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Oscillation of First Order Neutral Delay Differential Equations

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This paper is dedicated to László Hatvani on the occasion of his sixtieth birthday.

Abstract

In this paper, the authors established some new integral conditions for the oscillation of all solutions of nonlinear first order neutral delay differential equations. Examples are inserted to illustrate the results.

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1 Introduction

Consider the first order nonlinear neutral delay differential equation

$$(x(t)+px(t-\tau))' + q(t)f(x(t-\sigma)) = 0, \tag{1}$$

subject to the conditions

(C_1) $p \in R$, τ, and σ are positive constants;

(C_2) $q : [t_0, \infty) \to \mathbb{R}$ is a continuous function with $q(t) > 0$;

(C_3) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $uf(u) > 0$ for $u \neq 0$, and there is a positive constant M such that $f(u)/u^\alpha \geqslant M > 0$ where α is a ratio of odd positive integers.

If we let $\rho = \max\{\tau, \sigma\}$ and $T \geqslant t_0$, then by a *solution* of equation (1), we mean a continuous function $x : [T-\rho, \infty) \to \mathbb{R}$ such that $x(t)+px(t-\tau)$ is continuously differentiable for $t \geqslant T$, and x satisfies equation (1) for all $t \geqslant T$. A solution of equation (1) is said to be *oscillatory* if it has arbitrarily large zeros and *nonoscillatory* otherwise.

In [4], Gopalsamy, Lalli, and Zhang considered the linear equation

$$(x(t)+px(t-\tau))' + q(t)x(t-\sigma) = 0, \tag{2}$$

where $-1 < p \leqslant 0$ and proved that if

$$\liminf_{t\to\infty} \int_{t-\sigma}^{t} q(s)ds > 1+p,$$

then all solutions of equation (2) are oscillatory. For additional results on the oscillatory behavior of solutions of the linear equation (2), we refer the reader to the monographs by Bainov and Mishev [2], Erbe, Kong, and Zhang [3], and Györi and Ladas [7] as well as the papers of Agarwal and Saker [1], Pahri [13], Saker and Elabbasy [15], Tanaka [16], and Zhou [19] and the references contained therein.

In [5], Graef et al. considered the nonlinear equation (1) with f nondecreasing, sublinear, and $-1 < p \leqslant 0$, and they proved that if

$$\int_{t_0}^{\infty} q(t)dt = \infty,$$

then every solution of equation (1) is oscillatory. They also proved a similar result for equation (1) when f is superlinear and $p < -1$. Mishra [12] considered equation (1) with $-1 < p \leqslant 0$, $\alpha = 1$, and $M = 1$; he proved that if

$$\liminf_{t\to\infty} \int_{t-\sigma}^{t} q(s)ds > \frac{1+p}{e},$$

then all solutions of equation (1) oscillate. In [17], Tanaka considered neutral equations of the form

$$(x(t) + h(t)x(t - \tau))' + q(t)|x(t - \sigma)|^{\gamma} \operatorname{sgn} x(t - \sigma) = 0, \tag{3}$$

where $0 < \gamma < 1$ and $h(t) > 0$, and proved that all solutions of (3) are oscillatory provided

$$\int^{\infty} \min\left\{\frac{q(s)}{1 + (h(s - \sigma + \tau))^{\gamma}}, \frac{q(s - \tau)}{1 + (h(s - \sigma))^{\gamma}}\right\} ds = \infty.$$

Li and Saker [10] considered equation (1) with $-1 < p \leqslant 0$ and $\lim_{u\to 0}[u/f(u)] = \beta > 0$; they proved that if

$$\liminf_{t\to\infty} \int_{t-\sigma}^{t} q(s)ds > \frac{\beta}{e(1 + p)},$$

then every solution of equation (1) oscillates.

Additional results on the oscillatory behavior of solutions of the nonlinear equation (1) can be found in the papers of Jaros and Kusano [8], Li and Saker [11], Mishra [12], and Yilmaz and Zafer [14] as well the monographs [2], [3], and [7]. In reviewing the literature, it becomes apparent that most results concerning the oscillation of all solutions of equation (1) are for the cases $-1 < p \leqslant 0$ or $p < -1$, and far fewer results are known for the situation in which p is positive. Here we wish to develop sufficient conditions for equation (1) to be oscillatory if $p > 1$. Sufficient conditions for all solutions of the first order equation (1), and in fact for odd order equations in general, to oscillate if $p \geq 0$ are somewhat rare. Known results often take the form that any solution is either oscillatory or converges to zero; see, for example, the paper by Graef et al. [6].

