A note on the periods of periodic solutions of some autonomous functional differential equations*†

ANDREI RONTO

Institute of Mathematics
National Academy of Sciences of Ukraine
3 Tereschenkovskaya St., 01601 Kiev, Ukraine
E-mail: ar@imath.kiev.ua

Abstract

Based upon Samoilenko's method of periodic successive approximations, some estimates for the periods of periodic motions in Lipschitzian dynamical systems are obtained.

Mathematical Subject Classification: 34G20, 34K30, 47B60

Keywords: Autonomous system, functional differential equation, periodic successive approximations, partially ordered space, cone

1 Introduction

In this paper, we present several recent results [15–17] concerning periodic solutions of *autonomous*, in a sense, functional differential equations. The exposition follows basically [15] and [16].

Let us first consider the autonomous ordinary differential system

$$x' = f(x) \tag{1}$$

with $x : \mathbb{R} \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\|f(x_1) - f(x_2)\| \leq l \, \|x_1 - x_2\| \tag{2}$$

*Supported in part by OMFB, Grant UK-3/1999.

† *This paper is in final form and no version of it will be submitted for publication elsewhere.*

for some $l \in (0, +\infty)$ and all $\{x_1, x_2\} \subset \mathbb{R}^n$. It is important that, in relation (2), the symbol $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^n$.

In 1969, James Yorke proved the following remarkable statement (see [22]):

Theorem 1 (Yorke, 1969) *If x is an ω-periodic solution of* (1) *and f satisfies the Lipschitz condition* (2)*, then either $\omega = 0$ or*

$$\omega \geq 2\pi/l. \tag{3}$$

It is interesting that the constant 2π in relation (3) is the best possible one, in the sense that the relation $\omega \geq (2\pi + \epsilon)/l$, where ϵ is positive, may not hold for some systems (1). This fact becomes even more surprising when we notice, following Yorke, that the strictness of estimate (3) can be justified by the simplest example of the linear oscillator equation.

Yorke's Theorem 1 generalises an unpublished result of Sibuya referred to in [22], which, under the same conditions, states the inequality $\omega \geq 2/l$.

Later on, Theorem 1 was subject to numerous generalisations. It was extended, in particular, to the case of an arbitrary Hilbert space by Lasota and Yorke [10], in which case the above-mentioned property of the constant 2π in estimate (3) had been retained. In [3], Busenberg, Fisher, and Martelli obtained a similar result for Eq. (1) considered in a general Banach space; in that case, the exact value of the constant in the corresponding inequality was shown to be 6—the fact justified by an essentially infinite-dimensional example, which leads one to another interesting problem [3]. Vidossich [21] proved statements similar to those from [10] in the case when the non-linear term in Eq. (1) may be non-Lipschitzian. In [11], Tien-Yien Lee used a result of Vidossich [21] and the infinite-dimensional version [10] of Yorke's Theorem 1 to establish similar propositions for non-linear delay differential equations. Some results, amongst which there is a generalisation of Theorem 1.3 from [3], are obtained by Medveď [13].

An approach to the problem of obtaining lower estimates for the periods of periodic motions in autonomous systems, different from those developed in the papers cited above, was suggested in [15–17]. We had discovered that statements of that kind can be established by applying the general scheme of the so-called method of periodic successive approximations (see, e. g., [19]).

In this paper, we show some results on the periods of periodic solutions of a class functional differential equations in a partially ordered Banach space. A new formalism for the two-sided Lipschitz condition has allowed us to obtain estimates sometimes more efficient than those involving the scalar Lipschitz conditions of the type (2) in terms of the (inner product) norm. All the necessary definitions can be found in Section 3.

2 Method of periodic successive approximations for equations with argument deviations

The results of this paper are based upon the statement which may be regarded as an abstract formulation of the equivalence principle of the so-called method of periodic successive approximations suggested by Samoilenko in 1960s (see, e. g., [19]).

Theorem 2 given below was borrowed from [15], while Lemma 2 of Section 4 had been established in [16] under similar assumptions.

Let us consider the abstract ω-periodic kth order boundary value problem

$$x^{(k)} = \mathcal{F}x, \tag{4}$$

$$x^{(\nu)}(0) = x^{(\nu)}(\omega), \quad \nu = 0, 1, \ldots, k-1, \tag{5}$$

where $x : [0, \omega] \to X$, X is a Banach space, and $\mathcal{F} : C^{k-1}([0, \omega], X) \to C([0, \omega], X)$ is a mapping. We outline that no other properties of $\mathcal{F}$ are required as yet.

