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Applications of stability criteria to time delay systems

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Abstract

Stability and stabilization of time delay systems (even of the linear ones) is again in the mainstream of the research. A most recent example is the stability analysis of feedback control loops containing a first order controlled object with pure delay and a standard PID controller, thus generating a system with a second degree quasi-polynomial as characteristic equation. Since the classical memoir of Čebotarev and Meiman (1949) up to the more recent monographs by Stepan (1989) and Górecki *et al* (1989) several approaches to this problem have been given, aiming to find the most complete Routh–Hurwitz type conditions for this case. In fact the main problem is here a missing case in the original memoir of Čebotarev and Meiman and its significance within the framework of the most recent analysis of Górecki *et al.* The present paper aims to a fairly complete analysis of the problem combined with some hints for the nonlinear case (Aizerman problem).

State feedback stabilization based on Artstein reduction of a system with input delay to a system without delay is also considered.

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1 State of the art

One of the most widely used linear model in process control is that described by the transfer function $H(s)\exp(-\tau s)$, where $H(s)$ is a strictly proper rational function. If a state representation of it is taken then we obtain the system

$$\begin{aligned} \dot{x} &= Ax + bu(t-\tau) \\ y &= c * x \end{aligned} \tag{1}$$

or, equivalently

$$\begin{aligned} \dot{x} &= Ax + bu(t) \\ y &= c * x(t-\tau) \end{aligned} \tag{2}$$

The control problems for such systems are obviously affected by the time delay. Along many decades of control applications several approaches have been proposed, analyzed, tested and implemented in industry. We may cite two types of approaches:

a) the use of classical standard PID controllers; such controllers were introduced for delayless processes and the main problem to be solved is how to cope with the de–stabilizing effects of the delay;

b) compensator design using “modern” (i.e. state space oriented) or “neo–classical” (i.e. frequency domain oriented) methods; we may cite here the design of the stabilizing state feedback (combined with a suitable state observer) or the H^∞ design.

The first approach is obviously connected to the problem of Routh and Hurwitz for quasi–polynomials. Starting from the classical results of Pontryagin (1942) and Čebotarev and Meiman (1949) many attempts have been performed to obtain simple conditions of stability, as closed as possible to the necessary and sufficient ones, allowing a quick parameter choice. The results on this direction are enclosed in such books as the ones of Górecki (1970), Stépân (1989), Górecki *et al* (1989). Worth mentioning that the problem is still actual: a group of most recent engineering papers (G. J. Silva *et al*, 2001,2002, 2003) offer an algorithmic approach to the parameter choice for the simplest, first order system (1) i.e with $H(s) = K(Ts+1)^{-1}$ combined in a “negative” feedback loop with a PID controller with the transfer function $H_c(s) = K_R(1 + 1/(T_i s) + T_d s)$. This structure leads to the following characteristic equation

$$KK_R(\frac{1}{T_i} + s + T_d s^2)e^{-\tau s} + s(1+Ts) = 0 \tag{3}$$

for which the stability conditions may be found in the books cited above; as mentioned, the papers of Silva *et al* offer a feasible approach based on the very first results due to Pontryagin (1942).

The second approach is only natural in the context of most recent development in mathematics of control. Within this approach we shall mention a single one which becomes more and more popular (see e.g. the papers of the CNRS–NSF Workshop "Advances in Time Delay Systems" held in Paris, January 2003). This approach is somehow inspired by the Smith predictor but its mathematical fundamentals are more recent. A rather simple transformation due to Artstein (1982) reduces many control problems for systems with input delays to the same problems for delayless systems. Stability may be ensured from the start by solving a finite dimensional assignment problem. The price to be paid is the infinite dimensional structure of the compensator which require an approximate implementation. As follows from several recent papers (see the recent Workshop mentioned above), standard implementations lead to NFDE (Neutral Functional Differential Equations) involving an essential spectrum belonging to the difference operator associated to the equation (Hale and Verduyn Lunel, 1993); since the design was not meant to have the essential spectrum in the left half plane, the implementation is often de-stabilizing.

The present paper deals with both these approaches. First the systems having (3) as characteristic equation are considered; the problem is to find stability domains allowing simple choice of the parameters K_R, T_i, T_d in both cases $T > 0$ and $T < 0$ (stable and unstable controlled process). We follow the line of [1] combined with the results of [5]. This kind of results gives a hint to the so–called Aizerman problem in the nonlinear case; in the case of systems with delay this problem is less studied (Răsvan, 2002)[11].

Second, stabilization using the transform due to Artstein is implemented using the technique of hybrid control (Halanay and Răsvan, 1977; Drăgan and Halanay, 1999) : by using piecewise constant control, a discrete finite dimensional system is associated; it is *this* system which is stabilized and destabilization by implementation is thus avoided. The resulting hybrid system is stable provided the discretization step is small enough.

2 Stability inequalities for PID controllers

Consider the characteristic equation (3) for which the Routh–Hurwitz problem (i.e. localization of its roots in the half plane $\Re(s) < 0$) is analyzed. The roots of (3) coincide with the roots of

$$\begin{aligned} KK_R(\frac{1}{T_i} + s + T_d s^2)\exp(-\tau s/2) + s(1+Ts)\exp(\tau s/2) = \\ \left[\frac{KK_R}{T_i} + (1+KK_R)s + (KK_RT_d + T)s^2\right]\cosh(\tau s/2) + \\ \left[-\frac{KK_R}{T_i} + (1-KK_R)s + (T - KK_RT_d)s^2\right]\sinh(\tau s/2) = 0 \end{aligned} \tag{4}$$

Introducing the new variable $z = \tau s/2$ the characteristic equation becomes

$$\left[\frac{KK_R}{T_i} + \frac{2}{\tau}(1 + KK_R)z + \frac{4}{\tau^2}(KK_RT_d + T)z^2\right]\cosh z + \\ \left[-\frac{KK_R}{T_i} + \frac{2}{\tau}(1 - KK_R)z + \frac{4}{\tau^2}(T - KK_RT_d)z^2\right]\sinh z = 0 \qquad (5)$$

With the notations

$$\gamma_p = KK_R\ ,\ \gamma_i = \frac{T}{T_i}\ ,\ \gamma_d = \frac{T_d}{T}\ ,\ \delta = \frac{2T}{\tau} \qquad (6)$$

the characteristic equation becomes

$$[\gamma_p\gamma_i + \delta(1 + \gamma_p)z + \delta^2(1 + \gamma_p\gamma_d)z^2]\cosh z + \\ [-\gamma_p\gamma_i + \delta(1 - \gamma_p)z + \delta^2(1 - \gamma_p\gamma_d)z^2]\sinh z = 0 \qquad (7)$$

with the left hand side belonging to the class of quasi-polynomials

$$p(z) = (a_2z^2 + a_1z + a_0)\cosh z + (b_2z^2 + b_1z + b_0)\sinh z \qquad (8)$$

which were considered in the memoir of Čebotarev and Meiman[1] from the point of view of the Routh–Hurwitz problem.

