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# Stability Results for Cellular Neural Networks with Delays

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*This paper is dedicated to Prof. László Hatvani on the occasion of his 60th birthday.*

**Abstract**

In this paper we give a sufficient condition to imply global asymptotic stability of a delayed cellular neural network of the form

$$\dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f(x_j(t)) + \sum_{j=1}^{n} b_{ij} f(x_j(t-\tau_{ij})) + u_i, \qquad t \geq 0, \quad i = 1, \ldots, n,$$

where $f(t) = \frac{1}{2}(|t+1| - |t-1|)$. In order to prove this stability result we need a sufficient condition which guarantees that the trivial solution of the linear delay system

$$\dot{z}_i(t) = \sum_{j=1}^{n} a_{ij} z_j(t) + \sum_{j=1}^{n} b_{ij} z_j(t-\tau_{ij}), \qquad t \geq 0, \quad i = 1, \ldots, n$$

is asymptotically stable independently of the delays $\tau_{ij}$.

**keywords:** delayed cellular neural networks, global asymptotic stability, M-matrix

## 1 Introduction

The notion of cellular neural networks (CNNs) was introduced by Chua and Yang ([5]), and since then, CNN models have been used in many engineering applications, e.g., in signal processing and especially in static image treatment [6]. As a generalization of CNNs, cellular neural networks with delays (DCNNs) were introduced by Roska and Chua [14].

In this paper we study the asymptotic stability of the DCNN model described by the system of nonlinear delay differential equations

$$\dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f(x_j(t)) + \sum_{j=1}^{n} b_{ij} f(x_j(t-\tau_{ij})) + u_i, \qquad t \geq 0, \quad i = 1, \ldots, n. \quad (1.1)$$

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Here $n$ is the number of cells; $x_i(t)$ denotes the potential of the $i$th cell at time $t$; $d_i$ represents the rate with which the $i$th unit resets its potential to the resting state when it is isolated from other cells and inputs; $a_{ij}$ and $b_{ij}$ denote the strengths of the $j$th unit on the $i$th unit at time $t$ and $t-\tau_{ij}$, respectively; $\tau_{ij}$ corresponds to transmission delay between the $i$th and $j$th cells; $f$ denotes an output function; $u_i$ is an external input to the $i$th cell.

The stability of (1.1) and more general classes of DCNNs has been intensively studied, see, e.g., [2]–[4], [11]–[13], [15]–[18], and the references therein. We will assume throughout this paper that the output function $f\colon \mathbb{R}\to\mathbb{R}$ is defined by

$$f(t)=\frac{1}{2}(|t+1|-|t-1|)=\begin{cases}1, & t>1,\\ t, & -1\le t\le 1,\\ -1, & t<-1.\end{cases}\tag{1.2}$$

This function is widely used in CNN and DCNN models.

In a recent paper Mohamad and Gopalsamy ([13]) have shown using fixed point method that if $f$ is defined by (1.2) and

$$d_i>\sum_{j=1}^{n}(|a_{ij}|+|b_{ij}|),\qquad i=1,2,\dots,n,\tag{1.3}$$

then (1.1) has a unique fixed point which is globally exponentially stable. In our Theorem 4 (see below) we show that the weaker assumption

$$d_i-a_{ii}>\sum_{\substack{j=1,\\ j\ne i}}^{n}|a_{ij}|+\sum_{j=1}^{n}|b_{ij}|,\qquad i=1,2,\dots,n,\tag{1.4}$$

together with another condition (see (3.11) below) implies the global asymptotic stability of the unique equilibrium of (1.1). We also conjecture (see Conjecture 1 below) that assumption (3.11) can be omitted, (1.4) itself, or even a weaker condition implies the global asymptotic stability of the equilibrium.

We remark that condition (1.4) is equivalent to saying that the matrix $K=(k_{ij})$ with elements

$$k_{ij}=\begin{cases}d_i-a_{ii}-|b_{ii}|, & \text{if } i=j,\\ -|a_{ij}|-|b_{ij}| & \text{otherwise}\end{cases}$$

is diagonally dominant and it has positive diagonal elements. We recall that an $n\times n$ matrix $K=(k_{ij})$ is (row) diagonally dominant, if

$$|k_{ii}|>\sum_{\substack{j=1,\\ j\ne i}}^{n}|k_{ij}|,\qquad i=1,\dots,n.$$

Our condition (1.4) is similar to that given by Takahashi in [15], where it was shown that if $d_1=d_2=\dots=d_n=1$ and the $n\times n$ matrix $W=(w_{ij})$ with elements

$$w_{ij}=\begin{cases}a_{ii}-1-|b_{ii}|, & \text{if } i=j,\\ -|a_{ij}|-|b_{ij}| & \text{otherwise}\end{cases}$$

is a nonsingular M-matrix (see definition below), then every solution of (1.1) tends to a constant equilibrium, i.e., the system is completely stable. Clearly, condition (1.4) implies that $d_i - a_{ii} > |b_{ii}|$, so in this case $W$ can not be an M-matrix. Similarly, if $W$ is an M-matrix, then (1.4) can not hold, therefore the two conditions cover disjoint cases. We comment that despite the similarities of the two conditions, the proof of our result requires a different technique than that used in [15]. Our results were motivated by the monotone technique we used in [9], where we studied the scalar version of (1.1) with $f$ defined by (1.2), and showed that the scalar version of (1.4) implies the global asymptotic stability of the unique equilibrium.

