THE ABSTRACT RENEWAL EQUATION

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Abstract. The abstract Perron-Stieltjes integral defined in the Kurzweil-Henstock sense is used for introducing Stieltjes convolutions. The corresponding facts on integration are given in [6], [7] and [8].

The approach is used for obtaining the basic existence result for the abstract renewal equation which was studied e. g. by Diekmann, Gyllenberg and Thieme in [1] and [2].

For a given Banach space X let $L(X)$ be the Banach space of all bounded linear operators $A : X \to X$ with the uniform operator topology.

For $B : L(X) \times X \to X$ given by $B(A, x) = Ax \in X$ for $A \in L(X)$ and $x \in X$, we obtain the bilinear triple $(L(X), X, X)$ because we have

$$\|B(A, x)\|_X \le \|A\|_{L(X)} \|x\|_X$$

for the bilinear form B. Similarly, if we define the bilinear form $B^* : L(X) \times L(X) \to L(X)$ by $B^*(A, C) = AC \in L(X)$ for $A, C \in L(X)$ where AC is the composition of the linear operators A and C we get the bilinear triple $(L(X), L(X), L(X))$ because

$$\|B^*(A, C)\|_{L(X)} \le \|AC\|_{L(X)} \le \|A\|_{L(X)} \|C\|_{L(X)}.$$

Assume that the interval $[0, b] \subset \mathbb{R}$ is bounded.

Given $A : [0, b] \to L(X)$, the function A is of *bounded variation on* $[0, b]$ if

$$\mathrm{var}_{[0,b]}(A) = \sup\{\sum_{j=1}^{k} \|A(\alpha_j) - A(\alpha_{j-1})\|_{L(X)}\} < \infty,$$

where the supremum is taken over all finite partitions

$$D : 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{k-1} < \alpha_k = b$$

of the interval $[0, b]$. The set of all functions $A : [0, b] \to L(X)$ with $\mathrm{var}_{[0,b]}(A) < \infty$ will be denoted by $BV([0, b]; L(X))$.

For $A : [0, b] \to L(X)$ and a partition D of the interval $[0, b]$ define

$$V_0^b(A, D) = \sup\{\| \sum_{j=1}^{k} [A(\alpha_j) - A(\alpha_{j-1})] x_j \|_X\},$$

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where the supremum is taken over all possible choices of $x_j \in X, j = 1, \dots, k$ with $\|x_j\|_X \le 1$.

Let us set

$$s\,\mathrm{var}_{[0,b]}(A) = \sup V_0^b(A, D)$$

where the supremum is taken over all finite partitions D of the interval $[0, b]$.

An operator valued function $A : [0, b] \to L(X)$ with $s\,\mathrm{var}_{[0,b]}(A) < \infty$ is called a *function of bounded semi-variation on* $[0, b]$ (cf. [4]).

We denote by $BSV([0, b]; L(X))$ the set of all functions $A : [0, b] \to L(X)$ with

$$s\,\mathrm{var}_{[0,b]}(A) < \infty.$$

Assume that $\eta \ge 0$ is given and define

$$\mathrm{var}^{(\eta)}_{[0,b]}(A) = \sup\{\sum_{j=1}^{k} e^{-\eta\alpha_{j-1}}\|A(\alpha_j) - A(\alpha_{j-1})\|_{L(X)}\}$$

where the supremum is taken over all finite partitions D of the interval $[0, b]$.

Similarly define

$$V_0^b(\eta, A, D) = \sup\{\|\sum_{j=1}^{k}[A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta\alpha_{j-1}}\|_X\},$$

where the supremum is taken over all possible choices of $x_j \in X, j = 1, \dots, k$ with $\|x_j\|_X \le 1$ and set

$$s\,\mathrm{var}^{(\eta)}_{[0,b]}(A) = \sup V_0^b(\eta, A, D)$$

where the supremum is taken over all finite partitions D of the interval $[0, b]$.