It is a well known fact that for polynomials the Routh–Hurwitz conditions are expressed through a finite set of inequalities and this was shown to be true for quasi-polynomials also, in the sense that the solution is obtained after a procedure with a finite number of steps. With respect to this we would like to mention that in the cited above papers of Silva *et al* [6, 7, 8] the number of the inequalities to be checked is infinite but it is claimed that one can reduce this number to a finite one and the procedure is somehow convergent. We shall not discuss the matter here but rather focuse on the quasi-polynomial (8). To fix the ideas let $a_0 > 0$. Then the following inequalities are necessary for localization of the roots of (8) in the half plane $\Re(s) < 0$:

$$a_1 + b_0 > 0\ ,\ a_2 + \frac{a_0}{2} + b_1 > 0\ ,\ a_2 > 0\ ,\ b_2 > 0 \qquad (9)$$

Further necessary and sufficient conditions are obtained for solving the Routh–Hurwitz problem. In [1] this is done using the Sturm approach. The analysis is much simplified using the following results

Proposition 1. *(Theorem 5 in [1]) If all the zeros of*

$$V(z) = a_2(\cos z)z^2 + b_1(\cos z)z - a_0\sin z$$

are real, then a_0 and a_2 have the same sign.

Proposition 2. *(Theorem 5a in [1]) If all the zeros of*

$$V_1(z) = -b_2(\sin z)z^2 + a_1(\cos z)z + b_0\sin z$$

are real, then b_0 and b_2 have the same sign.

These results are used to eliminate some sign combinations of the coefficients of (8); since we fixed $a_0 > 0$ we have $2^5 = 32$ sign combinations but after taking into account the two propositions above only 4 of them are left as able to give the required results. These are the so–called Cases I – IV of [1]

$$\begin{array}{lcl} I : b_0 > 0 \ ; \ a_1 > 0 \ , \ b_1 > 0 & ; & II : b_0 > 0 \ ; \ a_1 < 0 \ , \ b_1 < 0 \ ; \\ III : b_0 > 0 \ ; \ a_1 > 0 \ , \ b_1 < 0 & ; & IV : b_0 > 0 \ ; \ a_1 < 0 \ , \ b_1 > 0 \ . \end{array} \tag{10}$$

On the other hand the quasi-polynomial (7) does not fit these cases since a_0 and b_0 have opposite signs. The natural question would be : is always the feedback system composed of a first order "plant" with time delay and a PID controller unstable? The answer is negative since other methods of analyzing stability say so and we may send the reader to various references including the cited papers [6, 7, 8]. What then about the classical memoir [1] ? An answer will be given in the next section.

3 The forgotten cases

A. Let us remark that Proposition 1 does not give anything new but an already known necessary condition $a_2 > 0$ (since $a_0 > 0$ had been fixed from the beginning). *Proposition 2 gives more but is false.* We are going to prove this assertion by contradiction. Our main tool will be as in most studies on quasi-polynomials a result due to Pontryagin that we cite after Bellman and Cooke(1963)

Theorem 1. *Let $f(z, u, v)$ be a polynomial in z, u, v with real coefficients*

$$f(z, u, v) = \sum_{m=0}^{r} \sum_{n=0}^{s} z^m \varphi_m^{(n)}(u, v) \tag{11}$$

where $\varphi_m^{(n)}(u, v)$ are homogeneous polynomials of degree n with respect to u and v, with $z^r \varphi_r^{(s)}(u, v)$ the principal term and let

$$\Phi^{(s)}(z) = \sum_{n=0}^{s} \varphi_r^{(n)}(\cos z, \sin z) \tag{12}$$

If ε is such $\Phi^{(s)}(\varepsilon + \imath\omega) \neq 0$, $\forall \omega \in \mathbb{R}$ then $f(z, \cos z, \sin z)$ has only real roots iff for sufficiently large integers k it has exactly $4sk + r$ zeros within the band $-2k\pi + \varepsilon \leq \Re(z) \leq 2k\pi + \varepsilon$.

Proof. Consider now the polynomial in Proposition 2 namely

$$g(z, u, v) = -b_2 v z^2 + a_1 u z + b_0 v$$

hence $r = 2,\ s = 1$. Assume that b_0 and b_2 have opposite signs i.e. $b_0 < 0$ since we know from the necessary conditions that $b_2 > 0$. Were Proposition 2 true we should find at least one combination of the coefficients of $V_1(z)$ such that this quasi-polynomial had non-real roots. We write $V_1(z)=0$ as follows

$$(b_0-b_2z^2)\sin z+(a_1\cos z)z=0 \tag{13}$$

At its turn this equation may be written as

$$\tan z-\frac{a_1z}{|b_0|+b_2z^2}=0 \tag{14}$$

without loosing roots. Indeed we might have lost the imaginary roots $\pm\imath\sqrt{|b_0|/b_2}$ but these are not roots since $\cos(\pm\imath\sqrt{|b_0|/b_2})=-\cosh(\sqrt{|b_0|/b_2})\neq 0$. We might have lost also the real roots $\nu\pi+\pi/2$ of $\cos z=0$ but these also are not roots of (13) since $\sin(\nu\pi+\pi/2)=(-1)^\nu\neq 0$. It follows that the roots of (13) and (14) coincide.