In Section 2 we give a sufficient condition which implies asymptotic stability of a linear delay system for all delays. Such stability is called absolute stability in the engineering literature. We extend a known result [3] for the case we use in Section 3 to prove our stability results for (1.1). In Section 4 we give an example to illustrate the main result and we formulate a conjecture to generalize the result.

First we introduce some notations. Let $\mathbb{R}_+$ be the set of positive real numbers. We use the relation $\mathbf{x} \le \mathbf{y}$ ($\mathbf{x} < \mathbf{y}$, respectively) for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, if $x_i \le y_i$ ($x_i < y_i$, respectively) for all $i = 1, \dots, n$, where $\mathbf{x} = (x_1, \dots, x_n)^T$ and $\mathbf{y} = (y_1, \dots, y_n)^T$. We introduce the vectors $\mathbf{0} = (0, 0, \dots, 0)^T \in \mathbb{R}^n$ and $\mathbf{1} = (1, 1, \dots, 1)^T \in \mathbb{R}^n$.

For an $n \times n$ matrix $B$ the symbol $|B|$ denotes the corresponding $n \times n$ matrix with $ij$th element $|b_{ij}|$. Similarly, $|\mathbf{u}| = (|x_1|, \dots, |x_n|)^T$.

We say that an $n \times n$ matrix $K$ is an M-matrix, if all of its diagonal elements are nonnegative, and its off-diagonal elements are nonpositive, and all of its principal minors are nonnegative (see, e.g., [1], [3] or [7]). It is known (see, e.g., [1]) that if $K$ is a nonsingular M-matrix, then $\mathbf{x} \le \mathbf{y}$ implies $K^{-1}\mathbf{x} \le K^{-1}\mathbf{y}$.

**Remark 1** Let $K$ be a matrix such that the diagonal elements of $K$ are all positive and the off-diagonal elements are all nonpositive. Then it is known (see, e.g., Theorem 2.3 in [1]) that if $K$ is a diagonally dominant, then it is a nonsingular M-matrix, as well. Moreover, $K$ is a nonsingular M-matrix, if and only if, there exists a positive diagonal matrix $D$ such that $KD$ is a diagonally dominant matrix. We note that there are 50 conditions listed in [1] which are all equivalent to that a matrix is a nonsingular M-matrix.

## 2 Absolute Stability of a Linear System

Consider the autonomous linear delay system

$$\dot{z}_i(t) = \sum_{j=1}^{n} a_{ij} z_j(t) + \sum_{j=1}^{n} b_{ij} z_j(t - \tau_{ij}), \qquad t \ge 0, \quad i = 1, \dots, n, \tag{2.1}$$

where $\tau_{ij} \ge 0$ for $i, j = 1, \dots, n$.

We put the coefficients to the $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$. For the matrix $A$ we associate the $n \times n$ diagonal matrix $A_0 = \mathrm{diag}(a_{11}, a_{22}, \dots, a_{nn})$, i.e., the diagonal part of

$A$, and let $A_1 = A - A_0$ be the off-diagonal part of $A$. Then with this notation, which we use throughout this paper, we can rewrite $A$ as $A = A_0 + A_1$. Similarly, let $B_0$ be the diagonal part of $B$, and denote $B_1 = B - B_0$.

In the case when $A_1 = 0$ and $B_0 = 0$ the necessary and sufficient condition for the stability and asymptotic stability of (2.1) for all selection of the delays $\tau_{ij}$ was established in [10]. Following the methods of [10] this result was extended in [3] for the special case when only $A_1 = 0$, i.e., $A$ is a diagonal matrix in (2.1), and $B$ is an arbitrary matrix.

**Theorem 1 (see Theorem 2.6 in [3])** *Suppose $A = A_0$. Then the trivial solution of* (2.1) *is asymptotically stable for all delays $\tau_{ij} \geq 0$, if and only if $-A - |B|$ is an M-matrix and $A + B$ is a nonsingular matrix.*

Note that in the case when $B$ is a nonnegative matrix, this result follows from a more general theorem in [7], where such result was proved for quasilinear delay differential equations. In the case when $B$ is a nonnegative matrix, Theorem 1 also follows from an other generalization of it given in [8], where it was shown that if $\tau_k \geq 0$, $(k = 1, \dots, p)$, $D_k \geq 0$ are diagonal matrices for $k = 1, \dots, p$ such that $\sum_{k=1}^{p} D_k$ is invertible, $B_\ell$ are nonnegative $n \times n$ matrices for $\ell = 1, \dots, r$, and equation

$$\dot{\mathbf{u}}(t) = -\sum_{k=1}^{p} D_k \mathbf{u}(t - \tau_k)$$

has a positive fundamental solution, then the trivial solution of

$$\dot{\mathbf{x}}(t) = -\sum_{k=1}^{p} D_k \mathbf{x}(t - \tau_k) + \sum_{\ell=1}^{r} B_\ell \mathbf{x}(t - \sigma_\ell)$$

is asymptotically stable for all $\sigma_1, \dots, \sigma_\ell \geq 0$, if and only if

$$\sum_{k=1}^{p} D_k - \sum_{\ell=1}^{r} B_\ell$$

is a nonsingular M-matrix.

