right)^{p-1} = \frac{q-1}{2\mu_p},$$

to every $\varepsilon > 0$ one can find $\delta > 0$ such that

$$\left(\frac{q-1}{2\mu_p} - \varepsilon\right)v \le \frac{\partial G}{\partial v} \le \left(\frac{q-1}{2\mu_p} + \varepsilon\right)v \tag{15}$$

as $|v| < \delta$.

We assume that a solution of modified Riccati equation (13) is in the form

$$v(t) = \frac{2\mu_p \log(\log t) + 4\mu_p + z(t)}{\log t \log(\log t)}.$$

Then for its derivative we have

$$v'(t) = \frac{(2\mu_p\frac{1}{t\log t} + z'(t))\log t\log(\log t) - (2\mu_p\log(\log t) + 4\mu_p + z(t))(\frac{1}{t}\log(\log t) + \frac{1}{t})}{\log^2 t\log^2(\log t)}$$

and substituing into the modified Riccati equation (13) we get the equation

$$z'(t) - \frac{z(t)}{t\log t\log(\log t)} + \frac{-4\mu_p - 4\mu_p\log(\log t) - \mu_p\log^2(\log t) - z(t)\log(\log t) + (p-1)G(v)\log^2 t\log^2(\log t)}{t\log t\log(\log t)} = 0,$$

which can be rewritten as

$$z'(t) + \frac{z(t)}{t\log t\log(\log t)} + \frac{-4\mu_p - 4\mu_p\log(\log t) - \mu_p\log^2(\log t) - 2z(t) - z(t)\log(\log t) + (p-1)G(v)\log^2 t\log^2(\log t)}{t\log t\log(\log t)} = 0.$$

If we denote

$$r(t) = \exp\left\{\int^t \frac{1}{s\log s \log(\log s)}\, ds\right\}$$

then the previous equation is equivalent to

$$(r(t)z(t))' + r(t)\frac{1}{t\log t \log(\log t)} H(z,t) = 0, \tag{16}$$

where

$$H(z,t) = -4\mu_p - 4\mu_p \log(\log t) - \mu_p \log^2(\log t) - 2z(t) - z(t)\log(\log t) + (p-1)G(v)\log^2 t \log^2(\log t).$$

Let $C_0[T,\infty)$ denote the set of all continuous functions on $[T,\infty)$ tending to zero as $t \to \infty$; concrete T will be specified later. $C_0[T,\infty)$ is a Banach space with the norm $\|z\| = \sup\{|z(t)| : t \geq T\}$. We consider the integral operator

$$Fz(t) = -\frac{1}{r(t)} \int^t \frac{r(s)}{s\log s \log(\log s)} H(z,s)\, ds$$

on the set

$$V = \{z \in C_0[T,\infty) : |z(t)| < \varepsilon_1, t \geq T\},$$

where ε_1, T are suitably chosen (will be specified later). Now our aim is to show that the operator F is a contraction on the set V and maps V to itself.

First we show that $\int^t \frac{r(s)}{s\log s \log(\log s)}\, ds$ diverges. We have $r(t) \to \infty$ for $t \to \infty$ and

$$\lim_{t\to\infty} \int^t \frac{r(s)}{s\log s \log(\log s)}\, ds = \lim_{t\to\infty} [\log(\log s)]^t = \infty$$

Furthermore,

$$r'(t) = r(t)\frac{1}{t\log t \log(\log t)}$$

and by L'Hospital's rule we have

$$\lim_{t\to\infty} \frac{1}{r(t)} \int^t \frac{r(s)}{s\log s \log(\log s)}\, ds = \frac{\frac{r(t)}{t\log t \log(\log t)}}{r'(t)} = 1 > 0.$$

Let T_1 be large enough such that

$$\frac{1}{r(t)} \int^t \frac{r(s)}{s\log s \log(\log s)}\, ds < 2 \tag{17}$$

for $t \geq T_1$.

