Construction of Liapunov Functionals for Linear Volterra Integrodifferential Equations and Stability of Delay Systems

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Abstract Inspired by the construction of a new Liapunov functional for linear Volterra integrodifferential equations, we prove some general stability theorems for functional differential equations with infinite delay and weaken the usual requirement for positive definiteness of Liapunov functionals involved in stability theory.

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1 Introduction.

Consider a system of functional differential equations

$$x'(t) = F(t, x_t), \tag{1.1}$$

in which $F(t, \phi)$, $F(t, 0) = 0$, is a functional defined for $t \geq 0$ and $\phi \in C$, where C is the set of bounded continuous functions $\phi : R^- \to R^n$, $R^- = (-\infty, 0]$, with the supremum norm $\|\cdot\| = \sup\{|\phi(s)| : s \in R^-\}$, where $|\cdot|$ is the Euclidean norm on

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R^n. We assume that for each $t_0 \geq 0$ and each $\phi \in C_H,\ C_H = \{\phi \in C : \|\phi\| < H\}$ for some $H > 0$, there is at least one solution $x = x(t, t_0, \phi)$ of (1.1) defined on an interval $[t_0, \alpha)$ with $x_{t_0} = \phi$. We may also denote the solution by $x_t = x_t(t_0, \phi)$. Here $x_t(s) = x(t+s)$ for $s \leq 0$. Moreover, if the solution remains bounded, then $\alpha = +\infty$. We denote by $C(X, Y)$ the set of continuous functions $\phi : X \to Y$.

The object of this paper is to give conditions on Liapunov functionals to ensure the asymptotic stability of the zero solution of (1.1) and to provide a new method of constructing Liapunov functionals for the linear Volterra integrodifferential equation

$$x' = Ax(t) + \int_0^t C(t-s)x(s)ds \tag{1.2}$$

where A is a constant and $C \in C(R^+, R), R^+ = [0, +\infty)$. In fact, it was equation (1.2) and its nonlinear perturbations which inspired this investigation. There are many known results and applications on stability and basic theory of systems (1.1) and (1.2) in the literature. For reference and history, we refer to the books of Burton [3,4], Driver [8], Gripenberg, Londen, and Staffans [9], Hale [11], Hino, Murakami and Naito [12], Kolmanovskii [16], Krasovskii [17], Kuang [19], Lakshmikantham, Wen, and Zhang [20], Yoshizawa [24].

Let $V : R \times C \to R^+$ be continuous. The upper right-hand derivative along a solution of (1.1) is defined by

$$V'_{(1.1)}(t, \phi) = \limsup_{\delta \to 0^+} \{V(t+\delta, x_{t+\delta}(t, \phi)) - V(t, \phi)\}/\delta$$

Definition 1.1. The zero solution of (1.1) is said to be stable if for each $\varepsilon > 0$ and $t_0 \geq 0$, there is a $\delta = \delta(\varepsilon, t_0) > 0$ such that $[\phi \in C, \|\phi\| < \delta, t \geq t_0]$ imply $|x(t, t_0, \phi)| < \varepsilon$. The zero solution of (1.1) is uniformly stable if it is stable and δ is independent of t_0.

Definition 1.2. The zero solution of (1.1) is said to be asymptotically stable if it is stable and if for each $t_0 \geq 0$ there exists $\delta_0 > 0$ such that $\|\phi\| < \delta_0$ implies that $x(t, t_0, \phi) \to 0$ as $t \to +\infty$. The zero solution is uniformly asymptotically stable if it is uniformly stable and if there is $\delta_1 > 0$ and for each $\varepsilon > 0$ there exists $T > 0$ such that $[t_0 \geq 0, \phi \in C, \|\phi\| < \delta_1, t \geq T + t_0]$ imply that $|x(t, t_0, \phi)| < \varepsilon$.

Definition 1.3. $W : R^+ \to R^+$ is called a wedge if W is continuous and strictly increasing with $W(0) = 0$. Throughout of this paper W and $W_j (j = 1, 2, \cdots)$ will denote the wedges.

Definition 1.4. A continuous function $g : R^+ \to R^+$ is convex downward if $g[(t+s)/2] \le [g(t) + g(s)]/2$ for all $t, s \in R^+$.

Jensen's inequality. Let g be convex downward and let x, $p : [a, b] \to R^+$ be continuous with $\int_a^b p(s)ds > 0$. Then

$$\int_a^b p(s)ds\, g\Big[\int_a^b p(s)x(s)ds \Big/ \int_a^b p(s)ds\Big] \le \int_a^b p(s)g(x(s))ds.$$

For reference on Jensen's inequality and its applications in stability theory, we refer to Becker, Burton, and Zhang [1] and Natanson [22].