We extend the sufficient part of Theorem 1 for the case which we will need later. We assume $A \neq A_0$ , i.e., there are nonzero off-diagonal parts of $A$. The proof follows that of Theorem 1 (see [3]).

**Theorem 2** *Suppose $-A_0 - |A_1| - |B|$ is a nonsingular M-matrix. Then the trivial solution of* (2.1) *is asymptotically stable for all delays $\tau_{ij} \geq 0$.*

**Proof** Finding the solution of (2.1) in the form $e^{\lambda t}\mathbf{v}$ ($\mathbf{v} \neq 0$) leads to the characteristic equation

$$\det \begin{pmatrix} a_{11} + b_{11}e^{-\lambda\tau_{11}} - \lambda & a_{12} + b_{12}e^{-\lambda\tau_{12}} & \cdots & a_{1n} + b_{1n}e^{-\lambda\tau_{1n}} \\ a_{21} + b_{21}e^{-\lambda\tau_{21}} & a_{22} + b_{22}e^{-\lambda\tau_{22}} - \lambda & \cdots & a_{2n} + b_{2n}e^{-\lambda\tau_{2n}} \\ \vdots & \vdots & & \vdots \\ a_{n1} + b_{n1}e^{-\lambda\tau_{n1}} & a_{n2} + b_{n2}e^{-\lambda\tau_{n2}} & \cdots & a_{nn} + b_{nn}e^{-\lambda\tau_{nn}} - \lambda \end{pmatrix} = 0 \qquad (2.2)$$

of (2.1). It is known that the asymptotic stability of the trivial solution of (2.1) is equivalent to that all roots of (2.2) have negative real parts. Let $\lambda$ be a root of (2.2), then $\lambda$ is an eigenvalue of the matrix

$$G(\lambda) = \begin{pmatrix} a_{11} + b_{11}e^{-\lambda\tau_{11}} & a_{12} + b_{12}e^{-\lambda\tau_{12}} & \cdots & a_{1n} + b_{1n}e^{-\lambda\tau_{1n}} \\ a_{21} + b_{21}e^{-\lambda\tau_{21}} & a_{22} + b_{22}e^{-\lambda\tau_{22}} & \cdots & a_{2n} + b_{2n}e^{-\lambda\tau_{2n}} \\ \vdots & \vdots & & \vdots \\ a_{n1} + b_{n1}e^{-\lambda\tau_{n1}} & a_{n2} + b_{n2}e^{-\lambda\tau_{n2}} & \cdots & a_{nn} + b_{nn}e^{-\lambda\tau_{nn}} \end{pmatrix}.$$

Since $-A_0 - |A_1| - |B|$ is a nonsingular M-matrix, it is known (see, e.g., Theorem 2.3 in [1]) there exist positive constants $\gamma_1, \ldots, \gamma_n > 0$ such that

$$(-a_{ii} - |b_{ii}|)\gamma_i > \sum_{\substack{j=1,\\ j\neq i}}^{n} (|a_{ij}| + |b_{ij}|)\gamma_j, \qquad i = 1, \ldots, n. \qquad (2.3)$$

Let $\Gamma = \operatorname{diag}(\gamma_1, \ldots, \gamma_n)$. Then $\Gamma$ is nonsingular, therefore $\lambda$ is an eigenvalue of the matrix $\Gamma^{-1}G(\lambda)\Gamma$, as well. Therefore an application of Gersgorin's theorem for the matrix $\Gamma^{-1}G(\lambda)\Gamma$ yields

$$|\lambda - a_{ii} - b_{ii}e^{-\lambda\tau_{ii}}| \leq \sum_{\substack{j=1,\\ j\neq i}}^{n} \gamma_i^{-1}(|a_{ij}| + |b_{ij}||e^{-\lambda\tau_{ij}}|)\gamma_j$$

for some $i$. Therefore for this fixed $i$

$$\operatorname{Re}(\lambda) \leq \operatorname{Re}(a_{ii} + b_{ii}e^{-\lambda\tau_{ii}}) + \sum_{\substack{j=1,\\ j\neq i}}^{n} \gamma_i^{-1}(|a_{ij}| + |b_{ij}|e^{-(\operatorname{Re}\lambda)\tau_{ij}})\gamma_j.$$

Suppose $\operatorname{Re}(\lambda) \geq 0$. Then (2.3) yields

$$\operatorname{Re}(\lambda)\gamma_i \leq (a_{ii} + |b_{ii}|)\gamma_i + \sum_{\substack{j=1,\\ j\neq i}}^{n} (|a_{ij}| + |b_{ij}|)\gamma_j < 0,$$

which contradicts to the assumption, therefore $\operatorname{Re}(\lambda) < 0$ for all solutions of (2.2). □

The proof implies immediately the next technical result.

