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Exponential Estimates of Solutions of Difference Equations with Continuous Time

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Dedicated to Professor László Hatvani on his 60th birthday

Abstract

In this paper [1] we study the scalar difference equation with continuous time of the form

$$x(t) = a(t)x(t-1) + b(t)x(p(t)),$$

where $a, b : [t_0, \infty) \to \mathbf{R}$ are given real functions for $t_0 > 0$ and $p : [t_0, \infty) \to \mathbf{R}$ is a given function such that $p(t) \le t$, $\lim_{t\to\infty} p(t) = \infty$. Using the method of characteristic equation we obtain an exponential estimate of solutions of this equation which can be applied to the difference equations with constant delay and to the case $t - p_2 \le p(t) \le t - p_1$ for real numbers $1 < p_1 \le p_2$.

We generalize the main result to the equation with several delays of the form

$$x(t) = a(t)x(t-1) + \sum_{i=1}^{m} b_i(t)x(p_i(t)),$$

where $a, b_i : [t_0, \infty) \to \mathbf{R}$ are given functions for $i = 1, 2, ..., m$, and $p_i : [t_0, \infty) \to \mathbf{R}$ are given such that $p_i(t) \le t$, $\lim_{t\to\infty} p_i(t) = \infty$ for $i = 1, 2, ..., m$. We apply the obtained result to particular cases such as $p_i(t) = t - p_i$ for $i = 1, 2, ..., m$, where $1 \le p_1 < p_2 < ... < p_m$ are real numbers.

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1. Introduction

The characteristic equation is a useful tool in the qualitative analysis of the theory of differential and difference equations. Knowing the solutions or the behavior of

[1] This paper is in final form and no version of it will be submitted for publication elsewhere.

solutions of the characteristic equations we can obtain some properties of solutions of the considered differential or difference equations. This method is usually applied in the investigations of the theory of oscillation, asymptotic behavior, stability, etc. Using the solutions of the characteristic equation we can, from a new point of view, describe the asymptotic behavior of solutions of the difference equation. We apply the obtained results for some particular cases.

Assume that $t_0 > 0$ and $a, b : [t_0, \infty) \to \mathbf{R}$ are given real functions. Let $p : [t_0, \infty) \to \mathbf{R}$ be given function such that, for every $T > t_0$ there exists a $\delta > 0$ such that $p(t) \le t - \delta$ for every $t \in [t_0, T]$, and $\lim_{t\to\infty} p(t) = \infty$. Now, we investigate the scalar difference equation with continuous arguments

$$x(t) = a(t)x(t-1) + b(t)x(p(t)), \tag{1}$$

where $x(t) \in \mathbf{R}$.

Let $\mathbf{N}$ be the set of nonnegative integers, $\mathbf{R}$ the set of real numbers and $\mathbf{R}_+ = (0, \infty)$.

For given $m \in \mathbf{N}$, $t \in \mathbf{R}_+$ and a function $f : \mathbf{R} \to \mathbf{R}$ we use the standard notation

$$\prod_{\ell=t}^{t-1} f(\ell) = 1, \qquad \prod_{\ell=t-m}^{t} f(\ell) = f(t-m)f(t-m+1)...f(t)$$

and

$$\sum_{\tau=t}^{t-1} f(\tau) = 0, \qquad \sum_{\tau=t-m}^{t} f(\tau) = f(t-m) + f(t-m+1) + ... + f(t).$$

The difference operator Δ is defined by

$$\Delta f(t) = f(t+1) - f(t).$$

For a function $g : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}$, the difference operator Δ_t is given by

$$\Delta_t g(t, a) = g(t+1, a) - g(t, a).$$

Set

$$t_{-1} = \min\left\{\inf\{p(s) : s \ge t_0\}, t_0 - 1\right\}$$

and

$$t_m = \inf\{s : p(s) > t_{m-1}\} \quad \text{for} \quad \text{all} \quad m = 1, 2, ...$$

Then $\{t_m\}_{m=-1}^{\infty}$ is an increasing sequence such that

$$\lim_{m\to\infty} t_m = \infty, \qquad \bigcup_{m=1}^{\infty} [t_{m-1}, t_m) = [t_0, \infty)$$

$$\text{and} \quad p(t) \in \bigcup_{i=0}^{m} [t_{i-1}, t_i) \quad \text{for} \quad \text{all} \quad t_m \le t < t_{m+1}, \quad m = 0, 1, 2, ...$$

