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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SOME FUNCTIONAL DIFFERENTIAL EQUATIONS BY SCHAUDER'S THEOREM

Dedicated to Professor Lásló Hatvani on his 60th birthday

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ABSTRACT. In a series of papers (Burton-Furumochi [1-4]) we have studied stability properties of functional differential equations by means of fixed point theory. Here we obtain new stability and boundedness results for half-linear equations and integro-differential equations by using Schauder's first theorem and a weighted norm, and show some examples.

1. INTRODUCTION

In Part I of Burton-Furumochi [4], we studied asymptotic stability in half-linear equations by using Schauder's first theorem, Ascoli-Arzela like lemma with the concept of equi-convergence, and the uniform norm. In this paper, we generalize the stability results in [4], and obtain boundedness results for some functional differential equations by using Schauder's first theorem and a weighted norm instead of the uniform norm.

Let r_0 be a fixed nonnegative constant and let $h : [-r_0, \infty) \to [1, \infty)$ be any strictly increasing and continuous function with

$$h(-r_0) = 1$$

and

$$h(t) \to \infty \text{ as } t \to \infty.$$

For any $t_0 \in \mathbf{R}^+ := [0, \infty)$ let C_{t_0} be the space of continuous functions $\phi : [t_0 - r_0, \infty) \to \mathbf{R} := (-\infty, \infty)$ with

$$\|\phi\|_h := \sup\left\{ \frac{|\phi(t)|}{h(t - t_0)} : t \geq t_0 - r_0 \right\} < \infty.$$

Then, $\| \cdot \|_h$ is a norm on C_{t_0}, and $(C_{t_0}, \ \| \cdot \|_h)$ is a Banach space.

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First we state a lemma without proof (see Burton [5; p. 169]).

Lemma. If the set $\{\phi_k(t)\}$ of $\mathbf{R}$-valued functions on $[t_0-r_0,\infty)$ is uniformly bounded and equi-continuous, then there is a bounded and continuous function ϕ and a subsequence $\{\phi_{k_j}\}$ such that

$$\left\|\phi_{k_j}-\phi\right\|_h \to 0 \ \text{ as } \ j\to\infty.$$

2. STABILITY IN HALF-LINEAR EQUATIONS

Consider the scalar half-linear equation

$$x'(t) = -a(t)x(t) - b(t)g\big(x(t-r(t))\big), \ \ t\in\mathbf{R}^+, \tag{1}$$

where $a,\ b:\mathbf{R}^+\to\mathbf{R},\ g:\mathbf{R}\to\mathbf{R}$ and $r:\mathbf{R}^+\to\mathbf{R}^+$ are continuous. Let α be any fixed positive number. We assume that there are constants $\beta>0,\ \gamma>0$ and $r_0\geq 0$ so that

$$|g(x)|\leq \beta|x| \ \text{ for } \ x\in\mathbf{R} \ \text{ with } \ |x|\leq\alpha, \tag{2}$$

$$\sup\left\{e^{\int_\tau^t (a(s)-\beta\gamma|b(s)|)ds} : t\in\mathbf{R}^+\right\}\leq\gamma, \tag{3}$$

where $\tau=\tau(t):=\max\big(0,\ t-r(t)\big)$,

$$t-r(t)\geq -r_0, \tag{4}$$

$$\sigma=\sigma(t_0):=\sup\left\{\int_{t_0}^t \big(\beta\gamma|b(s)|-a(s)\big)ds : t\geq t_0\right\}<\infty, \tag{5}$$

and define a number $\eta=\eta(t_0)$ by

$$\eta:=\alpha e^{-\sigma}. \tag{6}$$

Corresponding to Eq. (1), consider the scalar linear equation

$$q'=\big(\beta\gamma|b(t)|-a(t)\big)q, \ \ t\in\mathbf{R}^+. \tag{7}$$

Let $q : [t_0 - r_0, \infty) \to \mathbf{R}^+$ be a continuous function such that

$$q(t) = \eta \text{ on } [t_0 - r_0, t_0],$$

and that $q(t)$ is the unique solution of the initial value problem

$$q' = \big(\beta\gamma|b(t)| - a(t)\big)q, \;\; q(t_0) = \eta, \;\; t \ge t_0.$$

Then $q(t)$ can be expressed in two ways as

$$q(t) = \eta e^{-\int_{t_0}^t a(s)ds} + \beta\gamma \int_{t_0}^t e^{-\int_s^t a(u)du}|b(s)|q(s)ds$$

$$= \eta e^{\int_{t_0}^t \left(\beta\gamma|b(s)|-a(s)\right)ds}, \;\; t \ge t_0, \tag{8}$$

which together with (5) and (6), implies

$$0 < q(t) \le \eta e^{\sigma} = \alpha, \;\; t \ge t_0. \tag{9}$$

Concerning the stability of the zero solution of Eq. (1), we have the following theorem.