**Corollary 3** *If* $-A_0 - |A_1| - |B|$ *is a nonsingular M-matrix, then* $A + B$ *is nonsingular, as well.*

**Proof** Let $A$ and $B$ satisfy the assumption, pick any $\tau_{ij} \geq 0$ $(i, j = 1, \ldots, n)$, and consider the corresponding system (2.1). The proof of Theorem 2 shows that $\mathbf{v}$ is a nonzero constant solution of system (2.1) if and only if $\lambda = 0$ is a solution of (2.2). But under this assumption all solutions of (2.2) satisfy $\mathrm{Re}(\lambda) < 0$, therefore the only constant solution of (2.1) is the zero solution. On the other hand, the constant $\mathbf{v}$ solutions of (2.1) satisfy $(A + B)\mathbf{v} = \mathbf{0}$, hence $A + B$ is nonsingular. □

# 3 Stability of a Delayed Neural Network System

Suppose $n$ is a fixed positive integer,

$$d_i > 0,\ \tau_{ij} \geq 0, \quad a_{ij}, b_{ij}, u_i \in \mathbb{R}\ (i, j = 1, \ldots, n), \quad \text{and} \quad f(t) = \frac{1}{2}(|t+1| - |t-1|). \tag{3.1}$$

We introduce the notations $D = \mathrm{diag}(d_1, \ldots, d_n)$, $A = (a_{ij})$, $B = (b_{ij})$, $\mathbf{u} = (u_1, \ldots, u_n)^T$. As in the previous section, we use the notation $A = A_0 + A_1$, where $A_0$ is the diagonal part, $A_1$ is the off-diagonal part of $A$.

Consider the DCNN model equations

$$\dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f(x_j(t)) + \sum_{j=1}^{n} b_{ij} f(x_j(t - \tau_{ij})) + u_i, \qquad t \geq 0, \quad i = 1, \ldots, n \tag{3.2}$$

with the initial conditions

$$x_i(t) = \varphi_i(t), \qquad t \in [-r, 0], \quad i = 1, \ldots, n, \tag{3.3}$$

where $r = \max\{\tau_{ij} \colon i, j = 1, \ldots, n\}$.

To (3.2) we associate an auxiliary system. For a given $\mathbf{c} > \mathbf{0}$ and $\psi_i \colon [-r, 0] \to \mathbb{R}_+$ $(i = 1, \ldots, n)$ consider the system

$$\dot{y}_i(t) = -d_i y_i(t) + a_{ii} f(y_i(t)) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| f(y_j(t)) + \sum_{j=1}^{n} |b_{ij}| f(y_j(t - \tau_{ij})) + c_i, \ \ t \geq 0,\ i = 1, \ldots, n \tag{3.4}$$

associated to (3.2), and the initial condition

$$y_i(t) = \psi_i(t) \qquad t \in [-r, 0], \quad i = 1, \ldots, n. \tag{3.5}$$

**Lemma 1** *Suppose* (3.1)*. Let* $\psi_i\colon [-r,0] \to \mathbb{R}_+$ *(*$i = 1,\dots,n$*),* $\mathbf{c} > \mathbf{0}$*, and let* $y_1,\dots,y_n$ *be the corresponding solution of* (3.4)-(3.5)*. Then there exists* $M > 0$ *such that*

$$0 < y_i(t) < M, \qquad t \ge 0, \quad i = 1,\dots,n.$$

**Proof** Since $y_i(0) > 0$ and $y_i$ is continuous on $[0,\infty)$ for all $i = 1,\dots,n$, $y_i(t) > 0$ for small enough $t \ge 0$. Suppose there exists $i$ and $T > 0$ such that

$$y_j(t) > 0 \quad \text{for } t \in [-r,T),\ j = 1,\dots,n, \quad \text{and} \quad y_i(T) = 0.$$

Then $\dot y_i(T-) \le 0$. On the other hand, (3.4) implies

$$\dot y_i(T) = \sum_{\substack{j=1,\\ j\ne i}}^{n} |a_{ij}| f(y_j(T)) + \sum_{j=1}^{n} |b_{ij}| f(y_j(T-\tau_{ij})) + c_i > 0,$$

which is a contradiction. Therefore $y_i(t) > 0$ for all $t > 0$ and $i = 1,\dots,n$.