Let $a_1>0$ to fix the ideas; the LHS (left hand side) of (14) is odd hence the analysis for $z>0$ is sufficient; if $a_1<0$ then we may consider the case $z<0$ and use the change of variable $a_1z=\zeta>0$. Denoting by $\psi(z)$ the rational function $a_1z/(|b_0|+b_2z^2)$ we find the properties

$$\psi(0)=0\ ;\ \lim_{z\to\infty}\psi(z)=0\ ;\ \psi'(z)=\frac{a_1(|b_0|-b_2z^2)}{(|b_0|+b_2z^2)^2}$$

hence $\psi(z)$ has a maximum corresponding to $z_M=\sqrt{|b_0|/b_2}$ namely

$$\psi(z_M)=\frac{a_1}{\sqrt{b_2|b_0|}+b_2}>0$$

Consider now some interval $(\nu\pi,(\nu+1)\pi)$, $\nu\geq 1$; within such an interval one may find a single root of (14) located between $\nu\pi$ and $\nu\pi+\pi/2$, the sub-interval where $\tan z>0$. Since $\tan z$ is monotonically increasing and $\psi(z)$ is monotonically decreasing for $z>z_M$, the root will be given by $\nu\pi+\delta_\nu$ where $\{\delta_\nu\}_\nu$ is a positive bounded sequence tending monotonically to 0. We deduce that there are always $2k-1$ roots of (14) within the interval $[\pi,2k\pi]$ hence there are other $2k-1$ ones within the symmetric interval $[-2k\pi,-\pi]$. Within the central interval $[-\pi,\pi]$ one may find the root $z=0$ and possibly 2 other ones, located between $(0,\pi/2]$ and $[-\pi/2,0)$ respectively. For the existence of these roots we need to show that $-\psi(z)>0$ in the neighborhood of 0, $z>0$; this will follow from $-\psi'(0)>0$; but $-\psi'(0)=a_1/|b_0|-1>0$ provided $a_1+b_0>0$; this last inequality has been assumed since it is a necessary condition for the location of the roots of (8) in $\mathbb{C}^-$ – see (9).

It follows that if the necessary conditions hold then there are exactly $4k+1$ roots within the interval $[-2k\pi,2k\pi]$ whatever $k>0$ would be. Let us consider now the shifted interval $[-2k\pi+\varepsilon,2k\pi+\varepsilon]$ with $\varepsilon>0$. Obviously if $\varepsilon>0$ is small enough all $4k+1$ roots still lie within this interval also. Let now $k>0$ be large enough, in order that $\delta_k<\varepsilon$; in this way the root of (14) from the interval $(2k\pi,(2k+1)\pi)$ will be "caught" within the shifted interval $[-2k\pi+\varepsilon,2k\pi+\varepsilon]$ for sufficiently large $k>0$. Applying the result of Pontryagin i.e. Theorem 1 we deduce that (14) hence (13) has only real roots in spite of our assumption that b_0 and b_2 have opposite signs. The assertion on *falsity* of Proposition 2 is proved. $\square$

We deduce now that the cases with $b_0 < 0$ cannot be eliminated from the stability analysis. If we take into account the sign combinations for a_1 and b_1 we obtain four additional cases. But the cases corresponding to $b_0 < 0$, $a_1 < 0$ have to be eliminated according to the necessary condition $a_1 + b_0 > 0$ which does not hold in these cases. We deduce that we have to consider additionally the following two cases

$$V : b_0 < 0 \ , \ a_1 > 0 \ ; \ b_1 > 0 \ ; \ VI : b_0 < 0 \ , \ a_1 > 0 \ ; \ b_1 < 0 \ . \tag{15}$$

which are exactly the cases mentioned in [5]. We shall analyze them separately.

B. Any analysis is based on counting the sign changes in the Sturm sequence whose number has to be according to the results of Čebotarev and Meiman, also $4k + 2$. First of all we count those sign changes that are independent of the analyzed case hence independent of the fact that now $b_0 < 0$. Substituting $z = \pm 2\nu\pi + \varepsilon$ and neglecting the higher order terms with respect to ε we obtain, as in the case of the cited memoir [1]

$$\begin{array}{lcl} V(\pm 2\nu\pi + \varepsilon) \approx a_2(2\nu\pi)^2 > 0 & ; & V_1(\pm 2\nu\pi + \varepsilon) \approx -b_2(2\nu\pi)^2\varepsilon > 0 \\ V_2(\pm 2\nu\pi + \varepsilon) \approx \mp a_1 a_2 b_2 (2\nu\pi)\varepsilon & ; & V_3(\pm 2\nu\pi + \varepsilon) \approx -b_2 a_1^2 a_2 a_0 \varepsilon^3 < 0 \end{array}$$

We deduce that the number of the sign losses on $[-2k\pi + \varepsilon, 2k\pi + \varepsilon]$ where $k > 0$ is large and $k\varepsilon > 0$ also large will be

$$P(-2k\pi + \varepsilon) - P(2k\pi + \varepsilon) = 2sgn\ a_1 \tag{16}$$

We compute now the sign losses $\ell_{i\nu}$ when crossing the zeros $\nu\pi$ of $\sin z$, the multiplier of $V_3(z)$ where $i = 1, 2$ according to the type of the root: $i = 1$ when the root is of the first type and introduces a sign gain ($\ell_{i\nu} = 1$) and $i = 2$ when the root is of the second type and introduces a sign loss ($\ell_{i\nu} = -1$). This analysis is also independent of b_0 hence we keep the result of [1]

$$P(-2k\pi + \varepsilon) - P(2k\pi + \varepsilon) - \sum_i \sum_\nu \ell_{i\nu} = 4k + 2sgn\ a_1 \tag{17}$$

C. The next multiplier in the Sturm sequence is given by

$$\Omega(z) = A\cos^4 z + B\sin^2 z\cos^2 z + C\sin^4 z \tag{18}$$

and its zeros count in the sign losses provided they are real. Here

$$A = a_0 a_1^2 a_2 > 0 \ , \ B = a_1 b_1 (a_0 b_2 + a_2 b_0) - (a_0 b_2 - a_2 b_0)^2 \ , \ C = b_0 b_1^2 b_2 < 0$$

