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ALMOST SURE SUBEXPONENTIAL DECAY RATES OF SCALAR ITÔ-VOLTERRA EQUATIONS

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ABSTRACT. The paper studies the subexponential convergence of solutions of scalar Itô-Volterra equations. First, we consider linear equations with an instantaneous multiplicative noise term with intensity σ. If the kernel obeys

$$\lim_{t\to\infty} k'(t)/k(t) = 0,$$

and another nonexponential decay criterion, and the solution X_σ tends to zero as $t \to \infty$, then

$$\limsup_{t\to\infty} \frac{\log|X_\sigma(t)|}{\log(tk(t))} = 1 - \Lambda(|\sigma|), \quad \text{a.s.}$$

where the random variable $\Lambda(|\sigma|) \to 0$ as $\sigma \to \infty$ a.s. We also prove a decay result for equations with a superlinear diffusion coefficient at zero. If the deterministic equation has solution which is uniformly asymptotically stable, and the kernel is subexponential, the decay rate of the stochastic problem is exactly the same as that of the underlying deterministic problem.

1. Introduction

In Appleby and Reynolds [2] the asymptotic stability of the scalar deterministic equation

$$x'(t) = -ax(t) + \int_0^t k(t-s)x(s)\,ds, \quad t \ge 0 \tag{1}$$

is studied when k is continuous, positive and integrable and k obeys the nonexponential decay criterion

$$\lim_{t\to\infty} \frac{k'(t)}{k(t)} = 0. \tag{2}$$

If the zero solution of (1) is asymptotically stable, then k must be integrable and we have

$$\liminf_{t\to\infty} \frac{|x(t)|}{k(t)} \ge \frac{|x(0)|}{a(a - \int_0^\infty k(s)\,ds)}.$$

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In Appleby and Reynolds [5] a lower bound is found on the decay rate of a stochastic version of (1). For the linear equation

$$dX(t) = \left(-aX(t) + \int_0^t k(t-s)X(s)\,ds\right)dt + \sigma X(t)\,dB(t), \tag{3}$$

the methods of [2] are extended to establish an almost sure lower bound on the decay rate of solutions of (3) under a weaker hypothesis on the kernel k than (2), namely

$$\lim_{t\to\infty} \sup_{0\le s\le T} \left|\frac{k(t-s)}{k(t)} - 1\right| = 0, \quad \text{for all } T > 0. \tag{4}$$

It was shown in [5] if (3) has a solution which tends to zero on a set A of positive probability, then

$$\limsup_{t\to\infty} \frac{|X(t)|}{k(t)} = \infty, \quad \text{a.s. on } A.$$

In this paper, we seek to use the approach of [5] to impose a sharper lower bound on the decay rate by reimposing the condition (2) on the kernel k. Then, if $X(t) \to 0$ on a set A of positive probability, we prove for any fixed $\varepsilon > 0$ that

$$\limsup_{t\to\infty} \frac{|X(t)|}{k(t)t^{\sigma^2/(\sigma^2+2a)-\varepsilon}} = \infty, \quad \text{a.s. on } A.$$

This result has an important corollary for solutions of (3). When $a > \int_0^\infty k(s)\,ds$, and k decays to zero polynomially according to

$$\lim_{t\to\infty} \frac{\log k(t)}{\log t} = -\alpha,$$

for some $\alpha > 1$, then almost sure decay rate of the solution as the noise intensity becomes arbitrarily large is approximately $tk(t)$, in the sense that

$$\limsup_{t\to\infty} \frac{\log|X_\sigma(t)|}{\log(tk(t))} = 1 - \Lambda(|\sigma|), \quad \text{a.s.}$$

where Λ is a bounded nonnegative random variable with

$$\lim_{|\sigma|\to\infty} \Lambda(|\sigma|) = 0 \quad \text{a.s.}$$

In the second half of the paper, we concentrate on understanding the asymptotic behaviour of scalar Itô-Volterra equations where the state-dependent diffusion term is a nonlinear function of the current state. Specifically, we consider the equation

$$dX(t) = \left(-aX(t) + \int_0^t k(t-s)X(s)\,ds\right)dt + \sigma(X(t))\,dB(t).$$

Intuitively, we might expect the linearisation of this equation to determine the asymptotic behaviour of solutions, and in terms of the conditions required to guarantee a.s. asymptotic stability, it suffices to study the stability of the deterministic linear equation. When the kernel is subexponential, the relationship between the size of the diffusion term close to zero and the speed at which the subexponential kernel decays seems to play a role in determining whether the solutions are a.s. subexponential. For instance, if k is regularly varying at infinity with index $-\alpha < -1$, and $\sigma(x) \sim C|x|^\beta$ for some $\beta > 1$, it is sufficient to have $\alpha > 1 + (2(\beta - 1))^{-1}$ to ensure that the solution is a positive subexponential function, with

$$\lim_{t\to\infty} \frac{X(t)}{k(t)} = \frac{\int_0^\infty X(s)\,ds}{a - \int_0^\infty k(s)\,ds}.$$

However, if k is a subexponential function which decays more quickly, obeying

$$\lim_{t\to\infty} \frac{\log k(t)}{\log t} = -\infty$$

(and another technical condition), there does not seem to be such a restriction on the size of β. We explore and comment upon these questions at greater length in Sections 5-8.

2. Main Results for the Linear Problem

In this paper, $(B(t))_{t\geq 0}$ is a standard one-dimensional Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}^B(t))_{\geq 0}, \mathbb{P})$, where the filtration is the natural one, viz., $\mathcal{F}^B(t) = \sigma(B(s) : 0 \leq s \leq t)$. When almost sure events are referred in this paper, they are always $\mathbb{P}$-almost sure. Consider the scalar (stochastic) Itô-Volterra equation

$$dX(t) = \left(-aX(t) + \int_0^t k(t-s)X(s)\,ds\right) dt + \sigma X(t)\,dB(t), \tag{5}$$

for $t \geq 0$, where a and $\sigma \neq 0$ are real constants. There is no loss of generality incurred by assuming that $\sigma > 0$. The kernel satisfies

$$k(t) \geq 0, \quad k \in L^1(0,\infty), \quad k \in C[0,\infty). \tag{6}$$

As (5) is linear, we may assume $X(0) = 1$ without loss. The fact that (5) has a unique strong solution follows from, for example, Theorem 2E of Berger and Mizel [6].

To ensure that k is not exponentially integrable, we impose, as in [2], the following additional condition:

$$k \in C^1[0,\infty), \quad k(t) > 0 \text{ for } t \geq 0, \quad \lim_{t\to\infty} \frac{k'(t)}{k(t)} = 0. \tag{7}$$

Consequently, for every $\gamma > 0$,

$$\lim_{t\to\infty} k(t)e^{\gamma t} = \infty. \tag{8}$$

The last condition of (7) implies (4), as was pointed out in [2].

We now state the main result of the paper concerning the linear equation (5), and comment upon it.

Theorem 1. *Let k satisfy (7), and $\sigma \neq 0$. Suppose that the unique strong solution of (5) satisfies $\lim_{t\to\infty} X(t) = 0$ on a set A of positive probability. Then, for every $\varepsilon > 0$*

$$\limsup_{t\to\infty} \frac{|X(t)|}{k(t)t^{\frac{1}{1+2a/\sigma^2}-\varepsilon}} = \infty \quad a.s.\ on\ A. \tag{9}$$

Theorem 1 has an important corollary. In order to state it, we first recall the definition of a subexponential function, introduced in [3].

Definition 2. Let $k \in C(\mathbb{R}^+;\mathbb{R}^+)\cap L^1(\mathbb{R}^+)$, with $k(t) > 0$ for all $t \geq 0$. We say that k is a *positive subexponential function* if

$$\lim_{t\to\infty} \frac{(k*k)(t)}{k(t)} = 2\int_0^\infty k(s)\,ds, \tag{US1}$$

$$\lim_{t\to\infty} \frac{k(t-s)}{k(t)} = 1 \quad \text{for each fixed } s > 0. \tag{US2}$$

The class of positive subexponential functions is denoted by $\mathcal{U}$.

A discussion of this class in contained in [3]. Note however that it contains for example, all positive and integrable functions which are regularly varying at infinity, as well as functions positive functions which obey $k(t) \sim Ce^{-t^\alpha}$, as $t \to \infty$, for some $C > 0$ and $\alpha \in (0,1)$. Also observe that a function which obeys (7) satisfies (US2) above.

In Appleby [1] it is shown that $a > \int_0^\infty k(s)\,ds$ implies $X(t) \to 0$ as $t \to \infty$ a.s. In Appleby and Reynolds [4], it is shown that when k is a subexponential function, then

$$\limsup_{t\to\infty} \frac{X(t)}{k(t)t^{1+\varepsilon}} = 0, \quad \text{a.s.}$$

Therefore, if k decays to zero polynomially, in the sense that

$$\lim_{t\to\infty} \frac{\log k(t)}{\log t} = -\alpha, \tag{10}$$

for $\alpha > 1$, then almost sure decay rate of the solution as the noise intensity increases is approximately $tk(t)$. We make this precise in the following theorem.

Theorem 3. *Suppose that k is a subexponential kernel which obeys (7) and (10) and suppose the zero solution of (1) is uniformly asymptotically stable. If the unique nontrivial strong solution of (5) is denoted by X_σ, then*

$$\limsup_{t\to\infty} \frac{\log |X_\sigma(t)|}{\log(tk(t))} = 1 - \Lambda(|\sigma|), \quad a.s. \tag{11}$$

where $\Lambda(|\sigma|)$ is a bounded nonnegative random variable with

$$\lim_{|\sigma|\to\infty} \Lambda(|\sigma|) = 0 \quad a.s.$$

Therefore, the solution decays to zero t times more slowly than the deterministic solution as the noise intensity increases.

This mimics a result obtained in [4] for the subclass of subexponential kernels called superpolynomial kernels, which decay to zero more quickly than a polynomial in the sense that

$$\lim_{t\to\infty} \frac{\log k(t)}{\log t} = -\infty.$$

In [4] it is shown for this class that

$$\limsup_{t\to\infty} \frac{\log X(t)}{\log k(t)} = 1, \quad \text{a.s.}$$

3. Proof of Theorem 1

In order to prove Theorem 1, we first need a technical result on the asymptotic behaviour of a scalar diffusion process. Introduce the process $\overline{Y}_\varepsilon = \{Y_\varepsilon(t) : t \geq 0\}$ which is the unique strong solution of the stochastic differential equation

$$d\overline{Y}_\varepsilon(t) = \big(1 - (a+\varepsilon)\overline{Y}_\varepsilon(t)\big)\, dt + \sigma \overline{Y}_\varepsilon(t)\, dB(t) \tag{12}$$

where $\overline{Y}_\varepsilon(0) = 1$.

