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ON NONNEGATIVE SOLUTIONS OF A CERTAIN BOUNDARY VALUE PROBLEM FOR FIRST ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Unimprovable efficient conditions are established for the existence and uniqueness of a nonnegative solution of the problem

$$u'(t) = \ell(u)(t) + q(t), \qquad u(a) = h(u) + c,$$

where $\ell : C([a, b]; \mathbb{R}) \to L([a, b]; \mathbb{R})$ is a linear bounded operator, $h : C([a, b]; \mathbb{R}) \to \mathbb{R}$ is a linear bounded functional, $q \in L([a, b]; \mathbb{R})$ and $c > 0$.

1. INTRODUCTION

The following notation is used throughout.

$\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$.

$[x]_- = \frac{1}{2}\Big(|x| - x\Big)$, $[x]_+ = \frac{1}{2}\Big(|x| + x\Big)$.

$C([a, b]; \mathbb{R})$ is the Banach space of continuous functions $u : [a, b] \to \mathbb{R}$ with the norm $\|u\|_C = \max\{|u(t)| : t \in [a, b]\}$.

$C([a, b]; \mathbb{R}_+) = \{u \in C([a, b]; \mathbb{R}) : u(t) \geq 0 \text{ for } t \in [a, b]\}$.

$C_h([a, b]; \mathbb{R}_+) = \{v \in C([a, b]; \mathbb{R}_+) : v(a) = h(v)\}$.

$\widetilde{C}([a, b]; \mathrm{D})$, where $\mathrm{D} \subseteq \mathbb{R}$, is the set of absolutely continuous functions $u : [a, b] \to \mathrm{D}$.

$L([a, b]; \mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : [a, b] \to \mathbb{R}$ with the norm $\|p\|_L = \int\limits_a^b |p(s)| ds$.

$L([a, b]; \mathbb{R}_+) = \{u \in L([a, b]; \mathbb{R}) : p(t) \geq 0 \text{ for almost all } t \in [a, b]\}$.

$\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell : C([a, b]; \mathbb{R}) \to L([a, b]; \mathbb{R})$.

$\mathcal{P}_{ab}$ is the set of linear bounded operators $\ell \in \mathcal{L}_{ab}$ transforming the set $C([a, b]; \mathbb{R}_+)$ into the set $L([a, b]; \mathbb{R}_+)$.

F_{ab} is the set of linear bounded functionals $h : C([a, b]; \mathbb{R}) \to \mathbb{R}$.

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$\mathcal{P}F_{ab}$ is the set of linear functionals $h \in F_{ab}$ transforming the set $C([a,b];\mathbb{R}_+)$ into the set $\mathbb{R}_+$.

We will say that $\ell \in \mathcal{L}_{ab}$ is an a-Volterra operator if for arbitrary $a_1 \in \,]a,b]$ and $v \in C([a,b];\mathbb{R})$ satisfying

$$v(t) = 0 \quad \text{for } t \in [a, a_1],$$

we have

$$\ell(v)(t) = 0 \quad \text{for almost all } t \in [a, a_1].$$

Consider the boundary value problem

$$u'(t) = \ell(u)(t) + q(t), \tag{1.1}$$

$$u(a) = h(u) + c, \tag{1.2}$$

where $\ell \in \mathcal{L}_{ab}$, $h \in F_{ab}$, $q \in L([a,b];\mathbb{R})$ and $c \in \mathbb{R}$. By a solution of the problem (1.1), (1.2) we understand a function $u \in \widetilde{C}([a,b];\mathbb{R})$ satisfying the equality (1.1) almost everywhere in $[a,b]$ and the condition (1.2).

Throughout the paper we will assume that the functional $h(v) - v(a)$ is not identicaly zero and $h \in \mathcal{P}F_{ab}$.

From the general theory of linear boundary value problems the following theorem is well-known (see, e.g., [1,2,5,6]).

Theorem 1.1. *The problem* (1.1), (1.2) *is uniquely solvable if and only if the corresponding homogeneous problem*

$$u'(t) = \ell(u)(t), \tag{1.1$_0$}$$

$$u(a) = h(u) \tag{1.2$_0$}$$

has only the trivial solution.

Definition 1.1. We will say that an operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $\widetilde{V}_{ab}^{+}(h)$ if every function $u \in \widetilde{C}([a,b];\mathbb{R})$ satisfying

$$u'(t) \geq \ell(u)(t) \quad \text{for} \quad t \in [a,b], \tag{1.3}$$

$$u(a) \geq h(u) \tag{1.4}$$

is nonnegative.

Remark 1.1. It is clear that if $\ell \in \widetilde{V}_{ab}^{+}(h)$, then the problem (1.1$_0$), (1.2$_0$) has only the trivial solution. Therefore in this case, according to Theorem 1.1, the problem (1.1), (1.2) is uniquely solvable for any $c \in \mathbb{R}$ and $q \in L([a,b];\mathbb{R})$. If moreover $c \in \mathbb{R}_+$ and $q \in L([a,b];\mathbb{R}_+)$, then the unique solution of the problem (1.1), (1.2) is nonnegative.

Remark 1.2. Sufficient conditions guaranteeing inclusion $\ell \in \widetilde{V}_{ab}^{+}(0)$ are contained in [3,4].

In this paper, we will establish optimal, in a certain sense, sufficient conditions guaranteeing inclusion $\ell \in \widetilde{V}_{ab}^{+}(h)$.

First let us mention some properties of the set $\widetilde{V}_{ab}^{+}(h)$.

Remark 1.3. It is not difficult to verify that $\mathcal{P}_{ab} \cap \widetilde{V}_{ab}^{+}(h) \neq \varnothing$ if and only if

$$h(1) < 1. \tag{1.5}$$

Indeed, let $\ell \in \mathcal{P}_{ab} \cap \widetilde{V}_{ab}^{+}(h)$. Then, according to Remark 1.1, the problem

$$\begin{aligned} u'(t) &= \ell(u)(t), \\ u(a) &= h(u) + 1 \end{aligned} \tag{1.6}$$

has a unique solution u and

$$u(t) \geq 0 \quad \text{for } t \in [a, b]. \tag{1.7}$$

By virtue of (1.7) and the condition $\ell \in \mathcal{P}_{ab}$ we have

$$u'(t) \geq 0 \quad \text{for } t \in [a, b]. \tag{1.8}$$

Thus

$$u(t) \geq u(a) \quad \text{for } t \in [a, b]. \tag{1.9}$$

Now (1.9) and the condition $h \in \mathcal{P}F_{ab}$ imply that

$$h(u) \geq u(a)h(1), \tag{1.10}$$

whence, together with (1.6), we obtain

$$u(a)(1 - h(1)) \geq 1. \tag{1.11}$$

Therefore, the inequality (1.5) holds.