Since for every $j = 1, \dots, k$ we have

$$e^{-\eta b} \le e^{-\eta\alpha_{j-1}} \le 1$$

we get

$$e^{-\eta b}\,\mathrm{var}_{[0,b]}(A) \le \mathrm{var}^{(\eta)}_{[0,b]}(A) \le \mathrm{var}_{[0,b]}(A) \tag{1}$$

and also

$$e^{-\eta b}V_0^b(A, D) \le V_0^b(\eta, A, D) \le V_0^b(A, D).$$

The last inequalities lead immediately to

$$e^{-\eta b}s\,\mathrm{var}_{[0,b]}(A) \le s\,\mathrm{var}^{(\eta)}_{[0,b]}(A) \le s\,\mathrm{var}_{[0,b]}(A). \tag{2}$$

Let us mention that

$$\mathrm{var}^{(0)}_{[0,b]}(A) = \mathrm{var}_{[0,b]}(A) \text{ and } s\,\mathrm{var}^{(0)}_{[0,b]}(A) = s\,\mathrm{var}_{[0,b]}(A).$$

It is well known that $BV([0, b]; L(X))$ with the norm

$$\|A\|_{BV} = \|A(0)\|_{L(X)} + \mathrm{var}_{[0,b]}(A)$$

is a Banach space and in [8] it was shown that with the norm

$$\|A\|_{SV} = \|A(0)\|_{L(X)} + s\,\mathrm{var}_{[0,b]}(A)$$

the space $BSV([0, b]; L(X))$ is also a Banach space.

Taking into account the inequalities (1) and (2) we get the following statement.

1. Proposition. *For every $\eta \geq 0$ the space $BV([0,b];L(X))$ with the norm*

$$\|A\|_{BV,\eta} = \|A(0)\|_{L(X)} + \operatorname{var}^{(\eta)}_{[0,b]}(A)$$

is a Banach space and the space $BSV([0,b];L(X))$ with the norm

$$\|A\|_{SV,\eta} = \|A(0)\|_{L(X)} + s\operatorname{var}^{(\eta)}_{[0,b]}(A)$$

is also a Banach space.

The norms $\|A\|_{BV,\eta}$ and $\|A\|_{BV}$ are equivalent on $BV([0,b];L(X))$ and the norms $\|A\|_{SV,\eta}$ and $\|A\|_{SV}$ are equivalent on $BSV([0,b];L(X))$.

Given $x:[0,b]\to X$, the function x is called *regulated on* $[0,b]$ if it has one–sided limits at every point of $[0,b]$, i.e. if for every $s\in[0,b)$ there is a value $x(s+)\in X$ such that

$$\lim_{t\to s+} \|x(t)-x(s+)\|_X = 0$$

and if for every $s\in(0,b]$ there is a value $x(s-)\in X$ such that

$$\lim_{t\to s-} \|x(t)-x(s-)\|_X = 0.$$

The set of all regulated functions $x:[0,b]\to X$ will be denoted by $G([0,b];X)$.

The space $G([0,b];X)$ endowed with the norm

$$\|x\|_{G([0,b];X)} = \sup_{t\in[0,b]} \|x(t)\|_X,\ x\in G([0,b];X)$$

is known to be a Banach space (see [4, Theorem 3.6]).

It is clear that the space $C([0,b];X)$ of continuous functions $x:[0,b]\to X$ is a closed subspace of $G([0,b];X)$, i.e.

$$C([0,b];X)\subset G([0,b];X).$$

We are using the concept of abstract Perron-Stieltjes integral based on the Kurzweil-Henstock definition presented via integral sums (for more detail see e.g. [5], [6], [7]).

A finite system of points

$$\{\alpha_0,\tau_1,\alpha_1,\tau_2,\ldots,\alpha_{k-1},\tau_k,\alpha_k\}$$

such that

$$0=\alpha_0<\alpha_1<\cdots<\alpha_{k-1}<\alpha_k=b$$

and

$$\tau_j\in[\alpha_{j-1},\alpha_j] \text{ for } j=1,\ldots,k$$

is called a *P–partition* of the interval $[0,b]$.

Any positive function $\delta:[0,b]\to(0,\infty)$ is called a *gauge on* $[0,b]$.