Fix $i$. To prove that $y_i$ is bounded from above, assume that $\limsup_{t\to\infty} y_i(t) = \infty$. Then there exists a monotone increasing sequence $t_n$ such that

$$\lim_{n\to\infty} t_n = \infty, \qquad \lim_{n\to\infty} y_i(t_n) = \infty, \qquad \text{and} \qquad y_i(t_n) = \max\{y_i(t)\colon\ t \in [-r,t_n]\}.$$

Then $\dot y_i(t_n-) \ge 0$, which contradicts to the relations

$$\begin{aligned}
\dot y_i(t_n) &= -d_i y_i(t_n) + a_{ii} f(y_i(t_n)) + \sum_{\substack{j=1,\\ j\ne i}}^{n} |a_{ij}| f(y_j(T)) + \sum_{j=1}^{n} |b_{ij}| f(y_j(t_n-\tau_i)) + c_i \\
&\le -d_i y_i(t_n) + \sum_{j=1}^{n} |a_{ij}| + \sum_{j=1}^{n} |b_{ij}| + c_i \\
&< 0
\end{aligned}$$

for large enough $n$. □

**Remark 2** It is easy to check that the matrix $D - A_0 - |A_1| - |B|$ is a diagonally dominant matrix with positive diagonal elements, if and only if

$$\mathbf{0} < (D - A_0 - |A_1| - |B|)\mathbf{1}.$$

**Lemma 3** *Assume* (3.1)*,* $D - A_0 - |A_1| - |B|$ *is a diagonally dominant matrix, and*

$$\mathbf{0} < \mathbf{c} < (D - A_0 - |A_1| - |B|)\mathbf{1}. \tag{3.6}$$

*Let* $\psi_i\colon [-r,0] \to \mathbb{R}_+$ *(*$i = 1,\dots,n$*), and let* $\mathbf{y}(t) = (y_1(t),\dots,y_n(t))^T$ *be the corresponding solution of* (3.4)-(3.5)*. Then*

$$\lim_{t\to\infty} \mathbf{y}(t) = (D - A_0 - |A_1| - |B|)^{-1}\mathbf{c} < \mathbf{1}. \tag{3.7}$$

**Proof** It follows from Lemma 1 that

$$M_i = \limsup_{t\to\infty} y_i(t) \qquad m_i = \liminf_{t\to\infty} y_i(t)$$

are finite and $m_i \geq 0$. For a fixed $i$ there exists a sequence $t_n$ such that

$$t_n \to \infty \quad \text{as } n \to \infty, \qquad \dot{y}_i(t_n) \geq 0,\ n = 1, 2 \dots, \qquad \text{and} \qquad \lim_{n\to\infty} y_i(t_n) = M_i.$$

We may also assume that

$$\lim_{n\to\infty} y_j(t_n) = m_j^* \qquad \text{and} \qquad \lim_{n\to\infty} y_j(t_n - \tau_{ij}) = m_{ij}^{**}$$

for all $j = 1, \dots, n$ for some $m_j^*, m_{ij}^{**} \in [m_j, M_j]$, since otherwise we can select a subsequence of $t_n$ with this property. Then

$$\begin{aligned}
0 &\leq \lim_{n\to\infty} \dot{y}_i(t_n) \\
&= \lim_{n\to\infty} \Big( -d_i y_i(t_n) + a_{ii} f(y_i(t_n)) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| f(y_j(t_n)) + \sum_{j=1}^{n} |b_{ij}| f(y_i(t_n - \tau_{ij})) + c_i \Big) \\
&= -d_i M_i + a_{ii} f(M_i) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| f(m_j^*) + \sum_{j=1}^{n} |b_{ij}| f(m_{ij}^{**}) + c_i \\
&\leq -d_i M_i + a_{ii} f(M_i) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| f(M_j) + \sum_{j=1}^{n} |b_{ij}| f(M_j) + c_i.
\end{aligned}$$

Therefore for all $i = 1, \dots, n$

$$\begin{aligned}
c_i &\geq d_i M_i - a_{ii} f(M_i) - \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| f(M_j) - \sum_{j=1}^{n} |b_{ij}| f(M_j) \\
&\geq d_i M_i - a_{ii} f(M_i) - \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| - \sum_{j=1}^{n} |b_{ij}|. \qquad (3.8)
\end{aligned}$$

Suppose $M_i \geq 1$ for some $i$. Then (3.8) implies

$$c_i \geq d_i - a_{ii} - \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| - \sum_{j=1}^{n} |b_{ij}|$$

which contradicts to assumption (3.6), which yields

$$0 < c_i < d_i - a_{ii} - \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| - \sum_{j=1}^{n} |b_{ij}|.$$

Therefore $0 \le M_i < 1$ for all $i = 1, \ldots, n$. This means there exists $t_1 > 0$ such that for $t \ge t_1$ (3.4) is equivalent to the linear system

$$\dot{y}_i(t) = (-d_i + a_{ii})y_i(t) + \sum_{\substack{j=1,\\ j\ne i}}^{n} |a_{ij}|y_j(t) + \sum_{j=1}^{n} |b_{ij}|y_j(t-\tau_{ij}) + c_i, \qquad t \ge t_1. \tag{3.9}$$