Lemma 1.1. Let W_1 be a wedge. For any $L > 0$, define $W_0(r) = \int_0^r W_1(s)ds/L$ on $[0, L]$. Then W_0 is a convex downward wedge such that $W_0(r) \le W_1(r)$ for all $r \in [0, L]$.

It is well-known (see [7]) that conditions

$$\begin{aligned} W_1(|\phi(0)|) &\le V(t, \phi), \;\; V(t, 0) = 0, \\ V'_{(1.1)}(t, \phi) &\le 0 \end{aligned} \tag{1.3}$$

guarantee the stability of the zero solution of (1.1) and

$$\begin{aligned} W_1(|x(t)|) \le V(t, x_t) &\le W_2(|x(t)|) + W_3\Big(\int_0^t \Phi(t-s)W_4(|x(s)|)ds\Big), \\ V'_{(1.1)}(t, x_t) &\le -W_5(|x(t)|) \end{aligned} \tag{1.4}$$

yield the uniform asymptotic stability, where $\Phi : R^+ \to R^+$ is continuous, bounded, and $\Phi \in L^1(R^+)$.

The goal here is to weaken the lower bound of $V(t, \phi)$ to

$$W_1\Big(\Big|x(t) + \int_0^t G(t-s)x(s)ds\Big|\Big) \le V(t, x_t) \tag{1.5}$$

and drop the boundedness condition on $\Phi(t)$. This will be done in section 2. As an application of the main theorem, we construct a new Liapunov functional for (1.2) in section 3.

2 A General Stability Theorem

Liapunov functional of the form

$$V(t,x_t) = |x(t)|^2 + \int_0^t \Phi(t-s)|x(s)|^2 ds, \quad \Phi(s) \geq 0$$

may work for (1.2), but the stability condition may not be the best. Many concrete examples and applications (see [5]) suggest that $V(t,\phi)$ should take the form

$$V(t,x_t) = V_1(t,x_t) + V_2(t,x_t)$$

where

$$V_1(t,x_t) = (x(t) + \int_0^t G(t-s)x(s)ds)^2$$

or a more general integral operator. A complete discussion on the construction of such functionals for (1.2) will be given in Section 3. The following lemmas are needed for our main theorem.

Lemma 2.1 ([9], p.14). Let $G, Z : R^+ \to R^{n\times n}$ be continuous and satisfy

$$Z(t) = G(t) - \int_0^t G(t-s)Z(s)ds. \tag{2.1}$$

If $y, x : R^+ \to R^n$ satisfy

$$y(t) = x(t) + \int_0^t G(t-s)x(s)ds,$$

then

$$x(t) = y(t) - \int_0^t Z(t-s)y(s)ds$$

for all $t \in R^+$.

Lemma 2.2. For all the functions given in Lemma 2.1, if $\int_0^{+\infty} |G(u)|du < 1$, then $\int_0^{+\infty} |Z(u)|du < +\infty$ and if $y(t) \to 0$ as $t \to +\infty$, then $x(t) \to 0$ as $t \to +\infty$.

Proof. It follows from (2.1) that

$$|Z(t)| \leq |G(t)| + \int_0^t |G(t-s)||Z(s)|ds.$$

Integrate the above inequality from 0 to ∞ and interchange variables of the second term to obtain

$$\int_0^{+\infty} |Z(t)|dt \leq \int_0^{+\infty} |G(t)|dt + \int_0^{+\infty} |Z(s)|ds \int_0^{+\infty} |G(t)|dt.$$

This yields $\int_0^{+\infty} |Z(t)|dt < +\infty$ since $\int_0^{+\infty} |G(t)|dt < 1$. The statement $y(t) \to 0$ as $t \to +\infty$ implies $x(t) \to 0$ as $t \to +\infty$ is a known result ([3], p.48). The proof is complete.

The following result has a general lower bound on $V(t, \phi)$ and removes the boundedness restriction on $\Phi(t)$ in ([7], p.144).

Theorem 2.1. Suppose that there exists a continuous functional $V : R^+ \times C \to R^+$, $V(t, 0) = 0$, functions $G, \Phi \in C(R^+, R)$ with $\int_0^{+\infty} |G(u)|du < 1$ and $\Phi \in L^1(R^+, R^+)$, and wedges W_j such that the following conditions hold for $t \geq 0$ and $\|x_t\| < H$,

(i) $W_1(|x(t) + \int_0^t G(t-s)x(s)ds|) \leq V(t, x_t)$,

(ii) $V(t, x_t) \leq W_2(|x(t)|) + W_3(\int_0^t \Phi(t-s)W_4(|x(s)|)ds)$

(iii) $V'_{(1.1)}(t, x_t) \leq -W_5(|x(t)|)$.

