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ON PARAMETRIZED PROBLEMS WITH NON-LINEAR BOUNDARY CONDITIONS

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Abstract. We consider a parametrized boundary-value problem containing an unknown parameter both in the non-linear ordinary differential equations and in the non-linear boundary conditions. By using a suitable change of variables, we reduce the original problem to a family of those with linear boundary conditions plus some non-linear algebraic determining equations. We construct a numerical-analytic scheme suitable for studying the solutions of the transformed boundary-value problem.

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1. Introduction

The parametrized boundary value problems (PBVPs) were studied analytically earlier mostly in the case when the parameters are contained only in the differential equation (see, e.g. [1], [2]).

The analysis of the literature concerning the theory of boundary value problems (BVPS) shows that a lot of numerical methods (shooting, collocation, finite difference methods) are used for finding the solutions of BVPs and PBVPs as well. However, we note that the numerical methods appear only in the context when the existence of a solution of the given BVP or PBVP is supposed (see, e.g. [3], [4], [5], [6], [7]).

The boundary value problems with parameters both in the non-linear differential equations and in the linear boundary boundary conditions were investigated in [8], [9], [10], [11], [12], [13] by using the so called numerical-analytic method based upon successive approximations [8], [13].

According to the basic idea of the method mentioned the given boundary-value problem (BVP) is replaced by a problem for a "perturbed" differential equation containing some new artificially introduced parameter, whose numerical value should be determined later. The solution of the modified problem is sought for in the analytic form by successsive iterations with all iterations depending upon both the artificially introduced parameter and the parameter containing in the given BVP.

As for the way how the modified problem is constructed, it is essential that the form of the "perturbation term", depending on the original differential equation

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and boundary condition, yields a certain system of algebraic or transcendental "determining equations", which give the numerical values as well as for the artificially introduced parameters and for the parameters of the given BVP.

By studying these determining equations, one can establish existence results for the original PBVP. The numerical-analytic techniqueed described above was used to different types of parametrized boundary-value problems. Namely, in [8], [13] were studied the following two-point PBVPs :

$$\begin{cases} \dfrac{dx}{dt} = f(t,x), \ t \in [0,T], \ x, f \in \mathbb{R}^n, \\ Ax(0) + \lambda Cx(T) = d, \ \det C \neq 0, \ \lambda \in \mathbb{R}, \\ x_1(0) = x_{10}, \end{cases}$$

the PBVPs with nonfixed right boundary :

$$\begin{cases} \dfrac{dx}{dt} = f(t,x), \ t \in [0,\lambda], \ x, f \in \mathbb{R}^n, \\ Ax(0) + Cx(\lambda) = d, \ \det C \neq 0, \ \lambda \in (0,T], \\ x_1(0) = x_{10}, \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt} = f(t,x), \ t \in [0,\lambda_2], \ x, f \in \mathbb{R}^n, \\ \lambda_1 Ax(0) + Cx(\lambda_2) = d, \ \det C \neq 0, \ \lambda_1 \in \mathbb{R}, \ \lambda_2 \in (0,T], \\ x_1(0) = x_{10}, \ x_2(0) = x_{20}, \end{cases}$$

and the PBVP of form

$$\begin{cases} \dfrac{dx}{dt} = f(t,x), \ t \in [0,T], \ x, f \in \mathbb{R}^n, \\ \lambda_1 Ax(0) + \lambda_2 Cx(T) = d, \ \det C \neq 0, \ \lambda_1, \lambda_2 \in \mathbb{R}, \\ x_1(0) = x_{10}, \ x_2(0) = x_{20}, \end{cases}$$

The paper [9] deals with the two-point PBVP

$$\begin{cases} \dfrac{dx}{dt} = f(t,x) + \lambda_1 g(t,x), \ t \in [0,T], \ x, f \in \mathbb{R}^n, \\ Ax(0) + \lambda_2 Cx(T) = d, \ \det C \neq 0, \ \lambda_1, \lambda_2 \in \mathbb{R}, \\ x_1(0) = x_{10}, \ x_2(0) = x_{20}. \end{cases}$$

In [10], [11] a scheme of the numerical-analytic method of successive approximations was given for studying the solutions of PBVP

$$\begin{cases} \dfrac{dx}{dt} = f(t,x,\lambda_1), \ t \in [0,\lambda_2], \ x, f \in \mathbb{R}^n, \\ \lambda_1 Ax(0) + C(\lambda_1)x(\lambda_2) = d(\lambda_2), \ \det C \neq 0, \ \lambda_1 \in \mathbb{R}, \ \lambda_2 \in (0,T], \\ x_1(0) = x_{10}, \ x_2(0) = x_{20}. \end{cases}$$

In the paper [12] it was studied the three-point PBVP of the form

$$\begin{cases} \dfrac{dx}{dt} = f(t,x,\lambda_1), \ t \in [0,\lambda_2], \ x, f \in \mathbb{R}^n, \\ Ax(0) + A_1 x(t_1) + Cx(\lambda_2) = d(\lambda_1), \ \det C \neq 0, \ \lambda_1 \in \mathbb{R}, \ \lambda_2 \in (0,T], \\ x_1(0) = x_{10}, \ x_2(0) = x_{20}. \end{cases}$$

It should be noted, that the PBVPs mentioned above are subjected to linear boundary conditions. In [18], [8], [13] the methodology of the numerical-analytic method was extended in order to make it possible to study the non-linear two-point boundary value problem of the form

$$\begin{cases} \dfrac{dy}{dt} = f(t, y(t)),\ t \in [0,T],\ y, f \in \mathbb{R}^n, \\ g(y(0), y(T)) = 0,\ g \in \mathbb{R}^n, \end{cases}$$

with non-linear boundary conditions, for which purpose a general non-linear change of variable was introduced in the given equation.

In the paper [14], it was suggested to use a simpler substitution, which, as was shown, essentially facilitates the application of the numerical-analytic method based upon successive approximations. In particular all the assumptions for the applicability of the method are formulated in terms of the original problem, and not the transformed one. It was established, that for the non-linear boundary-value problem with separated non-linear boundary conditions of the form

$$\begin{cases} \dfrac{dx}{dt} = f(t, x(t)),\ t \in [0,T],\ x, f \in \mathbb{R}^n, \\ x(T) = a(x(0)),\ a \in \mathbb{R}^n, \end{cases}$$

the numerical-analytic method can be applied without any change of variables.

The similar results were obtained in [15] for problems with separated non-linear boundary conditions of form

$$\begin{cases} \dfrac{dx}{dt} = f(t, x(t)),\ t \in [0,T],\ x, f \in \mathbb{R}^n, \\ x(0) = b(x(T)),\ b \in \mathbb{R}^n. \end{cases}$$

Naturally, the latter non-linear BVP by the trivial change $t = T - \tau$ of the independent variable can be reduced to the last but one BVP. However, in [15] it was shown that the appropriate version of the numerical-analytic method based upon successive approximations can be applied directly without any change of variable.

Following to the method from [14], [15], in [16], [17] it was suggested how one can construct a numerical-analytic scheme suitable for studying the PBVPs with parameters both in the non-linear differential equation and in the non-linear two-point boundary conditions of the form

$$\begin{cases} \dfrac{dy}{dt} = f(t, y, \lambda_1, \lambda_2),\ t \in [0,T],\ y, f \in \mathbb{R}^n, \\ g(y(0), y(T), \lambda_1, \lambda_2) = 0,\ \lambda_1 \in [a_1, b_1], \lambda_2 \in [a_2, b_2], \\ y_1(0) = y_{10},\ y_2(0) = y_{20}. \end{cases}$$

Here we give a possible approach how one can handle, by using the numerical-analytic method, some PBVPs with boundary conditions of more general form then mentioned above.

2. Problem setting

We consider the non-linear two-point parametrized boundary-value problem

$$\frac{dy}{dt} = f(t, y\,(t)\,, \lambda),\ t \in [0,T] \tag{2.1}$$

$$g\,(y\,(0)\,,\ y\,(T)\,, \lambda) = 0, \tag{2.2}$$

$$y_1(0) = h(\lambda,\ y_2(0), y_3(0), ..., y_n(0)) \tag{2.3}$$

containing the scalar parameter λ both in Eq.(2.1) and in conditions (2.2), (2.3).

Here, we suppose that the functions

$$f : [0, T] \times G \times [a, b] \to \mathbb{R}^n,\ (n \geq 2),$$

$$g : G \times G \times [a, b] \to \mathbb{R}^n$$

and

$$h : [a, b] \times G_1 \to \mathbb{R}$$

are continuous, where $G \subset \mathbb{R}^n$, $G_1 \subset \mathbb{R}^{n-1}$are a closed, connected, bounded domains and $\lambda \in J := [a, b]$ is an unknown scalar parameter (the domain G_1 is chosen so that $G_1 \subset G$).

Assume that, for $t \in [0, T]$ and $\lambda \in J$ fixed, the function f satisfies the Lipschitz condition in the form

$$|f(t, u, \lambda) - f(t, v, \lambda)| \leq K\,|u - v| \tag{2.4}$$

for all $\{u, v\} \subset G$ and some non-negative constant matrix $K = (K_{ij})_{i,j=1}^n$. In (2.4), as well as in similar relations below the signs $|\cdot|$, $\leq$, $\geq$ are understood componentwise.

The problem is to find the values of the control parameter λ such that the problem (2.1), (2.2) has a classical continuously differentiable solution satisfying the additional condition (2.3). Thus, a solution is the pair $\{y, \lambda\}$ and, therefore, (2.1)-(2.3) is similar, in a sence, to an eigen-value or to a control problem.