In Section 2, we present some basic lemmas that are needed to prove our main results; in Section 3, we give some new integral conditions for the oscillation of all solutions of equation (1). We include examples to illustrate our main theorems.

2 Some Basic Lemmas

In this section, we establish some lemmas for the case $\alpha = 1$. These lemmas will be used to prove our main results.

Lemma 1 *Assume that $\sigma > \tau$, $p \in (1, \infty)$, $\alpha = 1$, and*

$$\limsup_{t\to\infty} \int_{t}^{t+\sigma-\tau} q(s)ds > 0. \tag{4}$$

If $x(t)$ is an eventually positive solution of equation (1), then

$$\liminf_{t\to\infty} \frac{z(t - \sigma + \tau)}{z(t)} < \infty, \tag{5}$$

where $z(t) = x(t) + px(t - \tau)$.

Proof. From our hypotheses, we see that $z(t) > 0$ eventually, and from equation (1), we have that $z(t)$ is decreasing. Then,

$$px(t-\tau) = z(t) - x(t) \tag{6}$$

and

$$z(t+\tau) = x(t+\tau) + px(t).$$

Since $z(t)$ is decreasing, we have

$$z(t) > z(t+\tau) \geq px(t),$$

and so from (6) we obtain

$$p^2 x(t-\tau) \geq pz(t) - z(t).$$

Thus,

$$x(t-\tau) \geq \frac{p-1}{p^2} z(t),$$

or

$$x(t-\sigma) \geq \frac{p-1}{p^2} z(t+\tau-\sigma). \tag{7}$$

From (1) and (7), we have

$$z'(t) + \frac{M(p-1)}{p^2} q(t) z(t+\tau-\sigma) \leqslant 0, \tag{8}$$

and by Lemma 1 in [9], we obtain the desired result.

Lemma 2 *Assume that $\sigma > \tau$, $p \in (1, \infty)$, and $\alpha = 1$. If equation (1) has an eventually positive solution, then*

$$\int\limits_t^{t+\sigma-\tau} q(s)ds \leqslant \frac{p^2}{M(p-1)} \tag{9}$$

for sufficiently large t.

Proof. Proceeding as in the proof of Lemma 1, we again obtain (8). Integrating (8) from t to $t+\sigma-\tau$ and using the decreasing behavior of $z(t)$, we obtain

$$z(t+\sigma-\tau) + \left(\frac{M(p-1)}{p^2} \int_t^{t+\sigma-\tau} q(s)ds - 1 \right) z(t) \leqslant 0. \tag{10}$$

Since $z(t) > 0$ eventually, (10) implies

$$\frac{M(p-1)}{p^2} \int\limits_t^{t+\sigma-\tau} q(s)ds - 1 \leqslant 0 \tag{11}$$

for large t, and the desired result (9) follows from (11).

3 Oscillation Results

In this section, we obtain integral conditions for the oscillation of all solutions of equation (1). We first consider the case $\alpha = 1$.

Theorem 1 *Assume that $\sigma > \tau$, $p \in (1, \infty)$, $\alpha = 1$, and (4) holds. If*

$$\int_{t_0}^{\infty} q(t) \ln \left(\frac{eM(p-1)}{p^2} \int_{t}^{t+\sigma-\tau} q(s)ds \right) dt = \infty, \tag{12}$$

then every solution of equation (1) oscillates.

Proof. For the sake of obtaining a contradiction, assume that there is an eventually positive solution $x(t)$ of equation (1). Then, $z(t)$ is eventually positive and decreasing and satisfies the inequality

$$z'(t) + \frac{M(p-1)}{p^2} q(t) z(t + \tau - \sigma) \leqslant 0. \tag{13}$$

Let $\lambda(t) = -z'(t)/z(t)$; then, $\lambda(t)$ is continuous and nonnegative, so there exists $t_1 \geqslant t_0$ with $z(t_1) > 0$ such that

$$z(t) = z(t_1) \exp \left(- \int_{t_1}^{t} \lambda(s)ds \right).$$

Moreover, $\lambda(t)$ satisfies

$$\lambda(t) \geqslant \frac{M(p-1)}{p^2} q(t) \exp \left(\int_{t+\tau-\sigma}^{t} \lambda(s)ds \right). \tag{14}$$