Theorem 2 *A continuous function $x : [0, \omega] \to X$ is a solution of problem* (4), (5) *if, and only if*

$$x = \xi + (\mathcal{P}_\omega \mathcal{J})^k \mathcal{F}x, \tag{6}$$

$$\mathcal{Q}_\omega \mathcal{J} \mathcal{F} x = 0 \tag{7}$$

with $\xi = x(0)$, where

$$\mathcal{J}x := \int_0^{\cdot} x(\tau) d\tau, \tag{8}$$

$$[\mathcal{P}_\omega x](t) := x(t) - \frac{t}{\omega}\left[x(\omega) - x(0)\right], \qquad t \in [0, \omega], \tag{9}$$

and $\mathcal{Q}_\omega x := x - \mathcal{P}_\omega x$ for all $x \in C\left([0, \omega], X\right)$.

Remarks. — **1**. The pair $(x(\cdot), \xi) \in C([0, \omega], X) \times X$ is unknown in system (6), (7). Although it suffices to consider $\xi \in X$ as a parameter artificially introduced in Eq. (6), the scheme described by Theorem 2 can be shown to be in quite a natural relation with the well-known Lyuapunov–Schmidt method (see, e. g., [8, 12]).

2. It can be readily verified that the mapping $\mathcal{P}_\omega$ involved in Eq. (6) is a projection operator in $C([0,\omega],X)$. All the functions z from $\operatorname{im}\mathcal{P}_\omega$ thus satisfy the condition $z(0)=z(\omega)$, whence it follows that the mapping $(\mathcal{P}_\omega\mathcal{J})^k\mathcal{F}$ preserves $\ker\mathcal{Q}_\omega$.The latter circumstance is rather important, because it allows one to study Eq. (6) restricted to the kernel of $\mathcal{Q}_\omega$.

Considering the equivalence theorem above, we can formulate the idea of the *method of periodic successive approximations* as follows:

> Seek for a solution of the ω-periodic boundary value problem (4), (5) via solving Eq. (6) by iteration on $\ker\mathcal{Q}_\omega$ and then excluding the parameter ξ using Eq. (7).

The name of this method is explained by the fact that the functions $x_l:[0,\omega]\times X\to X$ $(l=1,2,\dots)$ defined by the formula

$$x_l(\cdot,\xi)=\xi+(\mathcal{P}_\omega\mathcal{J})^k\mathcal{F}x_{l-1}(\cdot,\xi)\qquad(\xi\in X,\ l=1,2,\dots)$$

satisfy the relations

$$x_l(0,\xi)=x_l(\omega,\xi)\qquad(x\in X,\ l=1,2,\dots)$$

whatever $x_0\in C([0,\omega],X)$ be. The latter property, in turn, is due to $\mathcal{P}_\omega$ being a projector of $C([0,\omega],X)$ onto the subspace of the (restrictions of) ω-periodic functions.

A statement justifying application of the techniques of such kind to differential functional equations with “sufficiently smooth” nonlinearities will be given in Section 4. Prior to this, several definitions and subsidiary results will be stated.

3 Some notions of the theory of cones

Here, we follow basically the survey of Krein and Rutman [9] and the book of Krasnosel'skii [7], which are common reference on the topic. The notion of *abstract modulus* introduced in [15] with axioms (m_1)–(m_5), as well as the related Theorem 3 of this section seem to be new (see Remark 4).

Let $\langle X,\preccurlyeq_X,\|\cdot\|_X\rangle$ be a partially ordered Banach space (POBS for short). Let X_+ denote the *positive cone* corresponding to the partial ordering $\preccurlyeq_X$:

$$X_+:=\{x\in X\ :\ x\succcurlyeq_X 0\}.$$

Recall that a non-empty subset X_+ of X is called a *cone* if it is closed in X and

(i) $X_+ + X_+ \subset X_+$;

(ii) $\lambda X_+ \subset X_+$ for every $\lambda \in [0, +\infty)$;

(iii) $(-X_+) \cap X_+ = \{0\}$.

