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# Forced Oscillations of Beams on Elastic Bearings

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**Abstract**

We study the existence of weak periodic solutions for certain damped and forced linear beam equations resting on semi-linear elastic bearings. Conditions for the periodic forcing term and semi-linear elastic bearings are derived which ensure either the existence or nonexistence of periodic solutions of the beam equation. Topological degree arguments are used to achieve these results.

**Keywords.** beam equations, periodic solutions, topological degree
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## 1 Introduction

In this note, we consider a periodically forced and damped beam resting on two different bearings with purely elastic responses. The length of the beam is $\pi/4$. The equation of vibrations is as follows

$$\begin{aligned} &u_{tt} + u_{xxxx} + \delta u_t + h(x,t) = 0\,, \\ &u_{xx}(0,\cdot) = u_{xx}(\pi/4,\cdot) = 0\,, \\ &u_{xxx}(0,\cdot) = -ku(0,\cdot) - f(u(0,\cdot))\,, \\ &u_{xxx}(\pi/4,\cdot) = ru(\pi/4,\cdot) + g(u(\pi/4,\cdot))\,, \end{aligned} \tag{1}$$

where $\delta > 0$, $r \geq 0$, $k \geq 0$ are constants, $h \in C([0,\pi/4] \times S^T)$, and $f, g \in C(\mathbb{R})$ have at most linear growth at infinity. Here $S^T$ is the circle $S^T = \mathbb{R}/\{T\mathbb{Z}\}$.

The undamped and unforced case of the form

$$\begin{aligned} &u_{tt} + u_{xxxx} = 0\,, \\ &u_{xx}(0,\cdot) = u_{xx}(\pi/4,\cdot) = 0\,, \\ &u_{xxx}(0,\cdot) = -f(u(0,\cdot))\,, \\ &u_{xxx}(\pi/4,\cdot) = f(u(\pi/4,\cdot)) \end{aligned} \tag{2}$$

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is studied in [7] and [9] by using variational methods, where among others the following results are proved.

**Theorem**. ([9]) *If the function $f(u)$ satisfies the following assumptions:*

(i) $f \in C^1(\mathbb{R})$, $f(-u) = -f(u)$ *for all* $u \in \mathbb{R}$.

(ii) *For any* $C > 0$ *there is a* $K(C)$ *such that* $f(u) \geq Cu - K(C)$ *for all* $u \geq 0$.

(iii) $\frac{1}{2}f(u)u - F(u) \geq c_1|f(u)| - c_2$ *for all* $u \in \mathbb{R}$, *where* $F(u) = \int\limits_0^u f(s)\,ds$ *and* $c_1, c_2$ *are positive constants.*

(iv) $f(0) = f'(0) = 0$.

*Then there is a sufficiently large positive integer $M$ such that equation* (2) *possesses at least one nonzero time periodic solution with the period* $2\pi M^2$.

**Theorem**. ([7]) *If the continuous function $f(u)$ is odd on $\mathbb{R}$, $C^1$-smooth near $u = 0$ with $f'(0) > 0$ and $\lim\limits_{|u|\to\infty} f(u)/u = 0$. Then equation* (2) *possesses infinitely many odd time periodic solutions with periods densely distributed in an interval $(a_1, 2a_1)$ for a constant $a_1 > 0$.*

It is pointed out in [9] that equation (2) is a simple analogue of a more complicated shaft dynamics model introduced in the works [5] and [6].

When the nonlinearities and parameters are small, i.e. (1) is of the form

$$\begin{aligned}
&u_{tt} + u_{xxxx} + \varepsilon\delta u_t + \varepsilon\mu h(x, \sqrt{\varepsilon}t) = 0\,,\\
&u_{xx}(0, \cdot) = u_{xx}(\pi/4, \cdot) = 0\,,\\
&u_{xxx}(0, \cdot) = -\varepsilon f(u(0, \cdot)), \quad u_{xxx}(\pi/4, \cdot) = \varepsilon f(u(\pi/4, \cdot))\,,
\end{aligned} \tag{3}$$

where $\varepsilon > 0$ and $\mu$ are small parameters, $\delta > 0$ is a constant, $f \in C^2(\mathbb{R})$, $h \in C^2([0, \pi/4] \times \mathbb{R})$ and $h(x, t)$ is 1-periodic in $t$. Then by using analytic methods, the following result is proved in [2] and [3].

**Theorem.** ([2], [3]) *Let the following assumptions hold:*

(I) $f(0) = 0$, $f'(0) < 0$ *and the equation* $\ddot{x} + f(x) = 0$ *has a homoclinic solution* $\gamma(t) \neq 0$ *that is a non trivial bounded solution such that* $\lim\limits_{t\to\pm\infty} \gamma(t) = 0$.

(II) *The homoclinic solution* $\gamma_1(t) := \frac{\sqrt{\pi}}{2}\gamma\Big(2\sqrt{\frac{2}{\pi}}t\Big)$ *is non-degenerate, that is the linear equation*

$$\ddot{v} + \frac{24}{\pi} f'\Big(\frac{2}{\sqrt{\pi}}\gamma_1(t)\Big)v = 0$$

*has no nontrivial bounded solutions.*

(III) $10.5705675493 \cdot |f'(0)| < \delta$.

*If $\eta \neq 0$ can be chosen in such a way that the equation*

$$\delta \int\limits_{-\infty}^{\infty} \dot{\gamma}_1(s)^2\, ds + \frac{2}{\sqrt{\pi}}\eta \int\limits_{-\infty}^{\infty} \int\limits_{0}^{\pi/4} \dot{\gamma}_1(s) h(x, s+\alpha)\, dx\, ds = 0$$

*has a simple root $\alpha$, then there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}]$ and $\mu = \sqrt{\varepsilon}\eta$, equation* (3) *has a unique bounded solution on $\mathbb{R}$ near $\gamma\Big(2\sqrt{\frac{2}{\pi}}\big(\sqrt{\varepsilon}t - \alpha\big)\Big)$ which is exponentially homoclinic to a unique small periodic solution of* (3). *Moreover, the Smale horseshoe (see* [12]*) can be embedded into the dynamics of* (3).

Finally, a damped case is studied in [8] of the form

$$\begin{aligned} &u_{tt} + u_{xxxx} + \delta u_t + h(x,t) = 0\,, \\ &u_{xx}(0,\cdot) = u_{xx}(\pi/4,\cdot) = 0\,, \\ &u_{xxx}(0,\cdot) = -f(u(0,\cdot))\,, \\ &u_{xxx}(\pi/4,\cdot) = g(u(\pi/4,\cdot))\,, \end{aligned} \tag{4}$$

where $\delta$ is a positive constant, $f$ and $g$ are analytic, the function $h \in C([0,\pi/4] \times S^T)$ is splitted as follows

$$h(x,t) = 8\frac{\theta_2 - 2\theta_1}{T\pi} + 96\frac{\theta_1 - \theta_2}{T\pi^2}x + p(x,t)$$

for $\theta_{1,2} \in \mathbb{R}$ and

$$\int_0^T \int_0^{\pi/4} p(x,t)\, dx\, dt = \int_0^T \int_0^{\pi/4} x p(x,t)\, dx\, dt = 0\,.$$

Conditions are found in [8] between the numbers $\theta_{1,2}$, the function $p(x,t)$ and the nonlinearities $f, g$ under which (4) has a $T$-periodic solution. It is also shown that under certain assumptions, constants $\theta_{1,2}$ are functions of $p(x,t)$ in order to get a $T$-periodic solution of (4).