Lemma 4. *Let $\varepsilon > 0$ and $a + \sigma^2/2 > 0$. Then the process $\overline{Y}_\varepsilon$ given by (12), with $\overline{Y}_\varepsilon(0) = 1$, obeys*

$$\limsup_{t\to\infty} \frac{\overline{Y}_\varepsilon(t)}{t^{1/(1+2(a+\varepsilon)/\sigma^2)}} = \infty, \quad a.s. \tag{13}$$

Proof. A scale function p for the strictly positive process $\overline{Y}_\varepsilon$ is

$$p(x) = e^{-2/\sigma^2} \int_1^x e^{\frac{2}{\sigma^2}\frac{1}{\xi}} \xi^{\frac{2(a+\varepsilon)}{\sigma^2}}\, d\xi, \quad x > 0.$$

Since $a + \sigma^2/2 > 0$, $\varepsilon > 0$, we have $2(a+\varepsilon)/\sigma^2 > -1$. Therefore $\lim_{x\to\infty} p(x) = \infty$. Using the substitution $\zeta = 1/\xi$, we get

$$\lim_{x\to 0^+} p(x) = -e^{-2/\sigma^2} \int_1^\infty e^{\frac{2}{\sigma^2}\zeta} \zeta^{-(2+2(a+\varepsilon)/\sigma^2)}\, d\zeta = -\infty.$$

If m is a speed measure for $\overline{Y}_\varepsilon$, we have

$$m(0,\infty) = \frac{2}{\sigma^2} e^{2/\sigma^2} \lim_{y\to 0^+} \int_y^\infty e^{-\frac{2}{\sigma^2}\frac{1}{\xi}} \xi^{-(2+2(a+\varepsilon)/\sigma^2)}\, d\xi.$$

Since $2(a+\varepsilon)/\sigma^2 + 2 > 1$, we have

$$\int_1^\infty e^{-\frac{2}{\sigma^2}\frac{1}{\xi}} \xi^{-(2+2(a+\varepsilon)/\sigma^2)}\, d\xi < \infty.$$

Moreover, using the substitution $\zeta = 1/\xi$ once more gives

$$\lim_{y\to 0^+} \int_y^1 e^{-\frac{2}{\sigma^2}\frac{1}{\xi}} \xi^{-(2+2(a+\varepsilon)/\sigma^2)}\, d\xi = \int_1^\infty e^{-\frac{2}{\sigma^2}\zeta} \zeta^{2(a+\varepsilon)/\sigma^2}\, d\zeta < \infty.$$

Thus $m(0,\infty) < \infty$.

Now by e.g., Proposition 5.5.22 in Karatzas and Shreve [8], as $\overline{Y}_\varepsilon$ is the strong solution of a scalar stochastic differential equation with time independent coefficients which obey the usual nondegeneracy and local integrability conditions, and $\overline{Y}_\varepsilon$ has a deterministic initial condition in $(0,\infty)$, the conditions $\lim_{x\to\infty} p(x) = \infty$, $\lim_{x\to 0^+} p(x) = -\infty$ imply that $\overline{Y}_\varepsilon$ is recurrent on $(0,\infty)$. Since it also has a finite speed measure $m(0,\infty) < \infty$, we may apply the result of Motoo [9] (see Itô and McKean [7], Chapter 4.12, equation 6) to the diffusion $\overline{Y}_\varepsilon$.

Motoo's result tells us that if there is a positive and increasing function h such that $h(t) \to \infty$ as $t \to \infty$, and

$$\int_1^\infty \frac{1}{p(h(t))}\, dt = \infty, \tag{14}$$

then

$$\limsup_{t\to\infty} \frac{\overline{Y}_\varepsilon(t)}{h(t)} > 1, \quad \text{a.s.}$$

To apply the result, note as $2(a+\varepsilon)/\sigma^2 + 1 > 0$, that L'Hôpital's rule gives

$$\lim_{x\to\infty} \frac{p(x)}{x^{2(a+\varepsilon)/\sigma^2+1}} = \frac{1}{e^{2/\sigma^2}\left(\frac{2(a+\varepsilon)}{\sigma^2} + 1\right)} > 0.$$

Let $M > 0$ and $h(t) = Mt^{1/(1+2(a+\varepsilon)/\sigma^2)}$. Then

$$\lim_{t\to\infty} \frac{p(h(t))^{-1}}{t^{-1}} = \frac{M^{1+2(a+\varepsilon)/\sigma^2}}{e^{2/\sigma^2}\,(2(a+\varepsilon)/\sigma^2 + 1)} > 0,$$

so with this choice of h, (14) holds. Therefore

$$\limsup_{t\to\infty} \frac{\overline{Y}_\varepsilon(t)}{t^{1/(1+2(a+\varepsilon)/\sigma^2)}} > M, \quad \text{a.s.}$$

Letting $M \to \infty$ through the integers yields (13). □

We can now turn to the proof of Theorem 1.

Proof of Theorem 1. On account of the linearity of (3), we may choose $X(0) = 1$ without loss of generality. Referring to the proof of Theorem 1 in [5], we can show there is a pair of finite positive random variables L_1, T_1 such that

$$\frac{X(t)}{k(t)} \geq L_1 \frac{\phi(t)}{k(t)} \int_{T_1}^{t} k(s)\phi(s)^{-1}\, ds, \quad t \geq T_1 \tag{15}$$

almost surely where $(\phi(t))_{t\geq 0}$ is the positive process given by

$$\phi(t) = e^{-(a+\sigma^2/2)t+\sigma B(t)}, \quad t \geq 0. \tag{16}$$

A careful reading of the proof of Theorem 1 in [5] shows that $X(t) \geq \phi(t)$. Therefore, if $X(t) \to 0$ as $t \to \infty$ on a nontrivial set, it follows that $\phi(t) \to 0$ as $t \to \infty$ on a nontrivial set. This implies that $a + \sigma^2/2 > 0$.

Next, define the process $(Y(t))_{t\geq 0}$ by

$$Y(t) = \phi(t)k(t)^{-1} \int_{0}^{t} k(s)\phi(s)^{-1}\, ds, \quad t \geq 0 \tag{17}$$

and the random function $\tilde{Y}$ by

$$\tilde{Y}(t) = \phi(t)k(t)^{-1} \int_{T_1}^{t} k(s)\phi(s)^{-1}\, ds, \quad t \geq T_1. \tag{18}$$

As $a + \sigma^2/2 > 0$, (8) implies $k(t)\phi(t)^{-1} \to \infty$ as $t \to \infty$, a.s., we therefore get

$$\lim_{t\to\infty} \frac{\tilde{Y}(t)}{Y(t)} = \lim_{t\to\infty} \frac{\int_{T_1}^{t} k(s)\phi(s)^{-1}\, ds}{\int_0^t k(s)\phi(s)^{-1}\, ds} = 1, \quad \text{a.s.} \tag{19}$$

Thus (15) and (17)–(19) now show that it suffices to prove

$$\limsup_{t\to\infty} \frac{Y(t)}{t^{\frac{1}{1+2a/\sigma^2}-\varepsilon}} = \infty, \quad \text{a.s.} \tag{20}$$

for every $\varepsilon > 0$ in order to assure the result. The proof of (20) is the subject of Lemma 5 below. □

Lemma 5. *Suppose k obeys (7) and $a + \sigma^2/2 > 0$. Then the process $(Y(t))_{t\geq 0}$ defined by (17), obeys (20) for every $\varepsilon > 0$.*

Proof. By (16) and (17), and the fact that k is positive and in $C^1(0, \infty)$, it follows that Y obeys the stochastic differential equation

$$dY(t) = \left(1 - \left[a + \frac{k'(t)}{k(t)}\right] Y(t)\right) dt + \sigma Y(t)\, dB(t)$$

with $Y(0) = 0$. By (7), there exists $0 \le c < \infty$ such that

$$c = \sup_{t \ge 0} \left| \frac{k'(t)}{k(t)} \right|.$$

Therefore, the process does not explode in finite time, a.s. Therefore, for any fixed deterministic time $T > 0$, it follows that $0 < Y(T) < \infty$, a.s., as Y is a strictly positive process on $(0, \infty)$.

Next, let $\varepsilon > 0$. By (7), there exists $T = T(\varepsilon) > 0$ such that

$$\left| \frac{k'(t)}{k(t)} \right| \le \frac{\varepsilon}{2}, \quad t \ge T(\varepsilon).$$

Next, define the process $Y_\varepsilon = \{Y_\varepsilon(t) \, : \, t \ge T(\varepsilon)\}$ so that

$$Y_\varepsilon(t) = Y(T(\varepsilon)) + \int_{T(\varepsilon)}^t 1 - (a+\varepsilon) Y_\varepsilon(s) \, ds + \int_{T(\varepsilon)}^t \sigma Y_\varepsilon(s) \, dB(s), \tag{21}$$

for $t \ge T(\varepsilon)$. We prove momentarily that

$$Y_\varepsilon(t) \le Y(t), \quad t \ge T(\varepsilon), \quad \text{a.s.} \tag{22}$$

Also recall that Lemma 4 implies (13). Now we show that

$$\lim_{t \to \infty} \frac{Y_\varepsilon(t)}{\overline{Y}_\varepsilon(t)} = 1, \quad \text{a.s.} \tag{23}$$

where $\overline{Y}_\varepsilon$ is defined by (12) with $\overline{Y}_\varepsilon(0) = 1$.

To see this, introduce the process $(\tilde{\phi}(t))_{t \ge T(\varepsilon)}$ which obeys

$$\tilde{\phi}(t) = e^{-(a+\sigma^2/2)(t - T(\varepsilon)) + \sigma(B(t) - B(T(\varepsilon)))}, \quad t \ge T(\varepsilon).$$

Then $\tilde{\phi}(t) = \phi(t)\phi(T(\varepsilon))^{-1}$, $t \ge T(\varepsilon)$, and the processes Y_ε, $\overline{Y}_\varepsilon$ defined by (21), (22) are explicitly given by

$$\begin{aligned}
\overline{Y}_\varepsilon(t) &= \phi(t) e^{-\varepsilon t} + \int_0^t e^{-\varepsilon(t-s)} \phi(t)\phi(s)^{-1} \, ds, \\
Y_\varepsilon(t) &= \tilde{\phi}(t) Y(T(\varepsilon)) e^{-\varepsilon(t - T(\varepsilon))} + \int_{T(\varepsilon)}^t e^{-\varepsilon(t-s)} \tilde{\phi}(t) \tilde{\phi}(s)^{-1} \, ds,
\end{aligned}$$

where $\overline{Y}_\varepsilon(t)$ is defined for all $t \ge 0$, and $Y_\varepsilon(t)$ for all $t \ge T(\varepsilon)$, respectively. Thus, for $t \ge T(\varepsilon)$

$$\frac{Y_\varepsilon(t)}{\overline{Y}_\varepsilon(t)} = \frac{Y(T(\varepsilon)) e^{\varepsilon T(\varepsilon)} + \int_{T(\varepsilon)}^t e^{\varepsilon s} \tilde{\phi}(s)^{-1} \, ds}{\phi(T(\varepsilon)) + \int_0^t e^{\varepsilon s} \tilde{\phi}(s)^{-1} \, ds}. \tag{24}$$

Since $a + \sigma^2/2 > 0$, $e^{\varepsilon t} \tilde{\phi}(t)^{-1} \to \infty$ as $t \to \infty$ for any $\varepsilon > 0$. Therefore, applying L'Hôpital's rule to the righthand side of (24) enables us to conclude (23).