Assume now that (1.5) is fulfilled. We will show that $0 \in \widetilde{V}_{ab}^{+}(h)$. Let the function $u \in \widetilde{C}([a, b]; \mathbb{R})$ satisfy (1.8) and (1.4). Clearly, (1.9) holds, as well. Hence, on account of the condition $h \in \mathcal{P}F_{ab}$, the inequality (1.10) is satisfied. By virtue of (1.4) and (1.10) we have

$$u(a)(1 - h(1)) \geq 0,$$

which, together with (1.5), implies $u(a) \geq 0$. Taking now into account (1.9) we get (1.7). Therefore, $0 \in \widetilde{V}_{ab}^{+}(h)$.

Remark 1.4. Define the functional $h_\lambda \in F_{ab}$ by

$$h_\lambda(v) \stackrel{\text{def}}{=} \lambda v(a),$$

where $\lambda \in \mathbb{R}$. From Definition 1.1 it immediately follows that $\widetilde{V}_{ab}^{+}(h) = \widetilde{V}_{ab}^{+}(0)$ provided $h = h_\lambda$ for some $\lambda \in \,]0,1[$. On the other hand, if $h \neq h_\lambda$ for $\lambda \in \,]0,1[$, then, in general $\widetilde{V}_{ab}^{+}(h) \neq \widetilde{V}_{ab}^{+}(0)$. To see this first we will show that if

$$h \neq h_\lambda \quad \text{for} \quad \lambda \in \mathbb{R}, \tag{1.12}$$

then there exists $w \in \widetilde{C}([a,b];\mathbb{R}_+)$ such that $w \not\equiv 0$,

$$w'(t) \geq 0 \quad \text{for} \quad t \in [a,b], \quad w(a) = 0, \tag{1.13}$$

and

$$h(w) \neq 0.$$

Assume on the contrary, that for each $w \in \widetilde{C}([a,b];\mathbb{R}_+)$ satisfying (1.13) we have $h(w) = 0$. Evidently, an arbitrary function $f \in \widetilde{C}([a,b];\mathbb{R})$ admits the representation

$$f(t) = f(a) + w_1(t) - w_2(t) \quad \text{for} \quad t \in [a,b],$$

where

$$w_1(t) = \int_a^t [f'(s)]_+ \, ds \quad w_2(t) = \int_a^t [f'(s)]_- \, ds \quad \text{for} \quad t \in [a,b].$$

By our assumption clearly $h(w_1) = 0$ and $h(w_2) = 0$. Thus

$$h(f) = f(a)h(1) \quad \text{for} \quad f \in \widetilde{C}([a,b];\mathbb{R}). \tag{1.14}$$

Therefore, $h = h_\lambda$ for $\lambda = h(1)$, which contradicts (1.12).

Suppose now that

$$h \neq h_\lambda \quad \text{for} \quad \lambda \in]0,1[$$

and $\mathcal{P}_{ab} \cap \widetilde{V}_{ab}^{+}(h) \neq \varnothing$. According to Remark 1.3 we have $h(1) < 1$. Therefore (1.12) holds as well. By virtue of above proved there exists a function $w \in \widetilde{C}([a,b];\mathbb{R}_+)$ satisfying (1.13) and

$$h(w) = 1 - h(1).$$

Put

$$p(t) \stackrel{\text{def}}{=} w'(t) \quad \ell(v)(t) \stackrel{\text{def}}{=} p(t)v(a) \quad \text{for} \quad t \in [a,b]. \tag{1.15}$$

By virtue of (1.13) clearly $p(t) \geq 0$ for $t \in [a,b]$. Hence $\ell \in \mathcal{P}_{ab}$. It is also evident that ℓ is an $a - Volterra$ operator. Consequently, by Corollary 1.1 in [4] $\ell \in \widetilde{V}_{ab}^{+}(0)$. On the other hand, it is not difficult to verify that the function $u(t) = 1 + w(t)$ for $t \in [a,b]$ is a nontrivial solution of the problem $(1.1_0), (1.2_0)$. Consequently, by virtue of Remark 1.1, $\ell \notin \widetilde{V}_{ab}^{+}(h)$.

2. Main results

In this section we will establish optimal sufficient conditions guaranteeing the inclusion $\ell \in \widetilde{V}_{ab}^{+}(h)$. Theorems 2.1 and 2.3 below concerns the case when ℓ is monotone operator, i.e., when $\ell \in \mathcal{P}_{ab}$, resp. $-\ell \in \mathcal{P}_{ab}$. Theorems 2.2 and 2.4 cover also the case when ℓ is not monotone.

Theorem 2.1. *Let $\ell \in \mathcal{P}_{ab}$. Then $\ell \in \widetilde{V}_{ab}^{+}(h)$ if and only if there exists a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying the inequalities*

$$\gamma'(t) \geq \ell(\gamma)(t) \quad \text{for} \quad t \in [a,b], \tag{2.1}$$

$$\gamma(a) > h(\gamma). \tag{2.2}$$

Corollary 2.1. *Let $\ell \in \mathcal{P}_{ab}$ be an a-Volterra operator and*

$$h(\gamma) < 1, \tag{2.3}$$

where $\gamma(t) = \exp\left[\int_a^t \ell(1)(s)\,ds\right]$ for $t \in [a,b]$. Then $\ell \in \widetilde{V}_{ab}^{+}(h)$.

Remark 2.1. Inequality (2.3) is optimal and it can not be replaced by the inequality $h(\gamma) \leq 1$. Indeed, let $\gamma(t) = \exp\left[\int_a^t p(s)\,ds\right]$ for $t \in [a,b]$, where $p \in L([a,b];\mathbb{R}_+)$ is such that $h(\gamma) = 1$. Clearly, the function γ is a nontrivial solution of the problem (1.1_0), (1.2_0) with $\ell(v)(t) \overset{def}{=} p(t)v(t)$. Therefore, according to Remark 1.1, $\ell \notin \widetilde{V}_{ab}^{+}(h)$.

Corollary 2.2. *Let $h(1) < 1$ and let there exist $m, k \in \mathbb{N}$ and a constant $\alpha \in]0,1[$ such that $m > k$ and*

$$\rho_m(t) \leq \alpha \rho_k(t) \quad for \quad t \in [a,b],$$

where $\rho_1 \equiv 1$,

$$\rho_{i+1}(t) \overset{def}{=} \frac{1}{1-h(1)} h(\varphi_i) + \varphi_i(t) \quad for \quad t \in [a,b],\ i \in \mathbb{N},$$

$$\varphi_i(t) \overset{def}{=} \int_a^t \ell(\rho_i)(s)\,ds \quad for \quad t \in [a,b],\ i \in \mathbb{N}.$$

Then $\ell \in \widetilde{V}_{ab}^{+}(h)$.