Theorem 1. *Suppose that the solutions of* Eq. (1) *are uniquely determined by continuous initial functions, and that* (2)-(5) *hold. Then we have:*
(i) *The zero solution of* Eq. (1) *is stable.*
(ii) *If we have*

$$\sigma^* := \sup\left\{\sigma(t) : t \in \mathbf{R}^+\right\} < \infty,$$

then the zero solution of Eq. (1) *is uniformly stable.*
(iii) *If we have*

$$\int_0^t \big(a(s) - \beta\gamma|b(s)|\big)ds \to \infty \text{ as } t \to \infty, \tag{10}$$

then the zero solution of Eq. (1) *is asymptotically stable.*
(iv) *In addition to $\sigma^* < \infty$, if we have*

$$\int_{t_0}^t \big(a(s) - \beta\gamma|b(s)|\big)ds \to \infty \text{ uniformly for } t_0 \in \mathbf{R}^+ \text{ as } t \to \infty, \tag{11}$$

then the zero solution of Eq. (1) *is uniformly asymptotically stable.*

Proof. (i) Assumption (5) implies that the zero solution of Eq. (7) is stable. Thus, for any $\epsilon \in (0, \alpha]$ and $t_0 \in \mathbf{R}^+$, there is a $\delta = \delta(\epsilon, t_0) \in (0, \eta]$ such that for any q_0 with $|q_0| \le \delta$, we have

$$|q(t,t_0,q_0)| < \epsilon \ \text{ for all } \ t \geq t_0.$$

For the t_0, let $(C_{t_0},\ \|\cdot\|_h)$ be the Banach space of continuous functions $\phi : [t_0 - r_0, \infty) \to \mathbf{R}$ with the weighted norm $\|\cdot\|_h$. For a continuous function $\psi : [t_0 - r_0, \infty) \to \mathbf{R}^+$ with

$$\sup\big\{|\psi(\theta)| : -r_0 \leq \theta \leq 0\big\} \leq \delta,$$

let S be a set of continuous functions $\phi : [t_0 - r_0, \infty) \to \mathbf{R}$ such that

$$\phi(t) = \psi(t - t_0) \ \text{ for } \ t_0 - r_0 \leq t \leq t_0,$$

$$|\phi(t)| \leq q(t) \ \text{ for } \ t \geq t_0,$$

and

$$|\phi(t_1) - \phi(t_2)| \leq L|t_1 - t_2| \ \text{ for } \ t_1,\ t_2 \in \mathbf{R}^+ \ \text{ with } \ t_0 \leq \tau_1 \leq t_1,\ t_2 \leq \tau_2,$$

where $q(t)$ is defined by (8) with $\eta = \delta$, and where $L : \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ is defined by

$$L = L(\tau_1, \tau_2) := \max\big\{(|a(t)| + \beta\gamma|b(t)|)\alpha : \tau_1 \leq t \leq \tau_2\big\}. \tag{12}$$

Since we have (9), we obtain

$$|q'(t)| \leq (|a(t)| + \beta\gamma|b(t)|)\alpha, \ \ t \geq t_0.$$

Thus the function $\xi(t)$ defined by

$$\xi(t) := \begin{cases} \psi(t - t_0), & t_0 - r_0 \leq t \leq t_0, \\ \frac{\psi(0)q(t)}{\delta}, & t > t_0, \end{cases}$$

is an element of S, and from Lemma, S is a compact convex nonempty subset of C_{t_0}.