The zeros of the multiplier are real provided the zeros of $A\lambda^2 + B\lambda + C$ are real. Since we discuss the case $b_0 < 0$ and $C < 0$ this polynomial has always two real roots of opposite sign. Since only the positive root counts we deduce that (18) has two real roots ($mod\ \pi$). Following [1] we use instead (18) the equation

$$C\tan^4 z + B\tan^2 z + A = 0 \tag{19}$$

with $C < 0$, $A > 0$. The biquadratic equation

$$C\lambda^4 + B\lambda^2 + A = 0$$

has two real roots corresponding to the positive real root of the associated second degree equation

$$\lambda_{1,2} = \pm\sqrt{\frac{B}{2|C|} + \sqrt{\left(\frac{B}{2|C|}\right)^2 + \frac{A}{|C|}}}$$

to which there correspond the roots of (19) namely

$$z_{1,\nu} = \nu\pi + \tau_1\ ,\ z_{2,\nu} = \nu\pi - \tau_1\ ,\ \nu = 0, \pm 1,\ \pm 2,\ \ldots \tag{20}$$

and

$$\tau_1 = \arctan\sqrt{\frac{B}{2|C|} + \sqrt{\left(\frac{B}{2|C|}\right)^2 + \frac{A}{|C|}}}\ ,\ 0 < \tau_1 < \pi/2 \tag{21}$$

Denoting $\tau_2 = \pi - \tau_1$ it follows that in each interval $(\nu\pi, (\nu+1)\pi)$ we find 2 roots of (19) – or (18) – namely $z_{1,\nu} = \nu\pi + \tau_1$ such that $\nu\pi < z_{1,\nu} < \nu\pi + \pi/2$ and $z_{2,\nu} = (\nu+1)\pi - \tau_1 = \nu\pi + \tau_2$ such that $\nu\pi + \pi/2 < z_{2,\nu} < (\nu+1)\pi$.

Generally speaking these values are not zeros of the quasi-polynomials $V,\ V_1, V_2$ of the Sturm sequence constructed according to [1] ; this happens only if the coefficients of (8) are subject to some very special equalities – which clearly are "non-robust" and called "limit cases".

D. In the general cases the sign losses $\ell_{i\nu}$ are determined by the behavior of the ratio V_2/V_3 given by

$$\frac{V_2(z)}{V_3(z)} = \frac{(a_1a_2 + b_1b_2\tan^2 z)z - (a_0b_2 - a_2b_0)\tan z}{\cos^2 z\sin^2 z(C\tan^4 z + B\tan^2 z + A)} \tag{22}$$

when $z_{i\nu} = \nu\pi - \tau_i,\ i = 1, 2\ ,\ \nu = 0, \pm 1,\ \ldots$ If this ratio changes from $-$ to $+$ then $\ell_{i\nu} = +1$ and $z_{i\nu}$ is called a root of V_3 of 1st type; if the ratio changes from $+$ to $-$ then $\ell_{i\nu} = -1$ and $z_{i\nu}$ is called a root of 2nd type of V_3 .

Consider first the sign changes of the ratio's denominator. Usual continuity arguments show that when crossing $z_{1\nu}$ the sign changes from $+$ to $-$ and when crossing $z_{2\nu}$ the change is from $-$ to $+$.

As known from [1], the behavior of the numerator $V_2(z)$ depends on each analyzed case.

Case V $(b_0 < 0\ ,\ a_1 > 0\ ;\ b_1 > 0)$. This case is somehow alike *Case I* already analyzed in [1] : the coefficient of z in the numerator is positive for all z and the free term of the numerator namely $-(a_0b_2 - a_2b_0)\tan z = -(a_0b_2 + a_2|b_0|)\tan z$

is negative for $z_{1\nu} = \nu\pi + \tau_1$ and positive for $z_{2\nu} = \nu\pi + \tau_2$. We deduce the following

a) for the roots $z_{2\nu} = \nu\pi + \tau_2$ the numerator is positive for all $\nu \geq 0$ and $\nu < 0$ of modulus sufficiently small; if $|\nu|$ for $\nu < 0$ increases, the term in z decreases and the numerator becomes negative; a $k_2 < 0$ may be defined from the change of the sign as satisfying the inequalities

$$\begin{aligned} (a_1a_2 + b_1b_2\tan^2(k_2\pi + \tau_2))(k_2\pi + \tau_2) & & \\ -(a_0b_2 - a_2b_0)\tan(k_2\pi + \tau_2) & > & 0, \\ (a_1a_2 + b_1b_2\tan^2((k_2-1)\pi + \tau_2))((k_2-1)\pi + \tau_2) & & \\ -(a_0b_2 - a_2b_0)\tan((k_2-1)\pi + \tau_2) & < & 0 \end{aligned}$$

which lead after some simple manipulation to

$$\begin{aligned} k_2 - \frac{\tau_1}{\pi} + \frac{1}{\pi}\cdot\frac{(a_0b_2 - a_2b_0)\tan\tau_1}{a_1a_2 + b_1b_2\tan^2\tau_1} < 0 < \\ k_2 + 1 - \frac{\tau_1}{\pi} + \frac{1}{\pi}\cdot\frac{(a_0b_2 - a_2b_0)\tan\tau_1}{a_1a_2 + b_1b_2\tan^2\tau_1} \end{aligned}$$

hence

$$k_2 = \left[\frac{\tau_1}{\pi} - \frac{1}{\pi}\cdot\frac{(a_0b_2 - a_2b_0)\tan\tau_1}{a_1a_2 + b_1b_2\tan^2\tau_1}\right]_e \tag{23}$$

b) for the roots $z_{1\nu} = \nu\pi + \tau_1$ the numerator is positive for $\nu > 0$ sufficiently large and negative for $\nu < 0$ and $\nu \geq 0$ sufficiently small; a $k_1 > 0$ may be defined from the change of the sign, finally given by

$$k_1 = \left[-\frac{\tau_1}{\pi} + \frac{1}{\pi}\cdot\frac{(a_0b_2 - a_2b_0)\tan\tau_1}{a_1a_2 + b_1b_2\tan^2\tau_1}\right]_e + 1 \tag{24}$$

In the following we shall count $\sum_i\sum_\nu \ell_{i\nu}$ as follows. We consider an interval $[-2k\pi + \varepsilon, 2k\pi + \varepsilon]$ with $k > 0$ sufficiently large i.e. larger than $\max\{k_1, -k_2\}$ and also than that k for which we showed that Proposition 2 was false; $\varepsilon > 0$ is such that $k\varepsilon$ is still very large e.g. $\varepsilon = k^{-1/7}$.