The following standard notation will be used in the sequel: for every pair $\{x_1, x_2\} \subset X$, the relation $x_1 \succcurlyeq_X x_2$ or, which is the same, $x_2 \preccurlyeq_X x_1$, by definition, means that $x_1 - x_2 \in X_+$.

A cone $X_+ \subset X$ is said to be:

- *reproducing* if $X_+ - X_+ = X$;
- *normal* if the number

  $$\nu_{X_+} := \inf\{\nu > 0 \ : \ 0 \preccurlyeq_X x_1 \preccurlyeq_X x_2 \text{ implies } \|x_1\|_X \leq \nu \|x_2\|_X\}$$

  is finite.

It is well-known (see, e. g., [7]) that when the cone X_+ has finite *normality constant* ν_{X_+}, every order bounded subset of X is also bounded with respect to the norm $\|\cdot\|_X$. (A set $M \subset X$ is called *order bounded* if there exist some $\{u, v\} \subset X$ such that $u \preccurlyeq_X x \preccurlyeq_X v$ for all $x \in M$.)

Given an operator $L : X \to X$, we say that L *preserves a set* $M \subset X$ if $L(M) \subset M$.

Let us fix two POBS, $\langle X, \preccurlyeq_X, \|\cdot\|_X \rangle$ and $\langle E, \preccurlyeq_E, \|\cdot\|_E \rangle$. In the sequel, an important rôle will be played by the mappings $\mathfrak{m} : X \to E_+$ which may be called "vector-valued norms." More precisely, the following definition will be used.

Definition 1 A mapping $\mathfrak{m} : X \to E_+$ is said to be a *modulus* if

(m_1) $\mathfrak{m}(x) = 0$ *implies* $x = 0$*;*

(m_2) $\mathfrak{m}(\lambda x) = |\lambda| \mathfrak{m}(x)$ *for all* $x \in X$ *and* $\lambda \in \mathbb{R}$*;*

(m_3) $\mathfrak{m}(x_1 + x_2) \preccurlyeq_E \mathfrak{m}(x_1) + \mathfrak{m}(x_2)$ *for all* $\{x_1, x_2\} \subset X$ *("subadditivity").*

Remarks. — **3**. Obviously, conditions (m_2) and (m_3) themselves imply that a mapping $\mathfrak{m}$ satisfying them has range in E_+.

4. The history of the notion described by Definition 1 is so long that it can hardly be traced back. We note that the conception of a K*-norm* extensively used by Professor Zabreiko and his collaborators is rather similar to that of the E_+-valued modulus as it is defined in this paper and in [15].

When studying the so-called *lattice normed spaces* [6], a mapping $\mathfrak{m}$ with properties similar to (m_1)–(m_3) is considered. However, it is then usually assumed that the partial ordering induced by the cone E_+ turns E to a *lattice*, which may not be the case here. When $E = X$, $E_+ = X_+$, and the cone X_+ is *minihedral* in the sense of M. Krein [9] (i.e., X is a conditionally complete lattice, in G. Birkhoff's terminology [2]), both conceptions reduce to the standard notion of modulus in such spaces. As to the collection of axioms (m_1)–(m_5), where (m_4) and (m_5) are listed below, we deem it to be new.

We also need a theorem which, under fairly general conditions, establishes the continuity of a subadditive mapping. The statement to be formulated below follows the line of some results of M. Krein and V. Shmul'yan which, according to the information provided in [7], had already been obtained in early 1940s. Theorem 3 below, which is in fact a generalised Theorem 2 of [1] in the case of a single partial ordering, is borrowed from our paper [15].

Theorem 3 *Assume that X_+ is reproducing, E_+ is normal, and, besides assumptions (m_1)–(m_3), the modulus $\mathfrak{m} : X \to E_+$ also satisfies the following conditions:*

(m_4) $0 \preccurlyeq_X x_1 \preccurlyeq_X x_2$ *implies* $\mathfrak{m}(x_1) \preccurlyeq_E \mathfrak{m}(x_2)$*;*

(m_5) $\alpha(\mathfrak{m}) := \inf_{x \in X \setminus \{0\}} \dfrac{\|x\|_X}{\|\mathfrak{m}(x)\|_E} < +\infty.$

Then the mapping $\mathfrak{m}$ is continuous.