3. Construction of an equivalent problem with linear boundary conditions

Let us introduce the substitution

$$y(t) = x(t) + w, \tag{3.1}$$

where $w = \mathrm{col}(w_1, w_2, ..., w_n) \in \Omega \subset \mathbb{R}^n$ is an unknown parameter. The domain Ω is chosen so that $D + \Omega \subset G$, whereas the new variable x is supposed to have range in D, the closure of a bounded subdomain of G. Using the change of variables (3.1), the problem (2.1)-(2.3) can be rewritten as

$$\frac{dx}{dt} = f(t, x(t) + w, \lambda),\ t \in [0, T], \tag{3.2}$$

$$g(x(0) + w, x(T) + w, \lambda) = 0, \tag{3.3}$$

$$x_1(0) = h(\lambda,\ x_2(0) + w_2, x_3(0) + w_3, ..., x_n(0) + w_n) - w_1. \tag{3.4}$$

Let us rewrite the boundary conditions (3.3) in the form

$$Ax(0) + Bx(T) + g(x(0) + w, x(T) + w, \lambda) = Ax(0) + Bx(T), \tag{3.5}$$

where A, B are fixed square n-dimentional matrices such that $\det B \neq 0$.

The artificially introduced parameter w is natural to be determined from the system of algebraic determining equations

$$Ax(0) + Bx(T) + g(x(0) + w, x(T) + w, \lambda) = 0. \tag{3.6}$$

Obviously, if (3.6) holds then from (3.5)

$$Ax(0) + Bx(T) = 0. \tag{3.7}$$

Thus, the essentially non-linear problem (2.1)-(2.3) with non-linear boundary conditions turns out to be equivalent to the collection of two-point boundary value problems

$$\frac{dx}{dt} = f(t, x(t) + w, \lambda), \ t \in [0, T], \tag{3.8}$$

$$Ax(0) + Bx(T) = 0, \tag{3.9}$$

$$x_1(0) = h(\lambda, \ x_2(0) + w_2, x_3(0) + w_3, ..., x_n(0) + w_n) - w_1, \tag{3.10}$$

parametrized by the unknown vector $w \in \mathbb{R}^n$ and considered together with the determining equation (3.6). The essential advantage obtained thereby is that the boundary condition (3.9) is linear already.

By virtue of (3.9), every solution x of the boundary-value problem (3.8)-(3.10) satisfies the condition

$$x(T) = -B^{-1}Ax(0). \tag{3.11}$$

Therefore, taking into account (3.11), the determining equation (3.6) can be rewritten as

$$g\left(x(0) + w, -B^{-1}Ax(0) + w, \lambda\right) = 0. \tag{3.12}$$

So, we conclude that the original non-linear boundary-value problem (2.1)-(2.3) is equivalent to the family of boundary-value problems (3.8)-(3.10) with linear conditions (3.9) considered together with the non-linear system of algebraic determining equations (3.12).

We note, that the family of boundary-value problems (3.8)-(3.10) can be studied by using the numerical-analytic method based upon successive approximations developed in [8], [13].

Assume, that the given PBVP (2.1)-(2.3) is such, that the subset

$$D_\beta := \{y \in \mathbb{R}^n : B(y, \beta(y)) \subset G\}$$

is non-empty

$$D_\beta \neq \oslash, \tag{3.13}$$

where

$$\beta(y) := \frac{T}{2}\delta_G(f) + \left|(B^{-1}A + I_n)y\right|, \tag{3.14}$$

$$\delta_G(f) := \frac{1}{2}\left[\max_{(t,y,\lambda)\in[0,T]\times G\times J} f(t, y, \lambda) - \min_{(t,y,\lambda)\in[0,T]\times G\times J} f(t, y, \lambda)\right],$$

I_n is an n-dimensional unit matrix and $B(y, \beta(y))$ denotes the ball of radius $\beta(y)$ with the center point y.

Moreover, we suppose that the spectral radius $r(K)$ of the matrix K in (2.4) satisfies the inequality

$$r(K) < \frac{10}{3T}. \tag{3.15}$$

Let us define the subset $U \subset \mathbb{R}^{n-1}$ such that

$$U := \left\{u = \mathrm{col}(u_2, u_3, ..., u_n) \in \mathbb{R}^{n-1} : z \in D_\beta\right\},$$

where

$$z = \mathrm{col}(h(\lambda,\ u_2 + w_2, u_3 + w_3, ..., u_n + w_n) - w_1, u_2, u_3, ..., u_n). \tag{3.16}$$

Let us connect with the boundary-value problem (3.8)-(3.10) the sequence of functions

$$\begin{aligned} x_{m+1}(t, w, u, \lambda) := z &+ \int_0^t f\left(s, x_m(s, w, u, \lambda) + w, \lambda\right) ds \\ &- \frac{t}{T}\int_0^T f\left(s, x_m(s, w, u, \lambda) + w, \lambda\right) ds \\ &- \frac{t}{T}\left[B^{-1}A + I_n\right] z, \\ m = 0, 1, 2, ...,\ & x_0(t, w, u, \lambda) = z \in D_\beta, \end{aligned} \tag{3.17}$$

depending on the artificially introduced parameters $w \in \Omega \subset \mathbb{R}^n$, $u \in U \subset \mathbb{R}^{n-1}$ and on the parameter $\lambda \in [a, b]$ containing in the problem (2.1)-(2.3), where the vector z has the form (3.16).

Note, that for the initial value of functions $x_m(t, w, u, \lambda)$ at the point $t = 0$ holds the following equality

$$x_m(0, w, u, \lambda) = z \tag{3.18}$$

for all $m = 0, 1, 2, ...,$ and arbitrary $w \in \Omega,\ u \in U,\ \lambda \in [a, b]$.

It can be verified also, that all functions of the sequence (3.17) satisfy the linear homogeneous two-point boundary condition (3.9) and an additional condition (3.10) for arbitrary $u \in U$ given by (3.16) and $w \in \Omega,\ \lambda \in [a, b]$.

We suggest to solve the PBVP (3.8)-(3.10) together with the determining equation (3.12) sequentially, namely first solve (3.8)-(3.10), and then try to find the values of parameters $w \in \Omega \subset \mathbb{R}^n,\ u \in U \subset \mathbb{R}^{n-1},\ \lambda \in [a, b]$ for which the equation (3.12) can simultaneously be fulfilled.

4. Investigation of the solutions of the transformed problem (3.8)-(3.10)

It was already pointed out that the transformed family of PBVPs (3.8)-(3.10) can be studied on the base of the numerical-analytic technique developed in [8], [13]. We shall follow it. However, we note, that the form of additional condition (3.10) requires an appropriate modification of the scheme of successive approximations and, consequently, demands to find the corresponding conditions granting the applicability of the method.

First we establish some results concerning the PBVP (3.8)-(3.10) with specially modified right-hand side function in Eq.(3.8).

Theorem 1. *Let us suppose that the functions* $f : [0, T] \times G \times [a, b] \to \mathbb{R}^n$, $g : G \times G \times [a, b] \to \mathbb{R}^n$, $h : [a, b] \times G_1 \to \mathbb{R}^n$ *are continuous and the conditions (2.4), (3.13)-(3.16) are satisfied.*

Then:

1. The sequence of functions (3.17) satisfying the boundary conditions (3.9),(3.10) for arbitrary $u \in U$, $w \in \Omega$ and $\lambda \in [a,b]$, converges uniformly as $m \to \infty$ with respect the domain

$$(t, w, u, \lambda) \in [0, T] \times \Omega \times U \times [a, b] \tag{4.1}$$

to the limit function

$$x^*(t, w, u, \lambda) = \lim_{m \to \infty} x_m(t, w, u, \lambda). \tag{4.2}$$

2. The limit function $x^*(\cdot, w, u, \lambda)$ having the initial value $x^*(0, w, u, \lambda) = z$ given by (3.16) is the unique solution of the integral equation

$$\begin{aligned} x(t) = z &+ \int_0^t f\left(s, x(s) + w, \lambda\right) ds \\ &- \frac{t}{T}\left[\int_0^T f\left(s, x(s) + w, \lambda\right) ds + \left(B^{-1}A + I_n\right) z\right], \end{aligned} \tag{4.3}$$

i.e. it is a solution of the modified (with regard to (3.8)) integro-differential equation

$$\frac{dx}{dt} = f(t, x + w, \lambda) + \Delta(w, u, \lambda), \tag{4.4}$$

satifying the same boundary conditions (3.9),(3.10), where

$$\Delta(w, u, \lambda) = -\frac{1}{T}\left[\left(B^{-1}A + I_n\right) z + \int_0^T f\left(s, x(s) + w, \lambda\right) ds\right]. \tag{4.5}$$

3.The following error estimation holds :

$$|x^*(t, w, u, \lambda) - x_m(t, w, u, \lambda)| \leq e(t, w, u, \lambda), \tag{4.6}$$

where

$$\begin{aligned} e(t, w, u, \lambda) := \frac{20}{9} t\left(1 - \frac{t}{T}\right) Q^{m-1}\left(I_n - Q\right)^{-1}&\left[Q\delta_G(t)\right. \\ &\left. + K\left|\left(B^{-1}A + I_n\right) z\right|\right], \end{aligned}$$

the vector $\delta_G(t)$ is given by Eq.(3.14) and the matrix $Q = \frac{3T}{10}K$.