Now, by (13), (22), and (23), as $\overline{Y}_\varepsilon(t) > 0$ for all $t \geq 0$, a.s., we have

$$\begin{aligned}\limsup_{t\to\infty} \frac{Y(t)}{t^{1/(1+2(a+\varepsilon)/\sigma^2)}} &\geq \limsup_{t\to\infty} \frac{Y_\varepsilon(t)}{t^{1/(1+2(a+\varepsilon)/\sigma^2)}} \\ &= \limsup_{t\to\infty} \frac{\overline{Y}_\varepsilon(t)}{t^{1/(1+2(a+\varepsilon)/\sigma^2)}} \frac{Y_\varepsilon(t)}{\overline{Y}_\varepsilon(t)} = \infty, \quad \text{a.s.}\end{aligned}$$

Clearly, if

$$\limsup_{t\to\infty} \frac{Y(t)}{t^{1/(1+2(a+\varepsilon')/\sigma^2)}} = \infty, \quad \text{a.s.}$$

for all $\varepsilon' > 0$, this implies that (20) holds for every $\varepsilon > 0$.

We return finally to the proof of (22). Introduce the strictly positive process $(\psi(t))_{t\geq T(\varepsilon)}$ which is the unique strong solution of

$$\psi(t) = 1 + \int_{T(\varepsilon)}^{t} \sigma\psi(s)\, dB(s), \quad t \geq T(\varepsilon),$$

and also define the processes $(Z(t))_{t\geq T(\varepsilon)}$ and $(Z_\varepsilon(t))_{t\geq T(\varepsilon)}$ by $Z(t) = Y(t)\psi(t)^{-1}$, $Z_\varepsilon(t) = Y_\varepsilon(t)\psi(t)^{-1}$. Then $Z(T(\varepsilon)) = Z_\varepsilon(T(\varepsilon))$, and Z, Z_ε are positive. Using integration by parts, we get

$$\begin{aligned} Z'(t) &= \psi(t)^{-1} - \left(a + \frac{k'(t)}{k(t)}\right) Z(t), \quad t > T(\varepsilon), \\ Z_\varepsilon'(t) &= \psi(t)^{-1} - (a+\varepsilon)\, Z(t), \quad t > T(\varepsilon). \end{aligned}$$

Consider $D(t) = Z(t) - Z_\varepsilon(t)$ for $t \geq T(\varepsilon)$. Clearly, $D(T(\varepsilon)) = 0$ and

$$D'(t) = -(a+\varepsilon)D(t) + \left(\varepsilon - \frac{k'(t)}{k(t)}\right) Z(t), \quad t > T(\varepsilon).$$

Since $(\varepsilon - k'(t)/k(t))Z(t) > 0$ for all $t \geq T(\varepsilon)$, it follows that $D(t) > 0$ for all $t > T(\varepsilon)$ a.s. Therefore $Z(t) \geq Z_\varepsilon(t)$ for $t \geq T(\varepsilon)$, so by construction, (22) holds. □

The result of Theorem 1 does not rely directly on the hypothesis that $X(t) \to 0$ on a set of positive probability. In fact, by studying the proof of Theorem 1, it is apparent that the hypothesis $a + \sigma^2/2 > 0$ may be used in place of the asymptotic stability of the solution. Therefore, we have the following corollary of Theorem 1.

Corollary 6. *Let k satisfy (7) and $\sigma \neq 0$. If X is a nontrivial strong solution of (3), and $a + \sigma^2/2 > 0$, then*

$$\limsup_{t\to\infty} \frac{|X(t)|}{k(t)t^{\frac{1}{1+2a/\sigma^2}-\varepsilon}} = \infty, \quad a.s.$$

for every $\varepsilon > 0$.

An interesting consequence of this result is the following: if k obeys (7) and there exists $\varepsilon > 0$ such that

$$\lim_{t\to\infty} k(t) t^{\frac{1}{1+2a/\sigma^2}-\varepsilon} = \infty, \tag{25}$$

then every nontrivial solution of (3) obeys

$$\limsup_{t\to\infty} |X(t)| = \infty, \quad \text{a.s.}$$

For $a > 0$, this result cannot arise for integrable kernels, as (25) is not consistent with k being integrable. However, if $a < 0$ (when the deterministic problem (1) is unstable) but $a + \sigma^2/2 > 0$ (so the stochastic problem without memory is almost surely asymptotically stable), the solution of (3) is unstable if the kernel decays too slowly.

This result also rules out a natural conjecture for the almost sure asymptotic stability of solutions of (3). All solutions of the deterministic equation $x'(t) = -ax(t)$ tend to zero if and only if $a > 0$, while all solutions of the stochastic equation

$$dX(t) = -aX(t)\,dt + \sigma X(t)\,dB(t)$$

tend to zero almost surely if and only if $a + \sigma^2/2 > 0$. A necessary and sufficient condition for all solutions of (1) to be uniformly asymptotically stable is $a > \int_0^\infty k(s)\,ds$. On the basis of these three stability results, one might therefore conjecture that all solutions of (3) would be asymptotically stable (on a set of positive probability) whenever $a + \sigma^2/2 > \int_0^\infty k(s)\,ds$. However, if k is a positive, continuously differentiable and integrable function which obeys

$$\lim_{t\to\infty} \frac{k'(t)}{k(t)} = 0, \quad \lim_{t\to\infty} k(t) t^{\frac{\sigma^2}{\sigma^2+2a}-\varepsilon} = \infty \quad \text{for some } \varepsilon > 0$$

while $a < 0$ and $a + \sigma^2/2 > \int_0^\infty k(s)\,ds > 0$, we have

$$\limsup_{t\to\infty} |X(t)| = \infty \quad \text{a.s.}$$

Therefore the condition

$$a + \sigma^2/2 > \int_0^\infty k(s)\,ds$$

is *not* sufficient to ensure the asymptotic stability of solutions of (3), even on a set of positive probability.

4. Proof of Theorem 3

We now prove Theorem 3, which uses the result of Theorem 1.

Proof of Theorem 3. If the deterministic solution of (1) is uniformly asymptotically stable, then $a > \int_0^\infty k(s)\,ds > 0$. This condition ensures that $\lim_{t\to\infty} X_\sigma(t) = 0$, a.s., where X_σ denotes the solution of (3). Since k is a subexponential function, a result in [4] tells us that

$$\limsup_{t\to\infty} \frac{|X_\sigma(t)|}{k(t)t^{1+\varepsilon}} = 0, \quad \text{a.s.}$$

Since k obeys (10), this implies

$$\limsup_{t\to\infty} \frac{\log|X_\sigma(t)|}{\log t} \leq -\alpha + 1 + \varepsilon, \quad \text{a.s.}$$

Letting $\varepsilon \downarrow 0$ through the rational numbers, and using (10) once again gives

$$\limsup_{t\to\infty} \frac{\log|X_\sigma(t)|}{\log(tk(t))} \leq 1, \quad \text{a.s.} \tag{26}$$

Since $X_\sigma(t) \to 0$ as $t \to \infty$ a.s., and k obeys (7), Theorem 3 implies

$$\limsup_{t\to\infty} \frac{|X_\sigma(t)|}{k(t)t^{\frac{1}{1+2a/\sigma^2}-\varepsilon}} = \infty, \quad \text{a.s.}$$

for every $\varepsilon > 0$. Thus, as k satisfies (10), the last equation implies

$$\limsup_{t\to\infty} \frac{\log|X_\sigma(t)|}{\log t} \geq -\alpha + \frac{1}{1+2a/\sigma^2} - \varepsilon, \quad \text{a.s.}$$

Letting $\varepsilon \downarrow 0$ through the rational numbers, and using (10) once again gives

$$\limsup_{t\to\infty} \frac{\log|X_\sigma(t)|}{\log(tk(t))} \geq \frac{1}{1+2a/\sigma^2}, \quad \text{a.s.} \tag{27}$$

Taking (26), (27) together, we see there exists a bounded non-negative random variable $\Lambda(\sigma)$ such that $0 \leq \Lambda(\sigma) \leq \frac{2a}{\sigma^2+2a}$ a.s. and

$$\limsup_{t\to\infty} \frac{\log|X_\sigma(t)|}{\log(tk(t))} = 1 - \Lambda(\sigma), \quad \text{a.s.}$$

while $\Lambda(\sigma) = \mathcal{O}(\sigma^{-2})$ as $|\sigma| \to \infty$, so $\Lambda(\sigma) \to 0$ as $\sigma \to \infty$ almost surely. $\square$

5. Subexponential solutions of scalar nonlinear equations

In [5], it was shown that the decay rate of (3) differs from that of (1) when k is a subexponential function; in particular the a.s. rate of decay of the stochastic equation is slower. Moreover, although we have presented sharp upper bounds for the rate of decay for the equation (3) for superpolynomial functions, and also for regularly varying functions in this paper, we have not exhibited exact rates of decay for stochastic equations with subexponential kernels. In this section, we ask whether it is possible for a class of scalar stochastic equations to exhibit a.s. subexponential asymptotic behaviour.

Intuitively, it would appear that the decay rate of (3) is slower than that of (1) when k is subexponential on account of the strength of the state-dependent stochastic perturbation as the solution approaches zero. Therefore, we might conjecture that the solution of a stochastic perturbation of (1) would have the same asymptotic behaviour as (1) if the state-dependent diffusion term is sufficiently small.

5.1. **Problem to be studied; main results.** In this section, we consider the scalar Itô-Volterra equation

$$dX(t) = \left(-aX(t) + \int_0^t k(t-s)X(s)\,ds\right) dt + \sigma(X(t))\,dB(t) \tag{28}$$

with $X(0) > 0$. Here we assume $\sigma(0) = 0$, σ is locally Lipschitz continuous and has a global linear bound. We also assume that σ does not have a linear leading order term at the origin by imposing

$$\lim_{x\to 0} \frac{\sigma(x)}{x} = 0. \tag{29}$$

By adapting results in [1], it is possible to show that the linearisation of (28), namely (1), has the same asymptotic behaviour as (28). This fact is made precise in the following theorem.