Being satisfied in many applications, condition (m_4) seems to be quite a natural one; it means that the restriction of $\mathfrak{m}$ to X_+ is isotonic with respect to the partial orderings $\preccurlyeq_X$ and $\preccurlyeq_E$. Note also that condition (m_5) is not so very restrictive as it may seem on the first sight. It is always fulfilled, e. g., when $X = E$, the cone $X_+ = E_+$ is normal, and there exists a $\delta \in (0, +\infty)$ such that $\mathfrak{m}(x) \succcurlyeq_X \delta x$ for all $x \in X$.

One may say not without reason that conditions (m_1)–(m_5) introduced above are in a good agreement with the usual understanding of the conception of modulus.

Corollary 1 *Under conditions of Theorem 3, for an arbitrary function $x \in C([0, \omega], X)$, one has*

$$\mathfrak{m}\left(\int_0^\omega x(\tau)d\tau\right) \preccurlyeq_E \int_0^\omega \mathfrak{m}(x(\tau))d\tau,$$

where the integral is understood in the Riemann sense.

Corollary 1 allows one, roughly speaking, to estimate the "modulus of an integral" by the "integral of the modulus." This statement is needed to guarantee the validity of Lemma 2 in Section 4.

We also quote here a version of the Banach fixed point theorem for an operator satisfying a kind of the two-sided Lipschitz condition with respect to some abstract modulus. The statement under consideration is essentially a consequence of Theorem 6.2 from [8]; the presentation here follows the paper [15].

Theorem 4 *Let the partially ordered normed space* $\langle X, \preccurlyeq_X, \|\cdot\|_X \rangle$ *be complete with respect to the norm* $|\cdot|_{\mathfrak{m}}$ *generated by the abstract modulus* $\mathfrak{m} : X \to E_+$ *according to the formula*

$$|x|_{\mathfrak{m}} := \|\mathfrak{m}(x)\|_E \qquad (x \in X). \tag{10}$$

Assume that, moreover, the cone E_+ *is normal, and a mapping* $T : X \to X$ *satisfies the abstract Lipschitz condition*

$$\mathfrak{m}(Tx_1 - Tx_2) \preccurlyeq_E L\mathfrak{m}(x_1 - x_2) \qquad \textit{for all } \{x_1, x_2\} \subset X \tag{11}$$

with some linear and continuous operator $L : E \to E$ *preserving the cone* E_+ *and having the property that*[1] $r(L) < 1$.

Then T *has a unique fixed point in* X.

The possibility to apply Theorem 4 in the situation considered below is justified by the following corollary of Theorem 3 (see [15]):

Lemma 1 *If, under conditions of Theorem 3, the modulus* $\mathfrak{m}$ *also possesses property* (m_5)*, then the norms* $\|\cdot\|_X$ *and* $|\cdot|_{\mathfrak{m}}$[2] *are equivalent.*

4 Convergence of periodic sucessive approximations

Based on Theorem 4 and Corollary 1, we are now able to establish the solvability of Eq. (6) corresponding to the kth order ω-periodic problem (4), (5).

We assume that, in (4), the non-linear mapping $\mathcal{F} : C([0,\omega], X) \to C([0,\omega], X)$ satisfies the abstract Lipschitz condition of the form

$$\mathfrak{M}(\mathcal{F}x_1 - \mathcal{F}x_2) \preccurlyeq_E \Lambda\mathfrak{M}(x_1 - x_2), \quad \{x_1, x_2\} \subset C([0,\omega], X). \tag{12}$$

Remark. — **5**. Here and below, one and the same symbol $\preccurlyeq_E$ is used to denote both the partial ordering of E and the natural, pointwise partial ordering of the space $C([0,\omega], E)$ induced by that of E:

[1]Here and below, the symbol $r(L)$, as usual, denotes the spectral radius of a linear, continuous operator L.

[2]See (10).

if $\{x_1, x_2\} \subset C([0,\omega], X)$, the relation $x_1 \preccurlyeq_X x_2$, by definition, means that $x_1(t) \preccurlyeq_X x_2(t)$ for all $t \in [0,\omega]$.

It is pretty unlikely that there may arise any confusion in connexion with this convention.

In formula (12), Λ is a linear mapping of $C([0,\omega], E)$ into itself, which preserves the cone $C([0,\omega], E_+)$, and $\mathfrak{M} : C([0,\omega], X) \to E_+$ is the "point-wise modulus" (see Definition 1) generated by a modulus $\mathfrak{m} : X \to E_+$ satisfying conditions (m_4) and (m_5):

$$[\mathfrak{M}(x)]\,(t) := \mathfrak{m}(x(t)) \quad \text{for all } t \in [0,\omega]. \tag{13}$$

It is supposed that the cone X_+ is reproducing, whereas E_+ is normal.

