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ON THE GLOBAL EXISTENCE OF MILD SOLUTIONS OF NONLINEAR DELAY SYSTEMS ASSOCIATED WITH CONTINUOUS AND ANALYTIC SEMIGROUPS

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Abstract.

In this paper we prove sufficient conditions for the existence of global solutions of nonlinear functional-differential evolution equations whose linear parts are infinitesimal generators of strongly continuous and analytic semigroups. We apply the obtained results to a diffusion problem.

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1. Introduction

Integral inequalities play an important role in the study of asymptotic behavior of integral and differential equations. In the paper [11] a singular version of the Bihari nonlinear integral inequality (see [1]) is proved. Some modifications of this inequality are applied in the study of the existence of global solutions of semilinear evolution equations and their stability properties in the papers [7, 11, 12, 13, 14] These results are closed to the results obtained in the paper [4] by using a comparison method. In the paper [15] the following delay system is studied:

$$\dot{x}(t) = Ax(t) + f(t, x(t), x_t), \quad t \geqq 0, \tag{1}$$
$$x(t) = \Phi(t), \quad t \in \langle -r, 0\rangle,$$

where $x(t) \in X$ (X is a Banach space), $x_t \in C := C(\langle -r, 0\rangle, X)$, $x_t(\Theta) = x(t+\Theta)$, $r \leqq \Theta \leqq 0, \Phi \in C$, $||u|| = \sup_{-r \leqq \Theta \leqq 0} ||u(\Theta)||$, $u \in C$.

Using the technique of integral inequalities sufficient conditions for the existence of global mild solutions of this problem are proved. This problem is studied also in the paper [19] by applying a result on linear integral inequality with singular kernel, where it is assumed that the map $f(t, u, v)$ is linearly bounded in the variables u, v. Our conditions on the map f are more general. In the last section we apply our results to a diffusion problem which is a modification of the problem for differential equations of diffusion type without a delay, studied in the paper [8], to a similar problem for functional-differential equations.

Key words and phrases. continuous and analytic semigroup, functional-differential equation, global solution, boundary value problem, parabolic equation.

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2. On the existence of global solutions

Let $\{S(t)\}_{t\in R^+}$, $S(t) \in L := L(X,X)$ be a strongly continuous semigroup (see e. g. [2, 10, 15, 16])

and

$$Ax := lim_{t\to 0^+}\frac{1}{t}[S(t)x - x],\ x \in D(A)$$

be generator of the semigroup

We denote $S(t) := e^{At}$. By [16, Theorem 2. 2], or [2, p. 22] there exist constants $M \geqq 1, \alpha \geqq 0$ such that

$$(2) \qquad ||e^{At}||_L \leqq Me^{\alpha t}, \quad t \geqq 0.$$

By a **mild solution** of the initial value problem (1) on an interval $\langle -r, T)$, $T > 0$ we mean a continuous function $x \in C(\langle -r, T), X)$ satisfying

$$x(t) = e^{At}\Phi(0) + \int_0^t e^{A(t-s)}f(s, x(s), x_s)ds, \quad 0 \leqq t < T,$$

$$x(t) = \Phi(t),\ -r \leqq t \leqq 0,$$

We assume that the mapping $f : R^+ \times X \times C \mapsto X$ satisfies the conditions:

(H1) There exist continuous, nonnegative functions $F_1(t), F_2(t), t \geqq 0$ and continuous, positive, nondecreasing functions $\omega_1, \omega_2 : R^+ \to R^+$ such that

$$||f(t, x, y)|| \leqq F_1(t)\omega_1(||x||) + F_2(t)\omega_2(||y||_C)$$

for all $t \in R^+, x \in X, y \in C$,

(H2)

$$\int_0^\infty \frac{d\sigma}{\omega_1(\sigma) + \omega_2(\sigma)} = \infty,$$

In the paper [15] we have assumed the condition (H2) with $\omega_1 = \omega_2$.

Definition. *A solution* $x : \langle 0, T) \to X$ *of the equations (1) is called* **nonextendable**, *if either* $0 < T < \infty$*and then* $\limsup_{t\to T^-} ||x(t)|| = \infty$, *or* $T = \infty$. *In the second case the solution is called* **global**.

We do not study the problem of the existence of solutions of the problem (1). In the papers [18, 19] the problem of the existence of maximal and noncontinuable (nonextendable) solutions for functional-differential equations with the right-hand sides of the form $f(t, x_t)$ is studied.

