ON GEOMETRIC DETECTION OF PERIODIC SOLUTIONS AND CHAOTIC DYNAMICS

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Abstract. In this note we sketch some results concerning a geometric method for periodic solutions of non-autonomous time-periodic differential equations. We give the definition of isolating chain and we provide a theorem on the existence of periodic solutions inside isolating chains. We recall some results on existence of chaotic dynamics which can be proved by the theorem. We provide several examples of equations in which the presented theorems can be applied.

1. Introduction

The question of existence of periodic solutions is one of the most fundamental problems of qualitative theory of differential equations. Usually, the class of non-autonomous time-periodic equations is considered and one looks for harmonic and subharmonic periodic solutions of an equation in that class. In the nonlinear case, the method of guiding functions, functional-analytic methods based on modifications of the Leray-Schauder degree, and variational methods are mostly applied in research; see the books [KZ, RM].

Another method, called here geometric, was introduced in [S1, S2]. It is based on proper location of the vector-field on the boundaries of some subsets, called periodic isolating segments (or periodic isolating blocks), of the extended phase space of the equation. A proper value of the Lefschetz number of some homomorphism associated to the segment guarantees the existence of fixed points of the Poincaré map of the equation (via the Lefschetz theorem) and, in consequence, also the required periodic solution inside the segment. The notion of periodic isolated segments arose as a modification of the concept of isolating block from the Conley index theory, compare [C, CE, Sm]. In

2000 *Mathematics Subject Classification.* Primary 37B10, 37C25; Secondary 34C25, 37C60, 37D45.

Key words and phrases. Isolating segment, isolating chain, Lefschetz number, fixed point index, chaos, shift.

Supported by KBN, Grant P03A 028 17.

The paper is in final form and no version of it will be published elsewhere.

[KS, S3], the geometric method was applied in results on planar polynomial (or rational) equations of the form $\dot{z} = \sum_{k,m,n} c_{k,m,n} e^{ikt} z^m \overline{z}^n$. Generalizations of some of those results (obtained by other methods) appeared in [M, MMZ]. Further applications of the geometric method concern the existence of chaotic dynamics of the Poincaré map (see [PW, S4, S5, SW, W1, W2, W3, WZ1]) and the existence of homoclinic solutions (see [WZ2]).

The aim of this note is to present, without proofs, an improvement of the geometric method given in [S6] as well as to recall some theorems on chaotic dynamics which were obtained by method. The improvement is based on the notion of periodic isolating chain, being a union of several isolating segments satisfying some concordance relations (see Section 2). The main result, Theorem 1 in Section 3, asserts that the Lefschetz number of a homomorphism in reduced homologies of a section of a periodic isolating chain is equal to the fixed point index of the corresponding restriction of the Poincaré map. We give some examples of practical applications of the theorem. Finally, in Section 4 we define the notion of chaotic equation and we recall two results (Theorems 2 and 3) on existence of them. By chaotic we call a time-periodic equation such that its Poincaré map is semi-conjugated to the shift on r symbols and the counterimage of a periodic point in the shift contains an initial point of a periodic solutions of the equation. We provide some examples of chaotic equations and give some comments on further development of the theory.

A few remarks concerning the notation which is used in the sequel. If X is a topological space and $A \subset X$ then X/A is the space obtained by collapsing A to a point provided $A \neq \emptyset$, and $X/\emptyset := X \cup \{*\}$, where $*$ is a point, $* \notin X$. By $\widetilde{H}$ we denote the singular homology functor with rational coefficients. If X is such that the graded vector space $\widetilde{H}(X) = \{\widetilde{H}_n(X)\}_{n\in\mathbb{Z}}$ is finitely dimensional then $\chi(X)$ denotes the Euler characteristic of X and $\Lambda(\phi)$, the *Lefschetz number* of an endomorphism $\phi = \{\phi_n\} =: \widetilde{H}(X) \to \widetilde{H}(X)$, is defined as

$$\Lambda(\phi) := \sum_{n=0}^{\infty} (-1)^n \operatorname{trace} \phi_n.$$

Let X be an ENR (*Euclidean Neighborhood Retract*) and let $f : X \to X$ be a continuous map. A set $K \subset X$ is called an *isolated set of fixed points of* X provided K is compact and there exists a neighborhood U of K such that all fixed points of $f|_U$ are in K. In that case, by

$\operatorname{ind}(f, K)$ we denote the *fixed point index*, see [D]. (The index in our notation corresponds to $I(f|_U)$ in the notation of [D].)