For a given nonnegative integer m, fix a point $t \ge t_0$, and define the natural numbers $k_m(t)$ such that

$$k_m(t) := [t - t_m], \quad m = 0, 1, 2...$$

Then, for $t \in [t_m, t_{m+1})$, we have

$$t - k_m(t) - 1 < t_m \quad \text{and} \quad t - k_m(t) \geq t_m, \quad m = 0, 1, 2, ...$$

and for the arbitrary $t \geq t_0$, $m = 0$ we have

$$t - k_0(t) - 1 < t_0 \quad \text{and} \quad t - k_0(t) \geq t_0.$$

Set

$$T_m(t) := \{t - k_m(t), t - k_m(t) + 1, ..., t - 1, t\}, \quad m = 0, 1, 2, ...$$

For a given function $\varphi : [t_{-1}, t_0) \to \mathbf{R}$, Equation (1) has the unique solution x^φ satisfying the *initial condition*

$$x^\varphi(t) = \varphi(t) \quad \text{for} \quad t_{-1} \leq t < t_0. \tag{2}$$

We present the characteristic equation associated with the initial value problem (1) and (2), and using them obtain an asymptotic estimate of solutions of Equation (1) which can be applied to the difference equations with constant delay and to the case $t - p_2 \leq p(t) \leq t - p_1$ for real numbers $1 < p_1 \leq p_2$.

After that we generalize the main result to the equation with several delays

$$x(t) = a(t)x(t-1) + \sum_{i=1}^{m} b_i(t)x(p_i(t)),$$

where $a, b_i : [t_0, \infty) \to \mathbf{R}$ are given functions for $i = 1, 2, ..., m$, and $p_i : [t_0, \infty) \to \mathbf{R}$ are given such that, for every $T > t_0$ there exists a $\delta > 0$ such that $p_i(t) \leq t - \delta$ for every $t \in [t_0, T]$, and $\lim_{t\to\infty} p_i(t) = \infty$ for $i = 1, 2, ..., m$. We apply the obtained results to particular cases such as $p_i(t) = t - p_i$ for $i = 1, 2, ..., m$, where $1 \leq p_1 < p_2 < ... < p_m$ are real numbers.

Let $x = x^\varphi$ be a solution of the initial value problem (1) and (2) such that $x(t) \neq 0$ for $t \geq t_0$. Then

$$1 = a(t)\frac{x(t-1)}{x(t)} + b(t)\frac{x(p(t))}{x(t)} \quad \text{for} \quad t \geq t_0.$$

Define the new function

$$\lambda(t) := \frac{x(t-1)}{x(t)} \quad \text{for} \quad t \geq t_0. \tag{3}$$

Since, now

$$x(t) = \varphi(t - k_0(t) - 1) \prod_{\ell = t - k_0(t)}^{t} \frac{1}{\lambda(\ell)} \quad \text{for} \quad t \geq t_0,$$

the function λ defined by (3) is a solution of the characteristic equation of the form

$$1 - a(t)\lambda(t) =$$

$$= b(t)\frac{\varphi(p(t) - k_0(p(t)) - 1)}{\varphi(t - k_0(t) - 1)} \prod_{\ell = t - k_0(t)}^{t} \lambda(\ell) \prod_{\ell = p(t) - k_0(p(t))}^{p(t)} \frac{1}{\lambda(\ell)}, \tag{4}$$

for $t \geq t_0$. Characteristic equation (4) associated with the initial value problem (1) and (2) is a generalization of the characteristic equation given in [3] and [4] for discrete difference equations.

2. Preliminaries

For given scalar functions $a, \rho : [t_0, \infty) \to \mathbf{R}$, $\rho(t) \neq 0$, for given initial function φ and nonnegative numbers n we define the numbers

$$R_n := \sup_{t_n \le t < t_{n+1}} \left\{ \rho(t) \sum_{\tau = t - k_n(t)}^{t} \frac{\Delta\rho(\tau - 1)}{\rho(\tau)\rho(\tau - 1)} \prod_{\ell = \tau + 1}^{t} a(\ell) \right\} \tag{5}$$

and

$$M_0 := \sup_{t_{-1} \le t < t_0} \rho(t)|\varphi(t)|.$$

We shall need the following hypotheses.