Now for the intervals $(\nu\pi, (\nu+1)\pi)$ with $\nu \leq k_2 - 1$ we find easily that $\ell_{1\nu} = +1$, $\ell_{2\nu} = -1$ hence the sum is 0. For the intervals with $k_2 \leq \nu \leq k_1 - 1$ we deduce $\ell_{i\nu} = 1$, $\nu = 1, 2$ hence $\sum_{k_2}^{k_1-1}\sum_i \ell_{i\nu} = 2(k_1 - k_2)$. For $\nu \geq k_1$ we deduce again that the sum is zero. Therefore the real roots of (18) introduce now $2(k_1 - k_2)$ sign changes and since $a_1 > 0$ the overall number of the sign changes will be

$$N_1 - N_2 = 4k + 2 - 2(k_1 - k_2)$$

while the Pontryagin type result requires $N_1 - N_2 \geq 4k + 2$. Therefore the necessary and sufficient condition will be $k_1 - k_2 = 0$ i.e. $k_1 = k_2$. Using (23) and (24) we deduce the necessary and sufficient condition

$$\left[-\frac{\tau_1}{\pi}+\frac{1}{\pi}\cdot\frac{(a_0b_2-a_2b_0)\tan\tau_1}{a_1a_2+b_1b_2\tan^2\tau_1}\right]_e+1=\left[\frac{\tau_1}{\pi}-\frac{1}{\pi}\cdot\frac{(a_0b_2-a_2b_0)\tan\tau_1}{a_1a_2+b_1b_2\tan^2\tau_1}\right]_e \tag{25}$$

or

$$\left[-\frac{\tau_1}{\pi}+\frac{1}{\pi}\cdot\frac{(a_0b_2-a_2b_0)\tan\tau_1}{a_1a_2+b_1b_2\tan^2\tau_1}\right]_e=\left[-\frac{\tau_2}{\pi}+\frac{1}{\pi}\cdot\frac{(a_0b_2-a_2b_0)\tan\tau_2}{a_1a_2+b_1b_2\tan^2\tau_2}\right]_e \tag{26}$$

Equality (26) is exactly the one given without proof by Górecki [5].

Case VI ($b_0<0$, $a_1>0$; $b_1<0$) In this case the coefficient of z in the numerator is positive for small $z>0$ and decreases on $(0,\pi/2)$ since $b_1<0$. To see the sign for $z=\tau_1$ we compare $\tan^2\tau_1$ which corresponds to the positive root of the second degree equation associated to (19) and $-(a_1a_2)/(b_1b_2)$ which makes the coefficient 0. We deduce easily that

$$|C|\left(-\frac{a_1a_2}{b_1b_2}\right)^2+B\frac{a_1a_2}{b_1b_2}-A=-\frac{a_1a_2}{b_1b_2}(a_0b_2-a_2b_0)^2>0$$

hence $-(a_1a_2)/(b_1b_2)>\tan^2\tau_1$. The coefficient of z in the numerator is thus positive in some neighborhood of the root $z_{i\nu}$ where the sign change is counted. The free term of the numerator is as previously. We deduce that the analysis coincides with the previous one, k_1 and k_2 are determined as previously and the stability conditions are as previously (25) and (26).

It follows that in this case the formulae of Górecki [5], given without proof, *are not correct*; one may suppose that they have been obtained from a supposed analogy of Case VI and Case III .

To end this section we shall consider (25) in some detail. From the well known equality

$$[x]_e+[-x]_e=-1$$

valid for non-entire x we obtain for (25)

$$\left[\frac{\tau_1}{\pi}-\frac{1}{\pi}\cdot\frac{(a_0b_2-a_2b_0)\tan\tau_1}{a_1a_2+b_1b_2\tan^2\tau_1}\right]_e=0$$

hence

$$-1<-\frac{\tau_1}{\pi}+\frac{1}{\pi}\cdot\frac{(a_0b_2-a_2b_0)\tan\tau_1}{a_1a_2+b_1b_2\tan^2\tau_1}<0$$

or

$$-\pi+\tau_1<\frac{(a_0b_2-a_2b_0)\tan\tau_1}{a_1a_2+b_1b_2\tan^2\tau_1}<\tau_1$$

But we already showed that the denominator of the ratio above is positive, as well as the numerator, while $-\pi+\tau_1<0$ since $0<\tau_1<\pi/2$. It follows that (25) is fulfilled provided

$$\frac{(a_0b_2-a_2b_0)\tan\tau_1}{a_1a_2+b_1b_2\tan^2\tau_1}<\tau_1 \tag{27}$$

4 Application to the case of PID controllers and the first order object with time lag

As already mentioned, the problem of the stability inequalities in this application, while classical, is again under research[6, 7, 8]. A countable set of inequalities is obtained, possibly converging to some finite domain in the parameter space; the theoretical basis of this approach is given by the original paper of Pontryagin[4] combined with some interesting remarks on decoupling the controller parameters within the set of inequalities.

Our approach will be the application of the inequalities of Čebotarev and Meiman[1] from the complete set of cases, as done by Górecki[5]; additionally we shall consider the case of the unstable object, as Silva *et al* [6, 7, 8] did.

A. Assume first that the simplest case of the controlled object – the stable case, when the following conditions are true : $K > 0$, $\tau > 0$, $T > 0$ – holds. If we turn to the notations of (6) and (7) we deduce $\delta > 0$, $\gamma_i > 0$, $\gamma_d > 0$. By forcing $\gamma_p \gamma_i$ in (7) we obtain the characteristic equation

$$\left[1 + \frac{\delta}{\gamma_i}\left(1 + \frac{1}{\gamma_p}\right) z + \frac{\delta^2}{\gamma_i}(1/\gamma_p + \gamma_d) z^2\right] \cosh z + \left[-1 + \frac{\delta}{\gamma_i}\left(1 - \frac{1}{\gamma_p}\right) z + \frac{\delta^2}{\gamma_i}(1/\gamma_p - \gamma_d) z^2\right] \sinh z = 0 \tag{28}$$