2. Isolating segments and chains

Let M be a differentiable manifold and let $f : \mathbb{R} \times M \to TM$ be a time-dependent vector-field on M. We assume that the equation

$$\dot{x} = f(t, x) \tag{1}$$

has the uniqueness property, i.e. for each $t_0 \in \mathbb{R}$ and $x_0 \in M$ there is a unique solution $t \to u_{(t_0,t)}(x_0)$ such that

$$u_{(t_0,t_0)}(x_0) = x_0.$$

The map $u : (s, t, x) \mapsto u_{(s,t)}(x)$ is called an *evolutionary operator* corresponding to (1); the induced map $u_{(s,t)}$ defined on an open (possibly empty) subset of M describes the evolution from the time s to t. The cartesian product $\mathbb{R} \times M$ is called the *extended phase space* of the equation,

For a subset Z of the extended phase space $\mathbb{R} \times M$ and $t \in \mathbb{R}$ we put

$$Z_t := \{x \in M : (t, x) \in Z\}$$

and by $\pi_1 : \mathbb{R} \times M \to \mathbb{R}$ and $\pi_2 : \mathbb{R} \times M \to M$ we denote the projections.

Let a and b be real numbers, $a < b$. A compact ENR $W \subset [a, b] \times M$ is called an *isolating segment* over $[a, b]$ provided there exist compact ENRs W^- and W^+ contained in W such that

(i) there exists a homeomorphism $h : [a, b] \times M \to [a, b] \times M$ such that $\pi_1 \circ h = \pi_1$ and

$$h([a, b] \times W_a) = W, \quad h([a, b] \times W_a^\pm) = W^\pm,$$

(ii) $\partial W_a = W_a^- \cup W_a^+$,

(iii) The sets W and $W^\pm$ are related to the evolutionary operator by the following equations:

$$W^- \cap ([a, b) \times M) =$$
$$\{(t, x) \in W : t \in [a, b), \exists\{\epsilon_n\}, 0 < \epsilon_n \to 0 : u_{(t,t+\epsilon_n)}(x) \notin W_{t+\epsilon_n}\},$$

$$W^+ \cap ((a, b] \times M) =$$
$$\{(t, x) \in W : t \in (a, b], \exists\{\epsilon_n\}, 0 < \epsilon_n \to 0 : u_{(t,t-\epsilon_n)}(x) \notin W_{t-\epsilon_n}\}.$$

Let W be an isolating segment over $[a,b]$. A homeomorphism h in (j) induces homeomorphism

$$m : (W_a/W_a^-, [W_a^-]) \to (W_b/W_b^-, [W_b^-])$$

of pointed spaces by the formula

$$m([x]) := [\pi_2 h(b, \pi_2 h^{-1}(a,x))].$$

We call m a *monodromy map* of the isolating segment W. Monodromy maps of the segment W are unique up to homotopy class, hence the isomorphism

$$\mu_W := \widetilde{H}(m) : \widetilde{H}(W_a/W_a^-) \to \widetilde{H}(W_b/W_b^-)$$

is an invariant of W.

Let $a < b < c$, let U be an isolating segment over $[a,b]$ and let V be an isolating segment over $[b,c]$. We call the segments U and V *contiguous* if

$$\begin{aligned}(\overline{U_b \setminus V_b} \cup U_b^-) \cap V_b &\subset V_b^-,\\ (\overline{V_b \setminus U_b} \cup V_b^+) \cap U_b &\subset U_b^+.\end{aligned}$$

Assume that U and V are contiguous. Define a map

$$n : (U_b/U_b^-, [U_b^-]) \to (V_b/V_b^-, [V_b^-])$$

by

$$n([x]) := \begin{cases} [x], & \text{if } x \in U_b \cap V_b,\\ [V_b^-], & \text{if } x \in U_b \setminus V_b.\end{cases}$$