For a given gauge δ on $[0,b]$ a P–partition $\{\alpha_0,\tau_1,\alpha_1,\tau_2,\ldots,\alpha_{k-1},\tau_k,\alpha_k\}$ of $[0,b]$ is called *δ–fine* if

$$[\alpha_{j-1},\alpha_j]\subset(\tau_j-\delta(\tau_j),\tau_j+\delta(\tau_j)) \text{ for } j=1,\ldots,k.$$

Definition. Assume that functions $A, C : [0,b] \to L(X)$ and $x : [0,b] \to X$ are given.

We say that the Stieltjes integral $\int_0^b d[A(s)]x(s)$ exists if there is an element $J \in X$ such that for every $\varepsilon > 0$ there is a gauge δ on $[0,b]$ such that for

$$S(\mathrm{d}A, x, D) = \sum_{j=1}^{k} [A(\alpha_j) - A(\alpha_{j-1})]x(\tau_j)$$

we have

$$\|S(\mathrm{d}A, x, D) - J\|_X < \varepsilon$$

provided D is a δ–fine P–partition of $[0,b]$. We denote $J = \int_0^b d[A(s)]x(s)$.

Analogously we say that the Stieltjes integral $\int_0^b d[A(s)]C(s)$ exists if there is an element $J \in L(X)$ such that for every $\varepsilon > 0$ there is a gauge δ on $[0,b]$ such that for

$$S(\mathrm{d}A, C, D) = \sum_{j=1}^{k} [A(\alpha_j) - A(\alpha_{j-1})]C(\tau_j)$$

we have

$$\|S(\mathrm{d}A, C, D) - J\|_{L(X)} < \varepsilon$$

provided D is a δ–fine P–partition of $[0,b]$.

Similarly we can define the Stieltjes integral $\int_0^b A(s)\mathrm{d}[C(s)]$ using Stieltjes integral sums of the form

$$S(A, \mathrm{d}C, D) = \sum_{j=1}^{k} A(\tau_j)\big[C(\alpha_j) - C(\alpha_{j-1})\big].$$

Assume that $U, V : [0,\infty) \to L(X)$ and $x : [0,\infty) \to X$ are given and define the convolutions

$$(U * x)(t) = \int_0^t \mathrm{d}[U(s)]x(t-s)$$

and

$$(U * V)(t) = \int_0^t \mathrm{d}[U(s)]V(t-s)$$

for $t \in [0,\infty)$.

Let us denote by $BSV_{loc}([0,\infty), L(X))$ the set of all $U : [0,\infty) \to L(X)$ for which $U \in BSV([0,b], L(X))$ for every $b > 0$.

In [8] it was shown that if $U, V \in G([0,\infty), L(X)) \cap (\mathcal{B})BV_{loc}([0,\infty), L(X))$ and $x \in G([0,\infty), X)$ then the convolutions $(U * x)(t)$ and $(U * V)(t)$ are well defined for every $t \in [0,\infty)$ when the abstract Perron-Stieltjes integral is used.

It was also shown in [8] that

$$\|(U * V)(t)\|_{L(X)} \le \|U\|_{SV} . \|V\|_{SV} \tag{3}$$

holds for every $t \ge 0$.

2. Lemma. *Assume that*

$$U \in G([0,\infty), L(X)) \cap BSV_{loc}([0,\infty), L(X)),\ f \in G([0,\infty), X)$$

and that $\eta \geq 0$ *is given.*

Then the integral $\int_0^b \mathrm{d}[U(s)]e^{-\eta s} f(s) \in X$ *exists for every* $b > 0$ *and*

$$\| \int_0^b \mathrm{d}[U(s)]e^{-\eta s} f(s) \|_X \leq s\,\mathrm{var}_{[0,b]}^{(\eta)}(U) \cdot \sup_{s\in[0,b]} \|f(s)\|_X \tag{4}$$

holds.

Proof. The existence of the integral $\int_0^b \mathrm{d}[U(s)]e^{-\eta s} f(s)$ is clear because the function $e^{-\eta s} f(s)$ is regulated on $[0,\infty)$ (c.f. [6, Proposition 15]).