Define a mapping P on S by $(P\phi)(t) := \psi(t - t_0)$ for $t_0 - r_0 \leq t \leq t_0$, and

$$(P\phi)(t) := \psi(0)e^{-\int_{t_0}^{t} a(s)ds} - \int_{t_0}^{t} e^{-\int_s^t a(u)du} b(s) g\big(\phi(s - r(s))\big) ds, \ \ t > t_0,$$

where $\phi \in S$. Then we have

$$(P\phi)(t) = \psi(t - t_0) \ \text{ for } \ t_0 - r_0 \leq t \leq t_0,$$

and from (2) and (8) we obtain

$$|(P\phi)(t)| \leq \delta e^{-\int_{t_0}^t a(s)ds} + \beta \int_{t_0}^t e^{-\int_s^t a(u)du} |b(s)| q\big(s - r(s)\big) ds$$

$$\leq \delta e^{-\int_{t_0}^t a(s)ds} + \beta\gamma \int_{t_0}^t e^{-\int_s^t a(u)du} |b(s)| q(s) ds = q(t), \;\; t \geq t_0.$$

Moreover, we have

$$(P\phi)'(t) = -a(t)(P\phi)(t) - b(t)g\big(\phi(t - r(t))\big), \;\; t > t_0,$$

which implies

$$|(P\phi)'(t)| \leq |a(t)|q(t) + \beta|b(t)|q\big(t - r(t)\big)$$

$$\leq (|a(t)| + \beta\gamma|b(t)|)q(t) \leq (|a(t)| + \beta\gamma|b(t)|)\alpha, \;\; t > t_0,$$

and hence, P maps S into S. In addition, P is continuous. Thus, by Schauder's first theorem, P has a fixed point ϕ in S and that is the solution $x(t, t_0, \psi)$ of Eq. (1) which satisfies

$$|x(t, t_0, \psi)| \leq q(t) = q(t, t_0, \delta) < \epsilon, \;\; t \geq t_0,$$

and hence, the zero solution of Eq. (1) is stable.

(ii)-(iv) If $\sigma^* < \infty$, then the zero solution of Eq. (7) is uniformly stable. Next, Assumption (10) implies that $q(t) \to 0$ as $t \to \infty$, and hence, the zero solution of Eq. (7) is asymptoticaly stable. Moreover, Assumptions $\sigma^* < \infty$ and (11) imply that the zero solution of Eq. (7) is uniformly asymptotically stable. Thus, the uniform stability, the asymptotic stability and the uniform asymptotic stability of the zero solution of Eq. (1) can be similarly proved as in the proof of (i). So we omit the details.

Now we show two examples.

Example 1. Let $g(x) \equiv x$ on $\mathbf{R}$, and define functions $a, \; b, \; r : \mathbf{R}^+ \to \mathbf{R}^+$ by

$$a(t) := 2 + |t \sin t|, \;\; t \in \mathbf{R}^+,$$

$$b(t) := \frac{\max(1, \; 1 + 2t \sin t)}{25}, \;\; t \in \mathbf{R}^+,$$

and

$$r(t) := \frac{1}{t+1}, \;\; t \in \mathbf{R}^+.$$

Then, (2)-(5) hold with $\beta = 1, \; \gamma = 25$ and $r_0 = 1$, and $\sigma^* = \infty$. Thus, concerning the stability of the zero solution of the equation

$$x'(t) = -a(t)x(t) - b(t)x(t - \frac{1}{t+1}), \;\; t \in \mathbf{R}^+, \tag{13}$$

Theorem 1 does not assure uniform stability, but assures stability.

Example 2. Let $a : \mathbf{R}^+ \to \mathbf{R}$ be a 13-periodic function satisfying

$$a(t) := \begin{cases} -1, & 0 \le t < 1, \\ 6t - 7, & 1 \le t < 2, \\ 5, & 2 \le t < 12, \\ 77 - 6t, & 12 \le t \le 13, \end{cases}$$

and let $r(t) \equiv r$ $(0 \le r \le (\ln 2)/5)$, $b(t) \equiv B$ $(0 < B \le 53/26)$, (2) hold with $\beta = 1$. Then (3)-(5) hold with $\beta = 1$, $\gamma = 2$ and $r_0 = r$, and $\sigma^* < \infty$. Moreover, if $0 < B < 53/26$, then (11) holds. Thus, by Theorem 1, the zero solution of Eq. (1) is uniformly stable. Moreover, if $0 < B < 53/26$, then the zero solution of Eq. (1) is uniformly asymptotically stable.