- (H_1) Let $a : [t_0, \infty) \to \mathbf{R}$ be a given real function satisfying $0 < a(t) < 1$, for all $t \ge t_0$ and $b : [t_0, \infty) \to \mathbf{R}$ an arbitrary given real function for all $t \ge t_0$.
- (H_2) Let $p : [t_0, \infty) \to \mathbf{R}$ be a given function such that, for every $T > t_0$ there exists a $\delta > 0$ such that $p(t) \le t - \delta$ for every $t \in [t_0, T]$, and $\lim_{t\to\infty} p(t) = \infty$.
- (H_*) There exists a real function $\rho : [t_{-1}, \infty) \to (0, \infty)$, which is bounded on the initial interval $[t_{-1}, t_0)$, and such that

  $$|b(t)|\rho(t - 1) \le (1 - a(t))\rho(p(t)) \quad \text{for} \quad \text{all} \quad t \ge t_0,$$

  where functions a and b are given in (H_1).
- (H_{**}) There exists a real number R such that

  $$\prod_{n=0}^{j} (1 + R_n) \le R, \tag{6}$$

  for all positive integers j, where the numbers R_n are defined by (5).

Theorem A. (Péics [5]) *Suppose that conditions (H_1), (H_2), (H_*) and (H_{**}) hold.*

Let $x = x^{\varphi}$ be the solution of the initial value problem (1) and (2) with bounded function φ. Then

$$|x(t)| \le \frac{M_0 R}{\rho(t)} \quad for \quad all \quad t \ge t_0.$$

Zhou and Yu in [6] obtained for estimating function the exponential function for the case when the lag function is between two constant delays. Applying Theorem A, by finding an appropriate estimating function ρ, we can determine the rate of convergence of the solutions of generalized difference equations for different types of the lag function. We will provide two examples.

If the lag function p is such that $p_1 t \le p(t) \le p_2 t$, for real numbers $0 < p_1 \le p_2 < 1$, then the estimating function is a power function, namely $\rho(t) = t^k$.

Corollary A. (Péics [5]) *Suppose that condition (H_1) holds. For given real numbers p_1, p_2 and t_0 such that $0 < p_1 \le p_2 < 1$ and $t_0 > p_1/(1-p_1)$, let p be given real function such that $p_1 t \le p(t) \le p_2 t$ for all $t \ge t_0$. Suppose that there exist real numbers Q and α such that $0 < Q \le 1$, $0 < \alpha < 1$,*

$$|b(t)| \le Q(1-a(t)), \quad \alpha \le 1-a(t) \quad \textit{for} \quad \textit{all} \quad t \ge t_0$$

and

$$\log \frac{1}{Q}\left(\log \frac{1}{p_1} - \log \frac{1}{p_2}\right) < \log \frac{1}{p_1} \log \frac{1}{p_2}.$$

Let $x = x^\varphi$ be a solution of the initial value problem (1) and (2) with bounded function φ, and let

$$k = \frac{\log Q}{\log p_1}, \quad M_0 = \sup_{t_{-1} \le t < t_0} \{t^k |\varphi(t)|\}.$$

Then

$$|x(t)| \le \frac{C}{\left(t - \frac{p_1}{1-p_1}\right)^k} \quad \textit{for} \quad \textit{all} \quad t \ge t_0,$$

where

$$C = M_0 \prod_{n=0}^{\infty} \left(1 + \frac{k t_0^k (1-p_1)^{k+1}}{\alpha p_1^k (t_0(1-p_1) - p_2^n)^{k+1}} \left(\frac{p_2^{k+1}}{p_1^k}\right)^n\right).$$

Remark 1. *We can prove similarly the above result by choosing $\rho(t) = t^k$. Then the statement of Theorem A holds for $t \ge t_0/p_1$ that does not disturb the asymptotic behavior of solutions but the comparableness with the function ρ is clearer and the rate of the convergence is better.*

If the lag function p is such that $\sqrt[p_2]{t} \le p(t) \le \sqrt[p_1]{t}$, for natural numbers $1 < p_1 \le p_2$, then the estimating function is a logarithm function. Namely, $\rho(t) = \log^k t$.