Remark. — **6**. A special class of equations (4) is introduced here, for which the mapping $\mathcal{F}$, instead of being considered as an operator with domain in $C^{k-1}([0,\omega], X)$ only, is assumed to be defined on *all* of $C([0,\omega], X)$. The class mentioned contains, in particular, equations with inner superpositions of the type

$$x^{(k)}(t) = f\left(x(\vartheta_1(t)), x(\vartheta_2(t)), \ldots, x(\vartheta_m(t))\right), \tag{14}$$

where $\vartheta_\nu : [0,\omega] \to [0,\omega]$ $(\nu = 1, 2, \ldots, m)$ are some continuous functions. The latter, in turn, include equations of the form

$$x^{(k)} = f(x) \tag{15}$$

with $x : [0,\omega] \to X$ and $f : X \to X$. We note that equations of such kind arise in some applications.

Lemma 2 *Let us suppose that $r(\mathcal{A}_\omega^k \circ \Lambda) < 1$, where the linear operator $\mathcal{A}_\omega : C([0,\omega], E) \to C([0,\omega], E)$ preserving the cone $C([0,\omega], E_+)$ is defined with the formula*

$$[\mathcal{A}_\omega y]\,(t) := \left(1 - \frac{t}{\omega}\right) \int_0^t y(\tau)d\tau + \frac{t}{\omega} \int_t^\omega y(\tau)d\tau, \quad t \in [0,\omega]. \tag{16}$$

Then Eq. (6) *can be uniquely solved by iteration for every value of the parameter $\xi \in X$.*

The assertion of Lemma 2 is not difficult to be proved [16] by applying Theorem 4, for which purpose the estimations should be carried out with respect to the "point-wise" abstract modulus, $\mathfrak{M}$, defined by formula (13). The comparison operator (16) arises due to the following statement [15]:

Lemma 3 *If, under the conditions listed above, the modulus* $\mathfrak{m}$ *satisfies condition* (m_4)*, then, for an arbitrary* $x \in C([0,\omega], X)$*, the pointwise order relation*

$$\mathfrak{m}(\mathcal{P}_\omega \mathcal{J} x) \preccurlyeq_E \mathcal{A}_\omega \mathfrak{m}(x) \tag{17}$$

holds true, where the operators $\mathcal{P}_\omega$*,* $\mathcal{J}$*, and* $\mathcal{A}_\omega$ *are given by formulæ* (9)*,* (8)*, and* (16)*, respectively.*

The *proof* of Lemma 3 is based upon conditions (m_1)–(m_4) and Corollary 1.

5 An estimate for the period ω

Keeping the notations and assumptions of the preceding sections, here we establish some results concerning the periodic solutions of functional differential equations with non-linearities satisfying a certain Lipschitz condition in the sense described in Section 3.

First, we introduce a definition. Let Ω be a linear set in $C([0,\omega], X)$.

Definition 2 An operator $\mathcal{F} : \Omega \to \Omega$ is said to be *autonomous* if it preserves the set of all *constant* functions belonging to Ω.

The autonomous Nemytskii operator, $[\mathcal{F}x](t) := f(x(t))$, $t \in [0,\omega]$, and the inner superposition, $[\mathcal{S}_\vartheta x](t) := x(\vartheta(t))$, $t \in [0,\omega]$, where $\vartheta : [0,\omega] \to [0,\omega]$, are typical examples of the mappings autonomous in the sense of Definition 2.

The following statement from [15] is crucial for the proof of Theorem 5 below.

Lemma 4 *For an arbitrary* $k \geq 1$*, one has* $\ker(\mathcal{P}_\omega \mathcal{J})^k \simeq X$*.*

Remark. — **7**. In Lemma 4, the isomorphism is that identifying the constant functions $[0,\omega] \to X$ with the corresponding elements of the space X.