The following theorem is proved in the paper [15].

THEOREM 1 ([15, Theorem 1]**).** *Let* $A : D(A) \to X$ *be the infinitesimal generator of a strongly continuous semigroup* $\{e^{At}\}_{t\geqq 0}$ *and the condition (H1), (H2) with* $\omega := \omega_1 = \omega_2$, *be satisfied. Then any nonextendable solution* $x : \langle 0, T) \to X$ *of the initial value problem (1) is global.*

Theorem 2. *Let the assumptions of Theorem 1 be fulfilled, where the condition (H2) is with* $\omega_1 \neq \omega_2$*, i. e.*

$$\int_0^\infty \frac{d\sigma}{\omega_1(\sigma) + \omega_2(\sigma)} = \infty.$$

Then any nonextendable solution $x : \langle 0, T) \to X$ *of the initial value problem (1) is global.*

If we put $F = \max\{F_1, F_2\}, \omega = \omega_1 + \omega_2$ then we obtain the assertion of theorem 2 as a consequence of Theorem 1. The formulation of Theorem 2 has a reason because the variables y, z of the mapping $f(t, y, z)$ belong to different spaces and the appearance of different functions ω_1, ω_2 in the condition (H_2) is natural.

Before giving an example of different functions ω_1, ω_2 satisfying the condition (H2), let us recall a useful result by A. Constantin [3] (see also [4, Lemma 1]).

Proposition. *Let* $\omega \in C(R_+, (0, \infty))$ *be nondecreasing and* $\int_0^\infty \frac{ds}{\omega(s)} = \infty$*, then* $\int_0^\infty \frac{ds}{s+\omega(s)} = \infty$.

Example. Let $\omega_1(u) := u$, $\omega_2(u)$ be a continuous, nondecreasing, positive function such that $\int_0^\infty \frac{ds}{\omega_2(s)} = \infty$. Then the condition (H2) is satisfied.

Now let us consider the case when the operator $A : D(A) \subseteq X \to X$ is sectorial. Consider the initial value problem (1) in the form

$$\dot{x}(t) + Ax(t) = f(t, x(t), x_t),\ t \geqq 0, \tag{6}$$

$$x(t) = \Phi(t),\ t \in \langle -r, 0\rangle,\ r > 0$$

in accordance with the books [6, 7] and some other books and papers dealing with the problem, where the linear part of the equation is a **sectorial operator**.The definition of the sectorial operator can be found i. e. in [7]. By [7, Theorem 1.3.4] if A is sectorial then $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-A}\}_{t\geqq 0}$ (analytic means that $t \mapsto e^{-At}x$ is an analytic function on $(0, \infty)$ for any $x \in X$.)

If A is sectorial on X and $Re\, \sigma(A) > 0$ and $\alpha \in \langle 0, 1)$ then one can define the operator

$$A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At} dt,$$

where $\Gamma(\alpha)$ is the Eulerian gamma function evaluated at α. The operator $A^{-\alpha}$ is bounded linear operator on X which is one-to-one and one can define A^α as the inverse of $A^{-\alpha}$ and we define $A^0 = I$. The operator A^α is closed and densely defined.

If A is sectorial on X then there is an $a \in R$ such that $A_1 = A + aI$ has $Re\, \sigma(A_1) > 0$ and thus the power A_1^α is defined. If we define $X^\alpha = D(A_1^\alpha)$ and $||x||_\alpha := ||A_1^\alpha x||$, $x \in X^\alpha$ then $(X^\alpha, ||\,.\,||_\alpha)$ is a Banach space and there are constants $M \geqq 1, \gamma > 0$ such that the following holds:

$$||e^{-At}x||_\alpha \leqq Me^{-\gamma t}||x||_\alpha, \tag{7}$$

$$||e^{-At}x||_\alpha \leqq Mt^{-\alpha}e^{-\gamma t}||x|| \tag{8}$$

for all $t > 0, x \in X^\alpha$, (see [6, Theorem 1.5.4], or [10, Proposition 2.1.1]).