Proof. We shall prove, that under the conditions assumed, sequence (3.17) is a Cauchy sequence in the Banach space $C([0,T], \mathbb{R}^n)$ equipped with the usual uniform norm. First, we show that $x_m(t, w, u, \lambda) \in D$ for all $(t, w, u, \lambda) \in [0,T] \times \Omega \times U \times [a, b]$ and $m \in \mathbb{N}$. Indeed, using the estimation

$$\begin{aligned} &\left|\int_0^t \left[f(\tau) - \frac{1}{T}\int_0^T f(s) ds\right] d\tau\right| \leq \\ &\qquad \leq \frac{1}{2}\alpha_1(t)\left[\max_{t \in [0,T]} f(t) - \min_{t \in [0,T]} f(t)\right] \end{aligned} \tag{4.7}$$

of Lemma 2.3 from [13] or its generalization in Lemma 4 from [15], relation (3.17) for $m = 0$ implies that

$$|x_1(t,w,u,\lambda) - z| \leq \left| \int_0^t \left[f(t, z+w, \lambda) - \frac{1}{T}\int_0^T f(s, z+w, \lambda) ds \right] dt \right| +$$
$$+ \left| \left[\left(B^{-1}A + I_n \right) z \right] \right| \leq \alpha_1(t)\delta_G(f) + \beta_1(z) \leq \beta(z) \tag{4.8}$$

where

$$\alpha_1(t) = 2t\left(1 - \frac{t}{T}\right), \ |\alpha_1(t)| \leq \frac{T}{2}, \tag{4.9}$$

$$\beta_1(z) = \left| \left[B^{-1}A + I_n \right] z \right|. \tag{4.10}$$

Therefore, by virtue of (3.13), (3.14), (4.8), we conclude that $x_1(t,w,u,\lambda) \in D$ whenever $(t,w,u,\lambda) \in [0,T] \times \Omega \times U \times [a,b]$. By induction, one can easily establish that all functions (3.17) are also contained in the domain D for all $m = 1,2,...,\ t \in [0,T],\ w \in \Omega,\ u \in U,\ \lambda \in [a,b]$. Now, consider the difference of functions

$$\begin{aligned} x_{m+1}(t,w,u,\lambda) - x_m(t,w,u,\lambda) = &\int_0^t \left[f(s, x_m(s,w,u,\lambda) + w, \lambda) - \right. \\ &- \left. f(s, x_{m-1}(s,w,u,\lambda) + w, \lambda) \right] ds - \\ &- \frac{t}{T}\int_0^T \left[f(s, x_m(s,w,u,\lambda) + w, \lambda) - \right. \\ &- \left. f(s, x_{m-1}(s,w,u,\lambda) + w, \lambda) \right] ds \end{aligned} \tag{4.11}$$

and introduce the notation

$$d_m(t,w,u,\lambda) := |x_m(t,w,u,\lambda) - x_{m-1}(t,w,u,\lambda)|, \ m = 1,2,... \ . \tag{4.12}$$

By virtue of identity (4.12) and the Lipschitz condition (2.4), we have

$$d_{m+1}(t,w,u,\lambda) \leq K \left[\left(1 - \frac{t}{T}\right) \int_0^t d_m(s,w,u,\lambda) ds + \frac{t}{T} \int_t^T d_m(s,w,u,\lambda) ds \right] \tag{4.13}$$

for every $m = 0,1,2,...$ .According to (4.8)

$$d_1(t,w,u,\lambda) = |x_1(t,w,u,\lambda) - z| \leq \alpha_1(t)\delta_G(f) + \beta_1(z), \tag{4.14}$$

where $\beta_1(z)$ is given by (4.10).

Now we need the following estimations of Lemma 2.4 from [13]

$$\alpha_{m+1}(t) \leq \left(\frac{3}{10}T\right) \alpha_m(t), \ \alpha_{m+1}(t) \leq \left(\frac{3}{10}T\right)^m \overline{\alpha}_1(t), \tag{4.15}$$

obtained for the sequence of functions

$$\alpha_{m+1}(t)=\left(1-\frac{t}{T}\right)\int\limits_0^t\alpha_m(s)ds+\frac{t}{T}\int\limits_t^T\alpha_m(s)ds,\ m=0,1,2,...$$

$$\alpha_0(t)=1,\ \alpha_1(t)=2t\left(1-\frac{t}{T}\right), \tag{4.16}$$

where $\overline{\alpha_1}(t)=\frac{10}{9}\alpha_1(t)$.

In view of (4.14), (4.16), for $m=1$ it follows from (4.13)

$$\begin{aligned}d_2(t,w,u,\lambda)\le K\delta_G(f)&\left[\left(1-\frac{t}{T}\right)\int\limits_0^t\alpha_1(s)ds+\frac{t}{T}\int\limits_t^T\alpha_1(s)ds\right]+\\&+K\beta_1(z)\left[\left(1-\frac{t}{T}\right)\int\limits_0^t ds+\frac{t}{T}\int\limits_t^T ds\right]\\&\le K\left[\alpha_2(t)\delta_G(f)+\alpha_1(t)\beta_1(z)\right].\end{aligned}$$

By induction, we can easily obtain

$$d_{m+1}(t,w,u,\lambda)\le K^m[\alpha_{m+1}(t)\delta_G(f)+\alpha_m(t)\beta_1(z)],m=0,1,2,...\ , \tag{4.17}$$

where $\alpha_{m+1}(t),\ \alpha_m(t)$ are calculated according to (4.16), $\delta_G(f)$, and $\beta_1(z)$ are given by (3.14) and (4.10). By virtue of the second estimate from (4.15), we have from (4.17)

$$\begin{aligned}d_{m+1}(t,w,u,\lambda)&\le\overline{\alpha}_1(t)\left[\left(\frac{3}{10}TK\right)^m\delta_G(f)+K\left(\frac{3}{10}TK\right)^{m-1}\beta_1(z)\right]=\\&=\overline{\alpha}_1(t)\left[Q^m\delta_G(f)+KQ^{m-1}\beta_1(z)\right],\end{aligned} \tag{4.18}$$

for all $m=1,2,...$, where the matrix

$$Q=\frac{3}{10}TK. \tag{4.19}$$

Therefore, in view of (4.18)

$$\begin{aligned}&|x_{m+j}(t,w,u,\lambda)-x_m(t,w,u,\lambda)|\le\\&\le|x_{m+j}(t,w,u,\lambda)-x_{m+j-1}(t,w,u,\lambda)|+\\&+|x_{m+j-1}(t,w,u,\lambda)-x_{m+j-2}(t,w,u,\lambda)|+...+\\&+|x_{m+1}(t,w,u,\lambda)-x_m(t,w,u,\lambda)|=\sum_{i=1}^{j}d_{m+i}(t,w,u,\lambda)\le\\&\le\overline{\alpha}_1(t)\left[\sum_{i=1}^{j}\left(Q^{m+i}\delta_G(f)+KQ^{m+i-1}\beta_1(z)\right)\right]=\\&=\overline{\alpha}_1(t)\left[Q^m\sum_{i=0}^{j-1}Q^i\delta_G(f)+KQ^m\sum_{i=0}^{j-1}Q^i\beta_1(z)\right].\end{aligned} \tag{4.20}$$

Since, due to (3.15), the maximum eigenvalue of the matrix Q of the form (4.19) does not exceed the unity, therefore

$$\sum_{i=0}^{j-1} Q^i \leq (I_n - Q)^{-1}$$

and

$$\lim_{m\to\infty} Q^m = [0].$$

We can conclude from (4.20) that, according to the Cauchy criteria, the sequence $x_m(t, w, u, \lambda)$ of the form (3.17) uniformly converges in the domain (4.1) and, hence, the assertion (4.2) holds.

Since all functions $x_m(t, w, u, \lambda)$ of the sequence (3.17) satisfy the boundary conditions (3.9), (3.10), the limit function $x^*(t, w, u, \lambda)$ also satisfies these conditions. Passing to the limit as $m \to \infty$ in equality (3.17), we show that the limit function satisfies the integral equation (4.3). It is also obvious from (4.3), that

$$x^*(T, w, u, \lambda) = -B^{-1}Az, \tag{4.21}$$

which means that $x^*(t, w, u, \lambda)$ is a solution of the integral equation (4.3) as well as the solution of the integro-differential equation (4.4). Estimate (4.6) is an immediate consequence of (4.20). □

Now we show that, in view of Theorem 1, the PBVP (3.8)-(3.10) can be formally interpreted as a family of initial value problems for differential equations with "additively forced" right-hand side member. Namely, consider the Cauchy problem

$$\frac{dx(t)}{dt} = f(t, x(t) + w, \lambda) + \mu,\ t \in [0, T], \tag{4.22}$$

$$x(0) = z = \mathrm{col}(h(\lambda,\ x_2\,(0) + w_2, ..., x_n\,(0) + w_n) - w_1, u_2, u_3, ..., u_n), \tag{4.23}$$

where $\mu \in \mathbb{R}^n,\ z \in D_\beta,\ w \in \Omega,\ \lambda \in [a, b]$ are parameters.

Theorem 2. *Under the conditions of Theorem 1, the solution $x = x(t, w, u, \lambda)$ of the initial value problem (4.22), (4.23) satisfies the boundary conditions (3.9), (3.10) if and only if*

$$\mu = \Delta(w, u, \lambda), \tag{4.24}$$

where $\Delta : \Omega \times U \times [a, b] \to \mathbb{R}^n$ is the mapping defined by (4.5).