Assume that $b > 0$ is fixed. By the existence of the integral, for any $\varepsilon > 0$ there is a gauge δ on $[0,b]$ such that for every δ- fine P- partition

$$D = \{0 = \alpha_0, \tau_1, \alpha_1, \tau_2, \dots, \alpha_{k-1}, \tau_k, \alpha_k = b\}$$

of $[0,b]$ the inequality

$$\| \int_0^b \mathrm{d}[U(s)]e^{-\eta s} f(s) - \sum_{j=1}^k [U(\alpha_j) - U(\alpha_{j-1})]e^{-\eta\tau_j} f(\tau_j)\|_X < \varepsilon$$

holds. Hence

$$\| \int_0^b \mathrm{d}[U(s)]e^{-\eta s} f(s)\|_X < \varepsilon + \| \sum_{j=1}^k [U(\alpha_j) - U(\alpha_{j-1})]e^{-\eta\tau_j} f(\tau_j)\|_X. \tag{5}$$

Let us choose a fixed δ- fine P- partition D of $[0,b]$ for which $\alpha_{j-1} < \tau_j$ for every $j = 1,\dots,k$. Then

$$\| \sum_{j=1}^k [U(\alpha_j) - U(\alpha_{j-1})]e^{-\eta\tau_j} f(\tau_j)\|_X =$$

$$= \| \sum_{j=1}^k [U(\alpha_j) - U(\alpha_{j-1})]e^{-\eta\alpha_{j-1}} e^{-\eta(\tau_j-\alpha_{j-1})} f(\tau_j)\|_X =$$

$$= \| \sum_{j=1}^k [U(\alpha_j) - U(\alpha_{j-1})]e^{-\eta\alpha_{j-1}} \frac{e^{-\eta(\tau_j-\alpha_{j-1})} f(\tau_j)}{\|f(\tau_j\|_X} \|f(\tau_j)\|_X \|_X$$

and we have

$$\| \frac{e^{-\eta(\tau_j-\alpha_{j-1})} f(\tau_j)}{\|f(\tau_j)\|_X} \|_X \leq 1$$

for $j = 1,\dots,k$.

Hence

$$\| \sum_{j=1}^k [U(\alpha_j) - U(\alpha_{j-1})]e^{-\eta\alpha_{j-1}} \frac{e^{-\eta(\tau_j-\alpha_{j-1})} f(\tau_j)}{\|f(\tau_j)\|_X} \|f(\tau_j\|_X \|_X \leq$$

$$\leq \sup_{j=1,\dots,k} \|f(\tau_j\|_X \cdot \| \sum_{j=1}^k [U(\alpha_j) - U(\alpha_{j-1})]e^{-\eta\alpha_{j-1}} \frac{e^{-\eta(\tau_j-\alpha_{j-1})} f(\tau_j)}{\|f(\tau_j)\|_X} \|_X \leq$$

$$\leq \sup_{s\in[0,b]} \|f(s)\|_X \cdot s\,\mathrm{var}_{[0,b]}^{(\eta)}(U)$$

and this together with (5) gives the result.

3. Proposition. *Assume that* $U, V \in G([0,\infty), L(X)) \cap BSV_{loc}([0,\infty), L(X))$ *and that* $U(0) = V(0) = 0$.

Then the convolution

$$(U * V)(t) = \int_0^t \mathrm{d}[U(s)]V(t-s) \in L(X)$$

is well defined for every $t \in [0,\infty)$, *and for every* $b > 0$, $\eta \geq 0$ *the inequality*

$$s\,\mathrm{var}_{[0,b]}^{(\eta)}(U * V) \leq s\,\mathrm{var}_{[0,b]}^{(\eta)}(U).s\,\mathrm{var}_{[0,b]}^{(\eta)}(V) \tag{6}$$

holds.

Proof.

Define

$$\tilde{V}(\sigma) = V(\sigma) \ \text{ for } \ \sigma \geq 0$$

and

$$\tilde{V}(\sigma) = 0 \ \text{ for } \ \sigma < 0.$$

Assume that $b \geq 0$ and let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_k = b$ be an arbitrary partition of $[0, b]$.