Applying the inequality

$$e^{rx} \geqslant x + \frac{\ln(er)}{r} \text{ for } x > 0 \text{ and } r > 0,$$

to (14), we have

$$\begin{aligned} \lambda(t) &\geqslant \frac{M(p-1)}{p^2} q(t) \exp \left(A(t) \frac{1}{A(t)} \int_{t+\tau-\sigma}^{t} \lambda(s)ds \right) \\ &\geqslant \frac{M(p-1)}{p^2} q(t) \left[\frac{1}{A(t)} \int_{t+\tau-\sigma}^{t} \lambda(s)ds + \frac{\ln(eA(t))}{A(t)} \right] \end{aligned}$$

where we take

$$A(t) = \frac{M(p-1)}{p^2} \int_{t}^{t+\sigma-\tau} q(s)ds.$$

It follows that

$$\lambda(t) \int\limits_{t}^{t+\sigma-\tau} q(s)ds - q(t) \int\limits_{t+\tau-\sigma}^{t} \lambda(s)ds \geqslant q(t) \ln \left(\frac{eM(p-1)}{p^2} \int\limits_{t}^{t+\sigma-\tau} q(s)ds \right).$$

Then, for $u > T + \sigma - \tau$, we have

$$\int\limits_{T}^{u} \lambda(t) \left(\int\limits_{t}^{t+\sigma-\tau} q(s)ds \right) dt - \int\limits_{T}^{u} q(t) \left(\int\limits_{t+\tau-\sigma}^{t} \lambda(s)ds \right) dt$$
$$\geqslant \int\limits_{T}^{u} q(t) \ln \left(\frac{eM(p-1)}{p^2} \int\limits_{t}^{t+\sigma-\tau} q(s)ds \right) dt. \quad (15)$$

Interchanging the order of integration, we obtain

$$\int\limits_{T}^{u} q(t) \int\limits_{t+\tau-\sigma}^{t} \lambda(s)dsdt \geqslant \int\limits_{T}^{u+\tau-\sigma} \lambda(t) \left(\int\limits_{t}^{t+\sigma-\tau} q(s)ds \right) dt. \quad (16)$$

From (15) and (16), it follows that

$$\int\limits_{u+\tau-\sigma}^{u} \lambda(t) \left(\int\limits_{t}^{t+\sigma-\tau} q(s)ds \right) dt \geqslant \int\limits_{T}^{u} q(t) \ln \left(\frac{eM(p-1)}{p^2} \int\limits_{t}^{t+\sigma-\tau} q(s)ds \right) dt. \quad (17)$$

Using (9) in (17), we have

$$\int\limits_{u+\tau-\sigma}^{u} \lambda(t)dt \geqslant \frac{M(p-1)}{p^2} \int\limits_{T}^{u} q(t) \ln \left(\frac{eM(p-1)}{p^2} \int\limits_{t}^{t+\sigma-\tau} q(s)ds \right) dt$$

or

$$\ln \frac{z(u+\tau-\sigma)}{z(u)} \geqslant \frac{M(p-1)}{p^2} \int\limits_{T}^{u} q(t) \ln \left(\frac{eM(p-1)}{p^2} \int\limits_{t}^{t+\sigma-\tau} q(s)ds \right) dt.$$

In view of (12), we must have

$$\lim_{t\to\infty} \frac{z(t+\tau-\sigma)}{z(t)} = \infty, \quad (18)$$

which contradicts (5) and completes the proof of the theorem.

Example 1 *Consider the neutral differential equation*

$$(x(t) + 2x(t-1))' + \frac{4}{e}\left(1 + \frac{1}{t}\right) x(t-2)(1 + x^2(t-2)) = 0, \quad t \geq 2. \quad (19)$$

Here, $p = 2$, $\tau = 1$, $\sigma = 2$, $q(t) = (4/e)(1 + 1/t)$, and $M = 1$. Clearly,

$$\int_2^\infty q(t) \ln\left(\frac{eM(p-1)}{p^2}\int_t^{t+1} q(s)ds\right) dt \geqslant \frac{4}{e}\int_2^\infty \ln\left(1 + \ln\left(1+\frac{1}{t}\right)\right) dt = \infty.$$

By Theorem 1, every solution of equation (19) oscillates. None of the results given in the references can be applied to equation (19) to yield this conclusion.