Proof. According to Picard-Lindelöf existence theorem it is easy to show that the Lipschitz condition (2.4) implies that the initial value problem (4.22), (4.23) has a unique solution for all

$$(\mu, w, u, \lambda) \in \mathbb{R}^n \times \Omega \times U \times [a, b].$$

It follows from the proof of Theorem 1 that, for every fixed

$$(w, u, \lambda) \in \Omega \times U \times [a, b] \tag{4.25}$$

the limit function (4.2) of the sequence (3.17) satisfies the integral equation (4.3) and, in addition, $x^*(t, w, u, \lambda) = \lim_{m\to\infty} x_m(t, w, u, \lambda)$ satisfies the boundary conditions (3.9), (3.10). This implies immediately that the function $x = x^*(t, w, u, \lambda)$ of the form (4.2) is the unique solution of the initial value problem

$$\frac{dx(t)}{dt} = f(t, x(t) + w, \lambda) + \Delta(w, u, \lambda), \; t \in [0, T], \tag{4.26}$$

$$x(0) = \mathrm{col}(h(\lambda, \; x_2\,(0) + w_2, ..., x_n\,(0) + w_n) - w_1, u_2, u_3, ..., u_n), \tag{4.27}$$

where $\Delta(w, u, \lambda)$ is given by (4.5). Hence, (4.26), (4.27) coincides with (4.22), (4.23) corresponding to

$$\mu = \Delta(w, u, \lambda) = -\frac{1}{T}\left[\left(B^{-1}A + I_n\right) z + \int_0^T f\,(s, x(s) + w, \lambda)\, ds\right]. \tag{4.28}$$

The fact that the function (4.2) is not a solution of (4.22), (4.23) for any other value of μ, not equal to (4.28), is obvious, e.g., from Eq.(4.24). □

The following statement shows what is the relation of the solution $x = x^*(t, w, u, \lambda)$ of the modified PBVP (4.3), (3.9), (3.10) to the solution of the unperturbed BVP (3.8)- (3.10).

Theorem 3. *If the conditions of Theorem 1 are satisfied, then the function $x^*(t, w, u^*, \lambda^*)$ is a solution of the PBVP (3.8)- (3.10) if and only if, the triplet*

$$\{w, u^*, \lambda^*\} \in \Omega \times U \times [a, b] \tag{4.29}$$

satisfies the system of determining equations

$$\left[B^{-1}A + I_n\right] z + \int_0^T f\,(s, x^*(s, w, u, \lambda) + w, \lambda)\, ds = 0, \tag{4.30}$$

where z is given by (4.27) and w is considered as a parameter.

Proof. It suffices to apply Theorem 2 and notice that the differential equation in (4.26) coincedes with (3.8) if and only if the triplet (4.29) satisfies the equation

$$\Delta(w, u^*, \lambda^*) = 0, \tag{4.31}$$

i.e., when the relation (4.30) holds, where w is considered as a parameter $w \in \Omega$. □

Now becomes clear, how one should choose the value $w = w^*$ of the artificially introduced parameter w in (3.1) in order to the function

$$y^*(t) = x^*(t, w^*, u^*, \lambda^*) + w^* \tag{4.32}$$

be a solution of the original PBVP (2.1)-(2.3).

Theorem 4. *If the conditions of Theorem 1 are satisfied, then, for function (4.32) to be a solution of the given PBVP (2.1)-(2.3) it is necessary and sufficient that the triplet*

$$\{w^*, u^*, \lambda^*\} \tag{4.33}$$

satisfies the system of algebraic determining equations

$$g(z+w,-B^{-1}Az+w,\lambda)=0, \tag{4.34}$$

where

$$z:=\operatorname{col}(h(\lambda^*,\ u_2^*+w_2,...,u_n^*+w_n)-w_1,u_2^*,u_3^*,...,u_n^*), \tag{4.35}$$

and the pair $\{u^*,\lambda^*\}$ *is a solution of the system (4.30), parametrized by* w.

Proof. It was established in Section 3, that the PBVP (2.1)-(2.3) is equivalent to the family of BVPs (3.8)-(3.10) considered together with the determinig equation (3.12). The vector parameter z in (4.35) can be interpreted as the initial value at $t=0$ of a possible solution of the problem (3.8)-(3.10). Therefore, Eq.(3.12) can be rewritten in the form (4.34). Taking into account the change of variables (3.1) and the equivalence (2.1)-(2.3) to (3.8)-(3.10) (3.12), we notice that the function $y^*(t)$ in (4.32) coincides with the solution of the PBVP (2.1)-(2.3) if and only if $w=w^*$ satisfies the equation (4.34). □

Corollary 1. *Under the conditions of Theorem 1 the function* $y^*(t)$ *of the form (4.32), (4.2) will be a solution of the PBVP (2.1)-(2.3) if and only if the triplet (4.33) satisfies the system of determining equations*

$$\left[B^{-1}A+I_n\right]z+\int_0^T f\left(s,x^*(s,w,u,\lambda)+w,\lambda\right)ds=0,$$

$$g(z+w,-B^{-1}Az+w,\lambda)=0,$$

$$z=\operatorname{col}(h(\lambda,\ u_2+w_2,...,u_n+w_n)-w_1,u_2,u_3,...,u_n)), \tag{4.36}$$

containing $2n$ *scalar algebraic equations, where* $x^*(t,w,u,\lambda)$ *is given by (4.2).*

Proof. It suffices to apply Theorem 3 and Theorem 4. □

Remark 1. *In practice, it is natural to fix some natural* m *and instead of (4.36) consider the "approximate determining system"*

$$\left[B^{-1}A+I_n\right]z+\int_0^T f\left(s,x_m(s,w,u,\lambda)+w,\lambda\right)ds=0,$$

$$g(z+w,-B^{-1}Az+w,\lambda)=0, \tag{4.37}$$

$$z=\operatorname{col}(h(\lambda,\ u_2+w_2,...,u_n+w_n)-w_1,u_2,u_3,...,u_n).$$

In the case when the system (4.37) has an isolated root, say

$$w=w_m,\ u=u_m,\ \lambda=\lambda_m, \tag{4.38}$$

in some open subdomain of

$$\Omega\times U\times[a,b],$$

one can prove that under certain additional conditions, the exact determining system (4.36) is also solvable :

$$w=w^*,\ u=u^*,\ \lambda=\lambda^*.$$

Hence, the given non-linear PBVP (2.1)-(2.3) has a solution of form (4.32), such that

$$x^*(t=0) = \mathrm{col}(h(\lambda^*,\ u_2^* + w_2^*, ..., u_n^* + w_n^*) - w_1^*, u_2^*, u_3^*, ..., u_n^*) \in D_\beta,$$

$$w^* \in \Omega,\ \lambda^* \in [a,b],\ u^* \in U,\ y^* \in G.$$

Furthermore, the function

$$y_m(t) := x_m(t, w_m, u_m, \lambda_m) + w_m,\ t \in [0,T] \tag{4.39}$$

can be regarded as the "m-th approximation" to the exact solution $y^*(t) = x^*(t, w^*, u^*, \lambda^*) + w^*$, (see estimation (4.6)). To prove the solvability of the system (4.36), one can use some topological degree techniques (cf.Theorem 3.1 in [13], p.43) or the methods oriented to the solution of non-linear equations in Banach spaces developed in [19] (see, e.g. Theorem 19.2 in [19], p.281). Here, we do not consider this problem in more detail.

Remark 2. *If we choose in (3.5), (3.7) for the matrix A a zero matrix, then the PBVP (3.8)-(3.10) is reduced to the parametrized initial value problem*

$$\frac{dx}{dt} = f(t, x(t) + w, \lambda),\ t \in [0,T], \tag{4.40}$$

$$x(T) = 0, \tag{4.41}$$

with the additional condition (3.10). In this case, instead of successive approximations (3.17) we obtain

$$\begin{aligned} x_{m+1}(t,w,u,\lambda) := z + \int_0^t f(s, x_m(s,w,u,\lambda) + w, \lambda)\, ds \\ - \frac{t}{T}\int_0^T f(s, x_m(s,w,u,\lambda) + w, \lambda)\, ds - \frac{t}{T}z \\ m = 0,1,2,...,\ x_0(t,w,u,\lambda) = z \in D_\beta, \end{aligned} \tag{4.42}$$

where $z = \mathrm{col}(h(\lambda,\ u_2 + w_2, ..., u_n + w_n) - w_1, u_2, u_3, ..., u_n)$, and the system of determining equations (4.36) is transformed into the system

$$\begin{aligned} z + \int_0^T f(s, x^*(s,w,u,\lambda) + w, \lambda)\, ds = 0, \\ g(z + w, w, \lambda) = 0, \\ z = \mathrm{col}(h(\lambda, u_2 + w_2, ..., u_n + w_n) - w_1, u_2, u_3, ..., u_n). \end{aligned} \tag{4.43}$$

In this case Theorem 3 guarantees the existence of the solution of the parametrized Cauchy problem (4.40), (4.41) with the additional condition (3.10) on the interval $[0,T]$.

Remark 3. *If one can obtain the solution* $x = \widetilde{x}^0(t, w, \lambda)$ *of the parametrized initial value problem (4.40), (4.41) on the interval* $[0, T]$*, i.e. by Picard's iterations*

$$\widetilde{x}^0(t, w, \lambda) = \lim_{m\to\infty} \widetilde{x}_m(t, w, \lambda) = \\ = \lim_{m\to\infty} \int_T^t f\left(s, \widetilde{x}_{m-1}(t, w, \lambda)\right) ds, \tag{4.44}$$

$m = 1, 2, ...,$ $\widetilde{x}_0(t, w, \lambda) = z$*, then for finding the values of the parameters*

$$w = w^0, \ \lambda = \lambda^0, \tag{4.45}$$

for which the function

$$y^0(t) = \widetilde{x}^0(t, w, \lambda) + w^0 \tag{4.46}$$

will be the solution of the original PBVP (2.1)-(2.3), we should solve, according to (3.12), (3.4), the determining system

$$g(\widetilde{x}^0(0, w, \lambda) + w, w, \lambda) = 0, \\ \widetilde{x}^0_1(0, w, \lambda) = h(\lambda, \widetilde{x}^0_2(0, w, \lambda) + w_2, ..., \widetilde{x}^0_n(0, w, \lambda) + w_n) - w_1, \tag{4.47}$$

containing $(n+1)$ *equations with respect to* $(n+1)$ *unknown values* $w = \mathrm{col}(w_1, w_2, ..., w_n)$ *and* λ.