Using the definition of $\tilde{V}$ we have for every $\alpha \in [0, b]$ the equality

$$\int_0^\alpha \mathrm{d}[U(s)]V(\alpha - s) = \int_0^b \mathrm{d}[U(s)]\tilde{V}(\alpha - s)$$

and therefore we obtain for any choice of $x_j \in X$, $\|x_j\|_X \leq 1$, $j = 1, \ldots, k$ the equalities

$$\|\sum_{j=1}^k [(U * V)(\alpha_j) - (U * V)(\alpha_{j-1})]x_j e^{-\eta\alpha_{j-1}}\|_X =$$

$$= \|\sum_{j=1}^k [\int_0^{\alpha_j} \mathrm{d}[U(s)]V(\alpha_j - s) - \int_0^{\alpha_{j-1}} \mathrm{d}[U(s)]V(\alpha_{j-1} - s)]x_j e^{-\eta\alpha_{j-1}}\|_X =$$

$$= \|\sum_{j=1}^k \int_0^b \mathrm{d}[U(s)][\tilde{V}(\alpha_j - s) - \tilde{V}(\alpha_{j-1} - s)]x_j e^{-\eta\alpha_{j-1}}\|_X =$$

$$= \|\int_0^b \mathrm{d}[U(s)]e^{-\eta s} \sum_{j=1}^k [\tilde{V}(\alpha_j - s) - \tilde{V}(\alpha_{j-1} - s)]x_j e^{-\eta(\alpha_{j-1} - s)}\|_X. \tag{7}$$

The function

$$s \mapsto \sum_{j=1}^k [\tilde{V}(\alpha_j - s) - \tilde{V}(\alpha_{j-1} - s)]x_j e^{-\eta(\alpha_{j-1} - s)} \in X$$

is evidently regulated on $[0, b]$ because $V \in G([0, b], L(X))$ and therefore by Lemma 2 we obtain

$$\| \int_0^b \mathrm{d}[U(s)] e^{-\eta s} \sum_{j=1}^{k} [\tilde{V}(\alpha_j - s) - \tilde{V}(\alpha_{j-1} - s)] x_j e^{-\eta(\alpha_{j-1} - s)} \|_X \leq$$

$$\leq s \operatorname{var}_{[0,b]}^{(\eta)}(U) \cdot \sup_{s \in [0,b]} \| \sum_{j=1}^{k} [\tilde{V}(\alpha_j - s) - \tilde{V}(\alpha_{j-1} - s)] x_j e^{-\eta(\alpha_{j-1} - s)} \|_X .$$

On the other hand, for every $s \in [0, b]$ we have

$$\| \sum_{j=1}^{k} [\tilde{V}(\alpha_j - s) - \tilde{V}(\alpha_{j-1} - s)] x_j e^{-\eta(\alpha_{j-1} - s)} \|_X \leq s \operatorname{var}_{[0,b]}^{(\eta)}(V)$$

and this gives

$$\| \sum_{j=1}^{k} [(U * V)(\alpha_j) - (U * V)(\alpha_{j-1})] x_j e^{-\eta \alpha_{j-1}} \|_X \leq$$

$$\leq s \operatorname{var}_{[0,b]}^{(\eta)}(U) \cdot s \operatorname{var}_{[0,b]}^{(\eta)}(V)$$

and by the definition also

$$s \operatorname{var}_{[0,b]}^{(\eta)}(U * V) \leq s \operatorname{var}_{[0,b]}^{(\eta)}(U) \cdot s \operatorname{var}_{[0,b]}^{(\eta)}(V).$$

This inequality yields by (2) also that

$$s \operatorname{var}_{[0,b]}(U * V) < \infty,$$

i.e. that

$$U * V \in BSV_{loc}([0, \infty), L(X)) \tag{8}$$

because $b \geq 0$ can be taken arbitrarily.

Analogously it can be proved that the following statement holds.