Theorem 7. *Suppose that k is a positive, continuous and integrable function. Let X be the unique nontrivial strong solution of (28) with $X(0) \neq 0$. Suppose the zero solution of (1) is uniformly asymptotically stable. If $\sigma(0) = 0$, σ is locally Lipschitz continuous and obeys a global linear bound, then*

$$X \in L^1(\mathbb{R}^+), \quad \lim_{t\to\infty} X(t) = 0, \quad a.s. \tag{30}$$

The proof is very similar to that of results presented in [1], so it is not given.

It is also possible to establish estimates on the decay rate of solutions of (28) when k is subexponential without making a stronger assumption on the nature of the nonlinearity of the function σ at zero.

Theorem 8. *Suppose that k is a positive, continuous and integrable function. Let X be the unique nontrivial strong solution of (28) with $X(0) \neq 0$. Suppose the zero solution of (1) is uniformly asymptotically stable. If $\sigma(0) = 0$, σ is locally Lipschitz continuous and obeys a global* *linear bound, then*

$$\limsup_{t\to\infty} \frac{X(t)}{k(t)} \geq \frac{\int_0^\infty X(s)\,ds}{a}, \quad a.s. \tag{31}$$

Moreover, if k is subexponential, then for every $\varepsilon > 0$

$$\limsup_{t\to\infty} \frac{X(t)}{k(t)t^{1+\varepsilon}} = 0, \quad a.s. \tag{32}$$

Again, since very similar results are proven in [4], we do not present a proof here.

A simple corollary of Theorem 8 is the following: if k is a superpolynomial function (31) and (32) can be combined to give the following sharp estimate on the asymptotic rate of decay of solutions of (28).

Corollary 9. *Let k is a positive subexponential function which obeys*

$$\lim_{t\to\infty} \frac{\log k(t)}{\log t} = -\infty, \tag{33}$$

and X be the unique nontrivial strong solution of (28) with $X(0) \neq 0$. Suppose the zero solution of (1) is uniformly asymptotically stable. If $\sigma(0) = 0$, σ is locally Lipschitz continuous and obeys a global linear bound, then

$$\limsup_{t\to\infty} \frac{\log X(t)}{\log k(t)} = 1, \quad a.s. \tag{34}$$

These results will be of great use in determining more precise decay rates of solutions of (28). In this section, the main emphasis is placed on determining conditions under which solutions of (28) have exactly the same asymptotic behaviour as (1), when k is a positive subexponential function. In other words, we determine sufficient conditions which ensure

$$\lim_{t\to\infty} \frac{X(t)}{k(t)} = \frac{\int_0^\infty X(s)}{a - \int_0^\infty k(s)\,ds}, \quad \text{a.s.} \tag{35}$$

and also that a.a. paths of X are positive subexponential functions. Since solutions of (28) are nowhere differentiable, it is impossible to show that paths of X are smooth subexponential functions, as are the solutions of (1). However, it is possible to show that

$$\lim_{t\to\infty} \frac{X(t)}{\int_t^\infty X(s)\,ds} = 0, \quad \text{a.s.} \tag{36}$$

To prove these results, we assume that σ has a polynomial leading order term at zero, in the sense that there is $C \in [0, \infty)$ and $\beta > 1$ such that

$$\limsup_{x\to\infty} \frac{|\sigma(x)|}{|x|^\beta} = C. \tag{37}$$

We prove results in the cases where (a) k is subexponential with polynomial asymptotic behaviour, obeying

$$\lim_{t\to\infty} \frac{\log k(t)}{\log t} = -\alpha, \tag{38}$$

for some $\alpha > 1$, and (b) when k is subexponential with superpolynomial asymptotic behaviour, and obeys (33).

Theorem 10. *Suppose k is a positive subexponential function which obeys (38) for some $\alpha > 1$. Let $\sigma(0) = 0$, σ be locally Lipschitz continuous, obey a global linear bound, and obey (37) for some $\beta > 1$. If $a > \int_0^\infty k(s)\,ds$,*

$$\alpha > 1 + \frac{1}{2(\beta - 1)}, \tag{39}$$

and X is the unique nontrivial strong solution of (28) with $X(0) > 0$, then it satisfies each of the following:

(i) *X obeys (35), a.s.,*
(ii) *$X \in \mathcal{U}$, a.s.,*
(iii) *X obeys (36), a.s.*

In the superpolynomial case, we have the following result.

Theorem 11. *Suppose k is a positive subexponential function which obeys (33) and also satisfies*

$$\limsup_{t\to\infty} \sup_{\gamma\geq 1} \frac{\log k(t)}{\log k(\gamma t)} \leq 1. \tag{40}$$

Let σ be locally Lipschitz continuous and globally linearly bounded function, with $\sigma(0) = 0$ which obeys (37) for some $\beta > 1$. If $a > \int_0^\infty k(s)\,ds$, and X is the unique nontrivial strong solution of (28) with $X(0) > 0$, it satisfies each of the following:

(i) *X obeys (35), a.s.,*
(ii) *$X \in \mathcal{U}$, a.s.,*
(iii) *X obeys (36), a.s.*

The condition (40) indicates that $-\log k(t)$ behaves similarly to a regularly varying function at infinity of order zero. (40) is obeyed by many important superpolynomial and subexponential functions, such as $k(t) \sim e^{-t^\alpha}$ for $\alpha \in (0,1)$. We note also that (40) is satisfied if k is ultimately nonincreasing i.e., there exists $T > 0$ such that k is nonincreasing on $[T,\infty)$.

An interesting open question is to ask to what extent the ancillary hypotheses in Theorems 10 and 11 are essential. For instance, is the constraint (39) purely technical, or does it reflect a requirement that exact asymptotic estimates are possible only when the noise perturbation is sufficiently small and the rate of decay of the kernel sufficiently fast. In particular, we do not know whether necessary and sufficient conditions of the form (39) could be developed in Theorem 10 so that the limit

$$\lim_{t\to\infty} \frac{X(t)}{k(t)}$$

exists.

5.2. **Preliminary analysis.** We start with some general observations which will be necessary in proving Theorems 10, 11.

By Theorem 7, the process X which is a solution of (28) is in $L^1(\mathbb{R}^+)$ and $X(t) \to 0$ as $t \to \infty$, a.s. Therefore, as $|\sigma(x)| \le L|x|$ for some $L \ge 0$ and all $x \in \mathbb{R}$, it follows that $|\sigma(X(t))| \le L|X(t)|$ for all $t \ge 0$. Since $X \in L^1(\mathbb{R}^+)$ and $X(t) \to 0$ as $t \to \infty$, it follows that $X \in L^2(\mathbb{R}^+)$ a.s. Therefore

$$\int_0^\infty \sigma(X(s))^2\, ds < \infty, \quad \text{a.s.}$$

and consequently, by the martingale time change theorem

$$\lim_{t\to\infty} \int_0^t \sigma(X(s))\, dB(s) = \int_0^\infty \sigma(X(s))\, dB(s), \quad \text{a.s.}$$

where the limit on the righthand side is a.s. finite. Define

$$\Omega_0 = \{\omega \in \Omega \,:\, X(\cdot,\omega) \in L^1(\mathbb{R}^+),\ \lim_{t\to\infty} X(t,\omega) = 0\},$$

and

$$\Omega_1 = \{\omega \in \Omega \,:\, \lim_{t\to\infty} \left(\int_0^t \sigma(X(s))\, dB(s) \right)(\omega) \ \text{exists}\}.$$

For $\omega \in \Omega_1$, we may define the random function $T(\omega) : \mathbb{R}^+ \to \mathbb{R}$ by

$$\begin{aligned} T(t,\omega) = -&\left(\int_0^\infty \sigma(X(s))\, dB(s) - \int_0^t \sigma(X(s))\, dB(s) \right)(\omega) \\ &=: -\left(\int_t^\infty \sigma(X(s))\, dB(s) \right)(\omega), \end{aligned}$$

so that $\lim_{t\to\infty} T(t,\omega) = 0$. Next, introduce the process Y such that

$$Y(t) = X(0) + \int_0^t \sigma(X(s))\, dB(s).$$

Then for $\omega \in \Omega_0 \cap \Omega_1$

$$\lim_{t\to\infty} Y(t,\omega) = X(0)(\omega) + \left(\int_0^\infty \sigma(X(s))\, dB(s) \right)(\omega) =: Y^*(\omega)$$

where $Y^*(\omega)$ is finite. For each $\omega \in \Omega_0 \cap \Omega_1$, we introduce

$$U(t,\omega) = X(t,\omega) - Y(t,\omega) + Y^*(\omega) = X(t,\omega) + T(t,\omega),$$

so that $\lim_{t\to\infty} U(t,\omega) = 0$ for $\omega \in \Omega_0 \cap \Omega_1$. Notice also that $T(t,\omega) = Y(t,\omega) - Y^*(\omega)$. The process Z defined by $Z(t) = X(t) - Y(t)$ obeys

$$Z(t) = Z(0) + \int_0^t -aX(s) + \int_0^s k(s-u)X(u)\,du\,ds.$$

Therefore, as k and $X(\omega)$ are continuous functions, the function $Z(\omega)$ is in $C^1(\mathbb{R}^+)$. Therefore, for each $\omega \in \Omega_0 \cap \Omega_1$, since $U(t,\omega) = Z(t,\omega) + Y^*(\omega)$, we have

$$\begin{aligned}
U'(t,\omega) &= Z'(t,\omega) \\
&= -aX(t,\omega) + \int_0^t k(t-s)X(s,\omega)\,ds \\
&= -a(U(t,\omega) + T(t,\omega)) + \int_0^t k(t-s)(U(s,\omega) + T(s,\omega))\,ds \\
&= -aU(t,\omega) + \int_0^t k(t-s)U(s,\omega)\,ds + f(t,\omega)
\end{aligned}$$

where

$$f(t,\omega) = -aT(t,\omega) + \int_0^t k(t-s)T(s,\omega)\,ds.$$

Since $\Omega_0 \cap \Omega_1$ is almost sure, we have proven the following result.