We apply the above techniques to the following PBVP.

5. Example of parametrized boundary value problem

Consider the second order parametrized two-point boundary-value problem

$$\frac{d^2y}{dt^2} - \frac{t}{8}\frac{dy}{dt} + \frac{\lambda^2}{2}\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}y(t) = \frac{9}{32} + \frac{t^2}{16}, \ t \in [0, 1], \tag{5.1}$$

$$y(0) = \left[\frac{dy(1)}{dt}\right]^2, \tag{5.2}$$

$$\frac{dy(0)}{dt} = \frac{dy(1)}{dt} - y(1) - \frac{\lambda}{16}, \tag{5.3}$$

satisfying an additional condition

$$y(0) = \frac{1}{16} + \lambda\left[\frac{dy(0)}{dt}\right]^2. \tag{5.4}$$

There is no method for finding its exact solution.However, the construction of the example allows us to check directly that the pair

$$\left\{y^*(t) = \frac{t^2}{8} + \frac{1}{16}, \quad \lambda = \lambda^* = 1\right\}$$

is an exact solution.

The approximate solution to be found will be compare with this exact one.

We note, that symbolic algebra tools are suitable for performing the necessary computations for the method described here, the authors have used Maple for them.

By setting $y_1 := y$ and $y_2 := \dfrac{dy}{dt}$ the PBVP (5.1)-(5.4) can be rewritten in the form of system (2.1)-(2.3) :

$$\begin{aligned} \frac{dy_1}{dt} &= y_2, \\ \frac{dy_2}{dt} &= \frac{9}{32} + \frac{t^2}{16} + \frac{t}{8} y_2 - \frac{\lambda^2}{2} y_2^2 - \frac{1}{2} y_1, \end{aligned} \tag{5.5}$$

$$\begin{aligned} y_1(0) &= [y_2(1)]^2, \\ y_2(0) &= y_2(1) - y_1(1) - \frac{\lambda}{16}, \end{aligned} \tag{5.6}$$

$$y_1(0) = \frac{1}{16} + \lambda \left[y_2(0)\right]^2. \tag{5.7}$$

Suppose that the PBVP (5.5)-(5.7) is considered in the domain

$$\begin{aligned} &(t, y, \lambda) \in [0,1] \times G \times [-1,1], \\ &G := \left\{ (y_1, y_2) : |y_1| \le 1,\ |y_2| \le \frac{3}{4} \right\}. \end{aligned} \tag{5.8}$$

One can verify that for the PBVP (5.5)
-(5.7), conditions (3.3), (3.13) and (3.15) are fulfiled in the domain (5.8) with the matrices

$$A := B := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, K := \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{7}{8} \end{bmatrix}.$$

Indeed, from the Perron theorem it is known that the greatest eigenvalue $\lambda_{\max}(K)$ of the matrix K in virtue of the nonnegativity of its elements is real, nonnegative and computations show that

$$\lambda_{\max}(K) \le \frac{21}{16}.$$

Moreover the vectors $\delta_G(f)$ and $\beta(y)$ in (3.14) are such

$$\delta_G(f) \le \begin{bmatrix} \frac{3}{4} \\ \frac{5}{4} \end{bmatrix}, \ \beta(y) := \frac{T}{2}\delta_G(f) + \left|\left(B^{-1}A + I_2\right) y\right| \le \begin{bmatrix} \frac{3}{8} \\ \frac{5}{8} \end{bmatrix} + 2\,|y|\,.$$

Substitution (3.1) brings the given system of differential equations (5.5) and the additional conditions (5.7) to the following form

$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_2(t) + w_2, \\ \frac{dx_2(t)}{dt} &= \frac{9}{32} + \frac{t^2}{16} + \frac{t}{8}(x_2(t) + w_2) - \\ &- \frac{\lambda^2}{2}(x_2(t) + w_2)^2 - \frac{1}{2}(x_1(t) + w_1), \end{aligned} \tag{5.9}$$

and

$$x_1(0) = \frac{1}{16} + \lambda \left[x_2(0) + w_2\right]^2 - w_1. \tag{5.10}$$

Thus we reduce the essentially non-linear PBVP (5.5)- (5.7) to the collection of two-point BVPs of view (3.8)- (3.10), namely to the system (5.9), which is considered under the linear two-point boundary condition

$$x(0) + x(1) = 0, \tag{5.11}$$

together with an additional condition (5.10) and algebraic determining system of equations of form (3.12)

$$x_1(0) + w_1 = (x_2(1) + w_2)^2,$$

$$x_2(0) + w_2 = (x_2(1) + w_2) - (x_1(1) + w_1) - \frac{\lambda}{16}.$$

Taking into account that according to (3.11)

$$x(1) = \mathrm{col}(x_1(1), x_2(1)) = -B^{-1}Ax(0) = \mathrm{col}(-x_1(0), -x_2(0)),$$

the determining system obtained above can be rewritten in the form

$$x_1(0) + w_1 = (-x_2(0) + w_2)^2,$$

$$2x_2(0) = x_1(0) - w_1 - \frac{\lambda}{16}. \tag{5.12}$$

In our case due to the equality (3.16),

$$z = \mathrm{col}(z_1, z_2) = \mathrm{col}\left(\frac{1}{16} + \lambda\,(u_2 + w_2)^2 - w_1, u_2\right), \tag{5.13}$$

and the components of the iteration sequence (3.17) for the PBVP (5.9) under the linear boundary conditions (5.10) have the form

$$x_{m+1,1}(t, w, u, \lambda) = \left[\frac{1}{16} + \lambda\,(u_2 + w_2)^2 - w_1\right] +$$

$$+\int_0^t [x_{m,2}(s, w, u, \lambda) + w_2]\, ds- \tag{5.14}$$

$$-t\int_0^1 [x_{m,2}(s, w, u, \lambda) + w_2]\, ds - 2t\left[\frac{1}{16} + \lambda\,(u_2 + w_2)^2 - w_1\right],$$

$$x_{m+1,2}(t, w, u, \lambda) = u_2 + \int_0^t \left[\frac{9}{32} + \frac{s^2}{16} + \frac{s}{8}\,(x_{m,2}(s, w, u, \lambda) + w_2) -\right.$$

$$\left.- \frac{\lambda^2}{2}\,(x_{m,2}(s, w, u, \lambda) + w_2)^2 - \frac{1}{2}\,(x_{m,1}(s, w, u, \lambda) + w_1)\right] ds- \tag{5.15}$$

$$-t\int_0^1 \left[\frac{9}{32} + \frac{s^2}{16} + \frac{s}{8}\,(x_{m,2}(s, w, u, \lambda) + w_2) -\right.$$

$$\left.- \frac{\lambda^2}{2}\,(x_{m,2}(s, w, u, \lambda) + w_2)^2 - \frac{1}{2}\,(x_{m,1}(s, w, u, \lambda) + w_1)\right] ds - 2tu_2,$$

where $m = 0, 1, 2, ...,$and

$$x_0(t, w, u, \lambda) = z = \text{col}\left(\frac{1}{16} + \lambda\,(u_2 + w_2)^2 - w_1, u_2\right). \tag{5.16}$$

On the base of equalities (3.18) and (5.13) the determining equations (5.12), which are independent on the number of the iterations can be rewritten in the form

$$\frac{1}{16} + \lambda\,(u_2 + w_2)^2 = (w_2 - u_2)^2\,,$$

$$2u_2 = \frac{1}{16} + \lambda\,(u_2 + w_2)^2 - 2w_1 - \frac{\lambda}{16}. \tag{5.17}$$

The system of approximate determining equations depending on the number of iterations, which is given by the first equation in the system (4.37) together with (5.13), is written in component form as

$$2\left[\frac{1}{16} + \lambda\,(u_2 + w_2)^2 - w_1\right] + \int\limits_0^1 [x_{m,2}(s, w, u, \lambda) + w_2]\,ds = 0,$$

$$2u_2 + \int\limits_0^1 \left[\frac{9}{32} + \frac{s^2}{16} + \frac{s}{8}\,(x_{m,2}(s, w, u, \lambda) + w_2) - \right. \tag{5.18}$$

$$\left. - \frac{\lambda^2}{2}\,(x_{m,2}(s, w, u, \lambda) + w_2)^2 - \frac{1}{2}\,(x_{m,1}(s, w, u, \lambda) + w_1)\right] ds = 0.$$

Thus, for every $m \geq 1$, we have four equations (5.17), (5.18) in four unknowns w_1, w_2, u_2 and λ. Note, that in our case we can decrease the number of unknown values as follows.
Obviously, that from the first equation of (5.17)

$$\lambda = \frac{(w_2 - u_2)^2}{(w_2 + u_2)^2} - \frac{1}{16\,(w_2 + u_2)^2}. \tag{5.19}$$

Considering the auxiliarly equations (5.17) in the given domain, we find that

$$\frac{1}{16} + \lambda\,(u_2 + w_2)^2 = (w_2 - u_2)^2\,,$$

$$\frac{1}{16} + \lambda\,(u_2 + w_2)^2 = 2u_2 + 2w_1 + \frac{\lambda}{16},$$

from which

$$2w_1 = (w_2 - u_2)^2 - 2u_2 - \frac{\lambda}{16},$$

or by using (5.19), we obtain

$$w_1 = \frac{(w_2 - u_2)^2}{2} - u_2 - \tag{5.20}$$

$$-\frac{1}{32}\left[\frac{(w_2 - u_2)^2}{(w_2 + u_2)^2} - \frac{1}{16\,(w_2 + u_2)^2}\right].$$

So, by solving the determining system (5.12), which is independent on the number of iterations, we have already determined λ and w_1 in (5.19) and (5.20) as the functions of two other unknowns w_2 and u_2.