By a **mild solution** of the initial value problem (6)on the interval $\langle -r, T)$ we mean a continuous function $x : \langle 0, T) \to X^\alpha$ for which the function

$$f(\,.\,, x(\,.\,), x_.) : \langle -r, T) \to X,\, t \mapsto f(t, x(t), x_t)$$

is continuous, where

$$f : R^+ \times X^\alpha \times C_\alpha \to X$$

$(C_\alpha := C(\langle -r, 0), X^\alpha)$ with the norm $||\Phi||_\alpha := sup_{-r \leqq \Theta \leqq 0} ||\Phi(\Theta)||_\alpha)$ and

$$x(t) = e^{-At}\Phi(0) + \int_0^t e^{-A(t-s)} f(s, x(s), x_s) ds,\, t \in \langle 0, T),$$

$$x(t) = \Phi(t),\, t \in \langle -r, 0\rangle.$$

We assume

(G1)

$$||f(t, x, y)|| \leqq F_1(t)\omega_1(||x||_\alpha) + F_2(t)\omega_2(||y||_{C_\alpha})$$

for all $t \geqq 0, x \in X^\alpha, y \in C_\alpha$, where $F_1(t), F_2(t)$ and ω_1, ω_2 are as above.

(G2)

$$\int_0^\infty \frac{\tau^{q-1} d\tau}{\omega_1(\tau)^q + \omega_2(\tau)^q} = \infty,$$

where $q = q(\epsilon) = \frac{1}{\beta} + \epsilon$, $\epsilon > 0$, $\beta = 1 - \alpha$.

The definition of **nonextendable** and **global solution** is the same as for the initial value problem (1). In the paper [15] the following theorem is proved.

THEOREM 3 ([15, Theorem 3]**).** *Let the conditions (G1) and (G2) with $\omega_1 = \omega_2$ be fulfilled and the inequalities (7), (8) hold. Then any nonextendable solution $x : \langle 0, T) \to X$ of the initial value problem (6) is global.*

We shall prove the following theorem.

THEOREM 4. *Let the conditions (G1) and (G2) with $\omega_1 \neq \omega_2$, be fulfilled, i. e.*

$$\int_0^\infty \frac{\tau^{q-1} d\tau}{\omega_1(\tau)^q + \omega_2(\tau)^q} = \infty$$

and the conditions (7), (8) hold. Then any nonextendable solution $x : \langle 0, T) \to X^\alpha$ of the initial value problem (6) is global.

Proof. Let $x : \langle 0, T) \to X^\alpha$ be a nonextendable solution with $0 < T < \infty$. Then $\lim_{t \to T^-} ||x(t)||_\alpha = \infty$.

$$||x(t)||_\alpha \leqq ||e^{-At}\Phi(0)||_\alpha + \int_0^t ||e^{A(t-s)} f(s, x(s), x_s)||_\alpha ds.$$

Applying the property of f and the inequalities (7), (8) we obtain

$$||x(t)||_\alpha \leqq M e^{-\gamma t} ||\Phi(0)||_\alpha + M \int_0^t (t-s)^{-\alpha} e^{-\gamma s} ||f(s, x(s), x_s)|| ds \leqq$$

$$\leqq Me^{-\gamma t}||\Phi(0)||_\alpha + M\int_0^t (t-s)^{-\alpha}e^{-\gamma s}[F_1(s)\omega_1(||x(s)||_\alpha) + F_2(s)\omega_2(||x_s||_{C_\alpha})]ds.$$

Now we shall apply a desingularization method developed in the paper [11] for integral inequalities with singular kernels.

Let $\beta = 1-\alpha = \frac{1}{1+z}, z>0, p = \frac{1+z+\epsilon}{z+\epsilon}, q = q(\epsilon) = 1+z+\epsilon = \frac{1}{\beta}+\epsilon, \epsilon>0$ then $\frac{1}{p}+\frac{1}{q} = 1$ and using the Hölder inequality we obtain

$$\int_0^t (t-s)^{-\alpha}F_1(s)\omega_1(||x(s)||_\alpha)ds \leqq$$

$$\leqq \left(\int (t-s)^{-\alpha p}e^{ps}\right)^{\frac{1}{p}} \left(\int_0^t F_1(s)e^{-qs}\omega_1(||x(s)||_\alpha)^q ds\right)^{\frac{1}{q}}.$$