Then the zero solution of (1.1) is uniformly asymptotically stable.

Proof. Let $J = \int_0^{+\infty} \Phi(u)du$. For each $0 < B < H$, we choose $\delta > 0$ with $0 < \delta < B$ and

$$W_2(\delta) + W_3[JW_4(\delta)] < W_1[B(1 - \int_0^{+\infty} |G(u)|du)]. \tag{2.2}$$

Let $x(t) = x(t, t_0, \phi)$ be a solution of (1.1) with $\|\phi\| < \delta$ and $V(t) = V(t, x_t)$. Then for $t \geq t_0$, we have

$$\begin{aligned} W_1(|x(t) + \int_0^t G(t-s)x(s)ds|) &\leq V(t) \leq V(t_0) \\ &\leq W_2(\delta) + W_3(JW_4(\delta)). \end{aligned}$$

We claim that $|x(t)| < B$ for all $t \geq t_0$. Note that $|x(u)| < \delta < B$ for all $0 \leq u \leq t_0$. If there exists the first $\hat{t} > t_0$ such that $|x(\hat{t})| = B$ and $|x(s)| < B$ for $t_0 \leq s < \hat{t}$, then

$$\begin{aligned} W_1[B(1 - \int_0^{+\infty} |G(u)|du)] &\leq W_1(|x(\hat{t}) + \int_0^{\hat{t}} G(\hat{t}-s)x(s)ds|) \\ &\leq W_2(\delta) + W_3[JW_4(\delta)], \end{aligned}$$

which contradicts (2.2). Thus, the zero solution of (1.1) is uniformly stable. By (iii), we have $V(t)-V(\tau)\le -\int_\tau^t W_5(|x(s)|)ds$ for $t\ge\tau\ge t_0$. Since

$$V(\tau)\le W_2(B)+W_3(JW_4(B)) =: B^*$$

for any $\tau\ge t_0$, we have

$$\int_\tau^t W_5(|x(s)|)ds \le V(\tau)\le B^*.$$

This implies for each $\sigma>0$ there exists $L>0$ such that $W_5(|x(\tau^*)|)<\sigma$ for some $\tau^*\in[\tau,\tau+L]$. Now let $\sigma>0$ be fixed and find $h>0$ so that

$$(B+W_4(B))\int_h^{+\infty}\Phi(u)du<\sigma. \tag{2.3}$$

By the definition of L, we can choose a sequence $\{t_n\}$ satisfying the following conditions:

$$t_{n-1}+h\le t_n\le t_{n-1}+h+L$$

and

$$|x(t_n)|<\sigma$$

for $n=1,2,\cdots$. Define $K=\sup\{\Phi(u):0\le u\le h\}$. By (i), we have

$$\begin{aligned} V(t_j) &\le W_2(|x(t_j)|)\\ &\quad +W_3\Big(\int_0^{t_j-h}\Phi(t_j-s)W_4(|x(s)|)ds+\int_{t_j-h}^{t_j}\Phi(t_j-s)W_4(|x(s)|)ds\Big)\\ &< W_2(\sigma)+W_3\Big(W_4(B)\int_h^{+\infty}\Phi(s)ds+K\int_{t_j-h}^{t_j}W_4(|x(s)|)ds\Big)\\ &\le W_2(\sigma)+W_3\Big(\sigma+K\int_{t_j-h}^{t_j}W_4(|x(s)|)ds\Big). \end{aligned}$$

Since

$$V(t_n)-V(t_0)\le -\int_{t_0}^{t_n}W_5(|x(s)|)ds\le -\sum_{j=1}^{n}\int_{t_j-h}^{t_j}W_5(|x(s)|)ds,$$

we have

$$\sum_{j=1}^{+\infty}\int_{t_j-h}^{t_j}W_5(|x(s)|)ds\le V(t_0)\le B^*.$$

This implies there exists a positive integer N such that

$$\int_{t_N-h}^{t_N}W_5(|x(s)|)ds<\sigma.$$

By Lemma 1.1 , there exists a convex downward wedge W_6 with $W_6(r) \le W_5[W_4^{-1}(r)]$ for $0 \le r \le W_4(B)$. This yields

$$\int_{t_N-h}^{t_N} W_5(|x(s)|)ds = \int_{t_N-h}^{t_N} W_5(W_4^{-1})[W_4(|x(s)|)]ds \ge \int_{t_N-h}^{t_N} W_6[W_4(|x(s)|)ds.$$