Corollary 2.3. *Let $\ell \in \mathcal{P}_{ab}$ and let there exist $\bar{\ell} \in \mathcal{P}_{ab}$ such that*

$$h(\gamma_0) < 1 \tag{2.4}$$

and

$$\frac{h(\gamma_1)}{1 - h(\gamma_0)}\gamma_0(b) + \gamma_1(b) < 1, \tag{2.5}$$

where

$$\gamma_0(t) \overset{def}{=} \exp\left[\int_a^t \ell(1)(s)\, ds\right] \quad \text{for} \quad t \in [a, b],$$

$$\gamma_1(t) \overset{def}{=} \int_a^t \bar{\ell}(1)(s) \exp\left[\int_s^t \ell(1)(\xi)\, d\xi\right] ds \quad \text{for} \quad t \in [a, b].$$

Let, moreover on the set $C_h([a, b]; \mathbb{R}_+)$ the inequality

$$\ell(\theta(v))(t) - \ell(1)(t)\theta(v)(t) \leq \bar{\ell}(v)(t) \quad \text{for} \quad t \in [a, b] \tag{2.6}$$

hold, where

$$\begin{aligned} \theta(v)(t) &\overset{def}{=} \frac{1}{1 - h(1)} h(v_0) + v_0(t) \quad \text{for} \quad t \in [a, b], \\ v_0(t) &\overset{def}{=} \int_a^t \ell(v)(s)\, ds \quad \text{for} \quad t \in [a, b]. \end{aligned} \tag{2.7}$$

Then $\ell \in \widetilde{V}_{ab}^{+}(h)$.

Remark 2.2. From Theorem 2.1 it immediately follows that $\mathcal{P}_{ab} \cap \widetilde{V}_{ab}^{+}(h) \subseteq \mathcal{P}_{ab} \cap \widetilde{V}_{ab}^{+}(0)$. On the other hand, as it had been shown above (see Remark 1.4) in general $\mathcal{P}_{ab} \cap \widetilde{V}_{ab}^{+}(h) \neq \mathcal{P}_{ab} \cap \widetilde{V}_{ab}^{+}(0)$ (even in the case when $\mathcal{P}_{ab} \cap \widetilde{V}_{ab}^{+}(h) \neq \varnothing$). Therefore, without additional restrictions, the inclusion $\ell \in \mathcal{P}_{ab} \cap \widetilde{V}_{ab}^{+}(0)$ does not guarantee the inclusion $\ell \in \mathcal{P}_{ab} \cap \widetilde{V}_{ab}^{+}(h)$. However the following theorem is true.

Theorem 2.2. *Let $\ell \in \widetilde{V}_{ab}^{+}(0)$. Then $\ell \in \widetilde{V}_{ab}^{+}(h)$ if and only if there exists a function $\gamma \in \widetilde{C}([a, b]; \mathbb{R})$ satisfying the inequalities* (2.1), (2.2) *and*

$$\gamma(a) > 0. \tag{2.8}$$

Theorem 2.2 yields the following

Theorem 2.3. *Let $-\ell \in \mathcal{P}_{ab}$ be an a-Volltera operator and let*

$$h(1) < 1. \tag{2.9}$$

Then $\ell \in \widetilde{V}_{ab}^{+}(h)$ if and only if $\ell \in \widetilde{V}_{ab}^{+}(0)$.

Corollary 2.4. *Let $-\ell \in \mathcal{P}_{ab}$ be an a-Volterra operator and let* (2.9) *hold. Let, moreover, there exist a function $\gamma \in \widetilde{C}([a,b];\mathbb{R}_+)$ satisfying*

$$\begin{gathered} \gamma(t) > 0 \quad \textit{for} \quad t \in [a,b[, \\ \gamma'(t) \le \ell(\gamma)(t) \quad \textit{for} \quad t \in [a,b]. \end{gathered} \tag{2.10}$$

Then $\ell \in \widetilde{V}_{ab}^{+}(h)$.

Remark 2.3. Corollary 2.4 is unimprovable in a certain sense. More precisely the condition (2.10) cannot be replaced by the condition

$$\gamma(t) > 0 \quad \text{for} \quad t \in [a, b_1[, \tag{2.11}$$

where $b_1 \in]a,b[$ is an arbitrarily fixed point. Indeed, in [4] it is shown (see [4, Example 4.3]), that conditions (2.10) and (2.11) do not guarantee the inclusion $\ell \in \widetilde{V}_{ab}^{+}(0)$. Consequently, by virtue of Theorem 2.3, Corollary 2.4 is nonimprovable in above mentioned sense.

Corollary 2.5. *Let $-\ell \in \mathcal{P}_{ab}$ be an a-Volterra operator and let* (2.9) *hold. If, moreover,*

$$\int_a^b |\ell(1)(s)|\, ds \le 1, \tag{2.12}$$

then $\ell \in \widetilde{V}_{ab}^{+}(h)$.

Remark 2.4. Constant 1 on the right hand site of the condition (2.12) is the best possible (see Theorem 2.3 and [4, Example 4.4]).

Corollary 2.6. *Let $-\ell \in \mathcal{P}_{ab}$ be an a-Volterra operator and let* (2.9) *hold. Let, moreover,*

$$\int_a^b |\widetilde{\ell}(1)(s)| \exp\left(\int_a^s |\ell(1)(\xi)|\, d\xi \right) ds \le 1, \tag{2.13}$$

where $\widetilde{\ell} \in \mathcal{L}_{ab}$ is defined by

$$\widetilde{\ell}(v)(t) \overset{def}{=} \ell(\widetilde{\theta}(v))(t) - \ell(1)(t)\widetilde{\theta}(v)(t),$$

$$\widetilde{\theta}(v)(t) \overset{def}{=} \int_a^b \ell(\widetilde{v})(s)\,ds, \quad \widetilde{v}(t) \overset{def}{=} v(t)\exp\left[\int_a^t \ell(1)(s)\,ds\right].$$

Then $\ell \in \widetilde{V}_{ab}^{+}(h)$.

Remark 2.5. Constant 1 on the right hand site of the condition (2.13) is the best possible (see Theorem 2.3 and [4, Example 4.4]).

Theorem 2.4. *Let an operator* $\ell \in \mathcal{L}_{ab}$ *admit the representation* $\ell = \ell_0 - \ell_1$*, where* $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ *and*

$$\ell_0 \in \widetilde{V}_{ab}^{+}(h), \qquad -\ell_1 \in \widetilde{V}_{ab}^{+}(h). \tag{2.14}$$

Then $\ell \in \widetilde{V}_{ab}^{+}(h)$.