Corollary B. (Péics [5]) *Suppose that condition (H_1) holds. Let $t_0 \ge 1$ be given real number, p_1, p_2 be given natural numbers such that $1 < p_1 \le p_2$. Let p be given real function such that $\sqrt[p_2]{t} \le p(t) \le \sqrt[p_1]{t}$ for all $t \ge t_0$. Suppose that there exist real numbers Q and α such that $0 < Q \le 1$, $0 < \alpha < 1$ and*

$$|b(t)| \le Q(1-a(t)), \quad \alpha \le 1-a(t) \quad \textit{for} \quad \textit{all} \quad t \ge t_0.$$

Let $x = x^\varphi$ be a solution of the initial value problem (1) and (2) with bounded function φ, and let

$$k = -\frac{\log Q}{\log p_2}, \quad M_0 = \sup_{t_{-1} \le t < t_0} \left\{\log^k t |\varphi(t)|\right\}.$$

Then

$$|x(t)| \leq \frac{C}{\log^k t} \quad \textit{for} \quad \textit{all} \quad t \geq t_0,$$

where

$$C = M_0 \prod_{n=0}^{\infty} \left(1 + \frac{kp_2^k p_2^{nk} \log^k t_0}{\alpha(t_0^{p_1^n} - 1) \log^{k+1}(t_0^{p_1^n} - 1)}\right).$$

3. Main Result

Assume that

- (H_3) For the function a and b given in (H_1), there is a real function $\lambda : [t_0, \infty) \to (1, \infty)$ and there is an initial function φ in (2) such that

$$|b(t)| \frac{\varphi(p(t) - k_0(p(t)) - 1)}{\varphi(t - k_0(t) - 1)} \prod_{\ell = t - k_0(t)}^{t} \lambda(\ell) \prod_{\ell = p(t) - k_0(p(t))}^{p(t)} \frac{1}{\lambda(\ell)} \leq$$

$$\leq 1 - a(t)\lambda(t), \quad t \geq t_0. \tag{7}$$

In the next result we use the concept of the characteristic equation but it is not necessary to have the solution of characteristic equation (4). It is sufficient only to have a solution of Inequality (7) that is a much weaker condition.

Theorem 1. *Suppose that conditions (H_1), (H_2) and (H_3) hold. Let $x = x^{\varphi}$ be the solution of the initial value problem (1) and (2). Then*

$$|x(t)| \leq \left(\varphi(t - k_0(t) - 1) \sup_{t_{-1} \leq t \leq t_0} \lambda(t)\right) \prod_{\ell = t - k_0(t)}^{t} \frac{1}{\lambda(\ell)}, \quad t \geq t_0.$$

Proof. Introduce the transformation

$$y(t) := \frac{x(t)}{\varphi(t - k_0(t) - 1)} \prod_{\ell = t - k_0(t)}^{t} \lambda(\ell).$$

Then the function $y(t)$ satisfies the equation

$$\begin{aligned} y(t) &= a(t)\lambda(t)y(t-1) + \\ &+ b(t) \frac{\varphi(p(t) - k_0(p(t)) - 1)}{\varphi(t - k_0(t) - 1)} \prod_{\ell = t - k_0(t)}^{t} \lambda(\ell) \prod_{\ell = p(t) - k_0(p(t))}^{p(t)} \frac{1}{\lambda(\ell)} y(p(t)). \end{aligned}$$

Using hypothesis (7) we obtain that

$$|y(t)| \leq a(t)\lambda(t)|y(t-1)| + (1 - a(t)\lambda(t))|y(p(t))|.$$

Let $t \in [t_n, t_{n+1})$, $\tau \in T_n(t)$ and $|y(t)| = u(t)$. Then, the above inequality is equivalent to

$$\Delta_\tau \left(u(\tau-1) \prod_{\ell=t-k_n(t)}^{\tau-1} \frac{1}{a(\ell)\lambda(\ell)} \right) \leq$$
$$\leq (1 - a(\tau)\lambda(\tau))u(p(\tau)) \prod_{\ell=t-k_n(t)}^{\tau} \frac{1}{a(\ell)\lambda(\ell)}.$$

Summing up both sides of this inequality from $t - k_n(t)$ to t gives that

$$\begin{aligned} u(t) \;&\leq\; u(t-k_n(t)-1) \prod_{\ell=t-k_n(t)}^{t} a(\ell)\lambda(\ell) + \\ &+ \sum_{\tau=t-k_n(t)}^{t} (1-a(\tau)\lambda(\tau))u(p(\tau)) \prod_{\ell=\tau+1}^{t} a(\ell)\lambda(\ell). \end{aligned}$$