Define

$$\mathbf{e} = (D - A_0 - |A_1| - |B|)^{-1}\mathbf{c}.$$

Then $\mathbf{e} = (e_1, \ldots, e_n)^T$ is the unique equilibrium of the system (3.9), and it follows from (3.6) that $0 \le e_i \le M_i < 1$, so $\mathbf{0} \le \mathbf{e} < \mathbf{1}$. Introducing $\mathbf{z}(t) = \mathbf{y}(t) - \mathbf{e}$ we can rewrite (3.9) as

$$\dot{z}_i(t) = (-d_i + a_{ii})z_i(t) + \sum_{\substack{j=1,\\ j\ne i}}^{n} |a_{ij}|z_j(t) + \sum_{j=1}^{n} |b_{ij}|z_j(t-\tau_{ij}), \qquad t \ge t_1. \tag{3.10}$$

Since $D - A_0 - |A_1| - |B|$ is a nonsingular M-matrix by Remark 1, Theorem 2 yields the trivial solution of (3.10) is asymptotically stable (independently of the size of the delays), therefore (3.7) holds. □

**Theorem 4** *Assume* (3.1)*,* $D - A_0 - |A_1| - |B|$ *is a diagonally dominant matrix with positive diagonal elements, and* $\mathbf{u}$ *is such that*

$$|\mathbf{u}| < (D - A_0 - |A_1| - |B|)\mathbf{1}. \tag{3.11}$$

*Then any solution* $x$ *of* (3.2)-(3.3) *satisfies*

$$\lim_{t\to\infty} \mathbf{x}(t) = (D - A - B)^{-1}\mathbf{u}. \tag{3.12}$$

**Proof** Fix any initial functions $\psi_i \colon [-r, 0] \to \mathbb{R}_+$ such that

$$\psi_i(s) > |\varphi_i(s)|, \qquad s \in [-r, 0], \quad i = 1, \ldots, n,$$

and let $\mathbf{c} > |\mathbf{u}|$ be such that $\mathbf{c} < (D - A_0 - |A_1| - |B|)\mathbf{1}$. Let $\mathbf{y}$ denote the solution of the corresponding IVP (3.4)-(3.5). Since $\mathbf{y}(0) > |\mathbf{x}(0)|$, relation $|\mathbf{x}(t)| < \mathbf{y}(t)$ holds for sufficiently small $t > 0$. Suppose there exists $i$ and $T > 0$ such that

$$|x_j(t)| < y_j(t), \qquad t \in [-\tau, T), \quad j = 1, \ldots, n, \qquad \text{and} \quad |x_i(T)| = y_i(T). \tag{3.13}$$

It follows from Lemma 1 that $|x_i(T)| = y_i(T) \ne 0$, therefore $\frac{d}{dt}|x_i(t)|$ exists at $T$, and $\frac{d}{dt}(|x_i(t)|)_{|t=T} = \dot{x}_i(T)\,\mathrm{sign}\, x_i(T)$. Hence

$$\begin{aligned}\frac{d}{dt}&(|x_i(t)|)_{|t=T}\\ &= \Big(-d_i x_i(T) + \sum_{j=1}^{n} a_{ij} f(x_j(T)) + \sum_{j=1}^{n} b_{ij} f(x_j(T-\tau_{ij})) + u_i\Big)\,\mathrm{sign}\, x_i(T)\end{aligned}$$

$$
\begin{aligned}
&= -d_i|x_i(T)| + a_{ii}f(|x_i(T)|) + \sum_{\substack{j=1,\\ j\neq i}}^{n} a_{ij} f(x_j(T)) \operatorname{sign} x_i(T) \\
&\quad + \sum_{j=1}^{n} b_{ij} f(x_j(T-\tau_{ij})) \operatorname{sign} x_i(T) + u_i \operatorname{sign} x_i(T) \\
&< -d_i|x_i(T)| + a_{ii}f(|x_i(T)|) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| f(|x_j(T)|) + \sum_{j=1}^{n} |b_{ij}| f(|x_j(T-\tau_{ij})|) + c_i \\
&\le -d_i y_i(T) + a_{ii} f(y_i(T)) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| f(y_j(T)) + \sum_{j=1}^{n} |b_{ij}| f(y_j(T-\tau_{ij})) + c_i \\
&= \dot{y}_i(T).
\end{aligned}
$$

This contradicts to assumption (3.13), therefore $|x_i(t)| < y_i(t)$ holds for all $t > 0$ and $i = 1, \ldots, n$. Moreover, Lemma 3 yields

$$
\lim_{t\to\infty} \mathbf{y}(t) = (D - A_0 - |A_1| - |B|)^{-1}\mathbf{c} < \mathbf{1}
$$

holds, therefore there exists $t_1 > 0$ such that $|\mathbf{x}(t)| < \mathbf{1}$ for $t \ge t_1$. Then (3.2) is equivalent to

$$
\dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} x_j(t) + \sum_{j=1}^{n} b_{ij} x_j(t-\tau_{ij}) + u_i, \qquad t \ge t_1.
$$