3. Proofs of main results

First of all we will prove the following Lemma.

Lemma 3.1. *Let* $\ell \in \mathcal{P}_{ab}$*, the inequality* (1.5) *be fulfilled and let there do not exist a nontrivial function* $v \in \widetilde{C}([a,b];\mathbb{R}_+)$ *satisfying*

$$v'(t) \le \ell(v)(t) \quad \textit{for} \quad t \in [a,b], \quad v(a) = h(v). \tag{3.1}$$

Then $\ell \in \widetilde{V}_{ab}^{+}(h)$.

Proof. Let $u \in \widetilde{C}([a,b];\mathbb{R})$ satisfy (1.3) and (1.4). Evidently, (1.1) and (1.2) hold, as well, where

$$q(t) \overset{\text{def}}{=} u'(t) - \ell(u)(t) \ \text{ for } \quad t \in [a,b], \quad c \overset{\text{def}}{=} u(a) - h(u).$$

It is also evident that

$$q(t) \ge 0 \quad \text{for} \quad t \in [a,b], \quad c \ge 0. \tag{3.2}$$

Taking into account (1.1), (1.2), (3.2) and the assumption $\ell \in \mathcal{P}_{ab}$, we easily get

$$\begin{aligned}[u(t)]_-' &= \frac{1}{2}\Big(u'(t)\operatorname{sgn} u(t) - u'(t)\Big) = \frac{1}{2}\Big(\ell(u)(t)\operatorname{sgn} u(t) - \ell(u)(t)\Big) + \\ &+ \frac{1}{2}q(t)\Big(\operatorname{sgn} u(t) - 1\Big) \le \ell([u]_-)(t) \quad \text{for} \quad t \in [a,b],\end{aligned} \tag{3.3}$$

$$\begin{aligned}[u(a)]_- = \frac{1}{2}\Big(h(u)\operatorname{sgn} u(a) - h(u)\Big) + \frac{1}{2}c\Big(\operatorname{sgn} u(a) - 1\Big) \le \\ \le h([u]_-).\end{aligned} \tag{3.4}$$

Put

$$c_0 = \Big(1 - h(1)\Big)^{-1}\Big(h([u]_-) - [u(a)]_-\Big), \tag{3.5}$$

$$v(t) = [u(t)]_- + c_0 \quad \text{for} \quad t \in [a,b]. \tag{3.6}$$

On account of (1.5) and (3.4) we have

$$c_0 \geq 0. \tag{3.7}$$

On the other hand by virtue of (3.3), (3.5) and (3.7) we find that the function v satisfy (3.1). Taking now into account the assumption of lemma, (3.6) and (3.7) we get $[u]_- \equiv 0$. Thus, $u(t) \geq 0$ for $t \in [a,b]$. $\square$

Proof of Theorem 2.1. Let $\ell \in \widetilde{V}^+_{ab}(h)$. Then, according to Remark 1.1, the problem

$$\gamma'(t) = \ell(\gamma)(t), \tag{3.8}$$

$$\gamma(a) = h(\gamma) + 1 \tag{3.9}$$

has a unique solution γ and

$$\gamma(t) \geq 0 \quad \text{for} \quad t \in [a,b]. \tag{3.10}$$

It follows from (3.9), by virtue of (3.10) and the condition $h \in \mathcal{P}F_{ab}$, that

$$\gamma(a) > 1. \tag{3.11}$$

Now, on account of (3.10), (3.11), and the assumption $\ell \in \mathcal{P}_{ab}$, the equality (3.8) yields

$$\gamma(t) = \gamma(a) + \int_a^t \ell(\gamma)(s)\, ds \geq \gamma(a) > 0 \quad \text{for} \quad t \in [a,b].$$

Therefore, $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$. Clearly, (2.1) and (2.2) hold, as well.

Assume now that there exists a function $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ satisfying (2.1) and (2.2). According to Lemma 3.1 it is sufficient to show that there does not exist a nontrivial function $v \in \widetilde{C}([a,b];\mathbb{R}_+)$ satisfying (3.1). Assume the contrary, i.e., let there exist a nontrivial function $v \in \widetilde{C}([a,b];\mathbb{R}_+)$ satisfying (3.1).

Put

$$w(t) = \lambda\gamma(t) - v(t) \quad \text{for} \quad t \in [a,b],$$

where

$$\lambda = \max\left\{\frac{v(t)}{\gamma(t)} : t \in [a,b]\right\}.$$

Obviously,

$$\lambda > 0. \tag{3.12}$$

It is also evident that

$$w(t) \geq 0 \quad \text{for} \quad t \in [a,b]. \tag{3.13}$$

On account of (2.2), (3.12), (3.13), and the condition $h \in \mathcal{P}F_{ab}$ we have

$$w(a) = \lambda\gamma(a) - v(a) > h(w) \geq 0. \tag{3.14}$$

Taking into account (3.14), from the definition of number λ, it follows that there exists $t_0 \in]a, b]$ such that

$$w(t_0) = 0. \tag{3.15}$$

On the other hand, by virtue of (2.1), (3.1), (3.12), (3.13) and the condition $\ell \in \mathcal{P}_{ab}$ we get

$$w'(t) \geq \ell(w)(t) \geq 0 \quad \text{for} \quad t \in [a, b],$$

which together with (3.14) contradicts (3.15). $\square$

Proof of Corollary 2.1. From the definition of the function γ it follows that

$$\gamma(a) = 1 \tag{3.16}$$

and

$$\gamma'(t) = \ell(1)(t)\gamma(t) \quad \text{for} \quad t \in [a, b]. \tag{3.17}$$

Since $\ell \in \mathcal{P}_{ab}$ is an a-Volterra operator it is clear that

$$\ell(\gamma)(t) \leq \ell(1)(t)\gamma(t) \quad \text{for} \quad t \in [a, b].$$

Last inequality together with (3.17) yields that (2.1) is fulfilled. On the other hand, from (2.3) and (3.16) it follows that (2.2) holds. Therefore, by virtue of Theorem 2.1, $\ell \in \widetilde{V}_{ab}^{+}(h)$. $\square$

Proof of Corollary 2.2. It can be easily verified that the function

$$\gamma(t) \stackrel{\text{def}}{=} (1 - \alpha)\sum_{j=1}^{k} \rho_j(t) + \sum_{j=k+1}^{m} \rho_j(t) \quad \text{for} \quad t \in [a, b]$$

satisfies the assumptions of Theorem 2.1. $\square$

Proof of Corollary 2.3. According to (2.5) there exists $\varepsilon_0 > 0$ such that

$$\frac{h(\gamma_1) + \varepsilon_0}{1 - h(\gamma_0)}\gamma_0(b) + \gamma_1(b) < 1. \tag{3.18}$$