Let $\varepsilon_1 > 0$, such that

$$2\left(\varepsilon_1 + \frac{\varepsilon_1}{4\mu_p}\right) \leq 1 \tag{18}$$

and

$$2\left(\frac{\varepsilon_1}{2\mu_p}+\varepsilon_1\right)<\frac{1}{2}. \tag{19}$$

Let T_2 be such that $|z(t)|<\varepsilon_1^2$ for $t\geq T_2$.

In order to show that $H(z,t)\to 0$ for $t\to\infty$, the estimates for $H(z,t)$ are:

$$\begin{aligned}
|H(z,t)| &= \big|-4\mu_p-4\mu_p\log(\log t)-\mu_p\log^2(\log t)-2z(t)-z(t)\log(\log t)\\
&\qquad\qquad +(p-1)G(v)\log^2 t\log^2(\log t)\big|\\
&\leq \left|-4\mu_p-4\mu_p\log(\log t)-\mu_p\log^2(\log t)-2z(t)-z(t)\log(\log t)+\frac{v^2}{4\mu_p}\log^2 t\log^2(\log t)\right|\\
&\qquad +\left|(p-1)\log^2 t\; G(v)-\frac{v^2}{4\mu_p}\log^2 t\log^2(\log t)\right|\\
&= \left|\frac{z^2}{4\mu_p}\right|+\left|(p-1)\log^2 t\; G(v)-\frac{v^2}{4\mu_p}\log^2 t\log^2(\log t)\right|,
\end{aligned}$$

where the definition of v have been used in the step between the second and the third line of the above computation. Now, according to (14), the second term in the last expression is arbitrarily small for small v, i.e., as $v(t)\to 0$ for $t\to\infty$, there exists T_3 large enough such that for $t\geq T_3$

$$\left|\frac{z^2}{4\mu_p}\right|+\left|(p-1)\log^2 t\; G(v)-\frac{v^2}{4\mu_p}\log^2 t\log^2(\log t)\right|\leq\frac{\varepsilon_1^2}{4\mu_p}+\varepsilon_1^2$$

for $t\geq\max\{T_2,T_3\}$.

Similarly, we will need an estimate for the difference $|H(z_1,t)-H(z_2,t)|$. Using the mean value theorem (with $z\in V$ such that $\min\{z_1(t),z_2(t)\}\leq z(t)\leq\max\{z_1(t),z_2(t)\}$) we have (suppressing the argument t in the functions z,z_1,z_2)

$$\begin{aligned}
|H(z_1,s)-H(z_2,s)| &= \left|(z_2-z_1)(2+\log(\log t))+(p-1)\log^2 t\log^2(\log t)\frac{\partial G(v,z)}{\partial z}(z_1-z_2)\right|\\
&\leq \|z_1-z_2\|\left(\left|-(2+\log(\log t))+\frac{v}{2\mu_p}\log t\log(\log t)\right|\right.\\
&\qquad\left.+\left|(p-1)\log^2 t\log^2(\log t)\,\frac{\partial G(v,z)}{\partial z}-\frac{v}{2\mu_p}\log t\log(\log t)\right|\right)\\
&= \|z_1-z_2\|\left(\left|\frac{z}{2\mu_p}\right|+\left|(p-1)\log t\log(\log t)\,\frac{\partial G(v,z)}{\partial v}-\frac{v}{2\mu_p}\log t\log(\log t)\right|\right)\\
&\leq \|z_1-z_2\|\left(\left|\frac{z}{2\mu_p}\right|+\varepsilon_1\right)\leq\|z_1-z_2\|\left(\frac{\varepsilon_1}{2\mu_p}+\varepsilon_1\right)
\end{aligned}$$

for $t \geq \max\{T_2, T_4\}$, where T_4 is such that $|v(t)| < \delta$ for $t \geq T_4$ (such T_4 exists because of (15)).

We take $T = \max\{T_1, T_2, T_3, T_4\}$. Then

$$|Fz(t)| \leq \frac{1}{r(t)} \int_t^\infty \frac{r(s)}{s \log s \log(\log s)} |H(z,s)|\, ds \leq \left(\varepsilon_1^2 + \frac{\varepsilon_1^2}{4\mu_p}\right) \frac{1}{r(t)} \int_t^\infty \frac{r(s)}{s \log(\log s)}\, ds$$

$$< 2\left(\varepsilon_1^2 + \frac{\varepsilon_1^2}{4\mu_p}\right) \leq \varepsilon_1$$

using (18) and hence F maps V to itself.