If $g(x) \equiv x$ on $\mathbf{R}$ and $r(t) \equiv r$ on $\mathbf{R}^+$, then Eq. (1) becomes

$$x'(t) = -a(t)x(t) - b(t)x(t - r), \;\; t \in \mathbf{R}^+. \tag{14}$$

In Hale [6; p. 108], under the assumption

$$a(t) \ge \delta > 0, \;\; |b(t)| \le \theta\delta, \;\; \theta < 1, \tag{15}$$

where δ and θ are constants, the uniform asymptotic stability of the zero solution of Eq. (14) is discussed by using a Liapunov functional.

On the other hand, in Burton-Furumochi [1], under the assumption

$$\int_0^t e^{-\int_s^t a(u)du}|b(s)|ds \le \eta < 1 \;\text{ on }\; \mathbf{R}^+, \;\; \int_0^t a(s)ds \to \infty \;\text{ as }\; t \to \infty, \tag{16}$$

where η is a constant, the asymptotic stability of the zero solution of Eq. (14) is discussed by using the contraction principle.

But the functions $a(t)$ and $b(t) \equiv 1$ in Example 2 satisfy neither (15) nor (16).

Next consider the scalar integro-differential equation

$$x'(t) = -a(t)x(t) - \int_{t-r(t)}^t b(t,s)g\big(x(s)\big)ds, \;\; t \in \mathbf{R}^+, \tag{17}$$

where $a,\ r : \mathbf{R}^+ \to \mathbf{R},\ b : \mathbf{R}^+ \times \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are continuous. Let α be any fixed positive number. We assume that there are constants $\beta > 0,\ \gamma > 0$ and $r_0 \geq 0$ so that (2) and (4) hold and

$$\sup_{t\in\mathbf{R}^+} \left\{ \sup_{\tau\leq v\leq t} e^{\int_v^t (a(s)-\beta\gamma \int_{s-r(s)}^s |b(s,u)|du)ds} \right\} \leq \gamma, \tag{18}$$

where $\tau = \tau(t) := \max\big(0,\ t - r(t)\big)$, and

$$\sigma = \sigma(t_0) := \sup_{t\in\mathbf{R}^+} \int_{t_0}^t \big(\beta\gamma \int_{s-r(s)}^s |b(s,u)|du - a(s)\big)ds < \infty. \tag{19}$$

For this σ, define a number $\delta = \delta(t_0)$ by $\delta := \alpha e^{-\sigma}$.

Corresponding to Eq. (17), consider the scalar linear equation

$$q' = \big(\beta\gamma \int_{t-r(t)}^t |b(t,s)|ds - a(t)\big)q,\ \ t \in \mathbf{R}^+. \tag{20}$$

Let $q : [t_0 - r_0, \infty) \to \mathbf{R}$ be a continuous function such that

$$q(t) = \delta \ \text{ on } \ [t_0 - r_0, t_0],$$

and that $q(t)$ is the unique solution of the initial value problem

$$q' = \big(\beta\gamma \int_{t-r(t)}^t |b(t,s)|ds - a(t)\big)q,\ \ q(t_0) = \delta,\ \ t \geq t_0.$$

Then $q(t)$ can be expressed as

$$q(t) = \delta e^{\int_{t_0}^t \big(\beta\gamma \int_{s-r(s)}^s |b(s,u)|du - a(s)\big)ds},\ \ t \geq t_0,$$

which together with (19), implies (9) with $\eta = \delta$.

Concerning the stability of the zero solution of Eq. (17), we have the following theorem.

Theorem 2. *Suppose that the solutions of* Eq. (17) *are uniquely determined by continuous initial functions, and that* (2), (4), (18) *and* (19) *hold. Then we have:*

(i) *The zero solution of* Eq. (17) *is stable.*

(ii) *If* $\sigma^* := \sup\{\sigma(t) : t \in \mathbf{R}^+\} < \infty$, *then the zero solution of* Eq. (17) *is uniformly stable.*

(iii) *If we have*

$$\int_0^t \big(a(s) - \beta\gamma \int_{s-r(s)}^s |b(s,u)|du\big)ds \to \infty \ \text{ as } \ t \to \infty, \tag{21}$$

then the zero solution of Eq. (17) *is asymptotically stable.*

(iv) *In addition to* $\sigma^* < \infty$, *if we have*

$$\int_{t_0}^{t} \big(a(s) - \beta\gamma \int_{s-r(s)}^{s} |b(s,u)|du\big)ds \to \infty$$

$$\text{uniformly for } t_0 \in \mathbf{R}^+ \text{ as } t \to \infty, \tag{22}$$