4. Proposition. *Assume that $U, V \in BV_{loc}([0, \infty), L(X))$ and that $U(0) = V(0) = 0$.*

Then the convolution

$$(U * V)(t) = \int_0^t \mathrm{d}[U(s)] V(t - s) \in L(X)$$

is well defined for every $t \in [0, \infty)$ and for every $b > 0$, $\eta \geq 0$ the inequality

$$\operatorname{var}_{[0,b]}^{(\eta)}(U * V) \leq \operatorname{var}_{[0,b]}^{(\eta)}(U) \cdot \operatorname{var}_{[0,b]}^{(\eta)}(V) \tag{9}$$

holds.

In [8] the following result has been proved.

5. Proposition. *For every $b > 0$ the set of all $U : [0,b] \to L(X)$ with $U \in C([0,b], L(X)) \cap BSV([0,b], L(X))$ and $U(0) = 0$ is a Banach algebra with the Stieltjes convolution $U * V$ as multiplication and $s\,\mathrm{var}_{[0,b]}(U)$ as the norm.*

See [8 ,Theorem 15].

6. Remark. Unfortunately a statement of the form:

For every $b > 0$ the set of all $U : [0,b] \to L(X)$ with $U \in BV([0,b], L(X))$ and $U(0) = 0$ is a Banach algebra with the Stieltjes convolution

$$(U * V)(t) = \int_0^t \mathrm{d}[U(s)]V(t-s)$$

as multiplication and $\mathrm{var}_{[0,b]}(U)$ as the norm.

does not hold because in this case the multiplication given by the convolution is not associative.

It was also shown [8 ,Proposition 12 and 13] that the following two statements hold.

7. Proposition. *If $U, V \in BV_{loc}([0,\infty), L(X))$ and $U(0) = V(0) = 0$ then $U * V \in BV_{loc}([0,\infty), L(X))$.*

8. Proposition. *If $U, V \in C([0,\infty), L(X)) \cap BSV_{loc}([0,\infty), L(X))$ and $U(0) = V(0) = 0$ then $U * V \in C([0,\infty), L(X)) \cap BSV_{loc}([0,\infty), L(X))$.*

9. Lemma. *Assume that $A \in BSV([0,b], L(X))$ for some $b > 0$. Then for every $\eta \geq 0$ and $c \in (0,b]$ we have*

$$s\,\mathrm{var}^{(\eta)}_{[0,b]}(A) \leq s\,\mathrm{var}^{(\eta)}_{[0,c]}(A) + e^{-\eta c} s\,\mathrm{var}^{(\eta)}_{[c,b]}(A). \tag{10}$$

Proof. Assume that D is a partition of $[0,b]$ given by the points

$$0 = \alpha_0 < \alpha_1 < \cdots < \alpha_k = b$$

and that $x_j \in X$ with $\|x_j\|_X \leq 1$ for $j = 1, \ldots, k$. Then there is an index $l = 1, \ldots, k$ such that $c \in (\alpha_{l-1}, \alpha_l]$ and

$$\sum_{j=1}^{k} [A(\alpha_j) - A(\alpha_{j-1})] x_j e^{-\eta\alpha_{j-1}} =$$

$$= \sum_{j=1}^{l-1} [A(\alpha_j) - A(\alpha_{j-1})] x_j e^{-\eta\alpha_{j-1}} + [A(\alpha_l) - A(\alpha_{l-1})] x_l e^{-\eta\alpha_{l-1}} +$$

$$+ \sum_{j=l+1}^{k} [A(\alpha_j) - A(\alpha_{j-1})] x_j e^{-\eta\alpha_{j-1}}.$$

Taking into account that

$$[A(\alpha_l) - A(\alpha_{l-1})] x_l e^{-\eta\alpha_{l-1}} =$$

$$= [A(\alpha_l) - A(c)]x_l e^{-\eta\alpha_{l-1}} + [A(c) - A(\alpha_{l-1})]x_l e^{-\eta\alpha_{l-1}}$$

we obtain

$$\|\sum_{j=1}^{k}[A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta\alpha_{j-1}}\|_X =$$

$$= \|\sum_{j=1}^{l-1}[A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta\alpha_{j-1}} + [A(c) - A(\alpha_{l-1})]x_l e^{-\eta\alpha_{l-1}} +$$