In this note, we are interested in $T$-periodic vibrations of (1) by using topological degree arguments. We show the existence of $T$-periodic vibrations of (1) for $r > 0$, $k > 0$ and $f, g$ sublinear at infinity. Also a generic result is derived for this case when in addition $f, g \in C^1(\mathbb{R})$. If either $r = 0$ or $k = 0$, then we derive Landesman-Lazer type conditions on $h, f, g$ for showing either existence or nonexistence results of $T$-periodic vibrations of (1).

## 2 Setting of the Problem

By a weak $T$-periodic solution of (1), we mean any $u(x,t) \in C([0,\pi/4] \times S^T)$ satisfying the identity

$$\begin{aligned}&\int_0^T \int_0^{\pi/4} \Big[u(x,t)\Big\{v_{tt}(x,t) + v_{xxxx}(x,t) - \delta v_t(x,t)\Big\} + h(x,t)v(x,t)\Big]\,dx\,dt \\ &+\int_0^T \Big\{\Big(ku(0,t) + f(u(0,t))\Big)v(0,t) \\ &+\Big(ru(\pi/4,t) + g(u(\pi/4,t))\Big)v(\pi/4,t)\Big\}\,dt = 0\end{aligned} \tag{5}$$

for any $v(x,t) \in C^\infty([0,\pi/4] \times S^T)$ such that the following boundary value conditions hold

$$v_{xx}(0,\cdot) = v_{xx}(\pi/4,\cdot) = v_{xxx}(0,\cdot) = v_{xxx}(\pi/4,\cdot) = 0\,. \tag{6}$$

The eigenvalue problem

$$\begin{aligned}&w_{xxxx}(x) = \mu^4 w(x)\,,\\ &w_{xx}(0) = w_{xx}(\pi/4) = 0, \quad w_{xxx}(0) = w_{xxx}(\pi/4) = 0\end{aligned}$$

is known [9] to possesses a sequence of eigenvalues $\mu_k$, $k = -1, 0, 1, \cdots$ with

$$\mu_{-1} = \mu_0 = 0$$

and

$$\cos(\mu_k\pi/4)\cosh(\mu_k\pi/4) = 1, \quad k = 1, 2, \cdots\,. \tag{7}$$

The corresponding orthonormal in $L^2(0,\pi/4)$ system of eigenvectors is

$$\begin{aligned}&w_{-1}(x) = \tfrac{2}{\sqrt{\pi}}, \quad w_0(x) = \tfrac{16}{\pi}\left(x - \tfrac{\pi}{8}\right)\sqrt{\tfrac{3}{\pi}}\\ &w_k(x) = \tfrac{4}{\sqrt{\pi}W_k}\Big[\cosh(\mu_k x) + \cos(\mu_k x)\\ &\qquad - \tfrac{\cosh\xi_k - \cos\xi_k}{\sinh\xi_k - \sin\xi_k}\Big(\sinh(\mu_k x) + \sin(\mu_k x)\Big)\Big]\end{aligned}$$

where the constants $W_k$ are given by the formulas

$$W_k = \cosh(\xi_k) + \cos\xi_k - \tfrac{\cosh\xi_k - \cos\xi_k}{\sinh\xi_k - \sin\xi_k}\Big(\sinh\xi_k + \sin\xi_k\Big)$$

for $\xi_k = \mu_k\pi/4$. From (7) we get the asymptotic formulas

$$1 < \mu_k = 2(2k+1) + r(k) \quad \forall k \geq 1$$

along with

$$|r(k)| \leq \bar{c}_1 e^{-\bar{c}_2 k} \quad \forall k \geq 1\,,$$

where $\bar{c}_1$, $\bar{c}_2$ are positive constants. Moreover, the eigenfunctions $\{w_i\}_{i=-1}^{\infty}$ are uniformly bounded in $C([0,\pi/4])$.

## 3 Preliminary Results

Let $H_1(x,t) \in C([0,\pi/4] \times S^T)$, $H_2(t), H_3(t) \in C(S^T)$ be continuous $T$-periodic functions and consider the equation

$$\begin{aligned}&\int_0^T \int_0^{\pi/4} \Big[ z(x,t) \Big\{ v_{tt}(x,t) + v_{xxxx}(x,t) - \delta v_t(x,t) \Big\} + H_1(x,t) v(x,t) \Big]\, dx\, dt \\ &+ \int_0^T \Big\{ H_2(t) v(0,t) + H_3(t) v(\pi/4,t) \Big\}\, dt = 0\end{aligned} \tag{8}$$

for any $v(x,t) \in C^\infty([0,\pi/4] \times \mathbb{R})$ satisfying the boundary conditions (6) along with

$$\int_0^{\pi/4} v(x,t) dx = \int_0^{\pi/4} x v(x,t) dx = 0 \quad \forall t \in S^T. \tag{9}$$

Note that conditions (9) correspond to the orthogonality of $v(x,t)$ to $w_{-1}(x)$ and $w_0(x)$, for any $t \in S^T$. We look for $z(x,t)$ in the form

$$z(x,t) = \sum_{i=1}^{\infty} z_i(t) w_i(x)\,. \tag{10}$$

We formally put (10) into (8) to get a system of ordinary differential equations

$$\ddot{z}_i(t) + \delta \dot{z}_i(t) + \mu_i^4 z_i(t) = h_i(t)\,, \tag{11}$$

where

$$h_i(t) = -\Big( \int_0^{\pi/4} H_1(x,t) w_i(x)\, dx + H_2(t) w_i(0) + H_3(t) w_i(\pi/4) \Big)\,. \tag{12}$$

Let us put

$$M_1 = \sup_{i \geq 1, x} |w_i(x)|, \quad M_2 = 4 M_1 \sum_{i=1}^{\infty} 1/\mu_i^2 < \infty \quad M_3 = \sup_{i \geq 1} i^2/\mu_i^2 < \infty\,. \tag{13}$$

Since $\mu_i > 0$ for $i \geq 1$, equation (11) has a unique $T$-periodic solution $z_i(t)$, for $2\mu_i^2 > \delta$ given by

$$z_i(t) = \frac{2}{\bar{\omega}_i} \int_{-\infty}^{t} e^{-\delta(t-s)/2} \sin\Big( \frac{\bar{\omega}_i}{2}(t-s) \Big) \times h_i(s)\, ds\,, \tag{14}$$

where $\bar{\omega}_i = \sqrt{4\mu_i^4 - \delta^2}$, for $2\mu_i^2 = \delta$ given by

$$z_i(t) = \int_{-\infty}^{t} e^{-\delta(t-s)/2} (t-s) \times h_i(s)\, ds\,, \tag{15}$$

and for $2\mu_i^2 < \delta$ given by

$$z_i(t) = \int\limits_{-\infty}^{t} \frac{1}{\tilde{\omega}_i}\Big(e^{(-\delta+\tilde{\omega}_i)(t-s)/2} - e^{(-\delta-\tilde{\omega}_i)(t-s)/2}\Big) \times h_i(s)\, ds\,, \tag{16}$$

where $\tilde{\omega}_i = \sqrt{\delta^2 - 4\mu_i^4}$. Let $\|\cdot\|_\infty$ denote the maximum norm on $[0,T]$.