Analogously for the second integral. Since

$$\int_0^t (t-s)^{-\alpha p}e^{ps}ds = e^{pt}\int_0^t \tau^{-\alpha p}e^{-p\tau}d\tau < Qe^{pt},$$

where $Q = \frac{\Gamma(1-\alpha p)}{p^{1-\alpha p}}$ $(\alpha p = \frac{z(1+z+\epsilon)}{(1+z)(z+\epsilon)}) \neq 1)$. If we denote

$$a = Qe^T, C_i(t) = F_i(t)^q,$$

we obtain

$$||x(t)||_\alpha \leqq M||\Phi(0)||_\alpha +$$

$$+\left(\int_0^t C_1(s)\omega_1(||x(s)||_\alpha)^q ds\right)^{\frac{1}{q}} + \left(\int_0^t C_2(s)\omega_2(||x_s||_{C_\alpha})^q\right)^{\frac{1}{q}}.$$

Applying the inequality $(A+B+C)^q \leqq 3^{q-1}(A^q+B^q+C^q), A, B, C \geqq 0$ we have

$$(*)\qquad ||x(t)||_\alpha^q \leqq a + \int_0^t D_1(s)\omega_1(||x(s)||_\alpha)^q ds + \int_0^t D_2(s)\omega_2(||x_s||_{C_\alpha)})^q ds,$$

where $a = 3^{q-1}\left(M||\Phi(0)||_\alpha\right)^q, D_i = 3^{q-1}C_i, i = 1,2$. Now we can proceed as in the proof of [15, Theorem 3]. If we denote the right-hand side of the inequality (*) by $g(t)$ then

$$||x(t)||_\alpha^q \leqq g(t).$$

Without loss of generality we can choose $M \geqq 1$ so large that $||\Phi(t)||_C \leqq a$. Then we have

$$||x_t||_{C_\alpha}^q = \max\{||\Phi||_C^q, \sup_{0\leqq\tau\leqq t}||x(\tau)||^q\} \leqq \max\{a, g(t)\} = g(t).$$

Now one can proceed in the same way as in the proof of [15, Theorem 3] to obtain

$$\lim\sup_{t\to T^-}\Lambda(||x(t)||_\alpha^q) = \lim\sup_{t\to T^-}\int_0^{||x(t)||_\alpha^q}\frac{d\sigma}{\omega(\sigma^{\frac{1}{q}})} =$$

$$= \int_0^\infty \frac{d\sigma}{\omega(\sigma^{\frac{1}{q}})} = q\int_0^\infty \frac{\tau^{q-1}d\tau}{\omega(\tau)} = \infty,$$

where $\omega = \omega_1^q + \omega_2^q$. The contradiction.

Remark: If we put $\omega = \omega_1^q + \omega_2^q, F_1 = D_1, F_2 = D_2$ in the condition (G1)(see the proof of Theorem 4) we obtain Theorem 4 as a consequence of Theorem 3. However the proof of Theorem 4 says us why this function has the form $\omega = \omega_1^q + \omega_2^q$, if the condition (G1) is formulated with different functions ω_1, ω_2. The case of the strongly continuous semigroup, considered above is more clear. We remark that the variables y, z of the mapping $f(t, y, z)$ belong to different spaces.

Example:

Let

$$\omega_1(u) = u,\ \omega_2(u) = u^{\frac{q-1}{q}}(1 + \ln(1+u))^{\frac{1}{q}}.$$

Then

$$\int_0^\infty \frac{\sigma^{q-1}d\sigma}{\omega_1(\sigma)^q + \omega_2(\sigma)^q} = \int_0^\infty \frac{d\sigma}{\sigma + 1 + \ln(1+\sigma)} = \int_1^\infty \frac{d\tau}{\tau + \ln\tau} = \infty$$

and thus the condition (G2) is satisfied.

4. Application: Boundary value problem for a delay system of parabolic equations

The following application of our results concerning differential equations, whose linear parts are represented by sectorial operators, is motivated by the paper [8], where some stability results for systems of differential equations studied below, however without delay, are proved. We also use some notations from this paper. In the paper [9] there are many nice examples for equations, whose linear parts are represented by continuous semigroups. It would be possible to apply our results from the first part of this paper to this type of equations but we concentrate to the following application only.