Apply Jensen's inequality to obtain

$$\int_{t_N-h}^{t_N} W_6[W_4(|x(s)|)]ds \ge hW_6\Big[\frac{1}{h}\int_{t_N-h}^{t_N} W_4(|x(s)|)ds\Big].$$

Thus,

$$\int_{t_N-h}^{t_N} W_4(|x(s)|)ds < hW_6^{-1}(\sigma/h)$$

and

$$V(t) \le V(t_N) \le W_2(\sigma) + W_3\Big(\sigma + KhW_6^{-1}(\sigma/h)\Big)$$

for all $t \ge t_N$. Therefore,

$$W_1(|x(t) + \int_0^t G(t-s)x(s)ds|) \le W_2(\sigma) + W_3\Big(\sigma + KhW_6^{-1}(\sigma/h)\Big)$$

and

$$|x(t) + \int_0^t G(t-s)x(s)ds| \le W_1^{-1}\Big(W_2(\sigma) + W_3[\sigma + KhW_6^{-1}(\sigma/h)]\Big) =: \sigma^*.$$

Let $y(t) = x(t) + \int_0^t G(t-s)x(s)ds$. By Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
|x(t)| &= |y(t) - \int_0^t Z(t-s)y(s)ds| \\
&\le \sigma^* + \int_0^{t_N} |Z(t-s)||y(s)|ds + \int_{t_N}^t |Z(t-s)||y(s)|ds \\
&\le \sigma^* + 2B\int_{t_N}^t |Z(u)|du + \sigma^* \int_0^{+\infty} |Z(u)|du
\end{aligned}$$

for $t \ge t_N$. Note that $t_N \le t_0 + N(h+L)$. We also choose N large enough so that

$$2B\int_{N(h+L)}^{+\infty} |Z(u)|du < \sigma^*.$$

Define $\Gamma = \int_0^{+\infty} |Z(u)|du$. Then $|x(t)| < (2+\Gamma)\sigma^*$ for all $t \ge t_N$. For any $\varepsilon > 0$, choose $\sigma > 0$ so that $\sigma^* = \varepsilon/(2+\Gamma)$. Letting $T = N(h+L)$, we get $|x(t)| < \varepsilon$ for all $t \ge T + t_0$. This proves the zero solution of (1.1) is uniformly asymptotically stable.

Remark 2.1. It is clear from the proof that Theorem 2.1 can be extended to nonconvolution type with modifications on G and Φ. For more discussions on such extension, we refer the reader to Lakshmikantham, Wen, and Zhang [20].

3 Construction of a Liapunov functional

Consider the linear scalar Volterra integrodifferential equation

$$x'(t) = Ax(t) + \int_0^t C(t-s)x(s)ds \tag{3.1}$$

where A is a constant and $C : R^+ \to R$ is continuous. We will construct a Liapunov functional that is more general than those exist in the literature (see [6], [14]) and provides a stability region that is closer to the one given by the characteristic equation of (3.1). Our method here is inspired by the work of Knyazhishche and Shcheglov [15] who investigated the linear scalar differential equations with finite delay.

Theorem 3.1. Suppose there exists a constant $\alpha \leq 0$ such that $G_\alpha(t) = \int_t^{+\infty} C(u)e^{\alpha u}du e^{-\alpha t}$ exists with $\int_0^{+\infty} |G_\alpha(u)|du < 1$ and $\int_0^{+\infty} u|G_\alpha(u)|du < +\infty$. If

$$A + \int_0^{+\infty} C(u)e^{\alpha u}du + |\alpha| \int_0^{+\infty} |G_\alpha(u)|du < 0, \tag{3.2}$$

then the zero solution of (3.1) is uniformly asymptotically stable.