From (14) for $3\mu_i^4 > \delta^2$ we get

$$\begin{aligned} &\|z_i\|_\infty \le \tfrac{4}{\tilde{\omega}_i\delta}\|h_i\|_\infty \le \tfrac{4}{\mu_i^2\delta}\|h_i\|_\infty \\ &\|\dot{z}_i\|_\infty \le \tfrac{4}{\delta}\|h_i\|_\infty\,, \end{aligned} \tag{17}$$

and for $3\mu_i^4 < \delta^2 < 4\mu_i^4$ from (14) we get

$$\begin{aligned} &\|z_i\|_\infty \le \|h_i\|_\infty \int\limits_{-\infty}^{t} e^{-\delta(t-s)/2}(t-s)\, ds = \tfrac{4}{\delta^2}\|h_i\|_\infty \\ &\le \tfrac{4}{\sqrt{3}\mu_i^2\delta}\|h_i\|_\infty \le \tfrac{4}{\mu_i^2\delta}\|h_i\|_\infty\,, \\ &\|\dot{z}_i\|_\infty \le \tfrac{4}{\delta}\|h_i\|_\infty\,, \end{aligned} \tag{18}$$

where we use in derivation of (18) the inequality $|\sin x| \le |x|\ \forall x \in \mathbb{R}$. From (15) for $2\mu_i^2 = \delta$ we get

$$\begin{aligned} &\|z_i\|_\infty \le \|h_i\|_\infty \int\limits_{-\infty}^{t} e^{-\delta(t-s)/2}(t-s)\, ds = \tfrac{4}{\delta^2}\|h_i\|_\infty \\ &= \tfrac{2}{\mu_i^2\delta}\|h_i\|_\infty \le \tfrac{4}{\mu_i^2\delta}\|h_i\|_\infty\,, \\ &\|\dot{z}_i\|_\infty \le \tfrac{4}{\delta}\|h_i\|_\infty\,. \end{aligned} \tag{19}$$

From (16) for $2\mu_i^2 < \delta$ we get

$$\begin{aligned} &\|z_i\|_\infty \le \tfrac{1}{\mu_j^4}\|h_i\|_\infty \\ &\|\dot{z}_i\|_\infty \le \tfrac{\delta}{\mu_i^4}\|h_i\|_\infty \le \delta\|h_i\|_\infty\,. \end{aligned} \tag{20}$$

From (12) we get

$$\|h_i\|_\infty \le M_1\Big(\frac{\pi}{4}\|H_1\|_\infty + \|H_2\|_\infty + \|H_3\|_\infty\Big)\,. \tag{21}$$

We consider the Banach space $C([0,\pi/4]\times S^T)$ with the usual maximum norm $\|\cdot\|_\infty$. We need the following result.

**Proposition 1**. *A sequence $\{z^n(x,t)\}_{n=1}^\infty \subset C([0,\pi/4]\times S^T)$ is precompact if there is a constant $M > 0$ such that*

$$\sup_{i\ge1,n\ge1}\|z_i^n\|_\infty i^2 < M, \qquad \sup_{i\ge1,n\ge1}\|\dot{z}_i^n\|_\infty < M\,, \tag{22}$$

*where $z^n(x,t) = \sum\limits_{i=1}^{\infty} z_i^n(t)w_i(x)$.*

**Proof.** From (22) we get

$$|z_i^n(t)|i^2 \le M, \quad |\dot z_i^n(t)| \le M \quad \forall t \in S^T .$$

By the Arzela-Ascoli theorem, there is a subsequence $\{z_1^{n_k}\}_{k=1}^\infty$ of $\{z_1^n\}_{n=1}^\infty$ such that $z_1^{n_k}(t) \to z_1^0(t)$ uniformly on $S^T$. Similarly we have a subsequence $\{z_2^{n_{k_s}}\}_{s=1}^\infty$ of $\{z_2^{n_k}\}_{k=1}^\infty$ such that $z_2^{n_{k_s}}(t) \to z_2^0(t)$ uniformly on $S^T$. Then we follow this construction. By using the Cantor diagonal procedure, we find an increasing sequence $\{m_k\}_{k=1}^\infty \subset \mathbb{N}$ such that $z_i^{m_k}(t) \to z_i^0(t)$ uniformly on $S^T$ for any $i \ge 1$. Clearly we have

$$\sup_{i\ge 1} \|z_i^0\|_\infty i^2 \le M .$$

Hence $z^0(x,t) = \sum_{i=1}^\infty z_i^0(t) w_i(x) \in C([0,\pi/4]\times S^T)$. Let $\varepsilon > 0$ be given. Then we choose $i_1 \in \mathbb{N}$ so large that $2M_1 M \sum_{i=i_1}^\infty i^{-2} < \varepsilon/2$. We estimate

$$\begin{aligned}
&\|z^{m_k} - z^0\|_\infty \le M_1 \sum_{i=1}^\infty \|z_i^{m_k} - z_i^0\|_\infty \\
&\le 2M_1 M \sum_{i=i_1}^\infty i^{-2} + M_1 \sum_{i=1}^{i_1} \|z_i^{m_k} - z_i^0\|_\infty < \tfrac{\varepsilon}{2} + M_1 \sum_{i=1}^{i_1} \|z_i^{m_k} - z_i^0\|_\infty .
\end{aligned}$$

For $1 \le i < i_1$, we have

$$\|z_i^{m_k} - z_i^0\|_\infty \to 0$$

as $k \to \infty$. Hence $\|z^{m_k} - z^0\|_\infty < \varepsilon/2$ for $k$ large . This implies $z^{m_k} \to z^0$ in $C([0,\pi/4]\times S^T)$. The proof is finished.

Now if $h_i(t)$, $i \ge 1$ is given by (12) and $T$-periodic $z_i(t)$ are defined by (11), then $z(x,t)$ given by (10) satisfies $z(x,t) \in C([0,\pi/4]\times S^T)$. Indeed, from $\sum_{i=1}^\infty \mu_i^{-2} < \infty$ and (17) - (20) we have that the series (10) is uniformly convergent. Hence $z(x,t) \in C([0,\pi/4]\times S^T)$ and (17) - (20) also imply

$$\begin{aligned}
&\|z\|_\infty \le M_1 \sum_{2\mu_i<\delta} \tfrac{1}{\mu_i^4}\|h_i\|_\infty + M_1 \sum_{2\mu_i\ge\delta} \tfrac{4}{\mu_i^2\delta}\|h_i\|_\infty \\
&\le M_1 \sum_{i=1}^\infty \Big(\tfrac{1}{\mu_i^2} + \tfrac{4}{\mu_i^2\delta}\Big)\|h_i\|_\infty \\
&\le M_2 \Big(\tfrac{1}{\delta} + \tfrac{1}{4}\Big)\Big(\tfrac{\pi}{4}\|H_1\|_\infty + \|H_2\|_\infty + \|H_3\|_\infty\Big) .
\end{aligned}$$

Moreover, we derive

$$\sup_{i\ge 1} \|z_i\|_\infty i^2 \le M_3 \Big(\frac{4}{\delta} + 1\Big)\Big(\frac{\pi}{4}\|H_1\|_\infty + \|H_2\|_\infty + \|H_3\|_\infty\Big) \tag{23}$$

and

$$\sup_{i\ge 1} \|\dot z_i\|_\infty \le \Big(\frac{4}{\delta} + \delta\Big)\Big(\frac{\pi}{4}\|H_1\|_\infty + \|H_2\|_\infty + \|H_3\|_\infty\Big) . \tag{24}$$

We also know from [3] that such $z(x,t)$ satisfies (8). On the other hand, if $z(x,t) \in C([0,\pi/4]\times S^T)$ satisfies (8), then $z(x,t) = \sum\limits_{i=1}^{\infty} z_i(t)w_i(x)$ in $L^2(0,\pi/4)$ for a.e. $t \in S^T$. By inserting $v(x,t) = \phi(t)w_i(x)$ in (8) with $\phi \in C^\infty(S^T)$, we get (11) with (12). So $z(x,t)$ has the above properties.