Put

$$\gamma(t) = \frac{h(\gamma_1) + \varepsilon_0}{1 - h(\gamma_0)}\gamma_0(t) + \gamma_1(t) \quad \text{for} \quad t \in [a, b].$$

Obviously, $\gamma \in \widetilde{C}([a,b];]0,+\infty[)$ and γ is a solution of the problem

$$\gamma'(t) = \ell(1)(t)\gamma(t) + \bar{\ell}(1)(t), \tag{3.19}$$

$$\gamma(a) = h(\gamma) + \varepsilon_0. \tag{3.20}$$

Since $\ell, \bar{\ell} \in \mathcal{P}_{ab}$ and $\gamma(t) > 0$ for $t \in [a,b]$, the equality (3.19) yields $\gamma'(t) \geq 0$ for $t \in [a,b]$. Thus in view of (3.18) we obtain

$$\gamma'(t) \geq \ell(1)(t)\gamma(t) + \bar{\ell}(\gamma)(t) \quad \text{for} \quad t \in [a,b], \quad \gamma(a) > h(\gamma).$$

Consequently, by Theorem 2.1 we find

$$\widetilde{\ell} \in \widetilde{V}_{ab}^{+}(h), \tag{3.21}$$

where

$$\widetilde{\ell}(v)(t) \overset{\text{def}}{=} \ell(1)(t)v(t) + \bar{\ell}(v)(t) \quad \text{for} \quad t \in [a,b]. \tag{3.22}$$

According to Lemma 3.1 it is sufficient to show that the problem (3.1) has no nontrivial nonnegative solution. Let $v \in \widetilde{C}([a,b];\mathbb{R}_+)$ satisfy (3.1). Put

$$w(t) = \theta(v)(t) \quad \text{for} \quad t \in [a,b], \tag{3.23}$$

where θ is defined by (2.7). Obviously,

$$w'(t) = \ell(v)(t) \quad \text{for} \quad t \in [a,b], \tag{3.24}$$

$$w(a) = h(w). \tag{3.25}$$

It follows from (3.1), (3.24), (3.25) that

$$\bar{u}'(t) \geq 0 \quad \text{for} \quad t \in [a,b], \quad \bar{u}(a) \geq h(\bar{u}),$$

where $\bar{u}(t) = w(t) - v(t)$ for $t \in [a,b]$. As it was shown in Remark 1.3, $0 \in \widetilde{V}_{ab}^{+}(h)$ provided $h(1) < 1$. Therefore $\bar{u}(t) \geq 0$ for $t \in [a,b]$, i.e.,

$$0 \leq v(t) \leq w(t) \quad \text{for} \quad t \in [a,b]. \tag{3.26}$$

On the other hand, in view of (2.6), (3.22), (3.23), (3.24), (3.26) and the assumption $\ell, \bar{\ell} \in \mathcal{P}_{ab}$, we get

$$\begin{aligned} w'(t) = \ell(v)(t) \leq \ell(w)(t) = \ell(1)w(t) + \ell(\theta(v))(t) - \\ - \ell(1)(t)\theta(v)(t) \leq \ell(1)(t)w(t) + \bar{\ell}(v)(t) \leq \\ \leq \ell(1)(t)w(t) + \bar{\ell}(w)(t) = \widetilde{\ell}(w)(t) \quad \text{for} \quad t \in [a,b]. \end{aligned} \tag{3.27}$$

Now by (3.21), (3.25), (3.26) and (3.27) we get $w \equiv 0$. Consequently, by virtue of (3.26), $v \equiv 0$, as well . □

Proof of Theorem 2.2. Let $\ell \in \widetilde{V}_{ab}^{+}(h)$. Then, according to Remark 1.1, the problem (3.8), (3.9) has a unique solution γ. Moreover, the inequality (3.10) holds.

By virtue of (3.10) and the condition $h \in \mathcal{P}F_{ab}$, it follows from (3.9) that (2.8) is fulfilled. Clearly (2.1) and (2.2) are fulfilled as well.

Assume now that $\ell \in \widetilde{V}_{ab}^{+}(0)$ and there exists a function $\gamma \in \widetilde{C}([a,b];\mathbb{R})$ satisfying (2.1), (2.2) and (2.8). Suppose that $u \in \widetilde{C}([a,b];\mathbb{R})$ satisfies the inequalities (1.3) and (1.4). First we will show that

$$u(a) \geq 0. \tag{3.28}$$

Assume on the contrary, that

$$u(a) < 0. \tag{3.29}$$

Put

$$w(t) = \gamma(a)u(t) - u(a)\gamma(t) \quad \text{for} \quad t \in [a,b]. \tag{3.30}$$

On account of (1.3), (2.1), (2.8) and (3.29) it is evident that

$$w'(t) \geq \ell(w)(t) \quad \text{for} \quad t \in [a,b], \quad w(a) = 0.$$

Hence, by virtue of the condition $\ell \in \widetilde{V}_{ab}^{+}(0)$, the inequality (3.13) holds. From (3.13) and the condition $h \in \mathcal{P}F_{ab}$ we get

$$h(w) \geq 0. \tag{3.31}$$

On the other hand, it follows from (3.30), by virtue of (1.4), (2.2),(2.8) and (3.29), that

$$h(w) = \gamma(a)h(u) - u(a)h(\gamma) < \gamma(a)h(u) - u(a)\gamma(a) \leq 0,$$

which contradicts (3.31). Therefore, the inequality (3.28) holds. Now, by virtue of (3.28) and the condition $\ell \in \widetilde{V}_{ab}^{+}(0)$, we get that the inequality

$$u(t) \geq 0 \quad \text{for} \quad t \in [a,b]$$

is fulfiled, as well. □

To prove Theorem 2.3 we will need the following Lemma (see [4, Theorem 1.2]).

Lemma 3.2. *Let $-\ell \in \mathcal{P}_{ab}$ be an a-Volterra operator and let there exist $\gamma \in \widetilde{C}([a,b];\mathbb{R}_+)$ satisfying the inequalities*

$$\gamma(t) > 0 \quad \textit{for} \quad t \in [a,b[,$$
$$\gamma'(t) \leq \ell(\gamma)(t) \quad \textit{for} \quad t \in [a,b].$$

Then $\ell \in \widetilde{V}_{ab}^{+}(0)$.