For finding the rest unknown values of w_2 and u_2 for each step of iterations (5.14) and (5.15), one should use the approximate determining equations (5.18). On the base of (5.14) and (5.15) as a result of the first iteration ($m = 1$) we get

$$\begin{aligned} x_{1,1}(t,w,u,\lambda) &= \lambda u_2^2 + 2\lambda u_2 w_2 + \lambda w_2^2 + \frac{1}{16} - w_1 - 2\lambda t u_2^2 \\ &\quad - 4\lambda t u_2 w_2 - 2\lambda t w_2^2 - \frac{1}{8}t + 2tw_1, \end{aligned} \tag{5.21}$$

$$x_{1,2}(t,w,u,\lambda) = u_2 + \frac{1}{48}t^3 + \frac{1}{16}t^2 u_2 + \frac{1}{16}t^2 w_2 - \frac{1}{48}t - \frac{33}{16}u_2 t - \frac{1}{16}w_2 t.$$

The system (5.18) on the base of the first iteration (5.21), now has the form

$$\begin{aligned} &\frac{1}{256}\frac{768u_2^3 + 1792u_2^2 w_2 + 1280u_2 w_2^2 + 256w_2^3 + 16u_2^2 - 32u_2 w_2 768u_2^3}{u_2^2 + 2u_2 w_2 + w_2^2} \\ &\quad + \frac{1}{256}\frac{1792u_2^2 w_2 + 1280u_2 w_2^2 + 256w_2^3 + 16u_2^2 - 32u_2 w_2}{u_2^2 + 2u_2 w_2 + w_2^2} \\ &\quad\quad + \frac{1}{256}\frac{16w_2^2 + 256u_2^4 - 512u_2^2 w_2^2 + 256w_2^4 - 1}{u_2^2 + 2u_2 w_2 + w_2^2} = 0, \end{aligned} \tag{5.22}$$

$$\begin{aligned} &\frac{13}{48} + \frac{33}{16}u_2 + \frac{1}{16}w_2 - \frac{1}{512}\frac{(-1 + 16u_2^2 - 32u_2 w_2 + 16w_2^2)^2 u_2^2}{(u_2^2 + 2u_2 w_2 + w_2^2)^2} \\ &- \frac{1}{256}\frac{(-1 + 16u_2^2 - 32u_2 w_2 + 16w_2^2)^2 u_2 w_2}{(u_2^2 + 2u_2 w_2 + w_2^2)^2} - \frac{1}{512}\frac{(-1 + 16u_2^2 - 32u_2 w_2 + 16w_2^2)^2 w_2^2}{(u_2^2 + 2u_2 w_2 + w_2^2)^2} \\ &- \frac{1}{32}\frac{(-1 + 16u_2^2 - 32u_2 w_2 + 16w_2^2) u_2^2}{(u_2^2 + 2u_2 w_2 + w_2^2)} - \frac{1}{16}\frac{(-1 + 16u_2^2 - 32u_2 w_2 + 16w_2^2) u_2 w_2}{(u_2^2 + 2u_2 w_2 + w_2^2)} \\ &\quad - \frac{1}{32}\frac{(-1 + 16u_2^2 - 32u_2 w_2 + 16w_2^2)^2 w_2^2}{(u_2^2 + 2u_2 w_2 + w_2^2)} = 0. \end{aligned} \tag{5.23}$$

whose solution, in the given domain is

$$w_{1,2} \approx 0.1179015870,\ u_{1,2} \approx -0.1338961033. \tag{5.24}$$

Note that there are other solutions in the other domains.From (5.19) and (5.20) one can easily obtain the values

$$\lambda_1 \approx 3.526154164,\ w_{1,1} \approx 0.05540481607. \tag{5.25}$$

Therefore, the first approximation to the first and second components of the solution according to (4.39) has the form

$$y_{1,1}(t) \approx x_{1,1}(t, w_{1,1}, w_{1,2}, u_{1,2}, \lambda_1) + w_{1,1} \approx 0.06340207685 - 0.01599452160t,$$

$$
\begin{aligned}
y_{1,2}(t) &\approx x_{1,2}(t, w_{1,1}, w_{1,2}, u_{1,2}, \lambda_1) + w_{1,2} \\
&\approx 0.02083333333t^3 - 0.999657268 \cdot 10^{-3}t^2 + 0.2479585306t - 0.0159945163.
\end{aligned} \tag{5.26}
$$

Proceeding analogously for the fourth approximation ($m = 4$) in (5.14) and (5.15) we find

$$
\begin{aligned}
x_{4,1}(t, w, u, \lambda) = &-w_1 + 2\lambda u_2 w_2 - 0.21374562 \cdot 10^{-4}t^5 - 0.69130099 \cdot 10^{-3}t^4 \\
&+0.65708464 \cdot 10^{-3}t^3 - 0.6326129976 \cdot 10^{-5}t^8 \\
&-0.3808172041 \cdot 10^{-5}t^7 + 0.1248805182t^2 - 0.42080288 \cdot 10^{-8}t^9 \\
&+0.1662064401 \cdot 10^{-10}t^{13} - 0.1900061164 \cdot 10^{-11}t^{16} \\
&+0.6860781119 \cdot 10^{-9}t^{11} + 0.4862489477 \cdot 10^{-12}t^{15} \\
&-0.5957506904 \cdot 10^{-10}t^{14} + 0.6093635263 \cdot 10^{-8}t^{12} \\
&-0.2471813047 \cdot 10^{-3}t^6 + 0.1704708784 \cdot 10^{-6}t^{10} + \lambda u_2^2 + \lambda w_2^2 \\
&+2w_1 t - 0.2495677846t - 2t\lambda u_2^2 - 2t\lambda w_2^2 - 4t\lambda u_2 w_2 + 0.0625,
\end{aligned} \tag{5.27}
$$

and

$$
\begin{aligned}
x_{4,2}(t,w,u,\lambda) = u^2 &- 0.647545596\cdot 10^{-3}t^5 - 0.1118770981\cdot 10^{-3}t^4\\
&+0.0133283149t^3 - 0.4407955994\cdot 10^{-6}t^8 - 0.2304845756\cdot 10^{-4}t^7\\
&-0.5845940655\cdot 10^{-2}t^2 - 0.0625w_2t - 0.5342567636\cdot 10^{-6}t^9\\
&+0.7031117611\cdot 10^{-9}t^{13} + 0.5698229856\cdot 10^{-13}t^{16} + 0.1937169073\cdot 10^{-7}t^{11}\\
&-0.695042472\cdot 10^{-11}t^{15} + 0.1929181894\cdot 10^{-11}t^{14} + 0.7861311699\cdot 10^{-10}t^{12}\\
&-0.2838727909\cdot 10^{-5}t^6 - 0.47340324\cdot 10^{-9}t^{10} + 0.1503988637\cdot 10^{-12}\lambda^2t^{18}\\
&-0.6913042073\cdot 10^{-15}\lambda^2t^{20} + 0.4913103636\cdot 10^{-13}\lambda^2t^{19}\\
&-0.2235366075\cdot 10^{-12}t^{17} - 0.006696109t\\
&-0.6332049769\cdot 10^{-16}\lambda^2t^{22} + 0.4509410252\cdot 10^{-17}\lambda^2t^{25}\\
&-0.9620951462\cdot 10^{-18}\lambda^2t^{24} + 0.7391221931\cdot 10^{-23}\lambda^2t^{30}\\
&-0.1490676615\cdot 10^{22}\lambda^2t^{31} + 0.5192983789\cdot 10^{-4}\lambda^2t^7\\
&-0.1008467555\cdot 10^{-13}\lambda^2t^{21} - 0.121375516\cdot 10^{-15}\lambda^2t^{23}\\
&+0.6939382414\cdot 10^{-19}\lambda^2t^{27} - 0.4757705056\cdot 10^{-20}\lambda^2t^{26}+\\
&0.4518584297\cdot 10^{-21}\lambda^2t^{28} - 0.8752608579\cdot 10^{-21}\lambda^2t^{29}\\
&-0.1248805182\lambda^2t^2w_2 + 0.0625t^2w_2 + 0.0154415763\lambda^2t^2\\
&-0.2520690302\cdot 10^{-4}\lambda^2t^6 - .6093635262\cdot 10^{-8}\lambda^2t^2w_2 \qquad (5.28)\\
&-0.1704708784\cdot 10^{-6}\lambda^2t^{10}w_2 + 0.877965225\cdot 10^{-8}\lambda^2t^{10}\\
&-0.3437663379\cdot 10^{-8}\lambda^2t^{12} - 0.16620644\cdot 10^{-10}\lambda^2t^{13}w_2\\
&-0.1515754547\cdot 10^{-6}\lambda^2t^{11} - 0.6842671\cdot 10^{-8}\lambda^2t^{13}\\
&+0.247181304\cdot 10^{-3}\lambda^2t^6w_2 - 0.6860781118\cdot 10^{-9}\lambda^2t^{11}w_2\\
&+0.21374562\cdot 10^{-4}\lambda^2t^5w_2 + 0.380817204\cdot 10^{-5}t^7\lambda^2w_2\\
&-0.4862489477\cdot 10^{-12}t^{15}\lambda^2w_2 + 0.1105588996\cdot 10^{-9}t^{15}\lambda^2\\
&-0.0103155135\lambda^2t^3 + 0.1350964754\cdot 10^{-3}\lambda^2t^5\\
&+0.3785125161\cdot 10^{-6}t^8\lambda^2 - 0.2085655281\cdot 10^{-3}\lambda^2t^4\\
&+0.42080288\cdot 10^{-8}\lambda^2t^9w_2 + 0.6326129975\cdot 10^{-5}t^8\lambda^2w_2\\
&+0.00069130099\lambda^2t^4w_2 - 0.6570846399\cdot 10^{-3}\lambda^2t^3w_2\\
&+0.953475575\cdot 10^{-6}t^9\lambda^2 + 0.1177754495\cdot 10^{-10}\lambda^2t^{17}\\
&+0.3699654819\cdot 10^{-11}\lambda^2t^{16} + 0.5957506904\cdot 10^{-10}\lambda^2t^{14}w_2\\
&+0.1900061164\cdot 10^{-11}\lambda^2t^{16}w_2 - 0.1073892527\cdot 10^{-9}\lambda^2t^{14}\\
&-0.0050804956\lambda^2t + 0.1245677846\lambda^2tw_2 - 2tu_2.
\end{aligned}
$$