$$+[A(\alpha_l) - A(c)]x_l e^{-\eta\alpha_{l-1}} + \sum_{j=l+1}^{k}[A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta\alpha_{j-1}}\|_X \leq$$

$$\leq \|\sum_{j=1}^{l-1}[A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta\alpha_{j-1}} + [A(c) - A(\alpha_{l-1})]x_l e^{-\eta\alpha_{l-1}}\|_X +$$

$$+\|[A(\alpha_l) - A(c)]x_l e^{-\eta\alpha_{l-1}} + \sum_{j=l+1}^{k}[A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta\alpha_{j-1}}\|_X.$$

For the first term on the right hand side of this inequality we have evidently

$$\|\sum_{j=1}^{l-1}[A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta\alpha_{j-1}} + [A(c) - A(\alpha_{l-1})]x_l e^{-\eta\alpha_{l-1}}\|_X \leq$$

$$\leq s \operatorname{var}^{(\eta)}_{[0,c]}(A)$$

and for the second

$$\|[A(\alpha_l) - A(c)]x_l e^{-\eta\alpha_{l-1}} + \sum_{j=l+1}^{k}[A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta\alpha_{j-1}}\|_X =$$

$$= \|[A(\alpha_l) - A(c)]x_l e^{-\eta\alpha_{l-1}} + e^{-\eta c}\sum_{j=l+1}^{k}[A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta(\alpha_{j-1}-c)}\|_X \leq$$

$$\leq e^{-\eta c} V_c^b(\eta, A, D_+) \leq e^{-\eta c} \operatorname{var}^{(\eta)}_{[c,b]}(A)$$

(D_+ is the partition of $[c, b]$ given by the points $c \leq \alpha_l < \cdots < \alpha_k = b$). Hence

$$\|\sum_{j=1}^{k}[A(\alpha_j) - A(\alpha_{j-1})]x_j e^{-\eta\alpha_{j-1}}\|_X \leq$$

$$\leq s \operatorname{var}^{(\eta)}_{[0,c]}(A) + e^{-\eta c} \operatorname{var}^{(\eta)}_{[c,b]}(A)$$

and the lemma is proved.

Similarly it can be shown that the following statement is valid.

10. Lemma. *Assume that $A \in BV([0,b], L(X))$ for some $b > 0$. Then for every $\eta \geq 0$ and $c \in (0,b]$ we have*

$$\operatorname{var}^{(\eta)}_{[0,b]}(A) \leq \operatorname{var}^{(\eta)}_{[0,c]}(A) + e^{-\eta c} \operatorname{var}^{(\eta)}_{[c,b]}(A). \tag{11}$$

11. Proposition. *If $A \in C([0,b], L(X)) \cap BSV([0,b], L(X))$, $A(0) = 0$ and if there is a $c \in (0,b]$ such that*

$$s\operatorname{var}_{[0,c]}(A) < 1, \tag{12}$$

then there exists a unique $R \in C([0,b], L(X)) \cap BSV([0,b], L(X))$ with $R(0) = 0$ such that

$$R(t) - \int_0^t \mathrm{d}[A(s)]R(t-s) = A(t),\ t \in [0,b] \tag{13}$$

and

$$R(t) - \int_0^t \mathrm{d}[R(s)]A(t-s) = A(t),\ t \in [0,b]. \tag{14}$$

Proof. By Lemma 9, (2) and (12) we have

$$s\operatorname{var}^{(\eta)}_{[0,b]}(A) \leq s\operatorname{var}^{(\eta)}_{[0,c]}(A) + e^{-\eta c} s\operatorname{var}^{(\eta)}_{[c,b]}(A) \leq$$
$$\leq s\operatorname{var}_{[0,c]}(A) + e^{-\eta c} s\operatorname{var}_{[c,b]}(A)$$

and this yields that taking $\eta > 0$ sufficiently large we get

$$s\operatorname{var}^{(\eta)}_{[0,b]}(A) < 1. \tag{15}$$

Let us now define $A_0(t) = A(t)$ and $A_{n+1}(t) = (A * A_n)(t)$, $t \in [0,b]$ and put

$$R(t) = \sum_{n=0}^{\infty} A_n(t). \tag{16}$$

By (6) from Proposition 3 we get the inequalities

$$s\operatorname{var}^{(\eta)}_{[0,b]}(A_n) \leq (s\operatorname{var}^{(\eta)}_{[0,b]}(A))^n,\ n \in \mathbb{N}.$$