Proof of Theorem 2.3. Let $\ell \in \widetilde{V}_{ab}^{+}(0)$. Then by Remark 1.1 the problem

$$\gamma'(t) = \ell(\gamma)(t), \quad \gamma(a) = 1 \tag{3.32}$$

has a unique solution γ and the inequality (3.10) is fulfilled. From (3.32), by virtue of the inequality (3.10) and the condition $-\ell \in \mathcal{P}_{ab}$, it follows that

$$\gamma(t) \leq 1 \quad \text{for} \quad t \in [a, b].$$

Hence, on account of (2.9) and the condition $h \in \mathcal{P}F_{ab}$, we get

$$h(\gamma) \leq h(1) < 1.$$

Taking now into account (3.32) it is evident that the function γ satisfies (2.1), (2.2) and (2.8). Therefore, according to Theorem 2.2 we get $\ell \in \widetilde{V}_{ab}^{+}(h)$.

Now assume that $\ell \in \widetilde{V}_{ab}^{+}(h)$ and $\ell \notin \widetilde{V}_{ab}^{+}(0)$. Then evidently there exists $u \in \widetilde{C}([a, b]; \mathbb{R})$ such that

$$u'(t) \geq \ell(u)(t) \quad \text{for} \quad t \in [a, b]\ , \ u(a) = c, \tag{3.33}$$

$$u(t_0) < 0, \tag{3.34}$$

where $c > 0$ and $t_0 \in]a, b[$. Denote by u_0 a solution of the problem

$$u_0'(t) = \ell(u_0)(t), \tag{3.35}$$

$$u_0(a) = h(u_0) + 1 \tag{3.36}$$

(see Remark 1.1). Since $\ell \in \widetilde{V}_{ab}^{+}(h)$ we have

$$u_0(t) \geq 0 \quad \text{for} \quad t \in [a, b]. \tag{3.37}$$

Therefore, from (3.36), by virtue of the condition $h \in \mathcal{P}F_{ab}$, it follows that

$$u_0(a) > 0. \tag{3.38}$$

Since $\ell \notin \widetilde{V}_{ab}^{+}(0)$. From Lemma 3.2, on account of (3.37) and (3.38), it follows that there exists $a_0 \in]a, b[$ such that

$$u_0(t) > 0 \quad \text{for} \quad t \in [a, a_0[, \tag{3.39}$$

$$u_0(t) = 0 \quad \text{for} \quad t \in [a_0, b]. \tag{3.40}$$

Denote by ℓ_{a_0} the restriction of the operator ℓ to the space $C([a, a_0]; \mathbb{R})$. By virtue of (3.35), (3.39) and Lemma 3.2 we have $\ell_{a_0} \in \widetilde{V}_{aa_0}^{+}(0)$. It follows from (3.33) and (3.35) that

$$w'(t) \geq \ell_{a_0}(w)(t) \ \text{ for } \ t \in [a, a_0], \quad w(a) = 0, \tag{3.41}$$

where

$$w(t) = u(t) - \frac{c}{u_0(a)} u_0(t) \ \text{ for } \ t \in [a, a_0].$$

On account of condition $\ell \in \widetilde{V}^+_{aa_0}(0)$, the inequalities (3.41) yields that

$$w(t) \geq 0 \quad \text{for} \quad t \in [a, a_0].$$

Therefore,

$$u(t) \geq \frac{c}{u_0(a)} u_0(t) \quad \text{for} \quad t \in [a, a_0].$$

Taking now into account (3.34), (3.39) and (3.40) we conclude that

$$a_0 < t_0 < b. \tag{3.42}$$

Put

$$v(t) = u(t) + (h(u) - c)u_0(t) \quad \text{for} \quad t \in [a, b]. \tag{3.43}$$

It is clear that

$$v'(t) \geq \ell(v)(t) \quad \text{for} \quad t \in [a, b]\ , \ v(a) = h(v).$$

Hence, by virtue of the condition $\ell \in \widetilde{V}^+_{ab}(h)$, we have that the inequality $v(t) \geq 0$ for $t \in [a, b]$ holds. Consequently,

$$v(t_0) \geq 0. \tag{3.44}$$

On the other hand, from (3.43), on account of (3.34), (3.40) and (3.42), it follows that $v(t_0) < 0$, which contradicts (3.44). □

Proofs of Corollaries 2.4–2.6 Corollary 2.4 immediately follows from Theorem 2.3 and Lemma 3.2. Corollaries 2.5 and 2.6 follows from Theorem 2.3 and Theorem 1.3 in [4] resp., Corollary 1.2 in [4]. □

Proof of Theorem 2.4. Let $u \in \widetilde{C}([a, b]; \mathbb{R})$ satisfy (1.3) and (1.4). From Remark 1.1, on account of the condition $-\ell_1 \in \widetilde{V}^+_{ab}(h)$, it follows that the problem

$$v'(t) = -\ell_1(v)(t) - \ell_0([u]_-)(t), \tag{3.45}$$

$$v(a) = h(v) \tag{3.46}$$

has a unique solution v and

$$v(t) \leq 0 \quad \text{for} \ \ t \in [a, b]. \tag{3.47}$$

By virtue of (1.3), (1.4), (3.45), (3.46) and the condition $\ell_0 \in \mathcal{P}_{ab}$ it is clear that

$$w'(t) \geq -\ell_1(w)(t) \quad \text{for} \ \ t \in [a, b]\ , \ w(a) \geq h(w),$$

where

$$w(t) = u(t) - v(t) \quad \text{for} \ \ t \in [a, b].$$

Hence, by virtue of the condition $-\ell_1 \in \widetilde{V}^+_{ab}(h)$, we have

$$u(t) \geq v(t) \quad \text{for} \ \ t \in [a, b].$$

This inequality and (3.47) yields

$$-[u(t)]_- \geq v(t) \quad \text{for} \quad t \in [a,b]. \tag{3.48}$$

Therefore, from (3.45), on account of (3.47), (3.48) and the condition $\ell_1 \in \mathcal{P}_{ab}$, it follows that

$$v'(t) \geq \ell_0(v)(t) - \ell_1(v)(t) \geq \ell_0(v)(t) \quad \text{for} \quad t \in [a,b]. \tag{3.49}$$

By virtue of the condition $\ell_0 \in \widetilde{V}^+_{ab}(h)$, (3.46) and (3.49) yield

$$v(t) \geq 0 \quad \text{for} \quad t \in [a,b].$$

Last inequality and (3.47) result in $v \equiv 0$. Now it follows from (3.48) that $[u]_- \equiv 0$. Consequently, the inequality $u(t) \geq 0$ for $t \in [a,b]$ is fulfilled. □

4. Equations with deviating arguments

In this section we will concretize results of §2 for the case when the operator ℓ have one of the following form

$$\ell(v)(t) \stackrel{def}{=} p(t)v(\tau(t)), \tag{4.1}$$

$$\ell(v)(t) \stackrel{def}{=} -g(t)v(\mu(t)), \tag{4.2}$$

$$\ell(v)(t) \stackrel{def}{=} p(t)v(\tau(t)) - g(t)v(\mu(t)), \tag{4.3}$$

where $p, g \in L([a,b];\mathbb{R}_+)$ and $\tau, \mu : [a,b] \to [a,b]$ are measurable functions.