The determining system (5.18) for the fourth approximation is

$$0.5\cdot 10^{-9}\cdot\frac{126833963u_2^2-246332074u_2w_2+126833963w_2^2+0.1\cdot 10^{11}u_2^2w_2}{u_2^2+2u_2w_2+w_2^2}$$

$$+0.5\cdot 10^{-9}\cdot\frac{8\cdot 10^9u_2w_2^2+2\cdot 10^9w_2^3+2\cdot 10^9u_2^4-4\cdot 10^9u_2^2w_2^2}{u_2^2+2u_2w_2+w_2^2} \tag{5.29}$$

$$+0.5\cdot 10^{-9}\cdot\frac{2\cdot 10^9w_2^4+4\cdot 10^9u_2^3-7812500}{u_2^2+2u_2w_2+w_2^2}=0,$$

$$-0.1\cdot 10^{-13}\cdot\frac{358195910w_2+0.5\cdot 10^{14}u_2^5w_2+0.25\cdot 10^{14}u_2^4w_2^2-0.3\cdot 10^{15}u_2^3w_2^3}{(u_2^2+2u_2w_2+w_2^2)^2}$$

$$-0.1\cdot 10^{-13}\cdot\frac{0.275\cdot 10^{15}u_2^2w_2^4-0.25\cdot 10^{15}u_2^5+0.25\cdot 10^{14}u_2^6+0.75\cdot 10^{14}w_2^6}{(u_2^2+2u_2w_2+w_2^2)^2}$$

$$-0.1\cdot 10^{-13}\cdot\frac{(-0.1525366793)\cdot 10^{16}u_2^3w_2^2-0.1036949811\cdot 10^{16}u_2^2w_2^3}{(u_2^2+2u_2w_2+w_2^2)^2}$$

$$-0.1\cdot 10^{-13}\cdot\frac{(-0.2753667926)\cdot 10^{15}u_2w_2^4-0.615830184\cdot 10^{13}w_2^5-0.15\cdot 10^{15}w_2^5u_2}{(u_2^2+2u_2w_2+w_2^2)^2}$$

$$-0.1\cdot 10^{-13}\cdot\frac{(-0.100615)\cdot 10^{16}w_2u_2^4+0.6560293\cdot 10^{11}u_2^2-0.3277\cdot 10^{14}u_2^4}{(u_2^2+2u_2w_2+w_2^2)^2}$$

$$-0.1\cdot 10^{-13}\cdot\frac{(-0.1146226912)\cdot 10^{11}u_2^2w_2+0.229245382\cdot 10^{11}u_2w_2^2}{(u_2^2+2u_2w_2+w_2^2)^2} \tag{5.30}$$

$$-0.1\cdot 10^{-13}\cdot\frac{(-0.3902743497)\cdot 10^{14}w_2^4-0.1146226912\cdot 10^{11}w_2^3}{(u_2^2+2u_2w_2+w_2^2)^2}$$

$$-0.1\cdot 10^{-13}\cdot\frac{(-0.1269111518)\cdot 10^{15}u_2^3w_2-0.114411151\cdot 10^{15}u_2w_2^3+1001665969}{(u_2^2+2u_2w_2+w_2^2)^2}$$

$$-0.1\cdot 10^{-13}\cdot\frac{0.260915\cdot 10^{12}w_2^2-0.1904146\cdot 10^{15}u_2^2w_2^2+0.25941\cdot 10^{12}u_2w_2}{(u_2^2+2u_2w_2+w_2^2)^2}=0.$$

Solving numerically the system (5.18), taking into account (5.19), (5.20), we obtain the following values of the parameters:

$$w_{4,2}\approx 0.1264301453,\quad u_{4,2}\approx -0.1235847040,$$

$$\lambda_4\approx 0.9170414150,\ w_{4,1}\approx 0.1261810697. \tag{5.31}$$

The fourth approximation of the first and second components of the solution of PBVP (5.5)- (5.7) then has the form

$$y_{2,1}(t)\approx x_{2,1}(t,w_{2,1},w_{2,2},u_{2,2},\lambda_2)+w_{2,1}\approx \tag{5.32}$$

$$
\begin{gathered}
\approx -0.21374562 \cdot 10^{-4} t^5 - 0.0006913 t^4 + 0.0006570846 t^3 \\
-0.6326129976 \cdot 10^{-5} t^8 - 0.380817204 \cdot 10^{-5} t^7 + 0.1248805182 t^2 \\
-0.42080288 \cdot 10^{-8} t^9 + 0.16620644 \cdot 10^{-10} t^{13} - 0.1900061164 \cdot 10^{11} t^{16} \\
+0.6860781119 \cdot 10^{-9} t^{11} + 0.4862489477 \cdot 10^{-12} t^{15} - 0.5957506904 \cdot 10^{-10} t^{14} \\
+0.6093635263 \cdot 10^{-8} t^{12} - 0.2471813047 \cdot 10^{-3} t^6 + 0.1704708784 \cdot 10^{-6} t^{10} \\
+0.002779505 t + .00625074249,
\end{gathered}
$$

$$
y_{2,2}(t) \approx x_{2,2}(t, w_{2,1}, w_{2,2}, u_{2,2}, \lambda_2) + w_{2,2} \approx \tag{5.33}
$$

$$
\begin{gathered}
\approx -0.000531661 t^5 - 0.0021377198 t^4 + 0.0045834661 t^3 \\
+0.5501353148 \cdot 10^{-6} t^8 + 0.2102761382 \cdot 10^{4} t^7 + 0.0017640565 t^2 \\
+0.2680301942 \cdot 10^{-6} t^9 - 0.505310211 \cdot 10^{-8} t^{13} + 0.3370283168 \cdot 10^{-11} t^{16} \\
-0.1081709011 \cdot 10^{-6} t^{11} + 0.8597403591 \cdot 10^{-10} t^{15} - 0.8204719714 \cdot 10^{-10} t^{14} \\
-0.346023686 \cdot 10^{-8} t^{12} + 0.224428736 \cdot 10^{-5} t^6 - 0.1121505341 \cdot 10^{-7} t^{10} \\
+0.9680965973 \cdot 10^{-11} t^{17} + 0.2415433567 t + 0.6215758632 \cdot 10^{-23} t^{30} \\
-0.8090883031 \cdot 10^{-18} t^{24} + 0.379225599 \cdot 10^{-17} t^{25} - 0.532503196 \cdot 10^{-16} t^{22} \\
+0.4131747987 \cdot 10^{-13} t^{19} - 0.5813626128 \cdot 10^{-15} t^{20} + 0.1264801739 \cdot 10^{-12} t^{18} \\
-0.7360637096 \cdot 10^{-21} t^{29} + 0.3799971048 \cdot 10^{-21} t^{28} - 0.4001063227 \cdot 10^{-20} t^{26} \\
+0.5835777432 \cdot 10^{-19} t^{27} - 0.1020725556 \cdot 10^{-15} t^{23} - 0.8480858738 \cdot 10^{-14} t^{21} \\
-0.1253606795 \cdot 10^{-22} t^{31} + 0.002845441.
\end{gathered}
$$

As is seen from Figure 1, 2 and 3, 4, the graphs of the exact solution

$$
\left\{ y^*(t) = \frac{t^2}{8} + \frac{1}{16}, \quad \lambda = \lambda^* = 1 \right\}
$$

and the fourth approximation almost coincide, whereas the deviation of their derivatives does not exceed 0.0025.

References

[1] Luchka, A.Yu.: *Application of the iteration processes to boundary-value problems for differential equations with parameters,* Dokl. Akad.Nauk. Ukr.SSR, Ser. A, No.10, (1989), 22-27. (in Russian)

[2] Fečkan, M.: *Parametrized singular boundary-value problems,* J.Math. Anal. Appl., 188 (1994), 417-425.

[3] Keller, Herbert B. : *Numerical methods for two-point boundary-value problems,* Dover Publications,Inc., New York (1992).