Define

$$\mu_n := \sup_{t_{n-1} \leq t < t_n} u(t) \quad \text{and} \quad M_n := \max\{\mu_0, \mu_1, ..., \mu_n\} \tag{8}$$

for $n = 0, 1, 2, ...$ Then

$$\begin{aligned} u(t) \;&\leq\; M_n \left(\prod_{\ell=t-k_n(t)}^{t} a(\ell)\lambda(\ell) + \sum_{\tau=t-k_n(t)}^{t} (1-a(\tau)\lambda(\tau)) \prod_{\ell=\tau+1}^{t} a(\ell)\lambda(\ell) \right) \\ &= M_n \left(\prod_{\ell=t-k_n(t)}^{t} a(\ell)\lambda(\ell) + \sum_{\tau=t-k_n(t)}^{t} \Delta_\tau \left(\prod_{\ell=\tau}^{t} a(\ell)\lambda(\ell) \right) \right) \\ &= M_n. \end{aligned}$$

The above inequality implies that

$$M_{n+1} \leq M_n \quad \text{and} \quad u(t) = |y(t)| \leq M_0,$$

and the assertion of the theorem is valid. □

The following corollary represents an asymptotic estimate for the solutions of Equation (1) and gives information about the rate of the convergence of solutions to particular cases such as $t - p_2 \leq p(t) \leq t - p_1$, for real numbers $1 < p_1 \leq p_2$. We obtain that the solutions are exponentially decaying as in the results given by Zhou and Yu in [6].

Corollary 1. *Suppose that condition (H_1) holds. Let p_1, p_2 and t_0 be given real numbers such that $1 \leq p_1 < p_2$ and $t_0 \geq p_1$. Let $p(t) = t - \delta(t)$, with given real function δ such that $p_1 \leq \delta(t) \leq p_2$ for all $t \geq t_0$. Suppose that there exists a real number $\lambda > 1$ such that*

$$|b(t)| \leq \frac{1 - \lambda a(t)}{\lambda^{p_2}} \quad for \quad all \quad t \geq t_0. \tag{9}$$

Let $x = x^{\varphi}$ be the solution of the initial value problem (1) and (2), and

$$M_0 = \sup_{t_{-1} \le t < t_0} \{\lambda^t |\varphi(t)|\}.$$

Then

$$|x(t)| \le \frac{M_0}{\lambda^t} \quad \textit{for} \quad \textit{all} \quad t \ge t_0.$$

Proof. The relations

$$t_{n+1} - \delta(t_{n+1}) = t_n, \quad t - p_2 \le t - \delta(t) \le t - p_1$$

imply that

$$t_0 + np_1 \le t_n \le t_0 + np_2 \quad \text{for} \quad n = 1, 2, ...$$

Introduce the transformation $y(t) := x(t)\lambda^t$. Let $t \in [t_n, t_{n+1})$ and $\tau \in T_n(t)$. Then Equation (1) is equivalent to

$$\Delta_\tau \left(y(\tau - 1) \prod_{\ell = t - k_n(t)}^{\tau - 1} \frac{1}{\lambda a(\ell)} \right) = b(\tau)\lambda^{\delta(\tau)} y(\tau - \delta(\tau)) \prod_{\ell = t - k_n(t)}^{\tau} \frac{1}{\lambda a(\ell)}.$$

Summing up both sides of this equality from $t - k_n(t)$ to t gives that

$$\begin{aligned} y(t) &= y(t - k_n(t) - 1) \prod_{\ell = t - k_n(t)}^{t} \lambda a(\ell) + \\ &\quad + \sum_{\tau = t - k_n(t)}^{t} b(\tau)\lambda^{\delta(\tau)} y(\tau - \delta(\tau)) \prod_{\ell = \tau + 1}^{t} \lambda a(\ell). \end{aligned}$$