This implies (3.12) using an argument similar to that in the proof of Lemma 3. □

## 4 Examples

To illustrate our results consider the two-dimensional DCNN model equations

$$
\begin{aligned}
\dot{x}_1(t) &= -x_1(t) - 6f(x_1(t)) + f(x_2(t)) - 3f(x_1(t-1)) + f(x_2(t-2)) + u_1 && (4.1) \\
\dot{x}_2(t) &= -x_2(t) - f(x_1(t)) - 3f(x_2(t)) - f(x_1(t-1)) + f(x_2(t-2)) + u_2, && (4.2)
\end{aligned}
$$

where $f$ is defined by (1.2). It is easy to see that

$$
D - A_0 - |A_1| - |B| = \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}
$$

is a diagonally dominant matrix. Therefore Theorem 4 yields that if $|u_1| < 2$ and $|u_2| < 1$ then the trivial solution of this system is asymptotically stable. In Figure 1 we have plotted the two components of the solutions corresponding to $u_1 = -1$ and $u_2 = 0.5$ and to the initial functions

$$
\begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix} = \begin{pmatrix} t+1 \\ -t \end{pmatrix}, \quad \begin{pmatrix} \sin 2t \\ t^2 - 1 \end{pmatrix}, \quad \begin{pmatrix} \cos t + 1 \\ t+2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t^3 - 2 \\ -2\cos t \end{pmatrix}, \qquad (4.3)
$$

respectively.

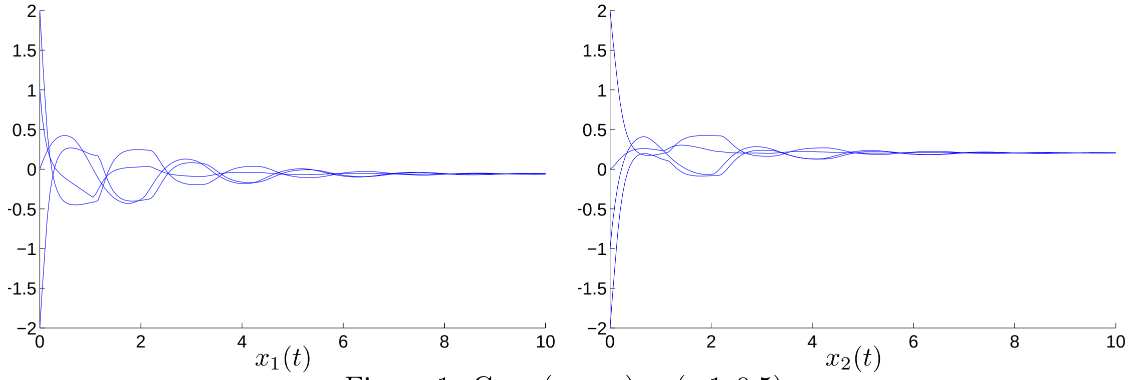


Figure 1. Case $(u_1, u_2) = (-1, 0.5)$.

We can observe that all solutions tend to the unique equilibrium $(-0.058824, 0.20588)^T$.

Note that the condition of Mohamad and Gopalsamy (1.3) is not satisfied for (4.1)-(4.2), and also the condition of Takahashi gives the matrix

$$W = \begin{pmatrix} -7 & -2 \\ -2 & -4 \end{pmatrix},$$

which is not an M-matrix. Therefore none of this two conditions can be applied for system (4.1)-(4.2).

By checking other input values outside the region $|u_1| < 2$ and $|u_2| < 1$ we observed in every cases we tried all solutions tended to the unique equilibrium $(v_1, v_2)^T$ of the system (not necessary satisfying $|v_1|, |v_2| < 1$). In Figure 2 we can see the graphs of solutions of (4.1)-(4.2) corresponding to $(u_1, u_2) = (3, 5)$ and to the initial functions (4.3). We can observe that all solutions tend to the unique equilibrium $(0.5, 2)^T$.

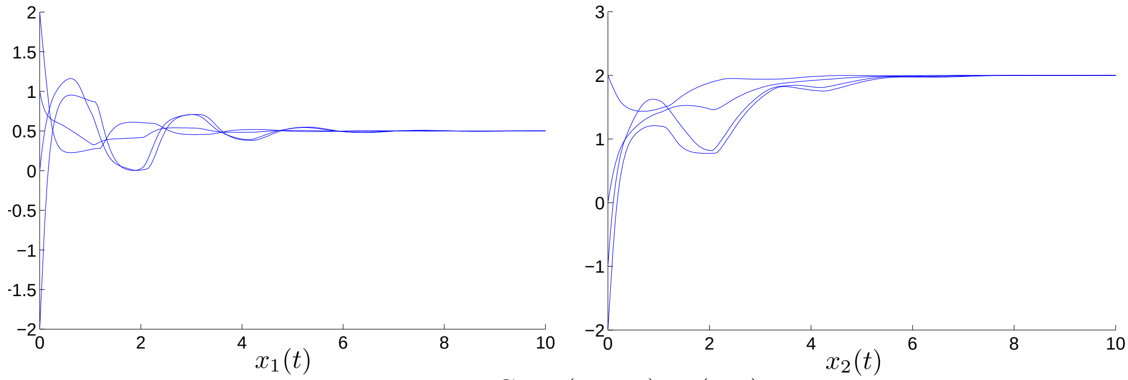


Figure 2. Case $(u_1, u_2) = (3, 5)$.