One can prove that n is correctly defined and continuous. We call n the *transfer map* of the contiguous isolating segments U and V. The transfer map induces the homomorphism

$$\nu_{UV} := \widetilde{H}(n) : \widetilde{H}(U_b/U_b^-) \to \widetilde{H}(V_b/V_b^-).$$

We denote the union of the above contiguous isolating segments U and V by UV. More generally, let $N \in \mathbb{N}$, $N \geq 1$, let $a_0 < a_1 < \ldots < a_N$ and let $U^1, \ldots, U^N$ be isolating segments, U^i over $[a_{i-1}, a_i]$. Assume that U^i and U^{i+1} are contiguous for every $i = 1, \ldots, N-1$. Denote by $U^1 \ldots U^N$ the union $\bigcup_{i=1}^N U^i$. Such a union of contiguous segments is called an *isolating chain* over $[a_0, a_N]$.

3. A theorem on periodic solutions

We assume that the vector-field f in (1) is T-periodic in t for some $T > 0$. An isolating segment W over $[a, a+T]$ is called a *periodic isolating segment* if $W_a = W_{a+T}$ and $W_a^{\pm} = W_{a+T}^{\pm}$. More generally, an isolating chain $U^1 \dots U^N$ over $[a, a+T]$ is called *periodic* provided U^N and $\tau_T(U^1)$ are contiguous, where the map τ_T is the translation; $\tau_T : (t, x) \mapsto (t+T, x)$.

Let $C := U^1 \dots U^N$ be a periodic isolating chain over $[a, a+T]$. We define a homomorphism

$$\rho_C : \widetilde{H}(U_a^1/U_a^{1-}) \to \widetilde{H}(U_a^1/U_a^{1-})$$

by

$$\rho_C := \nu_{U^N \tau_T(U^1)} \circ \mu_{U^N} \circ \dots \circ \nu_{U^2U^3} \circ \mu_{U^2} \circ \nu_{U^1U^2} \circ \mu_{U^1}.$$

Obviously, if W is a periodic isolating segment then $\rho_W = \mu_W$. The following theorem was proved in [S6]; it is a generalization of [S2, Th. 7.1] to the case of isolating chains.

Theorem 1. *If $C := U^1 \dots U^N$ is a periodic isolating chain over $[a, a+T]$ then*

$$F_C := \{x \in U_a^1 : u_{(a,a+T)}(x) = x, \forall t \in [a, a+T] : u_{(a,t)}(x) \in C_t\}$$

is an isolated set of fixed points of $u_{(a,a+T)}$ and

$$\operatorname{ind}(u_{(a,a+T)}, F_C) = \Lambda(\rho_C).$$

In particular, if $\Lambda(\rho_C) \neq 0$ then F_C is nonempty, hence $u_{(a,a+T)}$ has a fixed point, which means that the equation (1) has a T-periodic solution.

In Figure 1, it is shown a twisted prism with hexagonal base. If its

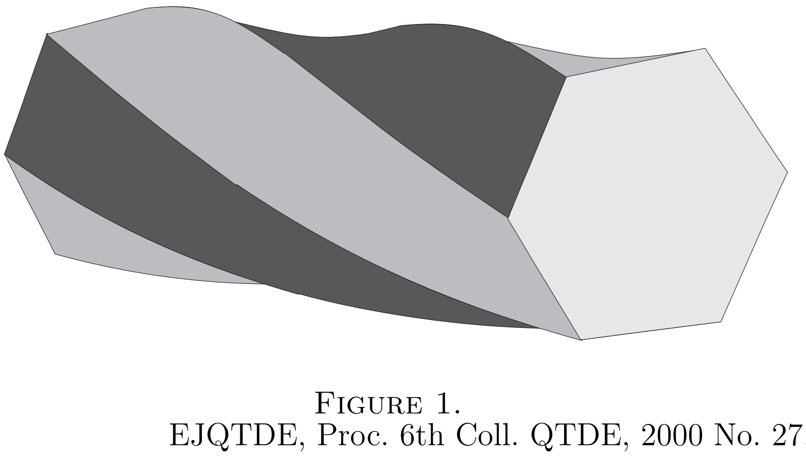

Figure 1.

size is large enough, it is a periodic isolating segment W over $[0, 2\pi]$ for the equation

$$\dot{z} = e^{it}\overline{z}^2 + 1, \tag{2}$$

where $z \in \mathbb{C}$. The set W^- is marked in dark grey. On can calculate that the corresponding number $\Lambda(\mu_W)$ is equal to 1, hence, by the above theorem, (2) has a 2π-periodic solution.