Define

$$\mu_n := \sup_{t_{n-1} \le t < t_n} |y(t)| \quad \text{and} \quad M_n := \max\{\mu_0, \mu_1, ..., \mu_n\} \tag{10}$$

for $n = 0, 1, 2, ...$ Since $|y(p(\tau))| \le M_n$ for $\tau \in T_n(t)$ and $t_n \le t < t_{n+1}$, by using the summation by parts formula, it follows that

$$\begin{aligned} |y(t)| &\le M_n \left(\prod_{\ell = t - k_n(t)}^{t} \lambda a(\ell) + \sum_{\tau = t - k_n(t)}^{t} (1 - \lambda a(\tau)) \prod_{\ell = \tau + 1}^{t} \lambda a(\ell) \right) \\ &= M_n \left(\prod_{\ell = t - k_n(t)}^{t} \lambda a(\ell) + \sum_{\tau = t - k_n(t)}^{t} \Delta_\tau \left(\prod_{\ell = \tau}^{t} \lambda a(\ell) \right) \right) \\ &= M_n. \end{aligned}$$

The above inequality implies that $M_{n+1} \le M_n$ and $|y(t)| \le M_0$. Therefore

$$|x(t)| \le \frac{M_0}{\lambda^t} \quad \text{for} \quad \text{all} \quad t \ge t_0$$

and the proof is complete. □

4. Generalization

In this section we generalize previous results to the difference equations with several delays and show the usefulness of the new results to the classical particular cases.

Assume that $t_0 > 0$ is a given real number, $a, b_i : [t_0, \infty) \to \mathbf{R}$ are given real functions for $i = 1, 2, ..., m$. Let $p_i : [t_0, \infty) \to \mathbf{R}$ be given functions such that $p_i(t) \le t$ for every $t \in [t_0, \infty]$, $i = 1, 2, ..., m$, and $\lim_{t\to\infty} p_i(t) = \infty$ for $i = 1, 2, ..., m$. Consider the difference equation with several delays

$$x(t) = a(t)x(t-1) + \sum_{i=1}^{m} b_i(t)x(p_i(t)). \tag{11}$$

Set

$$t_{-1} = \min\left\{t_0 - 1, \ \min_{1\le i\le m}\{\inf\{p_i(s), s \ge t_0\}\}\right\},$$

$$t_n = \min_{1\le i\le m} \inf\{s : p_i(s) > t_{n-1}\} \quad \text{for} \quad \text{all} \quad n = 1, 2, ...$$

Then

$$p_i(t) \in \bigcup_{j=0}^{n} [t_{j-1}, t_j) \quad \text{for} \quad t_n \le t < t_{n+1},$$

$i = 1, 2, ..., m$ and $n = 0, 1, 2, ...$

We shall need the following hypotheses.

- (H_4) $a, b_i : [t_0, \infty) \to \mathbf{R}$ are real functions $i = 1, 2, ..., m$, such that $0 < a(t) < 1$, for all $t \ge t_0$.
- (H_5) $p_i : [t_0, \infty) \to \mathbf{R}$ are real functions such that for every $T > t_0$ there exists a $\delta > 0$ such that $p_i(t) \le t - \delta$ for every $t \in [t_0, T]$, $i = 1, 2, ..., m$, and $\lim_{t\to\infty} p_i(t) = \infty$ for $i = 1, 2, ..., m$.

The next result gives an asymptotic estimate of solutions of Equation (11) and generalizes the result given in Theorem 1.

Theorem 2. *Suppose that conditions (H_4) and (H_5) hold. Suppose that there is a real function $\lambda : [t_0, \infty) \to (1, \infty)$ and there is an initial function φ in (2) such that*

$$\sum_{i=1}^{m} |b_i(t)| \frac{\varphi(p_i(t) - k_0(p_i(t)) - 1)}{\varphi(t - k_0(t) - 1)} \prod_{\ell=t-k_0(t)}^{t} \lambda(\ell) \prod_{\ell=p_i(t)-k_0(p_i(t))}^{p_i(t)} \frac{1}{\lambda(\ell)} \le$$

$$\le 1 - a(t)\lambda(t) \quad for \quad all \quad t \ge t_0. \tag{12}$$

Let $x = x^{\varphi}$ be the solution of the initial value problem (11) and (2). Then

$$|x(t)| \le \left(\varphi(t - k_0(t) - 1) \sup_{t_{-1}\le t\le t_0} \lambda(t)\right) \prod_{\ell=t-k_0(t)}^{t} \frac{1}{\lambda(\ell)} \quad for \quad all \quad t \ge t_0.$$