In our next theorem, we again consider the case $\alpha = 1$ and obtain a different type of sufficient condition for the oscillation of solutions of equation (1).

Theorem 2 *Assume that $\sigma > \tau$, $p \in (1, \infty)$, $\alpha = 1$, and there exists a constant $k > 0$ such that*

$$\frac{1}{e} \leqslant \int_{t-\sigma+\tau}^{t} q(s)ds < k. \tag{20}$$

Then every solution of equation (1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 1, we see that $z(t)$ is eventually positive, decreasing, and satisfies (13). Moreover, the generalized characteristic equation for (13) is given by

$$\lambda(t) \geqslant \frac{M(p-1)}{p^2} q(t) \exp\left(\int_{t+\tau-\sigma}^{t} \lambda(s)ds\right). \tag{21}$$

If we let $B(t) = \exp\left(e \int_{t+\tau-\sigma}^{t} q(s)ds\right)$, then we can rewrite this inequality as

$$B(t)\lambda(t) \geqslant \frac{M(p-1)}{p^2} B(t)q(t) \exp\left(\frac{B(t)}{B(t)}\int_{t+\tau-\sigma}^{t} \lambda(s)ds\right).$$

Applying the inequality

$$e^{x/r} \geqslant 1 + \frac{x}{r^2} \text{ for } x > 0 \text{ and } r > 1,$$

we obtain

$$B(t)\lambda(t) - q(t)\frac{M(p-1)}{p^2}\int_{t+\tau-\sigma}^{t} \lambda(s)ds \geqslant q(t)A(t),$$

where $A(t) = \frac{M(p-1)}{p^2} B(t)$. Then, for $u > T + \sigma - \tau$,

$$\int_T^u \lambda(t)B(t)dt - \int_T^u q(t)\frac{M(p-1)}{p^2}\left(\int_{t+\tau-\sigma}^{t} \lambda(s)ds\right) dt \geqslant \int_T^u q(t)A(t)dt. \tag{22}$$

Interchanging the order of integration and simplifying, we have

$$\int_T^u q(t) \int_{t+\tau-\sigma}^t \lambda(s)dsdt \geqslant \int_T^{u+\tau-\sigma} \lambda(t) \left(\int_{t+\tau-\sigma}^t q(s)ds \right) dt. \tag{23}$$

From (22) and (23), it follows that

$$\int_T^u \lambda(t)B(t)dt - \frac{M(p-1)}{p^2} \int_T^{u+\tau-\sigma} \lambda(t) \left(\int_{t+\tau-\sigma}^t q(s)ds \right) dt \geqslant \int_T^u q(t)A(t)dt,$$

and so

$$\int_T^u \lambda(t)B(t)dt + \int_{u+\tau-\sigma}^T \lambda(t)B(t)dt \geqslant \int_T^u q(t)A(t)dt \tag{24}$$

since

$$B(t) = \exp\left(e \int_{t+\tau-\sigma}^t q(s)ds \right) \geqslant \int_{t+\tau-\sigma}^t q(s)ds.$$

On the other hand, since

$$e \leqslant B(t) < k_1$$

for some $k_1 > 0$, (24) implies

$$\int_{u+\tau-\sigma}^u \lambda(t)dt \geqslant \frac{1}{k_1} \int_T^u q(t)A(t)dt.$$

Since (20) implies that the integral on the right hand side of the above inequality diverges as $u \to \infty$, the remainder of the proof is similar to that of Theorem 1 and so we omit the details. This completes the proof of the theorem.

Example 2 *Consider the neutral differential equation*

$$(x(t) + 5x(t-1))' + \frac{1}{4}x(t-3)(1 + x^2(t-3)) = 0, \quad t \geqslant 3. \tag{25}$$

Here we have $\tau = 1$, $\sigma = 3$, $q(t) = \frac{1}{4}$, and $M = 1$, and we see that $\frac{1}{e} \leqslant \int_{t-2}^t \frac{1}{4}ds = \frac{1}{2} < k = 1$. Also, $\int_{t_0}^{\infty} q(t) \exp\left(e \int_{t-\sigma+\tau}^t q(s)ds \right) = \int_{t_0}^{\infty} \frac{1}{4}e^{\frac{e}{2}}dt = \infty$. The hypotheses of Theorem 2 are satisfied so every solution of (25) *is oscillatory.*

In our final result, we consider equation (1) with $\alpha > 1$ since the case $0 < \alpha < 1$ has been studied by many other authors. The case $\alpha > 1$ is considered by Graef et al. [5] for $p < -1$. Here, we establish oscillation criteria for equation (1) with $p \in (1, \infty)$.