Finally, let us define the following Banach space

$$C_0([0,\pi/4]\times S^T) := \Big\{ z(x,t) \in C([0,\pi/4]\times S^T) \mid \int\limits_0^{\pi/4} z(x,t)\,dx = \int\limits_0^{\pi/4} z(x,t)x\,dx = 0 \quad \forall\, t \in S^T \Big\}$$

with the maximum norm $\|\cdot\|_\infty$ on $[0,\pi/4]\times S^T$. Summarizing, we arrive at the following result.

**Proposition 2**. *For any given functions $H_1(x,t) \in C([0,\pi/4]\times S^T)$, $H_2(t), H_3(t) \in C(S^T)$, equation* (8) *has a unique solution $z(x,t) \in C_0([0,\pi/4]\times S^T)$ of the form*

$$z(x,t) = \sum_{i=1}^{\infty} z_i(t)w_i(x)\,.$$

*Such a solution satisfies the condition* (9) *along with:*

**(a)** *$z(x,t) \in X$ for the Banach space*

$$X = \Big\{ z(x,t) \in C([0,\pi/4]\times S^T) \mid z(x,t) = \sum\limits_{i=1}^{\infty} z_i(t)w_i(x), \sup_{i\ge 1} \|z_i\|_\infty i^2 < \infty \Big\}\,.$$

**(b)** $\|z\|_\infty \le M_2\Big(\frac{1}{\delta}+\frac{1}{4}\Big)\big(\frac{\pi}{4}\|H_1\|_\infty + \|H_2\|_\infty + \|H_3\|_\infty\big)$.

**(c)** *The mapping $L : C([0,\pi/4]\times S^T)\times C(S^T)\times C(S^T) \to C_0([0,\pi/4]\times S^T)$ defined by $L(H_1,H_2,H_3) := z(x,t)$ is compact.*

**Proof.** Properties (a) and (b) are proved above. Property (c) follows from Lemma 1 and inequalities (23) and (24). The proof is finished.

# 4 Nonhomogeneous Linear Problems

In this section, we consider the linear problem

$$\begin{aligned}
&u_{tt} + u_{xxxx} + \delta u_t + h(x,t) = 0\,,\\
&u_{xx}(0,\cdot) = u_{xx}(\pi/4,\cdot) = 0\,,\\
&u_{xxx}(0,\cdot) = -ku(0,\cdot) - f_1(t)\,,\\
&u_{xxx}(\pi/4,\cdot) = ru(u(\pi/4,\cdot) + f_2(t)\,,
\end{aligned} \tag{25}$$

where $h(x,t) \in C([0,\pi/4] \times S^T)$, $f_1(t), f_2(t) \in C(S^T)$. Of course, we consider (25) in the sense of (5). Now we split $u(x,t)$ as follows

$$u(x,t) = y_1(t)w_{-1}(x) + y_2(t)w_0(x) + z(x,t)$$

with $z(x,t) \in C_0([0,\pi/4] \times S^T)$. Then (25) is equivalent to the system

$$\begin{aligned} &\ddot{y}_1(t) + \delta\dot{y}_1(t) + \tfrac{2}{\sqrt{\pi}} \int_0^{\pi/4} h(x,t)\,dx \\ &+\tfrac{4}{\pi}(k+r)y_1(t) + \tfrac{4\sqrt{3}}{\pi}(r-k)y_2(t) \\ &+\tfrac{2}{\sqrt{\pi}}\left(kz(0,t) + rz(\pi/4,t) + f_1(t) + f_2(t)\right) = 0\,, \end{aligned} \tag{26}$$

$$\begin{aligned} &\ddot{y}_2(t) + \delta\dot{y}_2(t) + \tfrac{16}{\pi}\sqrt{\tfrac{3}{\pi}} \int_0^{\pi/4} h(x,t)\Big(x - \tfrac{\pi}{8}\Big)\,dx \\ &+\tfrac{4\sqrt{3}}{\pi}(r-k)y_1(t) + \tfrac{12}{\pi}(k+r)y_2(t) \\ &+2\sqrt{\tfrac{3}{\pi}}\left(rz(\pi/4,t) - kz(0,t) + f_2(t) - f_1(t)\right) = 0\,, \end{aligned} \tag{27}$$

$$\begin{aligned} &\int_0^T \int_0^{\pi/4} \Big[z(x,t)\Big\{v_{tt}(x,t) + v_{xxxx}(x,t) - \delta v_t(x,t)\Big\} + h(x,t)v(x,t)\Big]\,dx\,dt \\ &+\int_0^T \Big\{\Big(k\left(\tfrac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\tfrac{3}{\pi}}y_2(t) + z(0,t)\right) + f_1(t)\Big)\,v(0,t) \\ &+\Big(r\Big(\tfrac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\tfrac{3}{\pi}}y_2(t) + z(\pi/4,t)\Big) + f_2(t)\Big)v(\pi/4,t)\Big\}\,dt = 0 \end{aligned} \tag{28}$$

where $v(x,t) \in C^\infty([0,\frac{\pi}{4}] \times S^T)$ satisfies the conditions (6), (9). Let us define

$$\begin{aligned} &L_1 : C(S^T) \times C(S^T) \to C_0([0,\pi/4] \times S^T)\,, \\ &L_2 : C_0([0,\pi/4] \times S^T) \to C_0([0,\pi/4] \times S^T)\,, \\ &H_4 \in C_0([0,\pi/4] \times S^T) \end{aligned}$$

by

$$\begin{aligned} &L_1(y_1,y_2) := L\Big(0, k\Big(\tfrac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\tfrac{3}{\pi}}y_2(t)\Big), r\Big(\tfrac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\tfrac{3}{\pi}}y_2(t)\Big)\Big)\,, \\ &L_2(z) := L\Big(0, kz(0,t), rz(\pi/4,t)\Big)\,, \\ &H_4(t) := L\Big(h(x,t), f_1(t), f_2(t)\Big)\,. \end{aligned}$$

Then according to Proposition 2, equation (28) has the form

$$z = L_2(z) + L_1(y_1,y_2) + H_4\,. \tag{29}$$

Moreover, operators $L_1$ and $L_2$ are compact. Furthermore, since for $r > 0$, $k > 0$ the matrix

$$A = \frac{4}{\pi}\begin{pmatrix} (k+r) & \sqrt{3}(r-k) \\ \sqrt{3}(r-k) & 3(r+k) \end{pmatrix}$$

is invertible, the system

$$\ddot{y} + \delta\dot{y} + Ay = \bar{h}(t) = (h_1(t), h_2(t)) \in C(S^T)^2 \tag{30}$$

has a unique $T$-periodic solution $y = (y_1, y_2) := L_3(h_1, h_2)$. Let us define

$$L_4 : C_0([0, \pi/4] \times S^T) \to C(S^T)^2, \quad H_5 \in C(S^T)$$

given by

$$\begin{aligned}
L_4(z) &:= L_3\Big(\tfrac{2}{\sqrt{\pi}}\left(kz(0,t) + rz(\pi/4,t)\right), 2\sqrt{\tfrac{3}{\pi}}\left(rz(\pi/4,t) - kz(0,t)\right)\Big)\,, \\
H_5(t) &:= L_3\Big(\tfrac{2}{\sqrt{\pi}}\int_0^{\pi/4} h(x,t)\,dx + \tfrac{2}{\sqrt{\pi}}(f_1(t) + f_2(t)), \\
&\quad \tfrac{16}{\pi}\sqrt{\tfrac{3}{\pi}}\int_0^{\pi/4} h(x,t)\Big(x - \tfrac{\pi}{8}\Big)\,dx + 2\sqrt{\tfrac{3}{\pi}}(f_2(t) - f_1(t))\Big)\,.
\end{aligned}$$