Proof. Introduce the transformation

$$y(t) := \frac{x(t)}{\varphi(t-k_0(t)-1)} \prod_{\ell=t-k_0(t)}^{t} \lambda(\ell).$$

Then the function $y(t)$ satisfies the equation

$$\begin{aligned} y(t) &= a(t)\lambda(t)y(t-1) + \\ &+ \sum_{i=1}^{m} b_i(t)y(p_i(t)) \frac{\varphi(p_i(t)-k_0(p_i(t))-1)}{\varphi(t-k_0(t)-1)} \times \\ &\times \prod_{\ell=t-k_0(t)}^{t} \lambda(\ell) \prod_{\ell=p_i(t)-k_0(p_i(t))}^{p_i(t)} \frac{1}{\lambda(\ell)}. \end{aligned}$$

Let $t \in [t_n, t_{n+1})$, $\tau \in T_n(t)$ and $|y(t)| = u(t)$. Therefore, it follows that

$$\begin{aligned} u(t) &\leq u(t-k_n(t)-1) \prod_{\ell=t-k_n(t)}^{t} a(\ell)\lambda(\ell) + \\ &+ \sum_{\tau=t-k_n(t)}^{t} \sum_{i=1}^{m} b_i(t)u(p_i(\tau)) \frac{\varphi(p_i(t)-k_0(p_i(t))-1)}{\varphi(t-k_0(t)-1)} \times \\ &\times \prod_{\ell=t-k_0(t)}^{t} \lambda(\ell) \prod_{\ell=p_i(t)-k_0(p_i(t))}^{p_i(t)} \frac{1}{\lambda(\ell)} \prod_{\ell=\tau+1}^{t} a(\ell)\lambda(\ell). \end{aligned}$$

Using notation (8) and hypothesis (12), the same argumentation as in Theorem 1 completes the proof. □

The next result is a special case of the previous theorem and generalizes the result given in Corollary 1.

Corollary 2. *Suppose that condition (H_4) holds. For given real number t_0 and for given natural numbers p_1, p_2,...,p_m such that $1 \leq p_1 < p_2 < ... < p_m$ and $t_0 \geq p_1$, let $p_i(t) = t - p_i$ for all $t \geq t_0$, $i = 1, 2, ..., m$. Suppose that there exists a real number $\lambda > 1$ such that*

$$\sum_{i=1}^{m} |b_i(t)|\lambda^{p_i} \leq 1 - \lambda a(t) \quad for \quad all \quad t \geq t_0. \tag{13}$$

Let $x = x^{\varphi}$ be the solution of the initial value problem (11) and (2). Then

$$|x(t)| \leq \frac{M_0}{\lambda^t} \quad \text{for} \quad \text{all} \quad t \geq t_0.$$

Proof. Introduce the transformation $y(t) := x(t)\lambda^t$. Let $t \in [t_n, t_{n+1})$ and $\tau \in T_n(t)$. Then, Equation (11) is equivalent to

$$\Delta_\tau \left(y(\tau-1) \prod_{\ell=t-k_n(t)}^{\tau-1} \frac{1}{\lambda a(\ell)} \right) = \sum_{i=1}^{m} b_i(\tau)\lambda^{p_i} y(\tau-p_i) \prod_{\ell=t-k_n(t)}^{\tau} \frac{1}{\lambda a(\ell)}.$$

Summing up both sides of this equality from $t-k_n(t)$ to t gives that

$$\begin{aligned} y(t) &= y(t-k_n(t)-1)\prod_{\ell=t-k_n(t)}^{t}\lambda a(\ell)+ \\ &+\sum_{\tau=t-k_n(t)}^{t}\sum_{i=1}^{m}b_i(\tau)\lambda^{p_i}y(\tau-p_i)\prod_{\ell=\tau+1}^{t}\lambda a(\ell). \end{aligned}$$

Using the notation (10) and the summation by parts formula, the same argumentation as in Corollary 1 completes the theorem. □

Remark 2. *Consider the difference equations of the form*

$$x(t)=a(t)x(t-h)+b(t)x(p(t))$$

with $h\in\mathbf{R}_+$. Then, using the transformation $y(\frac{t}{h})=\frac{1}{h}x(t)$ and $s=\frac{t}{h}$, we obtain the equation of the form (1) with the unknown function $y(s)$, and the above results can be applied.

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