Return now to the ω-periodic problem (4), (5). We have the following

Theorem 5 *Assume that, in Eq.* (4)*, the operator* $\mathcal{F}$ *is autonomous and satisfies the Lipschitz condition* (12)*, where* Λ *is a linear, continuous operator preserving the cone* $C([0,\omega], E_+)$*. Let the cone* X_+ *be reproducing,* E_+ *be normal and, furthermore, the mapping* $\mathfrak{m} : X \to E_+$ *satisfy conditions* (m_1)–(m_5)*.*

Then either problem (4), (5) *has no non-constant solutions or the inequality*[3]

$$r(\mathcal{A}_\omega^k \circ \Lambda) \geq 1 \tag{18}$$

holds true.

The *proof* of Theorem 5 is based upon Lemmata 2 and 4. This proof is similar to the argument used in [15, 17], and therefore is not given here. We note only that the operator $\mathcal{F}$ being *autonomous* allows one to claim that the standard iteration method for solving Eq. (6) does not "shift" the starting approximation unless the latter depends on time explicitly. It then suffices to make use of Lemma 4 in order to prove that there can exist no non-constant solutions of the ω-periodic problem (4)), (5) with ω *not* satisfying (18).

Corollary 2 *Let us suppose that, under the assumptions above, the mapping $\mathcal{F}$ is autonomous in the sense of Definition* 2 *and satisfies the Lipschitz condition* (12), *in which the operator Λ commutes with $\mathcal{A}_\omega$. Assume also that*

$$0 < r(\Lambda) < \left(\frac{\kappa}{\omega}\right)^k, \tag{19}$$

where

$$\kappa := 1/\max\left\{\lambda > 0 \ : \ \lambda = \int_0^{\frac{1}{2}} \exp\frac{\tau(\tau-1)}{\lambda}d\tau\right\} \approx 3.4161.$$

Then the ω-periodic problem (4), (5) *has no non-constant solutions.*

The *proof* of the latter result is obtained by using the well-known estimate for the spectral radius of the composition of two linear continues operators that commute (see, e. g., § 149 in [14]). Note that the assumption of Corollary 2 that the operators Λ and $\mathcal{A}_\omega$ should commute, is rather a technical one and may be significantly weakened. However, we do not consider this problem here.

Corollary 2 implies, in particular, that, under the conditions specified, the minimal period ω of every non-constant solution of problem (4), (5) is estimated from below as

$$\omega \geq \frac{\kappa}{\sqrt[k]{r(\Lambda)}} \tag{20}$$

[3] In relation (18), $\mathcal{A}_\omega^k$ stands for the kth iteration of the linear comparison operator (16).

provided that the spectrum of Λ contains non-zero points.

Remark. — **8**. The constant $\kappa \approx 3.4161$ from (19), (20) was used independently by Samoilenko and Laptinskii [18], Evkhuta and Zabreiko [4, 5], and E. Trofimchuk [20] in connexion with the analysis of convergence of the method of periodic successive approximations.

For the reader's convenience, we formulate separately the assertion of Corollary 2 in the particular case when Eq. (4) is the ordinary differential equation (1) considered in the POBS X.

Corollary 3 *If the mapping $f : X \to X$ satisfies the Lipschitz condition*

$$\mathfrak{m}(f(x_1) - f(x_2)) \preccurlyeq_E L\mathfrak{m}(x_1 - x_2) \quad \textit{for all } \{x_1, x_2\} \subset X \tag{21}$$

with a linear and continuous $L : E \to E$ such that $\sigma(L) \neq \{0\}$, and there exists an ω-periodic solution of Eq. (1) *different from constant, then necessarily*

$$\omega \geq \frac{\kappa}{r(L)}.$$

The latter result can also be derived from Theorem 2 of [17]. It states, roughly speaking, that, once the non-linear term in (1) is known to be smooth enough, no periodic solutions with "small" periods can be discovered in such a system.

Corollary 4 *Let us suppose that, under the assumptions above, $\mathcal{F}$ is autonomous and satisfies condition* (12)*, in which the operator Λ commutes with $\mathcal{A}_\omega$ and is quasi-nilpotent.*

Then problem (4)*,* (5) *does not have but constant solutions.*

The *proof* of this statement is easily obtained by using Corollary 2. Indeed, in this case, the fraction in the right-hand side of inequality (20) should have zero denominator, which formally corresponds to the period ω equal to $+\infty$.