Theorem 4.1. *Let $p \not\equiv 0$, $h(1) < 1$ and*

$$\sup\left\{\frac{h(\gamma) + (1-h(1))\gamma(t)}{h(\varphi) + (1-h(1))\varphi(t)} : t \in [a,b]\right\} < 1 - \frac{h(\varphi)}{1-h(1)}, \tag{4.4}$$

where

$$\varphi(t) = \int_a^t p(s)\,ds, \quad \gamma(t) = \int_a^t p(s)\left(\int_a^{\tau(t)} p(\xi)\,d\xi\right) ds \quad \textit{for} \quad t \in [a,b]. \tag{4.5}$$

Then the operator ℓ defined by (4.1) belongs to the set $\widetilde{V}^+_{ab}(h)$.

Proof. It follows from (4.4) that there exists $\alpha \in]0,1[$ such that for $t \in [a,b]$

$$\frac{h(\gamma)}{1-h(1)} + \gamma(t) \leq \left(\alpha - \frac{h(\varphi)}{1-h(1)}\right)\left(\frac{h(\varphi)}{1-h(1)} + \varphi(t)\right). \tag{4.6}$$

It is not difficult to verify that

$$\rho_2(t) = \frac{h(\varphi)}{1-h(1)} + \varphi(t) \quad for \quad t \in [a,b], \tag{4.7}$$

$$\rho_3(t) = \left(\frac{h(\varphi)}{1-h(1)}\right)^2 + \frac{h(\gamma)}{1-h(1)} + \frac{h(\varphi)}{1-h(1)}\varphi(t) + \gamma(t) \quad for \quad t \in [a,b], \tag{4.8}$$

On the other hand, (4.6), (4.7) and (4.8) yield that

$$\rho_2(t) \le \alpha\rho_3(t) \quad \text{for} \quad t \in [a,b].$$

Therefore, the assumptions of Corollary 2.2 are fulfilled for $k = 2$ and $m = 3$. □

Theorem 4.2. *Let the inequalities* (2.4) *and* (2.5) *hold, where*

$$\begin{gathered} \gamma_0(t) = exp\left[\int_a^t p(s)\,ds\right] \quad for \quad t \in [a,b], \\ \gamma_1(t) = \int_a^t p(s)\sigma(s)\left(\int_s^{\tau(s)} p(\xi)d\xi\right) exp\left[\int_a^s p(\eta)d\eta\right] ds \quad for \quad t \in [a,b] \end{gathered} \tag{4.9}$$

$$\sigma(t) = \frac{1}{2}\Big(1 + sgn(\tau(t) - t)\Big) \quad for \quad t \in [a,b]. \tag{4.10}$$

Then the operator ℓ defined by (4.1) *belongs to the set $\widetilde{V}_{ab}^{+}(h)$.*

Proof. Let ℓ be an operator defined by (4.1). Put

$$\bar{\ell}(v)(t) \overset{def}{=} p(t)\sigma(t)\int_t^{\tau(t)} p(s)v(\tau(s))\,ds, \tag{4.11}$$

where σ is defined by (4.10). Obviously $\bar{\ell} \in \mathcal{P}_{ab}$ and

$$\begin{aligned} \ell(\theta(v))(t) - \ell(1)(t)\theta(v)(t) = p(t)\int_t^{\tau(t)} p(s)v(\tau(s))\,ds \le \\ \le \bar{\ell}(v)(t) \quad for \quad t \in [a,b], \end{aligned} \tag{4.12}$$

where θ is defined by (2.7). Taking now into account (4.9), (4.10) and (4.11) we get that all the conditions of Corollary 2.3 are fulfilled. Consequently, $\ell \in \widetilde{V}_{ab}^{+}(h)$. □

Corollary 4.1. *Let*

$$\gamma_0(b)h(1) + \gamma_1(b) < 1, \tag{4.13}$$

where γ_0 *and* γ_1 *are defined by* (4.9) *and* (4.10)*. Then the operator* ℓ *defined by* (4.1) *belongs to the set* $\widetilde{V}_{ab}^{+}(h)$*.*

Proof. By virtue of the condition $h \in \mathcal{P}F_{ab}$ and the fact that the functions γ_0 and γ_1 are nondecreasing we get

$$h(\gamma_0) \le \gamma_0(b)h(1), \quad h(\gamma_1) \le \gamma_1(b)h(1). \tag{4.14}$$

It follows from (4.13) that

$$h(\gamma_0) < 1, \quad \gamma_1(b) < 1. \tag{4.15}$$

On account of (4.14) and (4.15) we have

$$\begin{aligned}\gamma_1(b) + \gamma_0(b)h(1) &= \gamma_1(b) + \gamma_0(b)h(1)\big(1 - \gamma_1(b)\big) + \\ &+ h(1)\gamma_1(b)\gamma_0(b) \ge \gamma_1(b) + h(\gamma_0)\big(1 - \gamma_1(b)\big) + \\ &+ \gamma_0(b)h(\gamma_1) = \gamma_1(b)\big(1 - h(\gamma_0)\big) + \gamma_0(b)h(\gamma_1) + h(\gamma_0).\end{aligned}$$

Hence, by virtue of (4.13), we get

$$\gamma_0(b)h(\gamma_1) + \gamma_1(b)\big(1 - h(\gamma_0)\big) < 1 - h(\gamma_0).$$

The last inequality, together with the first inequality in (4.15), yields that (2.5) is fulfilled. Consequently, according to Theorem 4.2, the operator ℓ defined by (4.1) belongs to the set $\widetilde{V}_{ab}^{+}(h)$. □