Consider the following delay system of parabolic equations with homogeneous Neumann boundary conditions:

$$\frac{\partial u(t,x)}{\partial t} = D\Delta u(t,x) + f(t, u(t,x), u_t),\ t \geqq 0,\ u \in R^n,\ x \in \Omega \subset R^N \text{ a bounded set,}$$

$$\frac{\partial u}{\partial \eta} = 0,\ x \in \partial\Omega,$$

$$u(t,x) = \Phi(t,x),\ -r \leqq t \leqq 0,\ x \in \Omega, \tag{9}$$

$$\frac{\partial \Phi}{\partial \eta} = 0,\ x \in \partial\Omega,$$

where $u_t \in C = C(\langle -r, 0\rangle \times \Omega, R^n), u_t(\Theta, x) = u(t+\Theta, x), N = 1, 2,$ or $3, D = diag\{d_1, d_2, \ldots, d_n\}$ is a diagonal matrix, $d_i > 0, i = 1, 2, \ldots, n,$

Let

$$X = Ł^2(\Omega) := L^2(\Omega, R),\ A : D(A) \subset X \to X,\ A\phi = -\Delta\phi,$$

$$D(A) = \{\phi \in H^2(\Omega, R) : \frac{\partial \phi}{\partial \eta} = 0, \text{ on } \partial\Omega\},$$

$H^2 := H^2(\Omega, R) := W^{2,2}(\Omega, R)$ is the Sobolev space of $g \in L^2(\Omega, R)$ possessing distributional derivatives $D^j g(x)$of order $\leqq 2$ with the norm

$$||g||_{H^2} = \sum_{0 \leqq |j| \leqq 2} \left(\int_\Omega ||D^j g(x)||^2 dx \right)^{\frac{1}{2}},$$

$j = (j_1, j_2, \ldots, N)), |j| = j_1 + j_2 + \cdots + j_N.$

This system, however without delay, is studied in the paper [8] and the approach we are using in the following considerations also comes from this paper.

The operator A has eigenvaues

$$0 = \lambda_1 < \lambda_2 < \cdots < \lambda_k \to \infty$$

with the property that λ_j has finite multiplicity γ_j (which equals to the dimension of the corresponding eigenspace). Further, there is a complete set $\{\Phi_{j,k}\}$ of eigenvectors of A. This yields that if $x \in D(A)$ then

$$Ax = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle x, \Phi_{j,k} \rangle \Phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j x,$$

where $\langle \, . \, , \, . \, \rangle$ is the inner product in X and

$$E_j x = \sum_{k=1}^{\gamma_j} \langle x, \Phi_{j,k} \rangle \Phi_{j,k}.$$

The family $\{E_j\}$ is a family of complete orthonormal projections in X and

$$x = \sum_{j=1}^{\infty} E_j x, \;\; x \in X.$$

The operator $-A$ generates an analytic semigroup $\{e^{-At}\}_{t \geqq 0}$ defined by

$$e^{-At} x = E_1 x + \sum_{j=2}^{\infty} e^{-\lambda_j t} E_j x.$$

For $a > 0, A_1 = A + aI, 0 < \alpha < 1$ the fractional space

$$X^\alpha = D(A_1^\alpha) = \{x \in X : \sum_{j=1}^{\infty} (\lambda_j + a)^{2\alpha} ||E_j x||^2 < \infty\}$$

is defined and

$$A_1^\alpha x = \sum_{j=1}^{\infty} (\lambda_j + a)^\alpha E_j x.$$

Let

$$Z = L^2(\Omega, R^n) = X^n = X \times X \times \cdots \times X,\ C := C(\Omega, R^n) = C(\Omega)^n$$

with the usual norms.

Define

$$A_D : D(A_D) \subset Z \to Z,\ \ A_D\psi = -D\Delta\psi = DA\psi,$$

where

$$D(A_D) = \{\phi \in H^2(\Omega, R^n) : \frac{\partial\phi}{\partial\eta} = 0,\ \text{ on}\, \partial\Omega\},$$

$-A_D$ is sectorial with the fractional power

$$Z^\alpha = D(A_{D1}^\alpha) = X^\alpha \times \ldots X^\alpha = [X^\alpha]^n,\ A_{D1} = A_D + aI,$$

$$||z||_\alpha = ||A_{D1}^\alpha z||,\ \ z \in Z^\alpha,$$

$$A_{D1}^\alpha z = \sum_{j=1}^{\infty} D^\alpha(\lambda_j + a)^\alpha P_j z,$$

$$D^\alpha = diag\{d_1^\alpha, d_2^\alpha, \ldots, d_n^\alpha\},\ \ P_j = diag\{E_j, E_j, \ldots, E_j\},\ \ n \times n\text{matrix}.$$

The operator generates the analytic semigroup $\{e^{-A_D t}\}_{t \geqq 0},$, where

$$e^{-A_D t} z = P_1 z + \sum_{j=2}^{\infty} e^{-\lambda_j D t} P_j z,\ \ z \in Z.$$