then the zero solution of Eq. (17) *is uniformly asymptotically stable.*

This theorem can be easily proved by taking the set S in the proof of Theorem 1 for the above function $q(t)$ and a function $L = L(\tau_1, \tau_2)$ with

$$\big(|a(t)| + \beta\gamma \int_{t-r(t)}^{t} |b(t,s)|ds\big)\alpha \le L \text{ for } \tau_1 \le t \le \tau_2,$$

and by defining a mapping P on S by $(P\phi)(t) := \psi(t - t_0)$ for $t_0 - r_0 \le t \le t_0$, and

$$(P\phi)(t) := \psi(0)e^{-\int_{t_0}^{t} a(s)ds} + \int_{t_0}^{t} e^{-\int_s^t a(u)du} \int_{s-r(s)}^{s} b(s,u)g\big(\phi(u)\big)duds$$

for $t > t_0$, where $\phi \in S$. So we omit the details of the proof.

Now we show an example.

Example 3. Let $a : \mathbf{R}^+ \to \mathbf{R}$ be the function defined in Example 2, and let $r(t) \equiv r$ $(r > 0)$, $b(t,s) \equiv B$ $(0 < B \le 53/(26r)$, $(5 - 2Br)r \le \ln 2)$, and $\beta = 1$. Then (2), (4), (18) and (19) hold with $\beta = 1$, $r_0 = r$ and $\gamma = 2$, and $\sigma^* < \infty$. Moreover, if $0 < B < 53/(26r)$, then (22) holds. Thus, by Theorem 2, the zero solution of Eq. (17) is uniformly stable. Moreover, if $0 < B < 53/(26r)$, then the zero solution of Eq. (17) is uniformly asymptotically stable.

In Burton-Furumochi [1], under the assumption

$$\text{there is an } \eta < 1 \text{ with } \int_{0}^{t} e^{-\int_s^t a(u)du} \int_{s-r(s)}^{s} |b(s,u)|duds \le \eta, \tag{23}$$

the asymptotic stability of the zero solution of Eq. (17) is discussed by using the contraction principle. But the functions $a(t)$ and $b(t,s) \equiv B$ in Example 3 do not satisfy (23) if $Br = 2$.

3. BOUNDEDNESS IN HALF-LINEAR EQUATIONS

First we discuss the boundedness of solutions of Eq. (1). In order to do so, we replace Assumption (2) by

$$|g(x)| \le \beta |x| \ \text{ for } \ x \in \mathbf{R}. \tag{2^*}$$

Then, concerning the boundedness of the solutions of Eq. (1), we have the following theorem.

Theorem 3. *Suppose that the solutions of* Eq. (1) *are uniquely determined by continuous initial functions, and that* (2^*) *and* (3)-(5) *hold. Then we have:*

(i) *The solutions of* Eq. (1) *are equi-bounded.*

(ii) *If* $\sigma^* < \infty$, *then the solutions of* Eq. (1) *are uniformly bounded.*

(iii) *If we have* (10), *then the solutions of* Eq. (1) *are equi-ultimately bounded for any bound* $A > 0$.

(iv) *If* $\sigma^* < \infty$, *and if we have* (11), *then the solutions of* Eq. (1) *are uniformly ultimately bounded for any bound* $A > 0$.

Proof. (i) Assumption (5) implies that the solutions of Eq. (7) are equi-bounded. Thus, for any $\alpha > 0$ and $t_0 \in \mathbf{R}^+$, there is an $A = A(\alpha, t_0) > 0$ such that for any q_0 with $|q_0| \le \alpha$, we have

$$|q(t, t_0, q_0)| < A \ \text{ for all } \ t \ge t_0.$$

For the t_0, let $(C_{t_0}, \ \|\cdot\|_h)$ be the Banach space as in the proof of Theorem 1(i). For a continuous function $\psi : [-r_0, 0] \to \mathbf{R}$ with $\sup\{|\psi(\theta)| : -r_0 \le \theta \le 0\} \le \alpha$, let S be a set of continuous functions $\phi : [t_0 - r_0, \infty) \to \mathbf{R}$ such that

$$\phi(t) = \psi(t - t_0) \ \text{ for } \ t_0 - r_0 \le t \le t_0,$$

$$|\phi(t)| \le q(t) \ \text{ for } \ t \ge t_0,$$

and

$$|\phi(t_1) - \phi(t_2)| \le L|t_1 - t_2| \ \text{ for } \ t_1, \ t_2 \ \text{ with } \ t_0 \le \tau_1 \le t_1, \ t_2 \le \tau_2,$$