In the next theorem, we will use the following notation

$$\tau^* = \operatorname{ess\,sup}\{\tau(t) : t \in [a,b]\}.$$

Theorem 4.3. *Let* (1.5) *hold,* $\int_a^{\tau^*} p(s)\,ds \ne 0$ *and*

$$\operatorname{ess\ sup}\left\{ \int_t^{\tau(t)} p(s)\,ds : t \in [a,b] \right\} < \eta^*, \tag{4.16}$$

where

$$\eta^* = \sup\left\{ \frac{1}{\lambda} \ln\left[\frac{\lambda \gamma_0^{\lambda}(\tau^*)}{\gamma_0^{\lambda}(\tau^*) - (1 - h(\gamma_0^{\lambda}))(1 - h(1))^{-1}} \right] : \lambda > 0,\ h(\gamma_0^{\lambda}) < 1 \right\} \tag{4.17}$$

and γ_0 *is a function defined by* (4.9)*. Then the operator* ℓ *defined by* (4.1) *belongs to the set* $\widetilde{V}^+_{ab}(h)$*.*

Proof. It follows from (4.16) and (4.17) that there exist $\varepsilon > 0$ and $\lambda_0 > 0$ such that for $t \in [a, b]$

$$\int_t^{\tau(t)} p(s)\,ds \le \frac{1}{\lambda_0} \ln\left[\frac{\lambda_0 \gamma_0^{\lambda_0}(\tau^*)}{\gamma_0^{\lambda_0}(\tau^*) - (1 - h(\gamma_0^{\lambda_0}) - \varepsilon)(1 - h(1))^{-1}}\right].$$

Hence,

$$\exp\left[\lambda_0 \int_t^{\tau(t)} p(s)\,ds\right] \le \frac{\lambda_0 \gamma_0^{\lambda_0}(\tau^*)}{\gamma_0^{\lambda_0}(\tau^*) - (1 - h(\gamma_0^{\lambda_0}) - \varepsilon)(1 - h(1))^{-1}} \le$$

$$\le \frac{\lambda_0 \gamma_0^{\lambda_0}(\tau(t))}{\gamma_0^{\lambda_0}(\tau(t)) - (1 - h(\gamma_0^{\lambda_0}) - \varepsilon)(1 - h(1))^{-1}} \quad for \quad t \in [a, b].$$

Consequently,

$$\lambda_0 \exp\left[\int_a^t p(s)\,ds\right] \ge \exp\left[\int_a^{\tau(t)} p(s)\,ds\right] - \delta \quad for \quad t \in [a, b], \tag{4.18}$$

where

$$\delta = (1 - h(\gamma_0^{\lambda_0}) - \varepsilon)(1 - h(1))^{-1}. \tag{4.19}$$

Put

$$\gamma(t) = \exp\left[\lambda_0 \int_a^t p(s)\,ds\right] - \delta \quad for \quad t \in [a, b]. \tag{4.20}$$

By virtue of the assumption $h \in \mathcal{P}F_{ab}$ we have $\delta < 1$ and therefore

$$\gamma(t) > 0 \quad for \quad t \in [a, b].$$

On account of (4.19) and (4.20) it is clear, that (2.2) holds. On the other hand, on account of (4.18) and (4.20), the condition (2.1) is fulfilled, as well. Consequently, by virtue of Theorem 2.1, the operator ℓ defined by (4.1) belongs to the set $\widetilde{V}^+_{ab}(h)$. $\square$

Corollary 4.2. *Let* (1.5) *hold and*

$$\text{ess sup}\left\{\int_t^{\tau(t)} p(s)\,ds : t\in[a,b]\right\} < \xi^*, \tag{4.21}$$

where

$$\xi^* = \sup\left\{\frac{\|p\|_L}{x}\ln\left[\frac{xe^x(1-h(1))}{\|p\|_L(e^x-1)}\right] : x>0\right\}. \tag{4.22}$$

Then the operator ℓ *defined by* (4.1) *belongs to the set* $\widetilde{V}_{ab}^{+}(h)$.

Proof. It is not difficult to verify that

$$\begin{aligned}
&\frac{\lambda\gamma_0^\lambda(\tau^*)}{\gamma_0^\lambda(\tau^*)-\big(1-h(\gamma_0^\lambda)\big)\big(1-h(1)\big)^{-1}} \geq \\
\geq\; &\frac{\lambda\gamma_0^\lambda(b)}{\gamma_0^\lambda(b)-\big(1-\gamma_0^\lambda(b)h(1)\big)\big(1-h(1)\big)^{-1}} = \\
&= \frac{\lambda\gamma_0^\lambda(b)\,(1-h(1))}{\gamma_0^\lambda(b)-1} \quad for \quad \lambda>0.
\end{aligned}$$

Hence, $\xi^*\leq\eta^*$ where η^* and ξ^* are defined by (4.17) and (4.22), respectively. Consequently (4.21) yields (4.16) and therefore, by virtue of Theorem 4.3, the operator ℓ defined by (4.1) belongs to the set $\widetilde{V}_{ab}^{+}(h)$. □

Theorem 4.4. *Let* $\mu(t)\leq t$ *for* $t\in[a,b]$, $h(1)<1$ *and let at least one of the following items be fulfilled:*

a)

$$\int_a^b g(s)\,ds \leq 1; \tag{4.23}$$

b)

$$\int_a^b g(s)\left(\int_{\mu(s)}^s g(\xi)\exp\left[\int_{\mu(\xi)}^s g(\eta)\,d\eta\right]d\xi\right)ds \leq 1; \tag{4.24}$$

c) $g \not\equiv 0$ and

$$\text{ess sup}\left\{\int\limits_{\mu(t)}^{t} g(s)\,ds : t \in [a,b]\right\} < \lambda^*, \tag{4.25}$$

where

$$\lambda^* = \sup\left\{\frac{1}{x}\ln\left[x + x\left(\exp\left[x\int\limits_a^b g(s)\,ds\right] - 1\right)^{-1}\right] : x > 0\right\}.$$

Then the operator ℓ defined by (4.2) *belongs to the set $\widetilde{V}_{ab}^{+}(h)$.*

Proof. As it follows from Theorem 1.10 in [4] each of the condition (4.23), (4.24) and (4.25) guarantee that the operator ℓ defined by (4.2) belongs to the set $\widetilde{V}_{ab}^{+}(0)$. Consequently, according to Theorem 2.3, $\ell \in \widetilde{V}_{ab}^{+}(h)$, as well. □

Theorem 4.5. *Let*

$$\mu(t) \le t \quad for \quad t \in [a,b], \quad h(1) < 1$$

and let at least one of the conditions a), b) or c) in Theorem 4.4 hold. Let, moreover, either (4.13) *be fulfilled, where γ_0 and γ_1 are defined by* (4.9) *and* (4.10)*, or* (4.21) *hold, where ξ^* defined by* (4.22)*. Then the operator ℓ defined by* (4.3) *belongs to the set $\widetilde{V}_{ab}^{+}(h)$.*

Proof. Theorem 4.5 immediately follows from Theorem 2.4, Theorem 4.4 and Corollaries 4.1 and 4.2. □

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