Since (15) holds, this implies the convergence of the series (16) in $BSV([0,b], L(X))$ and by Proposition 8 also the continuity of its sum $R(t)$, i. e. $R \in C([0,b], L(X)) \cap BSV([0,b], L(X))$ and clearly also $R(0) = 0$.

By the definitions we have

$$((\sum_{n=0}^{N} A_n) * A)(t) = (A * (\sum_{n=0}^{N} A_n))(t) = \sum_{n=1}^{N+1} A_n(t) = \sum_{n=0}^{N+1} A_n(t) - A(t)$$

for every $N \in \mathbb{N}$ and passing to the limit for $N \to \infty$ we obtain (13) and (14).

Concerning the uniqueness let us assume that

$$Q \in C([0,b], L(X)) \cap BSV([0,b], L(X))$$

also satisfies (13) and (14). Then

$$Q - A * Q = A \text{ and } R - R * A = A.$$

Using the associativity of convolution products we get

$$R = A + R * A = A + R * (Q - A * Q) = A + R * Q - R * A * Q =$$
$$= A + (R - R * A) * Q = A + A * Q = Q$$

and the unicity is proved.

12. Corollary. *Assume that* $A : [0,\infty) \to L(X)$, $A(0) = 0$. *If*

$$A \in C([0,\infty), L(X)) \cap BSV_{loc}([0,\infty), L(X))$$

and if there is a $c \in (0, b]$ *such that*

$$s\,\mathrm{var}_{[0,c]}(A) < 1$$

then there exists a unique $R : [0,\infty) \to L(X)$,

$$R \in C([0,\infty), L(X)) \cap BSV_{loc}([0,\infty), L(X))$$

with $R(0) = 0$ *such that for every* $b > 0$ (13) *and* (14) *hold.*

$R \in C([0,\infty), L(X)) \cap BSV_{loc}([0,\infty), L(X))$ given in Corollary 12 is called the *resolvent* of $A \in C([0,\infty), L(X)) \cap BSV_{loc}([0,\infty), L(X))$.

13. Theorem. *Assume that* $A : [0,\infty) \to L(X)$, $A(0) = 0$, $A \in C([0,\infty), L(X)) \cap BSV_{loc}([0,\infty), L(X))$ *and that there is a* $c \in (0, b]$ *such that*

$$s\,\mathrm{var}_{[0,c]}(A) < 1.$$

Then for every $F \in G([0,\infty), L(X))$ *and* $f \in G([0,\infty), X)$ *there exist unique solutions* $X : [0,\infty) \to L(X)$ *and* $x : [0,\infty) \to X$ *for the abstract renewal equations*

$$X(t) = F(t) + \int_0^t \mathrm{d}[A(s)]X(t-s) \tag{17}$$

and

$$x(t) = f(t) + \int_0^t \mathrm{d}[A(s)]x(t-s), \tag{18}$$

respectively, and the relations

$$X(t) = F(t) + \int_0^t \mathrm{d}[R(s)]F(t-s), \tag{19}$$

$$x(t) = f(t) + \int_0^t \mathrm{d}[R(s)]f(t-s) \tag{20}$$

hold for $t > 0$ *where* R *is the resolvent of* A.

Proof. The expression on the right hand side of (19) is well defined and it reads $X(t) = F(t) + (R * F)(t)$.

Hence using (13) we obtain

$$A*X(t) = A*F(t) + (A*(R*F))(t) = ((A + A*R)*F)(t) = (R*F)(t) = X(t) - F(t)$$

and this yields that by (19) a solution of (17) is given.

The analogous result for (18) can be shown similarly.

For renewal equations see also the excellent book [3].

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