Lemma 12. *Suppose that k is a positive and integrable function and that $a > \int_0^\infty k(s)\,ds$. Then for each ω in an almost sure set Ω^*, the realisation $X(\omega)$ can be represented as*

$$X(t,\omega) = U(t,\omega) + T(t,\omega) \tag{41}$$

where $T(\omega)$ is the function defined by

$$T(t,\omega) = -\left(\int_t^\infty \sigma(X(s))\,dB(s)\right)(\omega), \tag{42}$$

$U(\omega)$ solves the equation

$$U'(t,\omega) = -aU(t,\omega) + \int_0^t k(t-s)U(s,\omega)\,ds + f(t,\omega), \tag{43}$$

and $f(\omega)$ is the function

$$f(t,\omega) = -aT(t,\omega) + \int_0^t k(t-s)T(s,\omega)\,ds. \tag{44}$$

We use Lemma 12 to obtain the asymptotic behaviour of (28) as follows: Theorem 8 yields an a priori upper estimate on the almost sure decay rate of solutions of (28), so there is an estimate on the decay rate of

$$t \mapsto \int_t^\infty \sigma(X(s))^2 \, ds$$

as $\sigma(x)^2 \leq C'|x|^{2\beta}$ for x sufficiently small. This decay rate can then be linked to that of T by proving the following Lemma, whose proof is relegated to the Appendix.

Lemma 13. *Suppose that B is a standard Brownian motion with natural filtration $(\mathcal{F}^B(t))_{t\geq 0}$. Suppose that $A = \{A(t) : 0 \leq t < \infty; \mathcal{F}^B(t)\}$ has continuous sample paths, and satisfies $A(t) > 0$ for all $t \geq 0$, a.s., and $A \in L^2(\mathbb{R}^+)$, a.s. Then*

$$\limsup_{t\to\infty} \frac{\left|\int_t^\infty A(s)\, dB(s)\right|}{\sqrt{2\int_t^\infty A(s)^2\, ds \log\log\left(\int_t^\infty A(s)^2\, ds\right)^{-1}}} = 1, \quad a.s. \tag{45}$$

Once an upper estimate on the decay rate of T has been so obtained, an upper estimate on the decay rate of f given by (44) is known, and hence, by a variation of parameters argument, the decay rate of U obeying (43) is determined. Since upper bounds on the decay rate of U and T are now known, by (41), we have a new upper estimate on the a.s. decay rate of X. If the new estimate on the decay rate is faster, the argument can be iterated as often as necessary to obtain ever sharper estimates on the rate of decay of the process. If at any stage in this iteration it can be shown that the decay rate of T to zero is faster than that of k, it is then possible to prove that X enjoys the same decay rate as k.

The proof of Theorem 11 requires one iteration of this argument, while that of Theorem 10 may require several iterations.

6. Subexponential solutions

We start by proving the last claim above; namely, if T decays quickly enough, then $X(t)/k(t)$ tends to a well-defined finite limit as $t \to \infty$. This then enables us to conclude that X is an almost surely positive subexponential function, which obeys (36).

Lemma 14. *Let Ω^* be an almost sure set, and T defined by (42) obey*

$$\lim_{t\to\infty} \frac{T(t)}{k(t)} = 0, \tag{46}$$

on Ω^*. If k is a positive subexponential function and $a > \int_0^\infty k(s)\,ds$, then

$$\lim_{t\to\infty} \frac{X(t)}{k(t)} = \frac{\int_0^\infty X(s)\,ds}{a - \int_0^\infty k(s)\,ds}$$

on Ω^*.

Proof. If (46) holds then $T(\omega) \in L^1(\mathbb{R}^+)$ because $k \in L^1(\mathbb{R}^+)$. Thus by (44) and Theorem 4.1 in [3]
(47)

$$\lim_{t\to\infty} \frac{f(t,\omega)}{k(t)} = -a \lim_{t\to\infty} \frac{T(t,\omega)}{k(t)} + \lim_{t\to\infty} \frac{(k * T(\omega))(t)}{k(t)} = \int_0^\infty T(s,\omega)\,ds.$$

Now, by (44), as $T(\omega)$ and k are integrable, we have

$$\int_0^\infty f(s,\omega)\,ds = -(a - \int_0^\infty k(s)\,ds) \int_0^\infty T(s,\omega)\,ds. \tag{48}$$

Therefore, by Corollary 6.3 in [3] the solution of (43) obeys

$$\lim_{t\to\infty} \frac{U(t,\omega)}{k(t)} = \frac{1}{(a - \int_0^\infty k(s)\,ds)^2} \left(U(0,\omega) + \int_0^\infty f(s,\omega)\,ds \right) + \frac{L_k f(\omega)}{a - \int_0^\infty k(s)\,ds},$$

where $L_k f(\omega) = \lim_{t\to\infty} f(t,\omega)/k(t)$. Using (47), (48) this simplifies to give

$$\lim_{t\to\infty} \frac{U(t,\omega)}{k(t)} = \frac{U(0,\omega)}{(a - \int_0^\infty k(s)\,ds)^2},$$

which, on account of (46), yields

$$\lim_{t\to\infty} \frac{X(t)}{k(t)} = \frac{U(0,\omega)}{(a - \int_0^\infty k(s)\,ds)^2}. \tag{49}$$

Now
(50)

$$U(0,\omega) = X(0,\omega) - T(0,\omega) = X(0,\omega) + \left(\int_0^\infty \sigma(X(s))\,dB(s) \right)(\omega).$$

But, as $X \in L^1(\mathbb{R}^+)$, writing (28) in integral form, noting that

$$\lim_{t\to\infty} \int_0^t \sigma(X(s))\,dB(s)$$

exists a.s., and that $X(t) \to 0$ as $t \to 0$ a.s., we may let $t \to \infty$ to obtain

$$
\begin{aligned}
-X(0) = -a \int_0^\infty X(s)\,ds + \int_0^\infty k(s)\,ds \int_0^\infty X(s)\,ds \\
+ \int_0^\infty \sigma(X(s))\,dB(s), \quad \text{a.s.}
\end{aligned}
$$

Hence

$$
(a - \int_0^\infty k(s)\,ds) \int_0^\infty X(s)\,ds = X(0) + \int_0^\infty \sigma(X(s))\,dB(s), \tag{51}
$$

so by (49), (50), and (51), we have the required result. □

Once this Lemma is proved, we can show that X is a positive subexponential function which obeys (36).

Lemma 15. *Let Ω^* be an almost sure set, and T defined by (42) obey (46) on Ω^*. If k is a positive subexponential function, $a > \int_0^\infty k(s)\,ds$, and $X(0) > 0$, then all the conclusions of Theorem 10 and 11 hold.*

Proof. By considering the line of proof of Theorem 1 in [5], it is possible to show that whenever $X(0) > 0$, then $X(t) > 0$ for all $t \geq 0$, a.s. Therefore, as $X \in L^1(\mathbb{R}^+)$ (by Theorem 7) it follows that

$$
\int_0^\infty X(s)\,ds > 0, \quad \text{a.s.}
$$

and so

$$
\lim_{t\to\infty} \frac{X(t)}{k(t)} = \frac{\int_0^\infty X(s)\,ds}{a - \int_0^\infty k(s)\,ds} > 0, \quad \text{a.s.}
$$

Since k is a positive subexponential function, and

$$
L_k X = \lim_{t\to\infty} X(t)/k(t)
$$

exists a.s., by Lemma 4.3 in [3], X satisfies (US1) and (US2). Since X is positive and

$$
\lim_{t\to 0^+} \frac{X(t)}{k(t)} = \frac{X(0)}{k(0)} > 0
$$

Lemma 4.3 in [3] further enables us to conclude that X is positive subexponential, so (ii) of Theorem 10 and Theorem 11 follow. By Lemma 14, we know that part (i) of these Theorems also hold. It therefore is necessary to prove merely that part (iii) of Theorem 10 and 11 hold.

Since (35) holds, we have

$$
\lim_{t\to\infty} \frac{\int_t^\infty X(s)\,ds}{\int_t^\infty k(s)\,ds} = \frac{\int_0^\infty X(s)\,ds}{a - \int_0^\infty k(s)\,ds}, \quad \text{a.s.}
$$

Therefore, (36) holds if

$$\lim_{t\to\infty} \frac{k(t)}{\int_t^\infty k(s)\,ds} = 0. \tag{52}$$

Let $N \in \mathbb{N}$, so we have

$$\frac{1}{k(t+N)} \int_t^\infty k(s)\,ds \geq \int_t^{t+N} \frac{k(s)}{k(t+N)}\,ds = \int_0^N \frac{k(t+N-s)}{k(t+N)}\,ds.$$

By property (US1) of positive subexponential functions, for each $N \in \mathbb{N}$, there is $T(N) > 0$ such that $t > T(N)$ implies

$$\frac{k(t+N-s)}{k(t+N)} > \frac{1}{2}, \quad 0 \leq s \leq N.$$

Therefore, for $t > T(N)$

$$\frac{1}{k(t+N)} \int_t^\infty k(s)\,ds \geq \frac{N}{2}.$$

Thus, for fixed $N \in \mathbb{N}$, the fact that k obeys (US1) implies

$$\lim_{t\to\infty} \frac{k(t+N)}{k(t)} = 1.$$

and so

$$\liminf_{t\to\infty} \frac{\int_t^\infty k(s)\,ds}{k(t)} \geq \liminf_{t\to\infty} \frac{k(t+N)}{k(t)} \frac{1}{k(t+N)} \int_t^\infty k(s)\,ds \geq \frac{N}{2}.$$

Letting $N \to \infty$ proves (52), and hence the result. $\square$

7. Proof of Theorem 11

We are now in a position to establish (46) under the hypotheses of Theorem 11, which, by the analysis in the previous section, ensures that Theorem 11 holds.