Obviously $a_0 = 1 > 0$ but $b_0 = -1 < 0$ hence the only cases to be applied are V and VI, corresponding to $b_0 < 0$ and just analyzed above. We start by writing down the necessary conditions as given by (9)

$$\begin{array}{lcl} \dfrac{\delta}{\gamma_i}(1 + \dfrac{1}{\gamma_p}) - 1 > 0 & ; & \dfrac{\delta^2}{\gamma_i} + \dfrac{1}{2} + \dfrac{\delta}{\gamma_i}(1 - \dfrac{1}{\gamma_p}) > 0 \ ; \\ \dfrac{\delta^2}{\gamma_i}(1/\gamma_p + \gamma_d) > 0 & ; & \dfrac{\delta^2}{\gamma_i}(1/\gamma_p - \gamma_d) > 0 \ . \end{array} \tag{29}$$

To these conditions we have to add other necessary conditions that are more engineering–like : stability for the delay free system which ensures stability for small delays. We deduce from (7) or (28)

$$\frac{\delta}{\gamma_i}(1 + \frac{1}{\gamma_p}) > 0 \ ; \ \frac{\delta^2}{\gamma_i}(1/\gamma_p + \gamma_d) > 0 \tag{30}$$

But we have already mentioned that $\delta > 0$, $\gamma_i > 0$, $\gamma_d > 0$. These conditions combined with (30) will give

$$1 + 1/\gamma_p > 0 \ , \ 1/\gamma_p + \gamma_d > 0 \ ,$$

but the fourth inequality in (29) implies them both i.e.

$$1/\gamma_p > \gamma_d > 0 > -\min\{\gamma_d, 1\} \tag{31}$$

Remark that (31) imply fulfilment of the third inequality of (29) which is nothing more but second inequality of (30). We have to discuss the first two inequalities of (29).

The first one will give $\gamma_i < \delta(1 + 1/\gamma_p)$ while the second one will require to consider two cases : $1/\gamma_p \leq 1+\delta$ when it is automatically fulfilled and $1/\gamma_p 1+\delta$.

Summarizing we obtain under the assumption of positive parameters for the controller that the following conditions are necessary for the location of the roots of the characteristic equation in the LHS of $\mathbb{C}^-$

$$\gamma_d < 1/\gamma_p \ ; \ \max\{0, 2\delta(1/\gamma_p - 1 - \delta) < \gamma_i < \delta(1 + 1/\gamma_p) \tag{32}$$

The expressions connected with (27) could be rather complicated from analytical point of view. We give a single example. If $\gamma_d = 0$ is assumed – i.e. a PI controller having only two parameters to be chosen – then (27) reads as

$$\frac{\delta}{\gamma_i} \cdot \frac{2\tan\tau_1}{1 + 1/\gamma_p + (1 - 1/\gamma_p)\tan^2\tau_1} < \tau_1 \tag{33}$$

where

$$\tan\tau_1 = \frac{1}{|1 - 1/\gamma_p|}\sqrt{-\frac{2\gamma_i}{\gamma_p} + \sqrt{\left(\frac{2\gamma_i}{\gamma_p}\right)^2 + (1 - 1/\gamma_p^2)}} \tag{34}$$

It is interesting to remark that τ_1 is not dependent of the delay; in fact the only parameter to incorporate the delay is just δ. Therefore an estimate of the delay for stability purposes is easy to perform; moreover an optimization could be tried i.e. to find such a choice of γ_p and γ_i in order to maximize the upper bound for δ.

B. We shall consider now the unstable controlled object i.e. $K > 0$, $\tau > 0$, $T < 0$; we deduce $\delta < 0$, $\gamma_i < 0$, $\gamma_d < 0$.

For the necessary conditions we refer again to (28), where a_0 and b_0 remain the same. Since $\gamma_i < 0$ we need $1/\gamma_p + \gamma_d < 0$, $1/\gamma_p - \gamma_d < 0$ which are contradictory unless $\gamma_p < 0$; this last condition would require introducing "positive reaction" in the system; this will make the structure non-robust i.e. sensible to parameter perturbations. In fact this is one of the drawbacks of the classical PID structure; in the papers of Silva *et al* [6, 7, 8] these aspects seem to be neglected. The discussion being of pure engineering (technological) interest, we do not insist on this subject any longer. Worth mentioning nevertheless that exactly such drawbacks lead the researchers to the advanced techniques that we summarized next.

5 A nonlinear intermezzo: the Aizerman problem

This problem has a more than 60 years long history and it would be senseless to present it here. Following Răsvan(2002) [11] we shall state it for systems with delay as follows, starting from the simplest case. Given the time delay equation

$$\dot{x} + a_0x(t) + a_1x(t-\tau) = 0, \tau > 0 \tag{35}$$

the exponential stability is ensured provided the following inequalities hold:

$$1 + a_0\tau > 0, \qquad -a_0\tau < a_1\tau < \psi(a_0\tau) \tag{36}$$

where $\psi(\xi)$ is obtained by eliminating the parameter λ between the two equalities below

$$\xi = -\frac{\lambda}{\tan\lambda}, \qquad \psi = \frac{\lambda}{\sin\lambda} \tag{37}$$

Since these conditions contain the time delay τ such property is called *delay-dependent stability.* If one is interested in exponential stability conditions that hold for any delay $\tau > 0$, this property, called *delay-independent stability* is ensured provided the simple inequalities

$$a_0 > 0, \qquad |a_1| < a_0 \tag{38}$$

are fulfilled. It can be shown [15] that $\psi(\xi) > \xi$ for $\xi > 0$ hence the fulfilment of (38) implies the fulfilment of (36).

As already mentioned previously Čebotarev and Meiman pointed out that, according to Sturm theory, the Routh-Hurwitz conditions for quasi-polynomials have to be expressed as a finite number of inequalities that might be transcendental. The detailed analysis performed in their memoir for the 1st and 2nd degree quasi-polynomials showed two types of inequalities: one of them contained only algebraic inequalities while the other contained also transcendental inequalities; the first ones correspond to stability for arbitrary values of the delay τ while the second ones put some limitations on the values of $\tau > 0$ for which exponential stability of the linear system e.g. (35) holds. This system and conditions (36), (37) and (38) are good illustrations of this. The aspect is quite transparent in the examples analysis performed throughout author's book [16] as well as throughout the book of Stepan [2]. We may see here the difference operated between what will be called later *delay-independent* and *delay-dependent stability.*

Let us follow the way of Barbashin [17] to introduce a *stability problem in the nonlinear case: given system (35) for* $a_0 > 0$*, if we replace* a_0x *by* $\varphi(x)$ *where* $\varphi(x)x > 0$*, the equilibrium at the origin of the nonlinear time delay system should be globally asymptotically stable provided*

$$\frac{\varphi(\sigma)}{\sigma} > |a_1| \tag{39}$$

for the delay-independent stability, or provided

$$\frac{\varphi(\sigma)}{\sigma} > max\left\{-a_1, \frac{1}{\tau}\psi^{-1}(a_1\tau)\right\} \tag{40}$$

in the delay-dependent case.