Then (26) and (27) are equivalent to

$$y = L_4(z) + H_5(t) \tag{31}$$

and $L_4$ is compact. Consequently, in order to solve uniquely equations (29) and (31), we must show that if

$$\begin{aligned}
z &= L_2(z) + L_1(y_1, y_2), \quad z \in C_0([0, \pi/4] \times S^T) \\
y &= L_4(z), \quad y = (y_1, y_2) \in C(S^T)^2\,,
\end{aligned} \tag{32}$$

then $z = 0$ and $y = 0$. Equation (32) is equivalent to the system

$$\begin{aligned}
&\ddot{z}_j(t) + \delta\dot{z}_j(t) + \mu_j^4 z_j(t) \\
&+ \sum_{s=-1}^{\infty}\Big(kz_s(t)w_s(0)w_j(0) + rz_s(t)w_s(\pi/4)w_j(\pi/4)\Big) = 0
\end{aligned} \tag{33}$$

for $z_j \in C^2(S^T)$, $j \geq -1$ with $\sup\limits_{j\geq -1}\|z_j\|_\infty(j^2+1) < \infty$. Let us expand

$$z_j(t) = \sum_{m\in\mathbb{Z}} e^{\imath 2\pi mt/T} c_{mj}\,.$$

Note that $c_{mj} \sim j^{-2}$ as $j \to \infty$ uniformly for $m \in \mathbb{Z}$. Then by (33) we derive

$$\begin{aligned}
&c_{mj}\left(\mu_j^4 - \tfrac{4\pi^2 m^2}{T^2} + \imath\tfrac{2\delta\pi}{T}m\right) \\
&+ \sum_{s=-1}^{\infty}\Big(kc_{ms}w_s(0)w_j(0) + rc_{ms}w_s(\pi/4)w_j(\pi/4)\Big) = 0\,.
\end{aligned} \tag{34}$$

By taking $c_{mj} = a_{mj} + \imath b_{mj}$, from (34) we derive

$$\begin{aligned}
&a_{mj}\left(\mu_j^4 - \tfrac{4\pi^2 m^2}{T^2}\right) - \tfrac{2\delta\pi}{T}mb_{mj} \\
&+ \sum_{s=-1}^{\infty}\Big(kw_s(0)w_j(0) + rw_s(\pi/4)w_j(\pi/4)\Big)a_{ms} = 0\,, \\
\\
&a_{mj}\tfrac{2\delta\pi}{T}m + \left(\mu_j^4 - \tfrac{4\pi^2 m^2}{T^2}\right)b_{mj} \\
&+ \sum_{s=-1}^{\infty}\Big(kw_s(0)w_j(0) + rw_s(\pi/4)w_j(\pi/4)\Big)b_{ms} = 0\,.
\end{aligned}$$

Since $a_{mj}, b_{mj} \sim j^{-2}$ as $j \to \infty$, we get

$$\sum_{j=-1}^{\infty} \left(a_{mj}^2 + b_{mj}^2\right) \frac{2\pi\delta}{T} m = 0\,,$$

hence $a_{mj} = b_{mj} = 0$ for any $m \neq 0$ and $j$. For $m = 0$ we get

$$a_{0j}\mu_j^4 + \sum_{s=-1}^{\infty} \Big(kw_s(0)w_j(0) + rw_s(\pi/4)w_j(\pi/4)\Big)a_{0s} = 0\,, \tag{35}$$

$$b_{0j}\mu_j^4 + \sum_{s=-1}^{\infty} \Big(kw_s(0)w_j(0) + rw_s(\pi/4)w_j(\pi/4)\Big)b_{0s} = 0\,. \tag{36}$$

We put $a_{0j}(\mu_j^4 + 1) = e_j$ and from (35) we get

$$\sum_{j=-1}^{\infty} \tfrac{e_j^2}{\mu_j^4+1} \tfrac{\mu_j^4}{\mu_j^4+1} + k\Big(\sum_{s=-1}^{\infty} w_s(0)\tfrac{e_s}{\mu_s^4+1}\Big)^2 + r\Big(\sum_{s=-1}^{\infty} w_s(\pi/4)\tfrac{e_s}{\mu_s^4+1}\Big)^2 = 0\,. \tag{37}$$

From (37) for $r > 0$, $k > 0$ we immediately get $e_j = 0$ for $j \geq 1$ and

$$\frac{2}{\sqrt{\pi}}e_{-1} - 2\sqrt{\frac{3}{\pi}}e_0 = 0, \quad \frac{2}{\sqrt{\pi}}e_{-1} + 2\sqrt{\frac{3}{\pi}}e_0 = 0\,,$$

which imply also $e_{-1} = e_0 = 0$. Similar results hold for (36). Hence, (32) has the only solution $z = 0$ and $y = 0$. Consequently, (29) and (31) are uniquely solvable in $z, y$ for $r > 0$, $k > 0$. Summarizing, we arrive at the following result.

**Proposition 3**. *If $r > 0$, $k > 0$ then for any given functions $h(x,t) \in C([0,\pi/4] \times S^T)$ and $f_1(t)$, $f_2(t) \in C(S^T)$, equation (25) has a unique solution $u(x,t) \in C([0,\pi/4] \times S^T)$ of the form*

$$u(x,t) = \sum_{i=-1}^{\infty} z_i(t)w_i(x)\,.$$

*Such a solution satisfies:*

**(a)** *$u(x,t) \in Y$ for the Banach space*

$$Y = \Big\{u(x,t) \in C([0,\pi/4] \times S^T) \mid u(x,t) = \sum_{i=-1}^{\infty} z_i(t)w_i(x), \\ \|u\| := \sup_{i \geq -1} \|z_i\|_\infty (|i|+1)^2 < \infty\Big\}\,.$$

**(b)** $\|u\|$, $\|u\|_\infty \leq c\,(\|h\|_\infty + \|f_1\|_\infty + \|f_2\|_\infty)$ *for a constant $c > 0$.*

**(c)** *The mapping $\tilde{L} : C([0,\pi/4] \times S^T) \times C(S^T) \times C(S^T) \to C([0,\pi/4] \times S^T)$ defined by $\tilde{L}(h, f_1, f_2) := u(x,t)$ is compact.*

We also define a compact mapping $\bar{L} : C(S^T) \times C(S^T) \to C([0, \pi/4] \times S^T)$ by $\bar{L}(f_1, f_2) := \tilde{L}(0, f_1, f_2)$. We denote by $\|\bar{L}\|$ the norm of $\bar{L}$.