Actually, in the above result for the equation (2) we did not use Theorem 1 in full generality. In order to provide an example in which an essential isolating chain appear, we consider the class of planar equations

$$\dot{z} = \overline{z}^5 + \sin^2(\phi t)|z|^r\overline{z}. \tag{3}$$

Zero is a solution of the equation (3), hence one can look for another π/ϕ-periodic solution. If $0 \leq r < 4$ and $\phi > 0$ is small enough then (3) has two isolating chains over $[-\pi/(2\phi), \pi/(2\phi)]$ as shown in Figure 2. The isolating periodic segment Z being the prism with dodecagonal base on the upper picture is the first of them. The second of them is an essential periodic isolating chain UVW consisting of three isolating segments on the lower picture: the segments U over $[-\pi/(2\phi), -\pi/(3\phi)]$ and W over $[\pi/(3\phi), \pi/(2\phi)]$ have rectangles as the left and right faces, while the segment V over $[-\pi/(2\phi), \pi/(2\phi)]$ is again a prism with dodecagonal base. Moreover, $UVW \subset Z$. One can show that

$$\Lambda(\mu_Z) = -5, \quad \Lambda(\rho_{UVW}) = -1$$

hence the Poincaré map $u_{(-\pi/(2\phi), \pi/(2\phi))}$ has a nonzero fixed point, which means that in the considered range of values of ϕ and r the equation (3) has a nonzero π/ϕ-periodic solution. (For example, one can choose $r = 2$ and $0 < \phi \leq 0.001$).

4. On detecting of chaotic dynamics

Let r be a positive integer. Define

$$\Sigma_r := \{1, \dots r\}^{\mathbb{Z}},$$

the set of bi-infinite sequences of r symbols. Define the *shift map* as

$$\sigma : \Sigma_r \ni (\dots, s_{-1}.s_0, s_1, \dots) \to (\dots, s_0.s_1, s_2, \dots) \in \Sigma_r.$$

Assume, as in the previous section, that f is T-periodic in t. The equation (1) is called Σ_r-*chaotic* provided there is a compact set $I \subset M$, invariant with respect to the Poincaré map $u_{(a,a+T)}$ (for some $a \in \mathbb{R}$), and a map $g : I \to \Sigma_r$ such that:

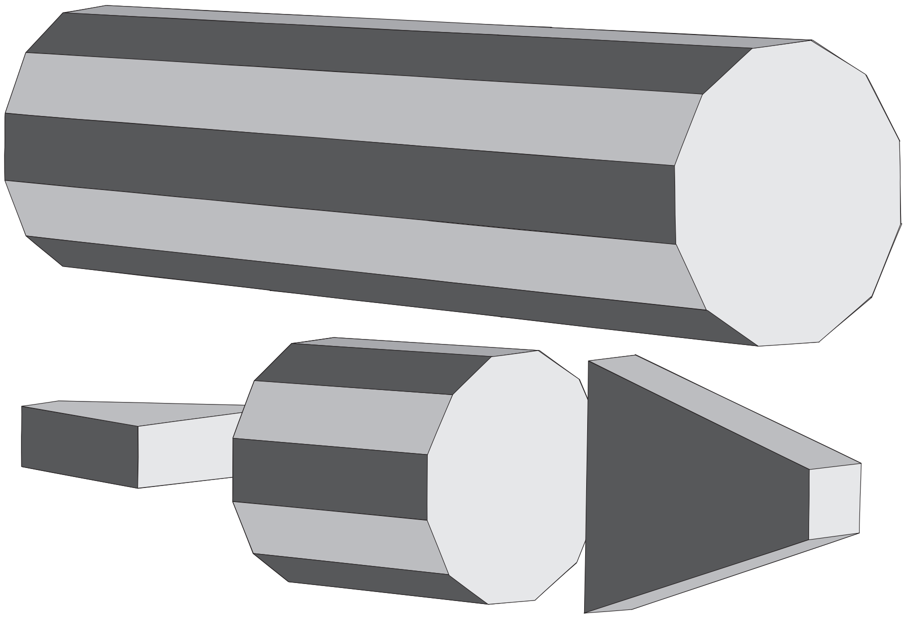

FIGURE 2.

(a) g is continuous and surjective,
(b) $\sigma \circ g = g \circ u_{(a,a+T)}$,
(c) for every k-periodic sequence $s \in \Sigma_r$ its counterimage $g^{-1}(s)$ contains at least one k-periodic point of $u_{(a,a+T)}$.

In particular, (c) implies that a Σ_r-chaotic equation has periodic solutions with minimal periods kT for every $k \in \mathbb{N}$. Chaotic dynamics in the above sense was considered in [Z1, Z2] and also in [MM], where the condition (c) was abandoned.

We recall here two results on chaotic equations which are consequences of Theorem 1 (actually, its simpler version which concerns periodic isolating segments, not chains). First of them was stated in [S4], its proof can be found in [S5]:

Theorem 2. *Assume that W and Z are periodic isolating segments for the equation* (1) *over* $[a, a+T]$ *and*

(i) $(W_a, W_a^-) = (Z_a, Z_a^-)$,
(ii) $\exists s \in (a, a+T) : W_s \cap Z_s = \emptyset$,
(iii) $\exists n \in \mathbb{N} : \widetilde{H}_n(W_a/W_a^-) = \mathbb{Q}$,
(iv) $\forall k \neq n : \widetilde{H}_k(W_a/W_a) = 0$.

Then (1) *is* Σ_2*-chaotic.*

The above theorem can be illustrated by the class of planar equations

$$\dot{z} = \frac{1}{2}e^{-i\phi t}z(\frac{1}{2}i\phi(z+1) + e^{i\phi t}(\overline{z}+1))(\frac{1}{2}i\phi(z-1) + e^{i\phi t}(\overline{z}-1)) \tag{4}$$

If $\phi > 0$ is small enough then there are two periodic isolating segments over $[0, 2\pi/\phi]$ as presented in Figure 3. It follows by Theorem 2 that

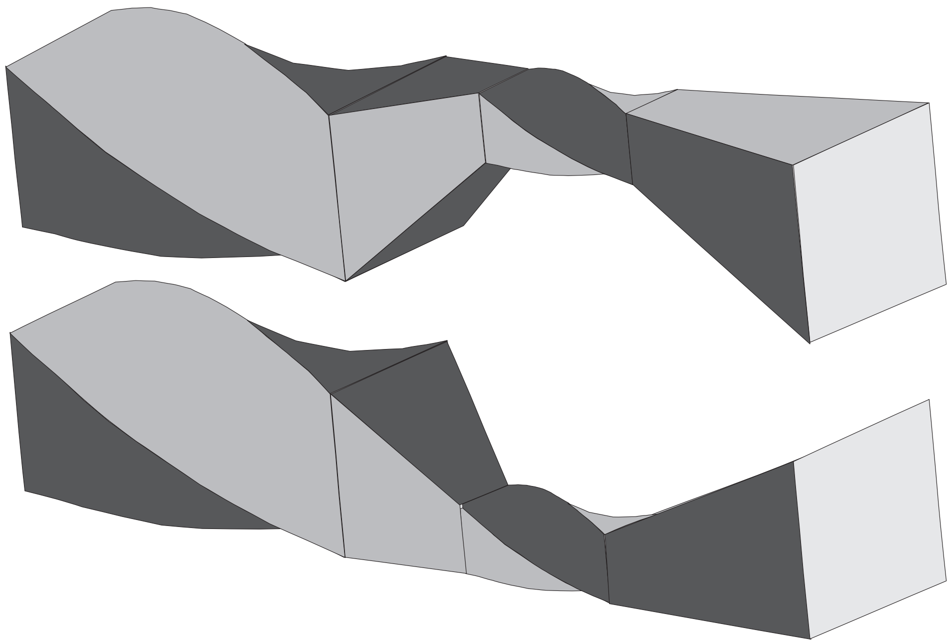

FIGURE 3.