We may view the above problem in a more general setting and state it as follows

Problem *Given the delay-(in)dependent exponential stability conditions for some time delay linearized system, are they valid in the case when the nonlinear system with a sector restricted nonlinearity i. e. satisfying*

$$\underline{\varphi}\sigma^2 < \varphi(\sigma)\sigma < \overline{\varphi}\sigma^2 \tag{41}$$

is considered instead of the linear one, or have they to be strengthened?

It is clear that we have gathered here both the delay-independent and delay-dependent cases, thus defining a stability problem in two different cases. This problem is called *Aizerman problem*, stated here as *delay dependent* (Aizerman problem) and *delay independent* (Aizerman problem).

Consider, for instance, the delay independent Aizerman problem defined above, for system (35) replaced by

$$\dot{x} + a_1 x(t-\tau) + \varphi\left(x(t)\right) = 0 \tag{42}$$

where $\varphi(\sigma)\sigma > 0$. Taking into account that (38) suggests $\varphi(\sigma) > |a_1|\,\sigma$ we introduce a new nonlinear function

$$f(\sigma) = \varphi(\sigma) - |a_1|\,\sigma$$

and obtain the transformed system (*via* a sector rotation):

$$\dot{x} + |a_1|\,x(t) + a_1 x(t-\tau) + f\left(x(t)\right) = 0 \tag{43}$$

For this system we apply the frequency domain inequality of Popov for $\overline{\varphi} = +\infty$ i.e. the inequality

$$Re(1 + \imath\omega\beta)H(\imath\omega) > 0, \quad \forall\omega \geq 0 \tag{44}$$

Here

$$H(s) = \frac{1}{s + |a_1| + a_1 e^{-s\tau}} \tag{45}$$

and the frequency domain inequality reduces to

$$\beta\omega^2 - (\beta a_1 \sin\omega\tau)\omega + |a_1| + a_1\cos\omega\tau \geq 0 \tag{46}$$

which is fulfilled provided the free Popov parameter β is chosen from

$$0 < \beta\,|a_1| < 2 \tag{47}$$

(more details concerning manipulation of the frequency domain inequality for time delay systems may be found in author's book [16]).

It follows that (43) is absolutely stable for the nonlinearities satisfying $f(\sigma)\sigma > 0$ i.e. $\varphi(\sigma)\sigma > |a_1|\sigma^2$: the just stated delay-independent Aizerman problem for (35) and (42) has been answered positively.

We have introduced this section in order to suggest how the rather sophisticated stability conditions from the linear cases with delay could be used in interaction with Popov frequency domain inequality aiming to obtain results concerning the problems of Aizerman for systems with delay - an interesting and useful sharpness measure for the sufficient stability conditions in the nonlinear case.

6 Stabilization by state feedback

In this section we turn back to system (1) which we want to stabilize by linear state feedback. Our main tool will be a state–control transform due to Artstein (1982)[9] that reduces 1 to a finite dimensional one. In this case the transform is

$$z(t) = x(t) + \int_{-\tau}^{0} e^{-A(\theta+\tau)} bu(t+\theta)d\theta \tag{48}$$

and leads to the system

$$\dot{z} = Az + e^{-A\tau} bu(t) \tag{49}$$

The following equivalence is valid

Proposition 3. *Let $(x(t), u(t); t > 0)$ be a solution (admissible pair) for (1), defined by some initial condition $(x_0, u_0(\cdot))$. Then $(z(t), u(t); t > 0)$ with $z(t)$ defined by (48) is a solution (admissible pair) for the system (49) with the initial condition $z_0 = z(0)$. Conversely, let $(z(t), u(t); t > 0)$ be a solution of (49) defined by some initial condition z_0. Then, given some $u_0(\cdot)$ defined on $(-\tau, 0)$ and taking*

$$x_0 = z_0 - \int_{-\tau}^{0} e^{-A(\theta+\tau)} bu_0(\theta)d\theta \tag{50}$$

the solution of (1) defined by these initial conditions and by $u(t), t > 0$ is given by

$$x(t) = z(t) - \int_{-\tau}^{0} e^{-A(\theta+\tau)} bu(t+\theta)d\theta \tag{51}$$

The proof of this result is straightforward. Further we may apply various control techniques to (49) and see their correspondent when the inverse transform (51) is applied[18, 19, 20, 21, 22]. We shall give below some of these results. Let f be a feedback vector such that the control function $u = f^*z$ is stabilizing for (49) i.e. the matrix $A+e^{-A\tau}bf^*$ has its eigenvalues with negative real parts. Since

$$\dot{z} = (A + e^{-A\tau}bf^*)z \tag{52}$$

is exponentially stable, the closed loop feedback system

$$\begin{aligned} &\dot{x}(t) = Ax(t) + bu(t-\tau) \\ &u(t) - \int_{-\tau}^{0} f^* e^{-A(\theta+\tau)} B_1 u(t+\theta)d\theta = f^*x(t) \end{aligned} \tag{53}$$

is also exponentially stable *via* the properties of (48). Remark nevertheless that the compensator described by the second equation of (53) contains an integral that has to be realized either as a device or as a programme. The standard technique used in many papers (e.g. the CNRS–NSF Workshop "Advances in Time Delay Systems" held in Paris, January 2003, mentioned previously) was to approximate the integral. The result was a compensator described by a difference equation with a finite number of lumped time delays what gives to the resulting feedback model a neutral character (i.e. the equations have the properties of the Neutral Functional Differential Equations NFDE). Their main feature is the essential spectrum – the spectrum of the difference operator. If this spectrum is not inside the unit disk of the complex plane, the system could be destabilized if perturbations of the delay are allowed – the system is non-robust and fragile. Note that the discretization is not connected to the essential spectrum and de-stabilization may be very possible. The role of the essential spectrum has been pointed out by J. K. Hale at the above mentioned Workshop and several solutions have been proposed. our approach is somehow different and will be presented in brief.