In the case when Eq. (4) has form (1), Corollary 4 takes the following form:

Corollary 5 *If, in Eq.* (1)*, $f : X \to X$ satisfies* (21) *with $\mathfrak{m}$ having properties* (m_1)–(m_5) *and $L : X \to X$ quasi-nilpotent, then Eq.* (1) *has no non-constant periodic solutions of any period.*

The latter statement may be regarded as an "easy test" for the absence of non-constant periodic solutions in the autonomous system

(4) with a sufficiently smooth right-hand side term. For example, given some $\{n, m\} \subset \mathbb{N}$, $m > 1$, consider the system

$$x'_\nu = \sum_{\mu=\nu+1}^{m} a_{\nu,\mu}(x_\mu), \qquad \nu = 1, 2, \ldots, m-1, \tag{22}$$

$$x'_m = 0 \tag{23}$$

with $a_{\nu,\mu} : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the component-wise relations

$$|a_{\nu,\mu}(y_1) - a_{\nu,\mu}(y_2)| \le L_{\nu,\mu}|y_1 - y_2|$$

for all $\{y_1, y_2\} \subset \mathbb{R}^n$ and $\nu, \mu = 1, 2, \ldots, m$ such that $\mu \ge \nu + 1$, where $L_{\nu,\mu}$ are square n-matrices with non-negative components. It follows, e. g., from Lemma 5 of [16] that the nm-matrix

$$\begin{bmatrix} 0 & L_{1,2} & L_{1,3} & \ldots & L_{1,m-1} & L_{1,m} \\ 0 & 0 & L_{2,3} & \ldots & L_{2,m-1} & L_{2,m} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & L_{m-1,m} \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

has no non-zero eigen-values. Corollary 4 then implies immediately that system (22), (23) has no any periodic solutions except equilibria.

It is worth mentioning that, theoretically, system (22), (23) with $n = 1$ can be integrated, because, in that case, the form of Eq. (23) allows one to split the variables. On the other hand, Corollary 5 gives an immediate negative answer to the question on non-trivial peridic orbits of this system, relieving one from the necessity of sequential application of Barrow's formula and thus reducing the amount of unproductive computational work to a minimum.

Remarks. — **9**. It follows from Theorem 3 of [21] that, for $f : X \to X$ bounded and satisfying the one-sided Lipschitz condition

$$[f(x_1) - f(x_2),\ x_1 - x_2]_- \le -\alpha(\|x_1 - x_2\|_X) \quad \text{for all } \{x_1, x_2\} \subset X,$$

where $\alpha(u) := \sigma(u^2)$ or $\alpha(u) := u\sigma(u)$ with $\sigma : (0, +\infty) \to (0, +\infty)$ continuous and such that $\sigma(u) = 0 \Leftrightarrow u = 0$, has no non-constant periodic solutions of any positive period. Corollary 5 thus can be regarded as an analogue of this statement in the case of a two-sided Lipschitz condition.

Since, in Corollary 4, the assumption that Λ should commute with $\mathcal{A}_\omega$ seems to be rather a technical one, motivated solely by Lemma 3, it is tempting to introduce the following

Conjecture 1 *Let us assume that the cone X_+ is reproducing, E_+ is normal, the modulus $\mathfrak{m} : X \to E_+$ possesses properties (m_4) and (m_5), and the mapping $\mathcal{F} : C([0,\omega], X) \to C([0,\omega], X)$, besides being autonomous in the sense of Definition 2, satisfies the Lipschitz condition (12) with a quasi-nilpotent Lipschitz operator Λ.*

Then, for an arbitrary positive ω, the ω-periodic problem (4), (5) has no non-constant solutions.

The conjecture formulated above is, of course, true when Eq. (4) has the form (15). (Indeed, in that case, the Lipschitz operator Λ for the corresponding mapping $\mathcal{F}$ is uniquely determined by a linear mapping $E \to E$; by (16), this implies that $\Lambda : C([0,\omega], E) \to C([0,\omega], E)$ commutes with $\mathcal{A}_\omega$, and it remains to apply Corollary 2.) We are able, however, neither to prove Conjecture 1 in full nor to construct any counterexamples at the moment.

6 Comments

Theorem 5 generalises a result from [17] to the case when the primary differential equation may contain argument deviations. The assertions of its Corollaries 2 and 4 can be extended to the case when the Lipschitz operator Λ for $\mathcal{F}$ may not commute with $\mathcal{A}_\omega$. The assumption that it should, however, will then be replaced by a weaker one, for it, probably, cannot be dropped completely. Here, we do not discuss this problem in more detail.