Proof of Theorem 11. By Lemma 13, we have $\int_0^\infty \sigma(X(s))^2\,ds < \infty$ a.s., so by the definition of T, we have

$$\limsup_{t\to\infty} \frac{|T(t)|}{\sqrt{2\int_t^\infty \sigma(X(s))^2\,ds \log\log\left(\int_t^\infty \sigma(X(s))^2\,ds\right)^{-1}}} = 1 \quad \text{a.s.}$$

Suppose this holds on the almost sure set Ω_2, which also contains the sets Ω_0 and Ω_1 defined in Section 5. Therefore (46) is true once

$$\lim_{t\to\infty} \frac{\int_t^\infty \sigma(X(s))^2\,ds \log\log\left(\int_t^\infty \sigma(X(s))^2\,ds\right)^{-1}}{k(t)^2} = 0, \quad \text{a.s.} \tag{53}$$

By (37) and the continuity of σ there is $x^* > 0$ such that $|x| < x^*$ implies

$$|\sigma(x)| \leq (1+C)|x|^\beta.$$

Since $X(t) \to 0$ as $t \to \infty$ on Ω_2, there is $T_1(\omega) > 0$ such that $|X(t,\omega)| < x^*$ for all $t > T_1(\omega)$ and $\omega \in \Omega_2$. Hence, for $t > T_1(\omega)$,

$$\sigma(X(t,\omega))^2 \leq (1+C)^2|X(t,\omega)|^{2\beta}.$$

By (32) in Theorem 8, for every $\varepsilon > 0$ there is an almost sure set Ω^ε such that for all $\omega \in \Omega^\varepsilon$ there is a $T_2(\omega,\varepsilon)$ such that $t > T_2(\omega,\varepsilon)$ implies

$$|X(t,\omega)| \leq k(t)t^{1+\varepsilon}.$$

Now consider the almost sure set $\tilde{\Omega}^\varepsilon = \Omega^\varepsilon \cap \Omega_2$. Then for $\omega \in \tilde{\Omega}^\varepsilon$, if we define $T_3(\omega,\varepsilon) = T_1(\omega) \vee T_2(\omega,\varepsilon)$, for all $t > T_3(\omega,\varepsilon)$, we have

$$\sigma(X(t,\omega))^2 \leq (1+C)^2 k(t)^{2\beta} t^{2\beta(1+\varepsilon)}. \tag{54}$$

Next, for all $\varepsilon \in (0,1)$ there is $x^{**}(\varepsilon) > 0$ such that $x \in (0, x^{**}(\varepsilon))$ implies

$$x \log\log\frac{1}{x} \leq x^{1-\varepsilon}.$$

Now, as $\int_t^\infty \sigma(X(s))^2\,ds \to 0$ as $t \to \infty$, for each $\omega \in \Omega_2$ there is a $T_4(\omega,\varepsilon) > 0$ such that $t > T_4(\omega,\varepsilon)$ implies $\int_t^\infty \sigma(X(s,\omega))^2\,ds < x^{**}$. Thus for $t > T_4(\omega,\varepsilon)$

$$\left(\int_t^\infty \sigma(X(s))^2\,ds \log\log\left(\int_t^\infty \sigma(X(s))^2\,ds\right)^{-1}\right)(\omega) \leq \left(\int_t^\infty \sigma(X(s))^2\,ds\right)^{1-\varepsilon}(\omega). \tag{55}$$

Thus, for $\omega \in \tilde{\Omega}^\varepsilon$ we may define $T_5(\omega,\varepsilon) = T_3(\omega,\varepsilon) \vee T_4(\omega,\varepsilon)$, so that, for $t > T_5(\omega,\varepsilon)$ we have

$$\left(\int_t^\infty \sigma(X(s))^2\,ds \log\log\left(\int_t^\infty \sigma(X(s))^2\,ds\right)^{-1}\right)(\omega) \leq (1+C)^{2(1-\varepsilon)}\left(\int_t^\infty k(s)^{2\beta}s^{2\beta(1+\varepsilon)}\,ds\right)^{1-\varepsilon}.$$

Therefore (53)— and consequently (46)— follows once

$$\lim_{t\to\infty}\frac{\left(\int_t^\infty k(s)^{2\beta}s^{2\beta(1+\varepsilon)}\,ds\right)^{1-\varepsilon}}{k(t)^2} = 0 \tag{56}$$

for some $\varepsilon \in (0,1)$. We finally prove that the conditions (33), (40) imply (56). This is the subject of Lemma 16 which follows this proof. Hence the proof of (46) is complete, and so, by Lemma 14, part (i) of Theorem 11 follows. The proofs of parts (ii), (iii) are given in Lemma 15, so Theorem 11 is established. $\square$

We return now to the proof of (56).

Lemma 16. *Suppose that k is a positive and integrable function which obeys (33) and (40). If $\beta > 1$, then there exists $\varepsilon \in (0,1)$ such that (56) holds.*

Proof. Since k obeys (33), for every $\varepsilon \in (0,1)$ there is $T_1(\varepsilon) > 0$ such that $t^{2\beta(1+\varepsilon)}k(t)^{2\beta\varepsilon} < 1$ for $t > T_1(\varepsilon)$. Thus, for $t > T_1(\varepsilon)$ we have $t^{2\beta(1+\varepsilon)}k(t)^{2\beta} \le k(t)^{2\beta(1-\varepsilon)}$. To prove (56) it is therefore enough to prove

$$\lim_{t\to\infty} \frac{\int_t^\infty k(s)^{2\beta(1-\varepsilon)}\,ds}{k(t)^{\frac{2}{1-\varepsilon}}} = 0 \tag{57}$$

for some $\varepsilon \in (0,1)$. Next define $\lambda(t) = -\log k(t)$ so $\lambda(t) \to \infty$ as $t \to \infty$. Therefore (40) implies that for every $\varepsilon \in (0,1)$ there is a $T_2(\varepsilon) > 0$ such that for all $s \ge t > T_2(\varepsilon)$ we have

$$\frac{\lambda(t)}{\lambda(s)} < 1 + \varepsilon,$$

Now define $\mu(\varepsilon) = \beta(1-\varepsilon)^2 - (1+\varepsilon)$ for $\varepsilon \in (0,1)$. Hence there exists $\varepsilon^* \in (0,1)$ so that $\mu(\varepsilon^*) = \frac{1}{2}(\beta - 1) > 0$. Now, for $s \ge t > T_2(\varepsilon^*)$, we have

$$\beta(1-\varepsilon^*)^2 - \frac{\lambda(t)}{\lambda(s)} > \mu(\varepsilon^*).$$

Therefore, because

$$\begin{aligned}\frac{1}{k(t)^{\frac{2}{1-\varepsilon}}} &\int_t^\infty k(s)^{2\beta(1-\varepsilon)}\,ds \\ &= \int_t^\infty \exp\left(-\frac{2\lambda(s)}{1-\varepsilon}\left\{\beta(1-\varepsilon)^2 - \frac{\lambda(t)}{\lambda(s)}\right\}\right)ds,\end{aligned}$$

for all $t > T_2(\varepsilon^*)$ we have

$$\frac{1}{k(t)^{\frac{2}{1-\varepsilon^*}}} \int_t^\infty k(s)^{2\beta(1-\varepsilon^*)}\,ds \le \int_t^\infty e^{-\frac{2\lambda(s)}{1-\varepsilon}\mu(\varepsilon^*)}\,ds.$$

Next fix $M = 2(1-\varepsilon^*)/(\beta - 1)$. For all $T > T_3(M) = T_3(\varepsilon^*)$ we have $\lambda(t) > M \log t$ by (33). Now let $T_4(\varepsilon^*) = T_2(\varepsilon^*) \vee T_3(\varepsilon^*)$. Therefore, for $t > T_4(\varepsilon^*)$

$$\frac{1}{k(t)^{\frac{2}{1-\varepsilon^*}}} \int_t^\infty k(s)^{2\beta(1-\varepsilon^*)}\, ds \le \int_t^\infty e^{-\frac{\lambda(s)(\beta-1)}{1-\varepsilon}}\, ds$$
$$\le \int_t^\infty e^{-\frac{\beta-1}{1-\varepsilon} M \log s}\, ds = \frac{1}{t}.$$

Therefore, by choosing $\varepsilon^* \in (0,1)$ such that $\beta(1-\varepsilon^*)^2 - (1+\varepsilon^*) = (\beta-1)/2$, we have proven (57) for $\varepsilon = \varepsilon^*$. □

8. Proof of Theorem 10

In this section, we turn to the proof of Theorem 10. We start with the proof of a technical lemma.

Lemma 17. *Suppose that g is a nonnegative and continuous function such that*

$$\limsup_{t\to\infty} \frac{\log g(t)}{\log t} \le -\lambda_1 \tag{58}$$

where $\lambda_1 > 0$, and there is a continuous nonnegative and integrable function f such that

$$\lim_{t\to\infty} \frac{\log f(t)}{\log t} = -\lambda_2 \tag{59}$$

where $\lambda_2 > 1$. Then

$$\limsup_{t\to\infty} \frac{\log (f*g)(t)}{\log t} \le -(\lambda_1 \wedge \lambda_2).$$

Proof. Let $\lambda = \lambda_1 \wedge \lambda_2$, and $0 < \varepsilon < \lambda$. Introduce $h(t) = (1+t)^{-\lambda+\varepsilon}$. Notice that if we can prove

$$\lim_{t\to\infty} \frac{(f*g)(t)}{h(t)} = 0, \tag{60}$$

the result follows immediately by taking logarithms and letting $\varepsilon \downarrow 0$.

We make some preliminary observations. First, the definition of h implies that

$$\sup_{\gamma\in[1/2,1]} \frac{h(\gamma t)}{h(t)} \le 2^{\lambda-\varepsilon}. \tag{61}$$

For any $t \ge 0$, notice that we can write

$$\frac{(f*g)(t)}{h(t)} = \int_0^{t/2} \frac{f(t-s)}{h(t-s)} \frac{h(t-s)}{h(t)} g(s)\, ds + \int_{t/2}^t f(t-s) \frac{h(s)}{h(t)} \frac{g(s)}{h(s)}\, ds. \tag{62}$$

Due to (59), for every $\varepsilon > 0$ there is a $T_2(\varepsilon) > 0$ such that

$$f(t) \le (1+t)^{-\lambda_2+\varepsilon/2}, \quad t > T_2(\varepsilon). \tag{63}$$

We now consider the cases $\lambda_1 \in (0,1]$ and $\lambda_1 > 1$ separately.

First, let $\lambda_1 \in (0,1]$. On account of (58), for every ε there exists $T_1(\varepsilon) > 0$ and $C(\varepsilon) > 0$ such that for all $t > T_1(\varepsilon)$

$$g(t) \leq (1+t)^{-\lambda_1+\varepsilon/2}, \quad \int_0^t g(s)\,ds \leq C(\varepsilon)(1+t)^{-\lambda_1+\varepsilon/2+1}. \tag{64}$$

Considering the first term on the righthand side of (62), and using (61), (63) and (64), for $t > 2T_1(\varepsilon) \vee 2T_2(\varepsilon)$ we get

$$\begin{aligned}
&\int_0^{t/2} \frac{f(t-s)}{h(t-s)} \frac{h(t-s)}{h(t)} g(s)\,ds \\
&\leq \sup_{t/2 \leq s \leq t} \frac{f(s)}{h(s)} \sup_{t/2 \leq s \leq t} \frac{h(s)}{h(t)} \int_0^{t/2} g(s)\,ds \\
&\leq (1+t/2)^{-\lambda_2+\frac{\varepsilon}{2}+\lambda-\varepsilon} \cdot 2^{\lambda-\varepsilon} \cdot C(\varepsilon)(1+t/2)^{-\lambda_1+\frac{\varepsilon}{2}+1} \\
&= C(\varepsilon) 2^{\lambda-\varepsilon}(1+t/2)^{-(\lambda_2-1)},
\end{aligned}$$

since $\lambda = \lambda_1$. Therefore the first term on the righthand side of (62) tends to zero as $t \to \infty$.