Theorem 3 *Assume that $\alpha > 1$, $\sigma > \tau$, and $p \in (1, \infty)$. In addition, assume that there exists a continuously differentiable function $\phi(t)$ such that*

$$\phi'(t) > 0, \quad \lim_{t\to\infty} \phi(t) = \infty, \tag{26}$$

$$\limsup_{t\to\infty} \frac{\phi'(t+\tau-\sigma)}{\phi'(t)} < \frac{1}{\alpha}, \tag{27}$$

and

$$\liminf_{t\to\infty} \left[M \left(\frac{p-1}{p^2} \right)^{\alpha} q(t) \frac{e^{-\phi(t)}}{\phi'(t)} \right] > 0. \tag{28}$$

Then every solution of equation (1) *oscillates.*

Proof. Proceeding as in the proof of Theorem 1, we see that $z(t)$ is eventually positive, decreasing, and satisfies the inequality

$$z'(t) + M \left(\frac{p-1}{p^2} \right)^{\alpha} q(t) z^{\alpha}(t+\tau-\sigma) \leqslant 0. \tag{29}$$

From (26) and (27), we see that

$$\limsup_{t\to\infty} \frac{\alpha\phi(t+\tau-\sigma)}{\phi(t)} < 1. \tag{30}$$

Now by (27) and (30), there exist $0 < \ell < 1$, $\varepsilon > 0$, and $T \geqslant t_0$, such that

$$\frac{(1+\varepsilon)\alpha\phi'(t+\tau-\sigma)}{\phi'(t)} \leqslant \ell \text{ and } \frac{(1+\varepsilon)\alpha\phi(t+\tau-\sigma)}{\phi(t)} \leqslant \ell \tag{31}$$

for $t \geqslant T$. In view of (28), we may choose $T_0 > T$ such that

$$M \left(\frac{p-1}{p^2} \right)^{\alpha} q(t) \geqslant \phi'(t) e^{\frac{\alpha\phi(t)}{1+\alpha}} \tag{32}$$

for $t \geqslant T_0$. Now set $p(t) = \phi'(t) e^{\frac{\alpha\phi(t)}{1+\alpha}}$. By Lemma 2 in [18], it suffices to consider the inequality

$$z'(t) + p(t) z^{\alpha}(t+\tau-\sigma) \leqslant 0 \tag{33}$$

instead of (29). In order to see that $z(t) \to 0$ as $t \to \infty$, first observe that $z(t+\tau-\sigma) \geq z(t)$. Hence,

$$z'(t) + p(t) z^{\alpha}(t) \leq z'(t) + p(t) z^{\alpha}(t+\tau-\sigma) \leq 0$$

and so

$$\frac{z'(t)}{z^{\alpha}(t)} \leq -p(t).$$

Integrating, we have

$$[z^{1-\alpha}(t) - z^{1-\alpha}(T)]/(1-\alpha) \to -\infty$$

as $t \to \infty$. This implies $z^{1-\alpha}(t) \to +\infty$ so $z(t) \to 0$. Thus, there exists a $T_1 > T_0$ such that

$$0 < z(t) < 1 \text{ and } z'(t) \leqslant 0$$

for $t \geqslant T_1$. Letting $y(t) = -\ln z(t)$ for $t \geqslant T_2 = T_1 + \sigma - \tau$, we see that $y(t) > 0$ for $t \geqslant T_2$ and (33) implies

$$y'(t) \geqslant p(t)e^{y(t)-\alpha y(t-\sigma+\tau)}$$

for $t \geqslant T_2$. The remainder of the proof is similar to the proof of Theorem 1 in [18] and will be omitted.

We conclude this paper with the following example.

Example 3 *Consider the neutral differential equation*

$$(x(t) + 2x(t-1))' + e^{3t+e^{2t}} x^3(t-2) = 0, \quad t \geqslant 2. \tag{34}$$

Here, $p = 2$, $\sigma = 2$, $\tau = 1$, $\alpha = 3$, $q(t) = e^{3t+e^{2t}}$, and $M = 1$. With $\phi(t) = e^{2t}$, all conditions of Theorem 3 are satisfied and so all solutions of (34) *are oscillatory.*

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