Proof. Let $x(t) = x(t, t_0, \phi)$ be a solution of (3.1) and define

$$V_1(t) = \frac{1}{2}\Big(x(t) + \int_0^t G_\alpha(t-s)x(s)ds\Big)^2 \tag{3.3}$$

for $t \geq t_0$. Taking the derivative along the solution, we obtain

$$\begin{aligned} V_1'(t) &= (x(t) + \int_0^t G_\alpha(t-s)x(s)ds)\Big(Ax(t) + \int_0^t C(t-s)x(s)ds \\ &+ \int_0^{+\infty} C(u)e^{\alpha u}du\, x(t) - \int_0^t C(t-s)x(s)ds - \alpha \int_0^t G_\alpha(t-s)x(s)ds\Big) \\ &= (x(t) + \int_0^t G_\alpha(t-s)x(s)ds)\Big(a_1 x(t) - \alpha \int_0^t G_\alpha(t-s)x(s)ds\Big) \\ &= a_1 x^2(t) + (a_1 - \alpha)x(t) \int_0^t G_\alpha(t-s)x(s)ds - \alpha\Big(\int_0^t G_\alpha(t-s)x(s)ds\Big)^2 \end{aligned} \tag{3.4}$$

where $a_1 = A + \int_0^{+\infty} C(u)e^{\alpha u}du$. It follows from Cauchy's inequality

$$\Big(\int_0^t G_\alpha(t-s)x(s)ds\Big)^2 \le \int_0^t |G_\alpha(t-s)|ds \int_0^t |G_\alpha(t-s)|x^2(s)ds.$$

Thus, for any constant $\delta > 0$ we have

$$\begin{aligned} V_1'(t) &\le a_1x^2(t) + (a_1-\alpha)x(t)\int_0^t G_\alpha(t-s)x(s)ds \\ &\quad +|\alpha| \int_0^t |G_\alpha(t-s)|ds\int_0^t |G_\alpha(t-s)|x^2(s)ds \\ &\le a_1x^2(t) + \frac{(a_1-\alpha)^2}{4\delta}x^2(t) \\ &\quad +(|\alpha|+\delta)\int_0^t |G_\alpha(t-s)|ds \int_0^t |G_\alpha(t-s)|x^2(s)ds. \end{aligned} \tag{3.5}$$

Next, define

$$V_2(t) = p\int_0^t \int_{t-s}^{+\infty} |G_\alpha(u)|dux^2(s)ds \tag{3.6}$$

where p is a positive constant and set

$$V(t) = V_1(t) + V_2(t).$$

Observe that

$$V_2'(t) = p\int_0^{+\infty} |G_\alpha(u)|dux^2(t) - p\int_0^t |G_\alpha(t-s)|x^2(s)ds.$$

Combining with (3.5), this yields

$$\begin{aligned} V'(t) &\le \Big(a_1 + \frac{(a_1-\alpha)^2}{4\delta} + p\int_0^{+\infty} |G_\alpha(u)|du\Big)x^2(t) \\ &+ [(|\alpha|+\delta)\int_0^{+\infty} |G_\alpha(u)|du - p]\int_0^t |G_\alpha(t-s)|x^2(s)ds. \end{aligned} \tag{3.7}$$

We will choose the best δ and p so that

$$a_1 + \frac{(a_1-\alpha)^2}{4\delta} + p\int_0^{+\infty} |G_\alpha(u)|du < 0 \tag{3.8}$$

and

$$(|\alpha|+\delta)\int_0^{+\infty} |G_\alpha(u)|du - p \le 0. \tag{3.9}$$

Notice that if $\int_0^{+\infty} |G_\alpha(u)|du \neq 0$, then (3.8) and (3.9) hold if and only if

$$(|\alpha|+\delta)\Big(\int_0^{+\infty} |G_\alpha(u)|du\Big)^2 \leq p\int_0^{+\infty} |G_\alpha(u)|du < |a_1| - \frac{(a_1-\alpha)^2}{4\delta}.$$

This implies that we choose $\delta > 0$ so that

$$(|\alpha|+\delta)\Big(\int_0^{+\infty} |G_\alpha(u)|du\Big)^2 < |a_1| - \frac{(a_1-\alpha)^2}{4\delta}. \tag{3.10}$$

Let $Q = (\int_0^{+\infty} |G_\alpha(u)|du)^2$. We may assume that $G_\alpha(u) \not\equiv 0$ since $G_\alpha(u) \equiv 0$ for all $u \in R^+$ implies $C(u) \equiv 0$ for all $u \in R^+$. Thus we assume that $Q > 0$. It is clear that (3.10) is equivalent to

$$4Q\delta^2 + 4(|\alpha|Q - |a_1|)\delta + (a_1-\alpha)^2 < 0. \tag{3.11}$$

The quadratic function of δ on the left-hand side of (3.11) has its minimum at

$$\delta_* = \frac{|a_1| - |\alpha|Q}{2Q}.$$

Since $\int_0^{+\infty} |G_\alpha(u)|du < 1$, we have $0 < Q < 1$ and

$$\begin{aligned} |a_1| - |\alpha|Q &= -A - \int_0^{+\infty} C(u)e^{\alpha u}du - |\alpha|Q \\ &\geq -\Big(A + \int_0^{+\infty} C(u)e^{\alpha u}du + |\alpha|\int_0^{+\infty} |G_\alpha(u)|du\Big) > 0 \end{aligned}$$