The family $\{P_j\}$ is a family of orthonormal projections in Z which is complete and therefore

$$z = \sum_{j=1}^{\infty} ||P_j z,\ \ ||z|| = \sum_{j=1}^{\infty} ||P_j z||^2,\ \ ||z||_\alpha = \sum_{j=1}^{\infty} ||P_j z||_\alpha^2,$$

For $\frac{3}{4} < \alpha < 1$ the following inequalities holds:

$$||e^{-A_D t} z||_\alpha \leqq M||z||_\alpha,\ \geqq 0,$$

$$||e^{-A_D t} z||_\alpha \leqq M t^{-\alpha}||z||,\ \ t > 0$$

([6, Theorem 1.6.1]) and the inclusions

$$Z^\alpha \subset C(\Omega, R^n),\ \ Z^\alpha \subset L^p(\Omega, R^n),\ \ p \geqq 2$$

are continuous (see [6]).

The abstract formulation of the problem:

$$\dot{z} + A_D y = g(t, z, z_t),\ \ t \geqq 0,$$

$$z(t) = \Phi(t),\ \ -r \leqq t \leqq 0,\ z \in Z^\alpha,$$

$$z(x)(t) = z(t, x), z(x)_t(\Theta) = y(t + \Theta, x),$$

$$g : R^+ \times Z^\alpha \times C_\alpha \to Y,\ g(t, z, y) = f(t, z(x), y(x)_t),\ \ x \in \Omega, y(x) \in C_\alpha.$$

From the continuous inclusion $Z^\alpha \subset C(\Omega)^n$ it follows that there exists $l > 1$ such that

$$\sup_{x\in\Omega} ||z(x)||_{R^n} \leqq l||z||_\alpha, \ z \in Z^\alpha.$$

If we assume

$$f(t,z,y) = f_1(t,z) + f_2(t,y)$$

and

$$||f_1(t,z)|| \leqq F_1(t)\omega_1(||z||), \ ||f_2(t,y)|| \leqq F_2(t)\omega_2(||y||_C),$$

then

$$g_1(t,z) = f_1(t,z(x)), \ g_2(t,y) = f_2(t,y)$$

and

$$\text{(11)} \qquad ||g_1(t,z)|| \leqq F_1(t)\omega_1(l||z||\alpha) \leqq F_1(t)\omega_1(k||z||_\alpha),$$

$$\text{(12)} \qquad ||g_2(t,y)|| \leqq F_2(t)\omega_2(Rl||y||_{C_\alpha}) \leqq F_2(t)\omega_2(k||y||_{C_\alpha})$$

where $k = \max\{l, Rl\}$. If we assume

$$\int_0^\infty \frac{\tau^{q-1} d\sigma}{\omega(\sigma)} = \infty,$$

where $\omega = \omega_1^q + \omega_2^q, q = \frac{1}{\beta} + \epsilon$ then also

$$\int_0^\infty \frac{\tau^{q-1} d\sigma}{\omega(k\sigma)} = \infty,$$

and Theorem 4 yields that the solution $x(t)$ of the boundary value problem has the property that if $0 < T < \infty$ then $\lim_{t\to T^-} ||x(t)||_\alpha < \infty$ i. e. any nonextendable solution of the boundary value problem is global.

We have proved the following theorem.

Theorem 5. *Let the operator A be as above and assume that the mapping f from the right-hand side of the diffusion equation has the form $f(t,z,y) = f_1(t,z) + f_2(t,y)$ and the following conditions are satisfied:*

(1)

$$||f_1(t,z)|| \leqq F_1(t)\omega_1(||z||),$$

$$||f_2(t,y)|| \leqq F_2(t)\omega_2(||y||_C),$$

for $t \geqq 0, (z,y) \in R^n \times C(\langle -r, 0\rangle \times \Omega, R^n)$, where $\omega_1, \omega_2 : \langle 0,\infty) \to (0,\infty)$ are continuous, positive and nondecreasing functions on R_+.

(2)

$$\int_0^\infty \frac{\tau^{q-1} d\tau}{\omega_1(\tau)^q + \omega_2(\tau)^q} = \infty,$$

where $q = \frac{1}{\beta} + \epsilon$, $\epsilon > 0, \frac{3}{4} < \alpha = 1 - \beta < 1$.

Then any nonextendable solution of the boundary value problem (9) is global.

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