(4) is Σ_2-chaotic for sufficiently small values of ϕ.

The other result comes from [SW]:

Theorem 3. *Assume that W and Z are periodic isolating segments for the equation* (1) *over* $[a, a+T]$ *and*

(j) $(W_a, W_a^-) = (Z_a, Z_a^-)$,
(jj) $Z \subset W$,
(jjj) $\mu_Z = \mu_W \circ \mu_W = \mathrm{id}_{\widetilde{H}(W_a/W_a^-)}$,
(jw) $\Lambda(\mu_W) \neq \chi(W_a) - \chi(W_a^-) \neq 0$.

Then (1) *is* Σ_2*-chaotic.*

The above theorem implies that the equation

$$\dot{z} = (1 + e^{i\phi t}|z|^2)\overline{z} \tag{5}$$

is Σ_2-chaotic if $0 < \phi \leq 0.495$. Indeed, for such ϕ there are two isolating segments W and Z over $[0, 2\pi/\phi]$ for (5) as shown in Figure 4. (On can deduce from the picture, that $\Lambda(\mu_W) = 1$, $\chi(W_0) = 1$, $\chi(W_0^-) = 2$ and the other required conditions are satisfied.) The above estimate on ϕ

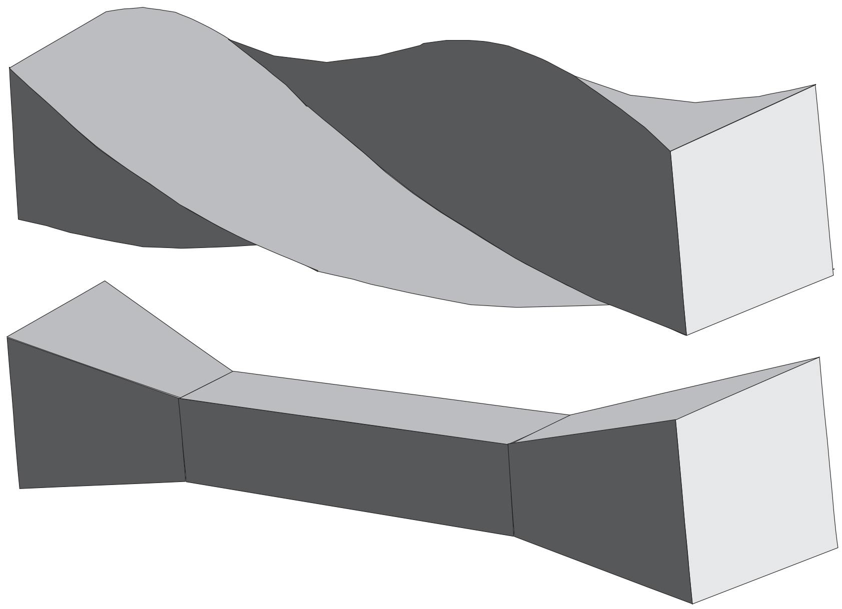

FIGURE 4.

was done in [WZ1] (originally, in [SW] it was assumed that $0 < \phi \leq 1/288$). Actually, in [WZ1] it was also proved that (5) is Σ_3 chaotic. Moreover, in the considered range of the parameter ϕ the equation (5) has infinitely many geometrically distinct homoclinic solutions to the zero one (this fact is proved in [WZ2]).

Some modifications and extensions of Theorem 3 are given in [PW, W1, W2, W3]. They concern also periodic isolating segments. The thesis [P] will contain results on existence of chaos in which Theorem 1 will be applied in full generality. In particular, there will be proved the existence of chaotic dynamics for some planar Fourier-Taylor polynomial differential equations of 5th degree. The use of isolating chains, which cannot be reduced to periodic isolating segments, will play an essential role in the arguments.

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