As to the second term on the righthand side of (62), as f is integrable, by employing (61), and (64), for $t > 2T_1(\varepsilon)$, we have

$$\begin{aligned}
\int_{t/2}^t f(t-s) \frac{h(s)}{h(t)} \frac{g(s)}{h(s)}\,ds. &\leq \sup_{t/2 \leq s \leq t} \frac{g(s)}{h(s)} \sup_{t/2 \leq s \leq t} \frac{h(s)}{h(t)} \int_0^{t/2} f(s)\,ds \\
&\leq (1+t/2)^{-\lambda_1+\frac{\varepsilon}{2}+\lambda-\varepsilon} \cdot 2^{\lambda-\varepsilon} \cdot \int_0^\infty f(s)\,ds \\
&= 2^{\lambda-\varepsilon} \int_0^\infty f(s)\,ds\,(1+t/2)^{-\varepsilon/2},
\end{aligned}$$

as $\lambda = \lambda_1$. Therefore the second term on the righthand side of (62) tends to zero as $t \to \infty$, and so (60) holds and the result follows.

Next, let $\lambda_1 > 1$. Then both f and g are integrable, and, since $\lambda = \lambda_2$, for $t > 2T_1(\varepsilon) \vee 2T_2(\varepsilon)$, we proceed to obtain the estimates

$$\begin{aligned}
\int_0^{t/2} \frac{f(t-s)}{h(t-s)} \frac{h(t-s)}{h(t)} g(s)\,ds &\leq \sup_{t/2 \leq s \leq t} \frac{f(s)}{h(s)} \sup_{t/2 \leq s \leq t} \frac{h(s)}{h(t)} \int_0^\infty g(s)\,ds \\
&\leq 2^{\lambda-\varepsilon} \int_0^\infty g(s)\,ds\,(1+t/2)^{-\varepsilon/2},
\end{aligned}$$

and

$$\int_{t/2}^t f(t-s) \frac{h(s)}{h(t)} \frac{g(s)}{h(s)}\,ds \leq 2^{\lambda-\varepsilon} \int_0^\infty f(s)\,ds\,(1+t/2)^{-\varepsilon/2}.$$

Hence each term on righthand side of (62) vanishes as $t \to \infty$, so (60) holds and the result follows. □

This result can now be used to obtain the proof of the following important lemma.

Lemma 18. *Suppose k is a positive subexponential function which satisfies (38) for some $\alpha > 1$, and T defined by (42) obeys*

$$\limsup_{t\to\infty} \frac{\log|T(t)|}{\log t} \leq -\gamma, \qquad a.s. \tag{65}$$

for some $\alpha > \gamma > 0$. Let $a > \int_0^\infty k(s)\,ds$, and suppose σ is locally Lipschitz, globally linearly bounded, obeys (37) for some $\beta > 1$, and has $\sigma(0) = 0$. If X is the unique nontrivial strong solution of (28) with $X(0) > 0$, then it obeys

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\log t} \leq -\gamma, \qquad a.s. \tag{66}$$

Proof. By applying Lemma 17 pathwise, (38), (65), and the fact that $\alpha > \gamma > 0$, together imply that

$$\limsup_{t\to\infty} \frac{\log(k * |T|)(t)}{\log t} \leq -\gamma, \quad \text{a.s.}$$

By (44), the last inequality, and (65), it follows that

$$\limsup_{t\to\infty} \frac{\log|f(t)|}{\log t} \leq -\gamma, \quad \text{a.s.} \tag{67}$$

Next consider the resolvent z with $z(0) = 1$ which obeys

$$z'(t) = -az(t) + \int_0^t k(t-s)z(s)\,ds, \quad t \geq 0. \tag{68}$$

Hence, as k is subexponential, and $a > \int_0^\infty k(s)\,ds$, it follows from [3] that

$$\lim_{t\to\infty} \frac{z(t)}{k(t)} = \frac{1}{(a - \int_0^\infty k(s)\,ds)^2}.$$

Therefore, by (38), we have

$$\lim_{t\to\infty} \frac{\log z(t)}{\log t} = -\alpha. \tag{69}$$

Next, the solution of (43) can be represented in terms of that of (68) according to

$$U(t) = U(0)z(t) + \int_0^t f(t-s)z(s)\,ds, \quad t \geq 0.$$

Applying Lemma 17 pathwise to the convolution term, and using (67), (69), and the fact that $0 < \gamma < \alpha$ yields

$$\limsup_{t\to\infty} \frac{\log(|f| * z)(t)}{\log t} \leq -\gamma, \quad \text{a.s.}$$

Therefore, by (69), and the fact that $0 < \gamma < \alpha$, we have

$$\limsup_{t\to\infty} \frac{\log |U(t)|}{\log t} \leq -\gamma, \quad \text{a.s.} \tag{70}$$

On account of (65), (70) and (41), we get (66), as required. $\square$

With this preparatory result established, we now state and prove the result that will enable us to successively improve estimates of the decay rate of solutions of (28).

Lemma 19. *Suppose that k is a subexponential function which obeys (38) for some $\alpha > 1$. Suppose $a > \int_0^\infty k(s)\,ds$, and that σ is a globally linearly bounded and locally Lipschitz continuous function which obeys (37) for some $\beta > 1$ and $\sigma(0) = 0$. Then the nontrivial strong solution of (28) obeys the following:*

(a)

$$\limsup_{t\to\infty} \frac{\log |X(t)|}{\log t} \leq -(\alpha - 1), \quad \textit{a.s.} \tag{71}$$

(b) *Suppose that $\alpha > 1 + (2\beta)^{-1}$ and there is $c_0 > (2\beta)^{-1}$ such that*

$$\limsup_{t\to\infty} \frac{\log |X(t)|}{\log t} \leq -c_0, \quad \textit{a.s.} \tag{72}$$

Then one of the following holds:

(i) *If $c_0\beta > 1/2 + \alpha$, then T defined by (42) obeys (46).*

(ii) *If $c_0\beta \leq 1/2 + \alpha$, then*

$$\limsup_{t\to\infty} \frac{\log |X(t)|}{\log t} \leq -(c_0\beta - 1/2), \quad \textit{a.s.} \tag{73}$$

Proof. The proof of part (a) follows from (32) in Theorem 8 and follows closely the argument of Theorem 3 thereafter, using the hypothesis (38).

To prove part (b), we will show that if (72) holds for some $c_0 > 1/(2\beta)$ then

$$\limsup_{t\to\infty} \frac{\log R(t)}{\log t} \leq -2c_0\beta + 1, \quad \text{a.s.} \tag{74}$$

where

$$R(t) = \int_t^\infty \sigma(X(s))^2\,ds \log\log\left(\int_t^\infty \sigma(X(s))^2\,ds\right)^{-1}.$$

We prove part (b) given (74) and return to its proof presently. If (74) holds, Lemma 13 implies

$$\limsup_{t\to\infty} \frac{|T(t)|}{\sqrt{2R(t)}} = 1 \quad \text{a.s.} \tag{75}$$

Therefore, by (74), (75), we get

$$\limsup_{t\to\infty} \frac{\log |T(t)|}{\log t} \le -(c_0\beta - 1/2), \quad \text{a.s.} \tag{76}$$

To conclude the proof of part (b), suppose first that $c_0\beta > 1/2 + \alpha$. Then

$$\limsup_{t\to\infty} \frac{\log |T(t)|}{\log t} \le -c_0\beta + 1/2 < -\alpha,$$

almost surely, so, by (38), (46) holds.

In the other case, when $c_0\beta \le 1/2 + \alpha$, because (76) holds and k obeys (38), Lemma 18 implies (73).

We now return to the proof of (74). If

$$\limsup_{t\to\infty} \frac{\log |X(t)|}{\log t} \le -c_0, \quad \text{a.s.},$$

then, for every $\varepsilon \in (0, c_0)$ there is $T_1(\omega, \varepsilon)$ such that for $t > T_1(\omega, \varepsilon)$ we have

$$X(t, \omega) \le t^{-c_0+\varepsilon}.$$

on an almost sure set. Revisiting the proof of Theorem 11, we see that there is $T_2(\omega)$ such that $t > T_2(\omega)$ implies $|\sigma(X(t, \omega))| \le (1 + C)|X(t, \omega)|^\beta$. Hence for every $t > T_3(\omega, \varepsilon) := T_1(\omega, \varepsilon) \vee T_2(\omega)$ in some almost sure set, we have

$$\sigma(X(t))^2 \le (1 + C)^2 t^{-2\beta c_0+2\beta\varepsilon}.$$

Since $\int_t^\infty \sigma(X(s))^2\, ds \to 0$ as $t \to \infty$, a.s., by again appealing to the proof of Theorem 11 for all $t > T_4(\omega, \varepsilon)$, (55) is true. Now, as $c_0 > 1/(2\beta)$, once $\varepsilon > 0$ can be chosen so small that $-2\beta c_0+2\beta\varepsilon < -1$, if we take $t > T_5(\omega, \varepsilon) = T_3(\omega, \varepsilon) \vee T_4(\omega, \varepsilon)$, it follows that for all ω in an almost sure set that we have

$$\begin{aligned}
R(t, \omega) &= \left(\int_t^\infty \sigma(X(s))^2\, ds \log\log \left(\int_t^\infty \sigma(X(s))^2\, ds \right)^{-1} \right)(\omega) \\
&\le \left(\int_t^\infty \sigma(X(s))^2\, ds \right)^{1-\varepsilon} (\omega) \\
&\le \left(\int_t^\infty (1 + C)^2 s^{-2\beta c_0+2\beta\varepsilon}\, ds \right)^{1-\varepsilon} \\
&= C(\varepsilon) t^{(-2\beta c_0+2\beta\varepsilon+1)(1-\varepsilon)},
\end{aligned}$$

where $C(\varepsilon)$ is a positive, ε-dependent constant. Hence for $t > T_5(\omega, \varepsilon)$,

$$\frac{\log R(t, \omega)}{\log t} \le \frac{\log C(\varepsilon)}{\log t} + (-2\beta c_0 + 2\beta\varepsilon + 1)(1 - \varepsilon).$$

Hence, for each $0 < \varepsilon < 1 \vee c_0 \vee (2\beta c_0 - 1)/(2\beta)$, there is an Ω^*_ε with $\mathbb{P}[\Omega^*_\varepsilon] = 1$ such that $\omega \in \Omega^*_\varepsilon$ implies

$$\limsup_{t\to\infty} \frac{\log R(t,\omega)}{\log t} \leq (-2\beta c_0 + 2\beta\varepsilon + 1)(1-\varepsilon).$$

Now let $\Omega^* = \cap_{n\in\mathbb{N}} \Omega^*_{1/n}$, so Ω^* is an almost sure set. This implies

$$\limsup_{t\to\infty} \frac{\log R(t,\omega)}{\log t} \leq (-2\beta c_0 + 2\beta \frac{1}{n} + 1)(1 - \frac{1}{n}), \quad \omega \in \Omega^*, n \in \mathbb{N}.$$

Letting $n \to \infty$ now yields (74). $\square$

The proof of Theorem 10 now proceeds by applying Lemma 19 part (b) iteratively.