Next we plotted the solutions corresponding to $(u_1, u_2) = (-8.5, -5.5)$ and to the initial functions (4.3) in Figure 3. Again, all solutions tend to the unique equilibrium $(-1.5, -1.5)^T$.

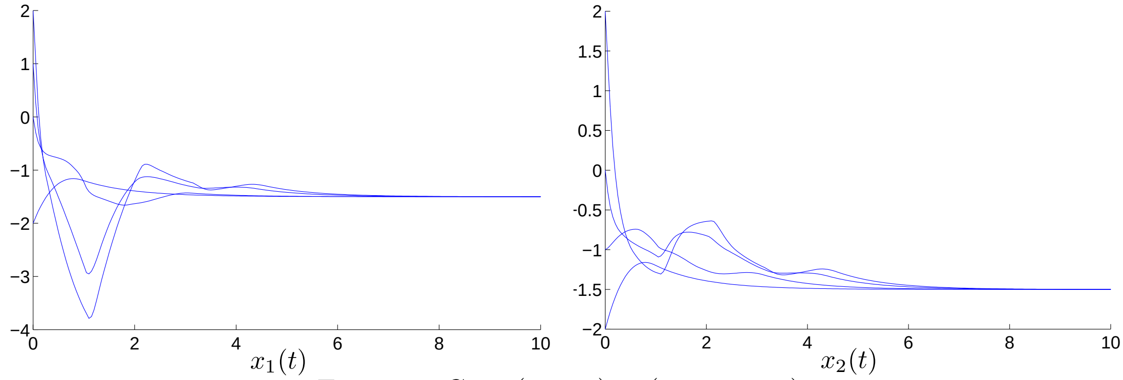


Figure 3. Case $(u_1, u_2) = (-8.5, -5.5)$.

Now change the coefficient of $f(x_2(t-2))$ in (4.1) to 4, i.e., consider the system

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) - 6f(x_1(t)) + f(x_2(t)) - 3f(x_1(t-1)) + 4f(x_2(t-2)) + u_1 \quad (4.4) \\ \dot{x}_2(t) &= -x_2(t) - f(x_1(t)) - 3f(x_2(t)) - f(x_1(t-1)) + f(x_2(t-2)) + u_2. \quad (4.5) \end{aligned}$$

We plotted the solutions corresponding to $(u_1, u_2) = (-6, 4)$ and to the initial functions (4.3) in Figure 4. As before, all solutions tend to the unique equilibrium, which is $(-0.1, 2.2)^T$ in this case. On the other hand,

$$D - A_0 - |A_1| - |B| = \begin{pmatrix} 4 & -5 \\ -2 & 3 \end{pmatrix}$$

is no longer a diagonally dominant matrix, but it is a nonsingular M-matrix.

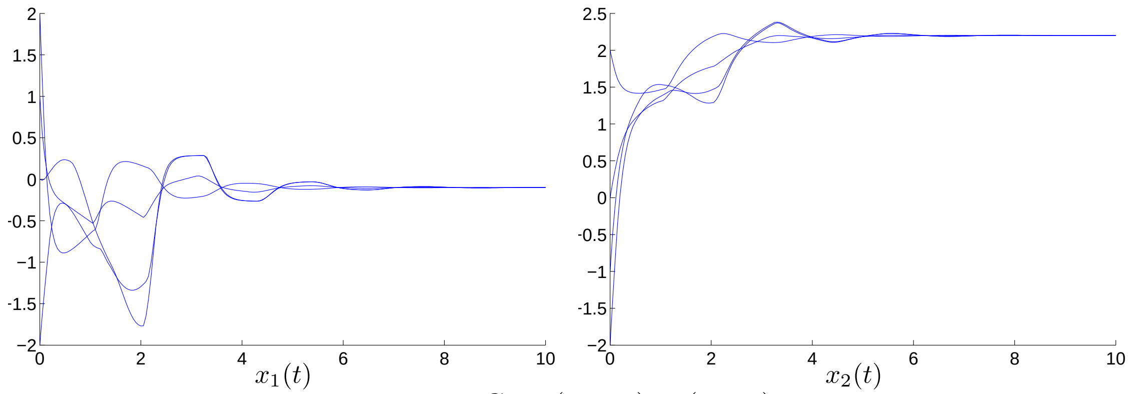


Figure 4. Case $(u_1, u_2) = (-6, 4)$.

Therefore our numerical experiments on these and other systems suggest the following conjecture.

**Conjecture 1** *Assume* (3.1) *and* $D - A_0 - |A_1| - |B|$ *is a nonsingular M-matrix. Then* (3.2) *has a unique equilibrium for any input vector* $\mathbf{u}$, *and any solution of* (3.2) *tends to this equilibrium.*

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