where $q(t)$ is defined by (8) with $\eta = \alpha$, and where $L = L(\tau_1, \tau_2)$ is a function given in (12) with $\alpha = A$. Then $q(t)$ satisfies

$$0 < q(t) < A \ \text{ for all } \ t \ge t_0.$$

As in the proof of Theorem 1, S is a compact convex nonempty subset of C_{t_0}. Let P be the mapping defined in the proof of Theorem 1. Then P maps S into S continuously. Thus, by Schauder's first theorem, P has a fixed point ϕ and that is the solution $x(t, t_0, \psi)$ of Eq. (1) which satisfies

$$|x(t, t_0, \psi)| \le q(t) = q(t, t_0, \alpha) < A, \ \ t \ge t_0,$$

and hence, the solutions of Eq. (1) are equi-bounded.

(ii)-(iv) Under the assumptions in (ii)-(iv), the solutions of Eq. (7) are uniformly bounded, equi-bounded and equi-ultimately bounded for any bound $A > 0$, and uniformly bounded and uniformly ultimately bounded for any bound $A > 0$, respectively. Thus, the uniform boundedness, the equi-ultimate boundedness for any bound $A > 0$ and the uniform ultimate boundedness for any bound $A > 0$ of the solutions of Eq. (1) can be similarly proved as in the proof of (i). So, we omit the details.

Now we revisit Examples 1 and 2.

Example 1^***.** Let $\alpha = \infty$. Then, (2^*) and (3)-(5) hold with $\beta = 1,\ \gamma = 25$ and $r_0 = 1$, and $\sigma^* = \infty$. Thus, concerning the boundedness of the solutions of Eq. (13), Theorem 3 does not assure uniform boundedness, but assures equi-boundedness.

Example 2^***.** Let $\alpha = \infty$. Then, (2^*) and (3)-(5) hold with $\beta = 1,\ \gamma = 2$ and $r_0 = 1$, and $\sigma^* < \infty$. Moreover, if $0 < B < 53/26$, then (11) holds. Thus, by Theorem 3, the solutions of Eq. (1) are uniformly bounded. Moreover, if $0 < B < 53/26$, then the solutions of Eq. (1) are uniformly ultimately bounded for any bound $A > 0$.

Next we revisit the scalar integro-differential equation

$$x'(t) = -a(t)x(t) - \int_{t-r(t)}^{t} b(t,s)g\big(x(s)\big)ds,\ \ t \in \mathbf{R}^+, \tag{17}$$

where we assume (2^*) instead of (2). Concerning the boundedness of the solutions of Eq. (17), we have the following theorem.

Theorem 4. *Suppose that the solutions of* Eq. (17) *are uniquely determined by continuous initial functions, and that* (2^*), (4), (18) *and* (19) *hold. Then we have:*

(i) *The solutions of* Eq. (17) *are equi-bounded.*

(ii) *If* $\sigma^* < \infty$, *then the solutions of* Eq. (17) *are uniformly bounded.*

(iii) *If we have* (21), *then the solutions of* Eq. (17) *are equi-ultimately bounded for any bound* $A > 0$.

(iv) *If we have* $\sigma^* < \infty$ *and* (22), *then the solutions of* Eq. (17) *are uniformly ultimately bounded for any bound* $A > 0$.

Since this theorem can be proved by a similar method used in the proof of Theorem 3, we omit the proof.

Finally we revisit Example 3.

Example 3^***.** Let $a : \mathbf{R}^+ \to \mathbf{R}$ be the function defined in Example 2, and let $r(t) \equiv r$ $(r > 0)$ and $b(t, s) \equiv B$ $(0 < B \leq 53/(26r),\ (5 - 2Br)r \leq \ln 2)$, and $\beta = 1$. Then (2^*), (4), (18) and (19) hold with $\beta = 1,\ r_0 = r$ and $\gamma = 2$, and $\sigma^* < \infty$. Moreover, if $0 < B < 53/(26r)$, then (22) holds. Thus, by Theorem 4, the solutions of Eq. (17) are uniformly bounded. Moreover, if $0 < B < 53/(26r)$, then the solutions of Eq. (17) are uniformly ultimately bounded for any bound $A > 0$.

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