[4] Ascher, Uri M., Mattheij, Robert M.M. and Russel, Robert D.: *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations* , SIAM, Philadelphia (1995).

[5] Bhattacharyya, T., Binding, P.A. and Seddigni, K.: *Multiparameter Sturm-Liouville problems with eigenparameter dependent boundary conditions,* J.Math.Anal. appl., 264, No.2, (2001), 560-576.

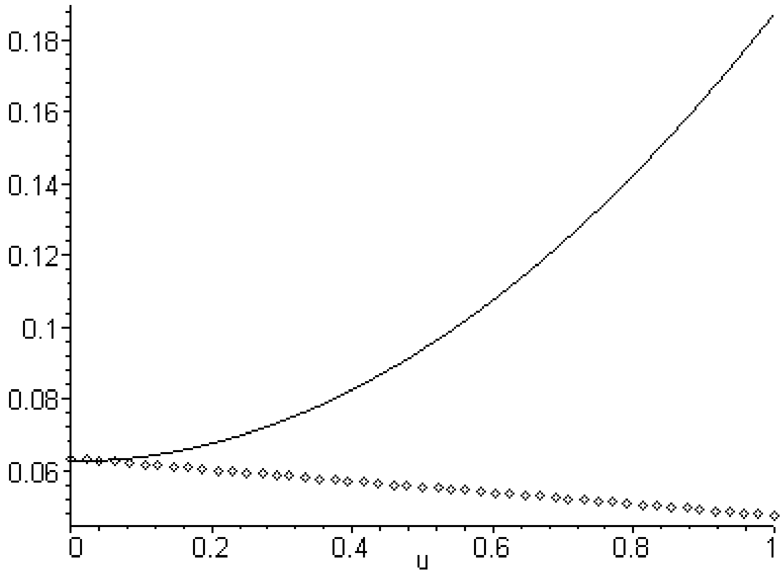

FIGURE 1. The first components of the exact solution (solid line) and its first approximation (drawn with dots).

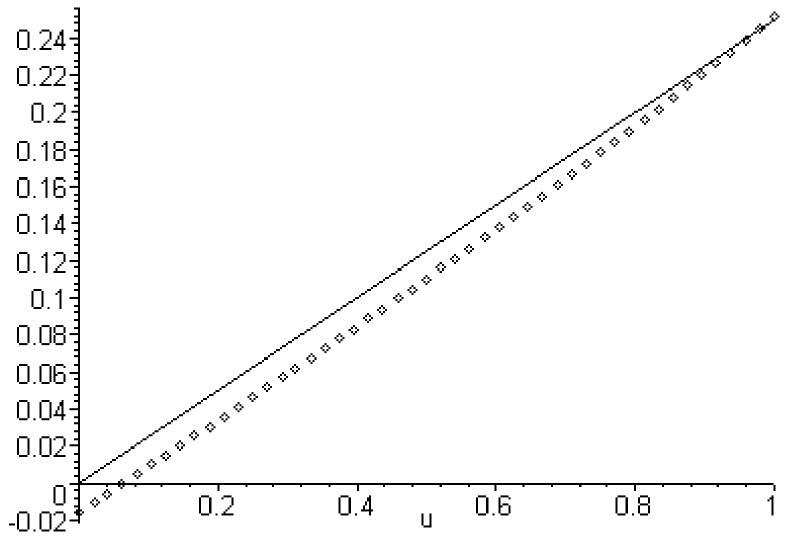

FIGURE 2. The second components of the exact solution (solid line) and its first approximation (drawn with dots)

[6] Abramov, A.A., Ul'yanova, V.I. and Yukhno, L.F. : *A method for solving a multiparameter spectral problem for some systems of differential equations,* Comput. Math. Math. Phys. 40, No. 1, (2000), 18-26.

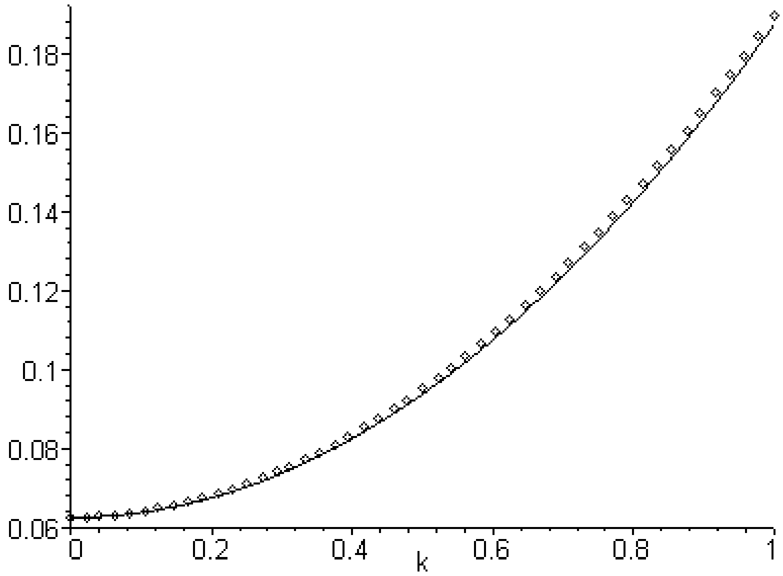

FIGURE 3. The first components of the exact solution (solid line) and its fourth approximation (drawn with dots)

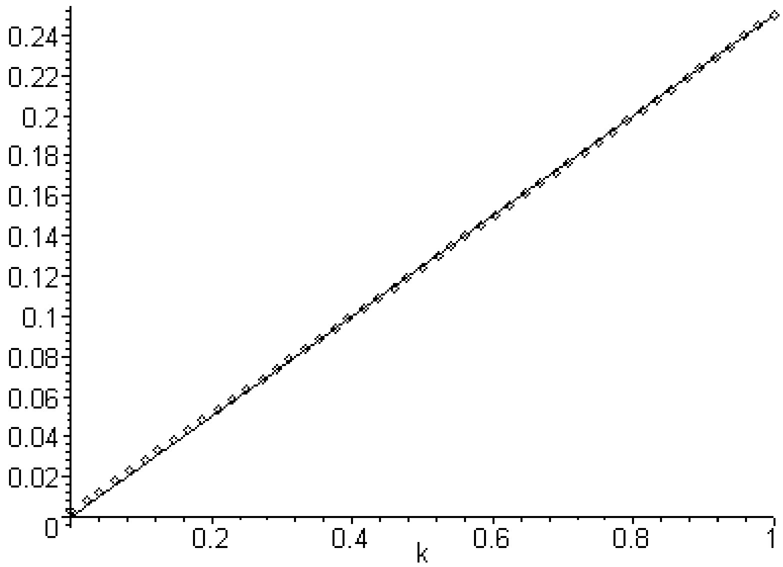

FIGURE 4. The second components of the exact solution (solid line) and its fourth approximation (drawn with dots)

[7] Chanane, B. and Boucherif A.: *Computation of the eigenpairs of two-parameter Sturm-Liouville problems with three-point boundary conditions,* Int.J. Differ. Equ. Appl. 7 , No.2, (2003), 203-210.

[8] Samoilenko, A.M. and Ronto, N.I.: *Numerical-Analytic Methods in the Theory of Boundary -Value Problems for ordinary Differential Equations,* Naukova Dumka, Kiev (1992). (in Russian).

[9] Ronto, N.I. and Korol, I.I.: *Investigating and solutions of parametrized boundary-value problems by numerical-analytic methods*, Ukrain. Math. Zh.,46, No.8 (1994), 1031-1042.

[10] Ronto, M.: *On numerical-analytic method for BVPs with parameters,* Publ.Univ. of Miskolc, Series D. Natural Sciences, vol.36, No.2. Mathematics (1996), 125-132.

[11] Ronto, M.: *On some existence results for parametrized boundary-value problems,* Publ.Univ. of Miskolc, Series D. Natural Sciences, vol.37. Mathematics (1997), 95-103.

[12] Ronto, M. and Tégen, M.: *Numerical-analytic methods for investigating three point boundary-value problems with parameters,* Publ.Univ. of Miskolc, Series D. Natural Sciences, vol.40. Mathematics (1999), 67-77.

[13] Ronto, M. and Samoilenko, A.M.: *Numerical-Analytic Methods in the Theory of Boundary -Value Problems,* World Scientific, Singapore (2000).

[14] Ronto, A. and Rontó, M.: *On the investigation of some boundary-value problems with non-linear conditions,* Mathematical Notes,Miskolc, vol.1., No.1., (2000), 43-45.

[15] Ronto, A. and Rontó, M.: *A note on the numerical-analytic method for non-linear two-point boundary-value problems,* Nonlinear Oscillations, vol.4., No.1., (2001), 112-128.

[16] Rontó, M.: *On non-linear boundary-value problems containing parameters,* Archivum Mathematicum (Brno), Tomus 36 (2000), 585-593.

[17] Rontó, M.: *On the investigation of parametrized non-linear boundary-value problems,* Nonlinear Analysis, 47 (2001), 4409-4420.

[18] Samoilenko, A.M. and Le Lyuong Tai : *On a method of investigating boundary-value problems with non-linear boundary conditions,* Ukrain. Math. Zh.,42, No.7 (1990), 951-957.1031-1042.

[19] Krasnosel'skii, M.A., Vainikko, G.M.,Zabreiko, P.P., Rutitskii, Ya.B. and Stecenko, V.Ya.: *Approximate Solution of Operator Equations* [in Russian], Nauka, Moscow (1969). English translation :Noordhoff, Groningen (1972).

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