Proof of Theorem 10. If $\alpha > (2\beta+1)/(2(\beta-1))$, then $\alpha > 1 + (2(\beta-1))^{-1} > 1 + (2\beta)^{-1}$. By (71), with $c_0 = \alpha - 1$, (72) holds, and $c_0\beta - 1/2 - \alpha = \frac{1}{2}(2\alpha(\beta-1) - (2\beta+1)) > 0$. Hence $c_0\beta > 1/2 + \alpha > 1/2$, so the alternative (i) in part (b) of Lemma 19 holds, and we have (46). Therefore, by Lemma 14 it now follows that (35) holds, and so Lemma 15 enables us to conclude that parts (ii), (iii) of Theorem 10 are also true.

Now, consider the case $1 + (2(\beta-1))^{-1} < \alpha \leq (2\beta+1)(2(\beta-1))^{-1}$. Notice that if $\alpha > 1 + (2(\beta-1))^{-1}$, then immediately $\alpha > 1 + (2\beta)^{-1}$. Let $c_0 = \alpha - 1 > (2(\beta-1))^{-1}$ and consider the sequence defined by $c_{n+1} = \beta c_n - 1/2$, $n \geq 0$. Note that $c_0\beta \leq 1/2 + \alpha$, and that the iteration for the sequence may be rewritten as $c_{n+1} - (2(\beta-1))^{-1} = \beta(c_n - (2(\beta-1))^{-1})$. Since $c_0 > (2(\beta-1))^{-1}$, and $\beta > 1$, the sequence is increasing and $\lim_{n\to\infty} c_n = \infty$. Therefore, there exists a minimal $n_0 > 1$ such that $c_{n_0}\beta > 1/2 + \alpha$, while

$$c_n\beta \leq \frac{1}{2} + \alpha, \quad n = 0, 1, \ldots, n_0 - 1.$$

Therefore, by Lemma 19, for $1 + (2\beta)^{-1} < 1 + (2(\beta-1))^{-1} < \alpha \leq (2\beta+1)(2(\beta-1))^{-1}$, we have

$$\limsup_{t\to\infty} \frac{\log X(t)}{\log t} \leq -c_0$$

and $c_0\beta \leq \frac{1}{2} + \alpha$. Hence, by Lemma 19 (b) alternative (ii), we have

$$\limsup_{t\to\infty} \frac{\log |X(t)|}{\log t} \leq -c_1, \quad \text{a.s.}$$

If $n_0 \neq 1$, since $c_1 > c_0 > (2\beta)^{-1}$, but $c_1\beta \leq \frac{1}{2} + \alpha$, we may apply Lemma 19(b) alternative (ii) again to give

$$\limsup_{t\to\infty} \frac{\log |X(t)|}{\log t} \leq -c_2, \quad \text{a.s.}$$

Continuing to iterate in this way leads to

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\log t} \le -c_{n_0-1}, \quad \text{a.s.}$$

where $c_{n_0-1}\beta \le \frac{1}{2}+\alpha$, so applying Lemma 19(b) alternative (ii) gives

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\log t} \le -c_{n_0}, \quad \text{a.s.}$$

However, because $c_{n_0}\beta > \frac{1}{2}+\alpha$, the next iteration of Lemma 19 (b) invokes alternative (i), thereby enabling us to conclude that (46) holds. The proof now concludes as in the case $\alpha > (2\beta+1)(2(\beta-1))^{-1}$ above, and Theorem 10 is proven in the case $\alpha \in (1+(2(\beta-1))^{-1}, (2\beta+1)(2(\beta-1))^{-1}]$ also. □

Appendix

Here, we prove a result which bounds the decay rate of $\int_t^\infty A(s)\,dB(s)$ as $t\to\infty$ for a scalar process $A=\{A(t),\mathcal{F}^B(t); 0\le t<\infty\}$ which is in $L^2(\mathbb{R}^+)$ a.s.

If $A\in L^2(\mathbb{R}^+)$ a.s. then

$$\lim_{t\to\infty}\int_0^t A(s)\,dB(s) \quad \text{exists a.s.}$$

by the martingale time change theorem. If we denote this limit by $\int_0^\infty A(s)\,dB(s)$, we may define, for each $t\ge 0$

$$\int_t^\infty A(s)\,dB(s) := \int_0^\infty A(s)\,dB(s) - \int_0^t A(s)\,dB(s)$$

This is a well-defined $\mathcal{F}^B(\infty)$-measurable random variable, for every $t\ge 0$; note, however, that it is not a stochastic process, but rather a one-parameter family of random variables. Observe that $\int_t^\infty A(s)\,dB(s)$ is obviously not $\mathcal{F}^B(t)$-measurable. However, in the sequel, it will suffice to understand the asymptotic behaviour of the random function

$$t\mapsto \left(\int_t^\infty A(s)\,dB(s)\right)(\omega) \quad \text{for all } \omega \text{ in an almost sure set.}$$

Proof of Lemma 13. By the comments preceding this Lemma, it is evident that both numerator and denominator in (45) exist for all $t\ge 0$, and, moreover, that both have limit zero, as $t\to\infty$, almost surely. Define for $t\ge 0$

$$W(t) = \begin{cases} tB(1/t), & t>0,\\ 0, & t=0.\end{cases}$$

Hence $W=\{W(t),\mathcal{F}^W(t); 0\le t<\infty\}$ is a standard one-dimensional Brownian motion. Moreover, its natural filtration can be expressed in terms of that of B; indeed $\mathcal{F}^W(t) = \mathcal{F}^B(1/t)$ for all t. If we define Y by

$$Y(t) = \begin{cases} A(1/t), & t > 0, \\ 0, & t = 0, \end{cases}$$

then $Y = \{Y(t), \mathcal{F}^W(t); 0 \leq t < \infty\}$. Furthermore, the process has continuous sample paths. Next, define the family of $\mathcal{F}^B(\infty)$-measurable random variables $\{M(\tau); \tau \geq 0\}$ by

$$M(\tau) = \begin{cases} \int_{\frac{1}{\tau}}^{\infty} A(s)\,dB(s), & \tau > 0, \\ 0, & \tau = 0. \end{cases}$$

Then the function $t \mapsto M(t,\omega)$ is continuous for almost all $\omega \in \Omega$, and $\lim_{\tau\to\infty} M(\tau)$ exists a.s.. Moreover, for $\tau > 0$, we have

$$\int_0^{\tau} \left(\frac{1}{u} Y(u)\right)^2 du = \int_{\frac{1}{\tau}}^{\infty} A(u)^2\,du < \infty, \quad \text{a.s.}$$

Furthermore, we have

$$\lim_{\tau\downarrow 0} \int_0^{\tau} \left(\frac{1}{u} Y(u)\right)^2 du = 0, \quad \text{a.s.}$$

Hence, for each $\tau > 0$, $\int_0^{\tau} \frac{1}{u} Y(u)\,dW(u)$ exists. Indeed, as

$$\int_{\frac{1}{\tau}}^{\infty} A(s)\,dB(s) = \int_0^{\tau} \frac{1}{u} Y(u)\,dW(u),$$

we can write

$$M(\tau) = \begin{cases} \int_0^{\tau} \frac{1}{u} Y(u)\,dW(u), & \tau > 0, \\ 0, & \tau = 0. \end{cases}$$

Consequently, $M = \{M(\tau), \mathcal{F}^W(\tau); 0 \leq \tau < \infty\}$ is a continuous local martingale with strictly increasing square variation, which, for $\tau > 0$, is given by

$$\langle M\rangle(\tau) = \int_0^{\tau} \left(\frac{1}{u} Y(u)\right)^2 du = \int_{\frac{1}{\tau}}^{\infty} A(u)^2\,du.$$

Then $\langle M\rangle$ satisfies $\lim_{\tau\downarrow 0}\langle M\rangle(\tau) = 0$, a.s., and $\lim_{\tau\to\infty}\langle M\rangle(\tau) < \infty$ a.s.. By the martingale time change theorem, there exists a standard Brownian motion $\tilde{B}$ such that

$$M(\tau) = \tilde{B}(\langle M\rangle(\tau)), \quad 0 \leq \tau < \infty \quad \text{a.s.}$$

The law of the iterated logarithm now gives

$$\limsup_{T\downarrow 0} \frac{|\tilde{B}(T)|}{\sqrt{2T \log\log(1/T)}} = 1, \quad \text{a.s.}$$

But as $\lim_{\tau\downarrow 0}\langle M\rangle(\tau) = 0$, a.s., and $\tau \mapsto \langle M\rangle(\tau)$ is continuous and strictly increasing, we have

$$\limsup_{\tau\downarrow 0} \frac{|\tilde{B}(\langle M\rangle(\tau)|}{\sqrt{2\langle M\rangle(\tau)\log\log(\langle M\rangle(\tau)^{-1})}} = 1, \quad \text{a.s.}$$

Therefore, as $\tilde{B}(\langle M\rangle(\tau)) = M(\tau)$, $\tau \geq 0$,

$$\limsup_{\tau\downarrow 0} \frac{|M(\tau)|}{\sqrt{2\langle M\rangle(\tau)\log\log(\langle M\rangle(\tau)^{-1})}} = 1, \quad \text{a.s.}$$

Therefore,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{\left|\int_t^\infty A(s)\,dB(s)\right|}{\sqrt{2\int_t^\infty A(s)^2\,ds \log\log\left(\int_t^\infty A(s)^2\,ds\right)^{-1}}} \\
&= \limsup_{\tau\downarrow 0} \frac{|\int_{\frac{1}{\tau}}^\infty A(s)\,dB(s)|}{\sqrt{2\int_{\frac{1}{\tau}}^\infty A(s)^2\,ds \log\log\left(\int_{\frac{1}{\tau}}^\infty A(s)^2\,ds\right)^{-1}}} \\
&= \limsup_{\tau\downarrow 0} \frac{|M(\tau)|}{\sqrt{2\langle M\rangle(\tau)\log\log(\langle M\rangle(\tau)^{-1})}} = 1, \quad \text{a.s.}
\end{aligned}$$

which establishes the desired result. □

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