by (3.2). Thus $\delta_* > 0$ and

$$\begin{aligned} & 4Q\delta_*^2 + 4(|\alpha|Q - |a_1|)\delta_* + (a_1-\alpha)^2 \\ = \ & \frac{(|a_1| - |\alpha|Q)^2}{Q} - \frac{2(|a_1| - |\alpha|Q)^2}{Q} + (a_1-\alpha)^2 \\ = \ & \frac{Q-1}{Q}[|a_1|^2 - Q|\alpha|^2] \\ = \ & \frac{Q-1}{Q}(|a_1| + \sqrt{Q}|\alpha|)(|a_1| - \sqrt{Q}|\alpha|) \\ = \ & \frac{1-Q}{Q}\Big(|a_1| + \sqrt{Q}|\alpha|)(A + \int_0^{+\infty} C(u)e^{\alpha u}du + |\alpha|\int_0^{+\infty} |G_\alpha(u)|du\Big) < 0. \end{aligned}$$

Let

$$\gamma_1 =: (|\alpha| + \delta_*)Q = (|\alpha| + \frac{|a_1| - |\alpha|Q}{2Q})Q = \frac{|a_1| + |\alpha|Q}{2}$$

and

$$\gamma_2 \quad =: \quad |a_1| - \frac{(a_1-\alpha)^2}{4\delta_*} = \frac{|a_1|^2 - \alpha^2 Q + (1-Q)|a_1|^2}{2(|a_1| - |\alpha|Q)}.$$

Define

$$p = \frac{\gamma_1+\gamma_2}{2\sqrt{Q}} = \frac{(1-Q)|a_1|^2 + 2|a_1|^2 - Q(1+Q)|\alpha|^2}{4(|a_1|-|\alpha|Q)\sqrt{Q}}. \tag{3.12}$$

We now obtain

$$a_1 + \frac{(a_1-\alpha)^2}{4\delta_*} + p\sqrt{Q} = -\frac{(1-Q)(|a_1|^2-\alpha^2 Q)}{4(|a_1|-|\alpha|Q)}$$

and

$$(|\alpha|+\delta_*)Q - p\sqrt{Q} = -\frac{(1-Q)(|a_1|^2-\alpha^2 Q)}{4(|a_1|-|\alpha|Q)}.$$

Define

$$\beta = \frac{(1-Q)(|a_1|^2-\alpha^2 Q)}{4(|a_1|-|\alpha|Q)} \min\{1, \frac{1}{\sqrt{Q}}\}.$$

Combining (3.3), (3.6), and (3.7), we have

$$\begin{aligned} V(t) &= \frac{1}{2}\Big(x(t) + \int_0^t G_\alpha(t-s)x(s)ds\Big)^2 + p\int_0^t \int_{t-s}^{+\infty} |G_\alpha(u)|du x^2(s)ds \\ &\leq x^2(t) + \int_0^t \Big(|G_\alpha(t-s)| + p\int_{t-s}^{+\infty} |G_\alpha(u)|du\Big)x^2(s)ds \\ &=: x^2(t) + \int_0^t \Phi(t-s)x^2(s)ds \end{aligned} \tag{3.13}$$

and

$$V'(t) \leq -\beta x^2(t) - \beta\int_0^t |G_\alpha(t-s)|x^2(s)ds \tag{3.14}$$

where p is given in (3.12) and

$$\Phi(t) = |G_\alpha(t)| + p\int_t^{+\infty} |G_\alpha(u)|du.$$

One can show by interchanging the order of integration that

$$\int_0^{+\infty}\int_s^{+\infty} |G_\alpha(u)|duds = \int_0^{+\infty} u|G_\alpha(u)|du < +\infty. \tag{3.15}$$

Thus, $\Phi \in L^1(R^+)$. By Theorem 2.1, the zero solution of (3.1) is uniformly asymptotically stable. This completes the proof.

When $\alpha = 0$, Theorem 3.1 takes the following form.