Now we study the case when $r = 0$ and $k > 0$ in (25). Then equation (29) remains, but the matrix $A$ is no more invertible. Equation (30) has a $T$-periodic solution if and only if

$$\int\limits_0^T (\sqrt{3}h_1(t) + h_2(t))\, dt = 0$$

and the linear equation

$$\ddot{y} + \delta\dot{y} + Ay = 0$$

has the only $T$-periodic solutions $y(t) = c(\sqrt{3}, 1)$, $c \in \mathbb{R}$. Consequently, we are still working with Fredholm operators of index 0 possessing forms of compact perturbations of identity operators [11]. Hence, in order to study (25) we consider like in (33) the equations

$$\ddot{z}_j(t) + \delta\dot{z}_j(t) + \mu_j^4 z_j(t) + k\sum_{s=-1}^{\infty} z_s(t)w_s(0)w_j(0) = 0 \tag{38}$$

$$\ddot{z}_j(t) + \delta\dot{z}_j(t) + \mu_j^4 z_j(t) + k\sum_{s=-1}^{\infty} z_s(t)w_s(0)w_j(0) + h_j(t) = 0 \tag{39}$$

for $h_j \in C(S^T)$, $z_j \in C^2(S^T)$, $j \geq -1$ with $\sup\limits_{j\geq -1} \|z_j\|_\infty (j^2 + 1) < \infty$. Like for (33), we get that $z_{-1}(t) = c\sqrt{3}$, $z_0(t) = c$, $c \in \mathbb{R}$, $z_j(t) = 0$, $j \geq 1$ for (38). According to the above comments, the set of all $\{h_j(t)\}_{j\geq -1}$ for which (39) is solvable must have a codimension 1. For this reason, we consider an adjoint equation to (38) of the form

$$\ddot{v}_j(t) - \delta\dot{v}_j(t) + \mu_j^4 v_j(t) + k\sum_{s=-1}^{\infty} v_s(t)w_s(0)w_j(0) = 0 \tag{40}$$

for $v_j \in C^2(S^T)$, $j \geq -1$ with $\sup\limits_{j\geq -1} \|v_j\|_\infty (j^2 + 1) < \infty$. Like above we get $v_{-1}(t) = c\sqrt{3}$, $v_0(t) = c$, $c \in \mathbb{R}$, $v_j(t) = 0$, $j \geq 1$ for (40). By multiplying (39) with $v_j(t)$ and using integration by parts, we get

$$\begin{aligned} &\int\limits_0^T z_j(t)\Big(\ddot{v}_j(t) - \delta\dot{v}_j(t) + \mu_j^4 v_j(t)\Big)\, dt \\ &+k\int\limits_0^T \Big(\sum_{s=-1}^{\infty} z_s(t)w_s(0)w_j(0)v_j(t)\Big)\, dt + \int\limits_0^T h_j(t)v_j(t)\, dt = 0\,. \end{aligned} \tag{41}$$

Inserting (40) to (41) we obtain

$$\begin{aligned} &k\int\limits_0^T \sum_{s=-1}^{\infty} \Big(z_s(t)w_s(0)w_j(0)v_j(t) - z_j(t)w_s(0)w_j(0)v_s(t)\Big)\, dt \\ &+\int\limits_0^T h_j(t)v_j(t)\, dt = 0\,. \end{aligned} \tag{42}$$

Since $v_j(t) \sim j^{-2}$, $z_j(t) \sim j^{-2}$ uniformly on $S^T$, (42) implies

$$0 = \sum_{s=-1}^{\infty} \int_0^T h_j(t) v_j(t)\, dt = \int_0^T (\sqrt{3} h_{-1}(t) + h_0(t))\, dt\,. \tag{43}$$

We recall that the set of all $\{h_j(t)\}_{j\geq -1}$ for which (39) is solvable has a codimension 1. Then condition (43) is necessary and also sufficient for solvability of (39). We note

$$\begin{aligned} h_{-1}(t) &= \tfrac{2}{\sqrt{\pi}} \int_0^{\pi/4} h(x,t)\, dx + \tfrac{2}{\sqrt{\pi}} \left(f_1(t) + f_2(t)\right)\,, \\ h_0(t) &= \tfrac{16}{\pi} \sqrt{\tfrac{3}{\pi}} \int_0^{\pi/4} h(x,t) \left(x - \tfrac{\pi}{8}\right) dx + 2\sqrt{\tfrac{3}{\pi}} \left(f_2(t) - f_1(t)\right)\,. \end{aligned}$$

Then condition (43) has the form

$$\frac{4}{\pi} \int_0^T \int_0^{\pi/4} h(x,t) x\, dx\, dt + \int_0^T f_2(t)\, dt = 0\,. \tag{44}$$

Finally, the corresponding kernel to (38) is spanned by the function

$$z_{-1}(t) w_{-1}(x) + z_0(t) w_0(x) = \frac{16}{\pi} \sqrt{\frac{3}{\pi}} x\,. \tag{45}$$

Summarizing we get the next result.

**Proposition 4**. *If $r = 0$, $k > 0$ then for any given functions $h(x,t) \in C([0,\pi/4] \times S^T)$ and $f_1(t)$, $f_2(t) \in C(S^T)$, equation (25) has a solution $u(x,t) \in C([0,\pi/4] \times S^T)$ if and only if condition (44) holds. Such a solution is unique if*

$$\int_0^T \int_0^{\pi/4} u(x,t) x\, dx\, dt = 0\,. \tag{46}$$

*Moreover, the mapping $K : C_1 \to C_2$ is compact where*

$$\begin{aligned} C_1 &:= \left\{ (h, f_1, f_2) \in C([0,\pi/4] \times S^T) \times C(S^T)^2 \mid \quad \text{condition (44) holds} \right\}, \\ C_2 &:= \left\{ u \in C([0,\pi/4] \times S^T) \mid \quad \text{condition (46) holds} \right\} \end{aligned}$$

*are Banach spaces endowed with the maximum norms and the mapping $K$ is defined by $K(h, f_1, f_2) := u(x,t)$.*

Similarly we derive the next results.

**Proposition 5**. *If $r>0$, $k=0$ then for any given functions $h(x,t)\in C([0,\pi/4]\times S^T)$ and $f_1(t)$, $f_2(t)\in C(S^T)$, equation (25) has a solution $u(x,t)\in C([0,\pi/4]\times S^T)$ if and only if*

$$\frac{4}{\pi}\int_0^T\int_0^{\pi/4} h(x,t)\Big(\frac{\pi}{4}-x\Big)\,dx\,dt+\int_0^T f_1(t)\,dt=0\,. \tag{47}$$

*Such a solution is unique if*

$$\int_0^T\int_0^{\pi/4} u(x,t)\Big(\frac{\pi}{4}-x\Big)\,dx\,dt=0\,. \tag{48}$$

*Moreover, the mapping $\tilde{K}:\tilde{C}_1\to\tilde{C}_2$ is compact where*

$$\begin{aligned}\tilde{C}_1&:=\Big\{(h,f_1,f_2)\in C([0,\pi/4]\times S^T)\times C(S^T)^2\mid \quad \text{condition (47) holds}\Big\}\,,\\ \tilde{C}_2&:=\Big\{u\in C([0,\pi/4]\times S^T)\mid \quad \text{condition (48) holds}\Big\}\end{aligned}$$

*are Banach spaces endowed with the maximum norms and the mapping $K$ is defined by $\tilde{K}(h,f_1,f_2):=u(x,t)$.*

**Proposition 6**. *If $r=k=0$ then for any given functions $h(x,t)\in C([0,\pi/4]\times S^T)$ and $f_1(t)$, $f_2(t)\in C(S^T)$, equation (25) has a solution $u(x,t)\in C([0,\pi/4]\times S^T)$ if and only if the both conditions (44) and (47) hold. Such a solution is unique if the both conditions (46) and (48) hold.*

*Moreover, the mapping $\bar{K}:\bar{C}_1\to\bar{C}_2$ is compact where*

$$\begin{aligned}\bar{C}_1&:=\Big\{(h,f_1,f_2)\in C([0,\pi/4]\times S^T)\times C(S^T)^2\mid \text{conditions (44), (47) hold}\Big\}\,,\\ \bar{C}_2&:=\Big\{u\in C([0,\pi/4]\times S^T)\mid \quad \text{conditions (46), (48) hold}\Big\}\end{aligned}$$

*are Banach spaces endowed with the maximum norms and the mapping $\bar{K}$ is defined by $\bar{K}(h,f_1,f_2):=u(x,t)$.*

# 5 Nonlinear Problems

In this section, we present the main results concerning equation (1).