Next we show that F is a contraction. We have (using the definition of F)

$$|Fz_1(t) - Fz_2(t)| = \frac{1}{r(t)} \int_t^\infty \frac{r(s)}{s \log(\log s)} |H(z_1, s) - H(z_2, s)|\, ds$$

$$\leq \|z_1 - z_2\| \left(\frac{\varepsilon_1}{2\mu_p} + \varepsilon_1\right) \frac{1}{r(t)} \int_t^\infty \frac{r(s)}{s \log s \log(\log s)}\, ds,$$

which is, according to (17) and (19), less than $\frac{1}{2}\|z_1 - z_2\|$ and hence F is a contraction.

By the Banach fixed point theorem, F has a fixed point z that satisfies $z = Fz$, i.e.

$$z(t) = -\frac{1}{r(t)} \int^t \frac{r(s)}{s \log s \log(\log s)} H(z,s)\, ds.$$

Differentiating the last equality we see that $z(t)$ is a solution of (16) and hence $v(t) = \frac{2\mu_p \log(\log t) + 4\mu_p + z(t)}{\log t \log(\log t)}$ is a solution of modified Riccati equation (13).

Now, for the solution of Riccati equation (11) $w(t)$, we have

$$w(t) = t^{1-p}\left(v + \left(\frac{p-1}{p}\right)^{p-1}\right) = t^{1-p}\left(\frac{2\mu_p \log(\log t) + 4\mu_p + z(t)}{\log t \log(\log t)} + \left(\frac{p-1}{p}\right)^{p-1}\right)$$

and the solution x of equation (1) is

$$x(t) = \exp \int^t \Phi^{-1}(w(s))\, ds.$$

Furthermore,

$$\Phi^{-1}(w) = \Phi^{-1}\left(t^{1-p}\left(\frac{p-1}{p}\right)^{p-1}\left[\left(\frac{p-1}{p}\right)^{1-p} v + 1\right]\right)$$

$$= \frac{1}{t}\frac{p-1}{p}\left[\left(\frac{p-1}{p}\right)^{1-p} v + 1\right]^{q-1} = \frac{1}{t}\frac{p-1}{p}\left[1 + (q-1)\left(\frac{p-1}{p}\right)^{1-p} v + o(v)\right]$$

$$= \frac{1}{t}\frac{p-1}{p} + \frac{1}{p}\left(\frac{p-1}{p}\right)^{1-p} \frac{2\mu_p \log(\log t) + 4\mu_p + z(t)}{\log t \log(\log t)} + o(\frac{v}{t})$$

$$= \frac{p-1}{p}\frac{1}{t} + \frac{\frac{1}{p}}{t\log t} + \frac{\frac{2}{p}}{t\log t\log(\log t)} + \frac{\frac{1}{p}\left(\frac{p-1}{p}\right)^{1-p} z(t) + o(2\mu_p \log(\log t) + 4\mu_p + z(t))}{t\log t \log(\log t)}.$$

Denote

$$\frac{1}{p}\left(\frac{p-1}{p}\right)^{1-p} z(t) + o(2\mu_p \log(\log t) + 4\mu_p + z(t)) = \varepsilon_2(t),$$

then the solution of (1) is in the form

$$x(t) = \exp \int^t \Phi^{-1}(w(s))\, ds = t^{\frac{p-1}{p}} (\log t)^{\frac{1}{p}} (\log(\log t))^{\frac{2}{p}} \exp\left\{\int^t \frac{\varepsilon_2(s)}{s\log s \log(\log s)}\, ds\right\}$$

and the statement is proved. □

Remark 1. Let us denote that functions $L_1(t), L_2(t)$ from Theorems 2, 3 are in some sence "more slowly varying" than standard slowly varying functions in the sence of Karamata. The function $L_1(t)$ remains slowly varying even after the substitution $u = \log s$ in the integrated term, and such subtitution can be used even twice in the function $L_2(t)$ without a change of the property of slowly variability.

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