Corollary 3.1. Suppose that

$$A + \int_0^{+\infty} C(u)du < 0,$$

$$\int_0^{\infty} \Big| \int_t^{\infty} C(u)du \Big| dt < 1,$$

and

$$\int_0^{\infty} s \Big| \int_s^{+\infty} C(u)du \Big| ds < +\infty. \tag{3.16}$$

Then the zero solution of (3.1) is uniformly asymptotically stable.

In ([6], p.162) Burton and Mahfoud proved the following theorem. Brauer [2], Jordan [13], and Krisztin [18] also investigated the same problem in one or higher dimensions using the characteristic equation of (3.1).

Theorem $\mathbf{B}_1$. Suppose that

$$A + \int_0^{+\infty} C(u)du < 0, \tag{3.17}$$

$$\int_0^{\infty} | \int_t^{\infty} C(u)du | dt < 1, \tag{3.18}$$

and

$$\int_0^{\infty} u|C(u)|du < +\infty. \tag{3.19}$$

Then the zero solution of (3.1) is uniformly asymptotically stable.

Although (3.19) is weaker than (3.16) for $C(u)$ with constant sign, one can show that (3.19) implies (3.16) for $\alpha < 0$. In fact, if (3.19) holds, we have

$$\int_0^{+\infty} u|G_\alpha(u)|du \quad \le \quad \int_0^{+\infty} u \int_u^{+\infty} |C(s)|e^{\alpha(s-u)} ds du$$

$$
\begin{aligned}
&= \int_0^{+\infty}\int_0^s ue^{-\alpha u}du|C(s)|e^{\alpha s}ds \\
&= \int_0^{+\infty}\Big(\frac{1}{|\alpha|}se^{-\alpha s}-\frac{1}{\alpha^2}e^{-\alpha s}+\frac{1}{\alpha^2}\Big)|C(s)|e^{\alpha s}ds \\
&\leq \alpha^*\int_0^{+\infty}(s+1)|C(s)|ds<+\infty
\end{aligned}
$$

where $\alpha^* = \max\{\frac{1}{|\alpha|}, \frac{1}{\alpha^2}\}$.

Corollary 3.2. Suppose (3.19) holds and there exists a constant $\alpha < 0$ such that

$$\int_0^{+\infty}|G_\alpha(u)|du < 1 \tag{3.20}$$

and

$$A+\int_0^{+\infty}C(u)e^{\alpha u}du+|\alpha|\int_0^{+\infty}|G_\alpha(u)|du<0, \tag{3.21}$$

then the zero solution of (3.1) is uniformly asymptotically stable.

It is known (see [10],[21]) that the zero solution of (3.1) is uniformly asymptotically stable if and only if

$$|z-A-C^*(z)|\neq 0$$

for $\mathrm{Re}(z)\geq 0$, where z is a complex number and C^* is the Laplace transform of C defined by $C^*(z)=\int_0^{+\infty}e^{-zt}C(t)dt$. Thus, condition (3.17) is necessary for uniform asymptotic stability. The following example shows that (3.18) does not hold, but (3.20) and (3.21) are satisfied for an appropriate α. Consider

$$x'(t)=0.06x(t)-\int_0^t e^{-0.9(t-s)}x(s)ds. \tag{3.22}$$

Let $A = 0.06$ and $C(t) = -e^{-0.9t}$. Then

$$\int_0^{+\infty}|\int_t^{+\infty}C(u)du|dt=\int_0^{+\infty}\frac{1}{0.9}e^{-0.9t}dt=\frac{1}{(0.9)^2}>1.$$

Thus, (3.18) fails. Letting $\alpha = -0.8$, we have

$$
\begin{aligned}
G_\alpha(t)=\int_t^{+\infty}C(u)e^{\alpha u}due^{-\alpha t}&=-\frac{1}{1.7}e^{-0.9t}, \\
\int_0^{+\infty}|G_\alpha(u)|du=\frac{1}{1.7}\int_0^{+\infty}e^{-0.9t}dt&=\frac{1}{1.53}<1,
\end{aligned}
$$

and

$$
\begin{aligned}
& A+\int_0^{+\infty} C(u)e^{\alpha u}du+|\alpha|\int_0^{+\infty}|G_\alpha(u)|du \\
= \; & 0.06-\int_0^{+\infty} e^{-1.7u}du+\frac{0.8}{1.53} \\
< \; & -0.005.
\end{aligned}
$$

It is obvious that $\int_0^{+\infty} u|G_\alpha(u)|du < +\infty$. By Theorem 3.1, the zero solution of (3.22) is uniformly asymptotically stable.