**Theorem 1**. *If $r>0,k>0$ and there are positive constants $c_{11},c_{12},c_{21},c_{22}$ along with*

$$c_{12}+c_{22}<1/\|\bar{L}\|$$

*and such that*

$$\begin{aligned}|f(u)|&\le c_{11}+c_{12}|u|,\quad \forall u\in\mathbb{R}\\ |g(u)|&\le c_{21}+c_{22}|u|,\quad \forall u\in\mathbb{R}\,,\end{aligned}$$

*then for any given function $h(x,t) \in C([0,\pi/4] \times S^T)$, equation* (1) *possesses a weak $T$-periodic solution $u(x,t) \in C([0,\pi/4] \times S^T)$.*

**Proof.** By using the above results, the proof is standard. According to Proposition 3, equation (1) is equivalent to

$$u = F(u) := \tilde{L}(h,0,0) + \bar{L}\Big(f(u(0,\cdot)), g(u(\pi/4,\cdot))\Big) . \tag{49}$$

Proposition 3 also implies the compactness of the mapping

$$F : C([0,\pi/4] \times S^T) \to C([0,\pi/4] \times S^T) .$$

From the assumptions of Theorem 1 and (b) of Proposition 3, we get

$$\begin{aligned} \|F(u)\|_\infty &\le c\|h\|_\infty + \|\bar{L}\|\Big(\|f(u(0,\cdot))\|_\infty + \|g(u(\pi/4,\cdot))\|_\infty\Big) \\ &\le c\|h\|_\infty + \|\bar{L}\|\Big(c_{11} + c_{21} + (c_{12} + c_{22})\|u\|_\infty\Big) . \end{aligned} \tag{50}$$

Since $\|\bar{L}\|(c_{12} + c_{22}) < 1$, there is a unique $\tau > 0$ such that

$$\tau = c\|h\|_\infty + \|\bar{L}\|\big(c_{11} + c_{21} + (c_{12} + c_{22})\tau\big) .$$

Consequently, (50) implies that the ball

$$B_\tau = \Big\{u \in C([0,\pi/4] \times S^T) \mid \quad \|u\| \le \tau\Big\}$$

is mapped to itself by the mapping $F$. The Schauder fixed point theorem ensures the existence of a fixed point $u \in B_\tau$ of $F$. This gives a weak $T$-periodic solution of (1). The proof is finished.

Of course, when $f, g$ have sublinear growth at infinity:

$$\lim_{|u|\to\infty} f(u)/u = 0, \quad \lim_{|u|\to\infty} g(u)/u = 0$$

and $r > 0$, $k > 0$, then the assumptions of Theorem 1 hold and equation (1) possesses a weak $T$-periodic solution $u(x,t) \in C([0,\pi/4] \times S^T)$ for any $h(x,t) \in C([0,\pi/4] \times S^T)$.

The implicit function theorem together with Proposition 3 gives the next result.

**Theorem 2**. *If $r > 0$, $k > 0$, $f(0) = f'(0) = g(0) = g'(0) = 0$ and $f, g \in C^1(S^T)$, then there are positive constants $K_1, \varepsilon_0$ such that for any given function $h(x,t) \in C([0,\pi/4] \times S^T)$ with $\|h\|_\infty < \varepsilon_0$, equation* (1) *possesses a unique small weak $T$-periodic solution $u(x,t) \in C([0,\pi/4] \times S^T)$ satisfying $\|u\|_\infty \le K_1\|h\|_\infty$.*

Now we suppose that $r = 0$ and $k > 0$ in equation (1). Let

$$P : C([0, \pi/4] \times S^T) \times C(S^T)^2 \to C_1$$

be a continuous projection and let

$$C_2 \oplus \mathbb{R}x = C([0, \pi/4] \times S^T)$$

be a continuous splitting $u(x, t) = v(x, t) + c\frac{4}{\pi}x$ with $v(x, t) \in C_2$ and $c \in \mathbb{R}$. Then according to Proposition 4, equation (1) is equivalent to the system

$$v = \lambda K\Big(P\Big(h, f(v(0, \cdot)), g(c + v(\pi/4, \cdot))\Big)\Big), \tag{51}$$

$$\frac{4}{\pi}\int_0^T\int_0^{\pi/4} h(x, t)x\,dx\,dt + \int_0^T g(c + \lambda v(\pi/4, t))\,dt = 0 \tag{52}$$

for $\lambda = 1$. Now we can prove the next result.

**Theorem 3**. *Let $r = 0$ and $k > 0$. If $\sup_{u\in\mathbb{R}} |f(u)| < \infty$, finite limits*

$$\lim_{u\to\pm\infty} g(u) := g_\pm$$

*exist and it holds*

$$\frac{4}{T\pi}\int_0^T\int_0^{\pi/4} h(x, t)x\,dx\,dt \in (-g_-, -g_+)\,. \tag{53}$$

*Then equation (1) possesses a weak $T$-periodic solution $u(x, t) \in C([0, \pi/4] \times S^T)$. On the other hand, if*

$$\Big|\frac{4}{T\pi}\int_0^T\int_0^{\pi/4} h(x, t)x\,dx\,dt\Big| > \sup_{u\in\mathbb{R}} |g(u)| \tag{54}$$

*then equation (1) has no weak $T$-periodic solutions.*

**Proof.** This is a Landesman-Lazer type result [4], [10]. We consider (51) and (52) for $0 \le \lambda \le 1$ on $C_2 \oplus \mathbb{R}$. Since $h, f, g$ are bounded, Proposition 4 implies that any solution of (51) must satisfy $\|v\|_\infty \le K_1$, for a constant $K_1 > 0$. We take the set

$$B = \Big\{(v, c) \in C_2 \oplus \mathbb{R} \mid \ \|v\|_\infty < K_1 + 1, \quad |c| < K_2\Big\}$$

for a fixed large $K_2 > 0$. If $(v, c) \in \partial B$ then either $\|v\|_\infty = K_1 + 1$ and then (51) does not hold, or $\|v\|_\infty \le K_1 + 1$ and $c = \pm K_2$, and then (52) does not hold according to (53). Hence we can apply Leray-Schauder degree to (51) and (52) on $B$ [4], [10]. For $\lambda = 0$ we get a function

$$c \to \frac{4}{T\pi} \int_0^T \int_0^{\pi/4} h(x,t) x \, dx \, dt + g(c) \,,$$

which according to (53) changes the sign on $[-K_2, K_2]$. Consequently, (51) and (52) are solvable on $B$. On the other hand, if (52) holds then clearly (54) can not be satisfied. The proof is finished.

Similarly we get the next result.