The implementation of the designed compensator requires memorizing of a trajectory segment i.e. a set of data that has infinite size. The practical implementation is finite and based on a suitable discretization. Following the line of the paper of Halanay and Răsvan (1977)[12] and of the book of Drăgan and Halanay (1999)[13] we shall use *piecewise constant control signals*, defined as follows

$$u(t) = u_k, \quad k\delta \le t < (k+1)\delta, \quad k = 0, 1, 2, \cdots \tag{54}$$

where $\delta = \tau/N$. For the system (1) we associate the discrete time system

$$x_{k+1} = \mathbf{A}(\delta)x_k + \mathbf{b}(\delta)u_{k-N} \tag{55}$$

where

$$\mathbf{A}(\delta) = e^{A\delta}, \quad \mathbf{b}(\delta) = \left(\int_0^\delta e^{A\theta}d\theta\right) b, \quad i = 0, 1 \tag{56}$$

Let $(x_0, u_0(\cdot))$ be the initial condition associated with (1). Since the discretized system is satisfied by $x_k = x(k\delta)$, $x(\cdot)$ being the solution of (1) with piecewise constant control, it is only natural to choose the discretized initial condition $(x_0; u^0_{-i} = u_0(-i\delta), i = \overline{0, N})$. We may define

$$z_k = x_k + \sum_{-N}^{-1} \mathbf{A}(\delta)^{-(N+j+1)}\mathbf{b}(\delta)u_{k+j} \tag{57}$$

which is the discrete analogue of Artstein transform and find the associate system

$$z_{k+1} = \mathbf{A}(\delta)z_k + \mathbf{A}(\delta)^{-N}\mathbf{b}(\delta)u_k \tag{58}$$

It is worth mentioning that (57) might be obtained by writing (48) at $t = k\delta$ and computing the integral for piecewise constant control signals.

Let $\mathbf{f}$ be a stabilizing feedback for (58), i.e. is such that $\mathbf{A}(\delta)+\mathbf{A}(\delta)^{-N}\mathbf{b}(\delta)\mathbf{f}^*$ has its eigenvalues inside the unit disk. We deduce that the compensator

$$u_k = \mathbf{f}^* x_k + \sum_{-N}^{-1} \mathbf{f}^*\mathbf{A}(\delta)^{-(N+j+1)}\mathbf{b}(\delta)u_{k+j} \tag{59}$$

is stabilizing for (58). On the other hand, if we consider the closed loop system

$$\begin{aligned} x_{k+1} &= \mathbf{A}(\delta)x_k + \mathbf{b}(\delta)u_{k-N} \\ u_k &= \mathbf{f}^* x_k + \sum_{-N}^{-1} f^*\mathbf{A}(\delta)^{-(N+j+1)}\mathbf{b}(\delta)u_{k+j} \end{aligned} \tag{60}$$

one may see that this is a feedback system with an augmented dynamics:

$$\begin{aligned} x_{k+1} &= \mathbf{A}(\delta)x_k + \mathbf{b}(\delta)v_k \\ v_{k+1} &= w^1_k \\ &\cdots \\ w^{N-1}_{k+1} &= u_k \\ u_k &= \mathbf{f}^*[x_k + \mathbf{A}(\delta)^{-1}\mathbf{b}(\delta)v_k + \cdots \\ &\quad +\mathbf{A}(\delta)^{-(N-1)}\mathbf{b}(\delta)w^{N-2}_k + \mathbf{A}(\delta)^{-N}\mathbf{b}(\delta)w^{N-1}_k] \end{aligned} \tag{61}$$

Since $w^{N-1}_k = u_{k-1}$ the corresponding initial condition is $w^{N-1}_0 = u_{-1} = u_0(-\delta)$; further, $w^{N-2}_0 = u_{-2} = u_0(-2\delta)$, $\cdots$, $w^1_0 = u_0(-(N-1)\delta)$, $v_0 = u_0(-N\delta)$. Obviously (61) is exponentially stable. This follows from the fact that $u = \mathbf{f}^* z$ is exponentially stabilizing system (58) and making use of (57). The result may be obtained also spectrally, as in [18].

The specific issue of the approach lies exactly in the choice of f as a stabilizing feedback for the discrete-time system; it is as the basic system is discretized, transformed *via* the discrete analogue of the Artstein transform and stabilized; the stabilization is performed over the discrete time system and, according to [12, 13], the property holds for the hybrid system composed of the continuous time controlled system and the discrete compensator that generates piecewise constant control signals using discrete-time state measurements (samples), provided the sampling step is small enough. Obviously the size of the sampling step is still object of theoretical estimates and simulation experiments.

7 Concluding remarks

We would like to point out a single but most important feature of our approach, feature that was confirmed also by simulation (nevertheless the proofs are rigorous and on a sound basis – see again [12, 13]). Most implementation approaches are based on the discretization of the integral what leads to continuous time compensators described by difference equations hence to systems of neutral type with an essential spectrum. Stability of such systems require this spectrum to be inside the unit disk which *is not automatically ensured even by a refined(with the step small enough) discretization*; consequently such systems often de-stabilize being either non-robust or fragile. The introduction of a Low Pass Filter changes the system into one of delayed type and may re-stabilize, the price paid being another dimension augmentation.

The method of this paper makes a difference in the sense that a specific control is used – the piecewise constant control. in this way a discrete-time system is associated and it is *this* system that is stabilized; its augmented dynamics replaces the discretized integral term. Under these circumstances the closed loop system (which is hybrid since it contains a continuous-time controlled plant and a sampled data compensator) is always stable provided the sampling step δ is small enough [12, 13]. The small sampling step is helpful in stabilization from another point of view also [12]: let $\mathbf{f}(\delta)$ be the stabilizing feedback for the discretized system. Using the asymptotic expansions [12] it is easily found that

$$\mathbf{f}(\delta) = f + f_1\delta + o(\delta)$$

where f is a stabilizing feedback for the continuous time system; one may use for implementation with piecewise constant control the gain f instead of $\mathbf{f}(\delta)$ and the stability is preserved provided δ is small enough.

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