Generalizations of Theorem B_1 to systems and noncovolution type of equations can be found in Burton and Mahfoud [6] and Zhang [24]. The author has not yet obtained a satisfactory extension of Theorem 3.1 to systems and will pursue further investigation. Finally we present a theorem on asymptotic stability.

Theorem 3.2. Suppose there exists a constant $\alpha \le 0$ such that (3.20) and (3.21) hold. Then the zero solution of (3.1) is asymptotically stable.

Proof. It is clear that the zero solution of (3.1) is stable by (3.13) and (3.14). Let $x(t) = x(t, t_0, \phi)$ be a solution of (3.1). We will show that $x(t) \to 0$ as $t \to +\infty$. Indeed, by (3.14) and Cauchy's inequality, we have

$$V'(t) \le -\frac{\beta}{2}\Big(x(t)+\int_0^t G_\alpha(t-s)x(s)ds\Big)^2. \tag{3.23}$$

This implies that

$$\gamma(t) =: \Big(x(t)+\int_0^t G_\alpha(t-s)x(s)ds\Big)^2 \in L^1(R^+). \tag{3.24}$$

By (3.4), we have

$$
\begin{aligned}
|\gamma'(t)| \; &= \; 2\Big|a_1x^2(t)+(a_1-\alpha)x(t)\int_0^t G_\alpha(t-s)x(s)ds-\alpha\Big(\int_0^t G_\alpha(t-s)x(s)ds\Big)^2\Big| \\
&\le \; (2|a_1|+|a_1-\alpha|)x^2(t)+(2|\alpha|+|a_1-\alpha|)\int_0^t |G_\alpha(t-s)|x^2(s)ds. \qquad (3.25)
\end{aligned}
$$

Combining (3.24) and (3.25), we get $|\gamma'(t)| \in L^1(R^+)$. Therefore, $\lim_{t\to+\infty}\gamma(t) = 0$. Let $y(t) = x(t) + \int_0^t G_\alpha(t-s)x(s)ds$. By Lemmas 2.1 and 2.2, we have

$$|x(t)| \;=\; |y(t)-\int_0^t Z(t-s)y(s)ds|$$

$$\leq \quad |y(t)| + |\int_0^t Z(t-s)y(s)ds| \to 0$$

since $Z \in L^1(R^+)$ and $y(t) \to 0$ as $t \to +\infty$. This completes the proof.

Corollary 3.3. If

$$A + \int_0^{+\infty} C(u)du < 0$$

and

$$\int_0^{\infty} |\int_t^{\infty} C(u)du|dt < 1,$$

then the zero solution of (3.1) is asymptotically stable.

Corollary 3.4. If

$$A + \int_0^{+\infty} |C(u)|du < 0, \tag{3.26}$$

then the zero solution of (3.1) is asymptotically stable.

Proof. If (3.26) holds, we can choose α sufficiently large so that conditions of Theorem 3.2 will be satisfied. In fact, for $\alpha < 0$ we have

$$\begin{aligned} |\alpha| \int_0^{+\infty} |G_\alpha(u)|du &\leq -\alpha \int_0^{+\infty} \int_t^{+\infty} |C(u)|e^{\alpha u} du e^{-\alpha t} dt \\ &= \Big(\int_t^{+\infty} |C(u)|e^{\alpha u} du \Big) e^{-\alpha t} \Big|_0^{+\infty} + \int_0^{+\infty} |C(u)|du \\ &= -\int_0^{+\infty} |C(u)|e^{\alpha u} du + \int_0^{+\infty} |C(u)|du. \end{aligned}$$

This yields

$$\int_0^{+\infty} |G_\alpha(u)|du \leq \frac{1}{|\alpha|} \int_0^{+\infty} |C(u)|du$$

for $\alpha < 0$. If $\int_0^{+\infty} |C(u)|du < |\alpha|$, then $\int_0^{+\infty} |G_\alpha(u)|du < 1$. By (3.26), we also have

$$\begin{aligned} & A + \int_0^{+\infty} C(u)e^{\alpha u} du + |\alpha| \int_0^{+\infty} |G_\alpha(u)|du \\ \leq \;& A + \int_0^{+\infty} C(u)e^{\alpha u} du - \int_0^{+\infty} |C(u)|e^{\alpha u} du + \int_0^{+\infty} |C(u)|du \\ \leq \;& A + \int_0^{+\infty} |C(u)|du < 0. \end{aligned}$$

By Theorem 3.2, the zero solution of (3.1) is asymptotically stable.

Burton and Mahfoud ([6], p.146) showed that (3.26) yields uniform asymptotic stability by proving that $x(t) \in L^1(R^+, R)$.

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