**Theorem 4**. *Let $r = 0$ and $k > 0$. If $\sup_{u \in \mathbb{R}} |f(u)| < \infty$, finite limits*

$$\lim_{u \to \pm\infty} g(u) := g_\pm$$

*exist and $g$ is monotonic on $\mathbb{R}$. Then equation* (1) *possesses a weak $T$-periodic solution if* (53) *holds and it has no weak $T$-periodic solutions if*

$$\frac{4}{T\pi} \int_0^T \int_0^{\pi/4} h(x,t) x \, dx \, dt \notin [-g_-, -g_+] \,.$$

*If in addition, $g$ is strictly monotonic on $\mathbb{R}$, then equation* (1) *possesses a weak $T$-periodic solution if and only if* (53) *holds.*

By using Proposition 5, similar arguments hold for the case $r > 0$ and $k = 0$. We state this result for the reader convenience.

**Theorem 5**. *Let $r > 0$ and $k = 0$. If $\sup_{u \in \mathbb{R}} |g(u)| < \infty$, finite limits*

$$\lim_{u \to \pm\infty} f(u) := f_\pm$$

*exist and it holds*

$$\frac{4}{T\pi} \int_0^T \int_0^{\pi/4} h(x,t) \Big( \frac{\pi}{4} - x \Big) \, dx \, dt \in (-f_-, -f_+) \,. \tag{55}$$

*Then equation* (1) *possesses a weak $T$-periodic solution $u(x,t) \in C([0,\pi/4] \times S^T)$. On the other hand, if*

$$\Big| \frac{4}{T\pi} \int_0^T \int_0^{\pi/4} h(x,t) \Big( \frac{\pi}{4} - x \Big) \, dx \, dt \Big| > \sup_{u \in \mathbb{R}} |f(u)| \tag{56}$$

*then equation* (1) *has no weak $T$-periodic solutions.*

Theorem 4 can be also modified for the case $r > 0$ and $k = 0$. Now we study the case that $r = k = 0$ in equation (1). This is a codimension two problem. The above approach to Theorem 3 can be used with the following modifications. Let

$$\bar{P} : C([0, \pi/4] \times S^T) \times C(S^T)^2 \to \bar{C}_1$$

be a continuous projection and let

$$\bar{C}_2 \oplus \mathbb{R}\Big(1 - \frac{4}{\pi}x\Big) \oplus \mathbb{R}x = C([0, \pi/4] \times S^T)$$

be a continuous splitting $u(x,t) = v(x,t) + c_1\Big(1 - \frac{4}{\pi}x\Big) + c_2\frac{4}{\pi}x$ with $v(x,t) \in \bar{C}_2$ and $c_1, c_2 \in \mathbb{R}$. Then according to Proposition 6, equation (1) is equivalent to the system

$$v = \lambda\bar{K}\Big(\bar{P}\Big(h, f(c_1 + v(0,\cdot)), g(c_2 + v(\pi/4,\cdot))\Big)\Big), \tag{57}$$

$$\frac{4}{\pi}\int_0^T\int_0^{\pi/4} h(x,t)\Big(\frac{\pi}{4} - x\Big)\,dx\,dt + \int_0^T f(c_1 + \lambda v(0,t))\,dt = 0 \tag{58}$$

$$\frac{4}{\pi}\int_0^T\int_0^{\pi/4} h(x,t)x\,dx\,dt + \int_0^T g(c_2 + \lambda v(\pi/4,t))\,dt = 0 \tag{59}$$

for $\lambda = 1$. By repeating the proof of Theorem 3 to (57)-(59), we get the next result.

**Theorem 6**. *Let $r = k = 0$. If finite limits*

$$\lim_{u\to\pm\infty} f(u) := f_\pm, \quad \lim_{u\to\pm\infty} g(u) := g_\pm$$

*exist and the both conditions* (53) *and* (55) *hold, then equation* (1) *possesses a weak $T$-periodic solution $u(x,t) \in C([0,\pi/4] \times S^T)$. On the other hand, if one of the conditions* (54) *and* (56) *is satisfied, then equation* (1) *has no weak $T$-periodic solutions.*

Now let us suppose that $f, g \in C^1(\mathbb{R})$. If we consider equation (8) for any $v(x,t) \in C^\infty([0,\pi/4] \times \mathbb{R})$ satisfying the boundary conditions (6) and also orthogonal to each $w_i(x)$, $i = -1, 0, \cdots, i_1$ for $i_1 \in \mathbb{N}$ large and fixed, like in (9). Then we look for $z(x,t)$ in the form

$$z(x,t) = \sum_{i=i_1+1}^{\infty} z_i(t)w_i(x)\,,$$

and we get a result similar to Proposition 2 with an estimate as (b) for $M_2 \to 0$ as $i_1 \to \infty$. Consequently, we can locally reduce by means of the Ljapunov-Schmidt method the solvability of (1) to finite-dimensional mappings. In this way, we can repeat the proof of the Sard-Smale theorem [4], [11] for (1). Moreover, by following a method of [11], we can prove the next result.

**Theorem 7**. *Let the assumptions of Theorem* 1 *hold along with that* $f, g \in C^1(\mathbb{R})$. *Then there is an open and dense subset* $C_3 \subset C([0, \pi/4] \times S^T)$ *such that for any given* $h(x,t) \in C_3$, *equation* (1) *possesses a finite nonzero number of weak* $T$*-periodic solutions* $u(x,t) \in C([0, \pi/4] \times S^T)$. *This number of solutions is constant on each connected components of* $C_3$.

Finally, we note that the question on the existence of a global bounded weak solution of (1) remains open when $h(x,t)$ is only bounded on $[0, \pi/4] \times S^T$. A combination of methods of [1] and this paper would be hopeful.

# References


[1] ALONSO, J. M., MAWHIN, J. and ORTEGA, R. M. Bounded solutions of second order semilinear evolution equations and applications to the telegraph equation, *J. Math. Pures Appl.* **78** (1999), 49-63.

[2] BATTELLI, F. and FEČKAN, M. Chaos in the beam equation, *preprint* (2003).

[3] BATTELLI, F. and FEČKAN, M. Homoclinic orbits of slowly periodically forced and weakly damped beams resting on weakly elastic bearings, *Adv. Differential Equations* **8** (2003), 1043-1080.

[4] BERGER, M. S. Nonlinearity and Functional Analysis, Academic Press, New York 1977.

[5] CAPRIZ, G. Self-excited vibrations of rotors, *Int. Union Theor. Appl. Mech., Symp. Lyngby/Denmark 1974*, Springer-Verlag 1975.

[6] CAPRIZ, G. and LARATTA, A. Large amplitude whirls of rotors, *Vibrations in Rotating Machinery*, Churchill College, Cambridge 1976.

[7] FEČKAN, M. Free vibrations of beams on bearings with nonlinear elastic responses, *J. Differential Equations* **154** (1999), 55-72.

[8] FEČKAN, M. Periodically forced damped beams resting on nonlinear elastic bearings, *Mathematica Slovaca* (to appear).

[9] FEIREISEL, E. Nonzero time periodic solutions to an equation of Petrovsky type with nonlinear boundary conditions: Slow oscillations of beams on elastis bearings, *Ann. Scu. Nor. Sup. Pisa* **20** (1993), 133-146.

[10] MAWHIN, J. Equivalence theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector spaces, *J. Differential Equations* **12** (1972), 610-636.

[11] ŠEDA, V. Fredholm mappings and the generalized boundary value problem, *Diff. Int. Equations* **8** (1995), 19-40.

[12] WIGGINS, S. Global Bifurcations and Chaos, Analytical Methods, *Applied Mathematical Sciences* **73**, Springer-Verlag, New York, Heidelberg, Berlin 1988.

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