Unlike [3,10,22], the estimates provided by Theorem 5 and Corollary 2 of this paper are, unfortunately, not the best possible ones. The reason for this lies in the inequality (17) of Lemma 3, which is essentially used in the proofs (more precisely, in the choice of the comparison operator $\mathcal{A}_\omega$). However, it is not difficult to show by examples that even in the present form the estimates mentioned may give more substantial information on the period in question, because the spectral radius of a linear operator (here, the Lipschitz operator, L), in general, is less than its norm.

As is indicated in [15], the assertion of Corollary 3 concerning ordinary differential equations can be significantly improved by extending Theorem 1.3 of [3] to the general case considered here. It seems that such an extension is impossible for a functional differential equation (at least, under similar assumptions). A statement of that kind was obtained, e. g., in [13]; however, some additional conditions should be assumed for it to be true. In particular, an estimate for the period of a solution of Eq. (14) can be established by using the method of [13] only under condition that the argument transformations $\vartheta_1, \vartheta_2, \ldots, \vartheta_m$ be strictly monotone, continuously differentiable, and such that $|\vartheta_\nu'(t)| \leq 1$ for all t and ν. Our results thus seem to complement those mentioned above.

Although we still cannot claim that the estimates obtained are exact (and they actually are not), it is reasonable to develop the approach suggested still further, because the only "deficiency" of the method of proof of Theorem 5 consists in the use of Lemma 3. Thus, having replaced inequality (17) in Lemma 3 by a sharper one, we shall immediately obtain an improved version of Theorem 5 with no significant changes in the proof. Another advantage of this approach is that it is closely connected with the method of successive periodic approximations, which may prove useful by itself in many situations.

The approach suggested in [15–17] and exploited here allows one to obtain similar results to other kinds of equations. These generalisations will be described elsewhere.

References

[1] *I. A. Bakhtin, M. A. Krasnosel'skii & V. Ya. Stetsenko*, "On the continuity of linear positive operators," *Sib. Mat. Zh.*, **3**, No. 1 (1962), 156–160.

[2] *G. Birkhoff*, Lattice Theory, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 25, Amer. Math. Soc., New York, 1948.

[3] *S. Busenberg, D. Fisher & M. Martelli,* "Minimal periods of discrete and smooth orbits," *Amer. Math. Monthly,* **96**, 5–17 (1989).

[4] *N. A. Evkhuta & P. P. Zabreiko,* "On A. M. Samoilenko's method for finding periodic solutions of quasi-linear differential equations in a Banach space," *Dokl. Akad. Nauk Belor. SSR,* **29**, No. 1, 15–18 (1985).

[5] *N. A. Evkhuta & P. P. Zabreiko,* "The Poincaré method and Samoilenko method for the construction of periodic solutions to ordinary differential equations," *Math. Nachr.,* **153**, 85–99 (1991).

[6] *L. V. Kantorovich & G. P. Akilov,* Functional Analysis [in Russian], Nauka, Moscow (1977).

[7] *M. A. Krasnosel'skii,* Positive Solutions of Operator Equations, Noordhoff, Groningen (1964).

[8] *M. A. Krasnosel'skii et al.,* Approximate Solution of Operator Equations, Noordhoff, Groningen (1972).

[9] *M. G. Krein & M. A. Rutman,* "Linear operators leaving invariant a cone in a Banach space," Amer. Math. Soc. Transl. No. 26, Amer. Math. Soc., Providence, RI (1950).

[10] *A. Lasota & J. Yorke*, "Bounds for periodic solutions of differential equations in Banach spaces," *J. Differential Equations,* **10**, 83–91 (1971).

[11] *Tien-Yien Lee,* "Bounds for the periods of periodic solutions of differential delay equations," *J. Math. Anal. Appl.*, **49**, No. 1 (1975), 124–129.

[12] *J. Mawhin,* Topological Degree Methods in Nonlinear Boundary Value Problems, CBMS Series, vol. 40, Amer. Math. Soc., Providence, RI (1979).

[13] *M. Medveď,* "On minimal periods of functional-differential equations and difference inclusions," *Ann. Polon. Math.*, **3**, 263–270 (1991).

[14] *F. Riesz & B. Szökefalvi–Nagy,* Leçons d'Analyse Fonctionelle, Akadémiai Kiadó, Budapest (1972).

[15] *A. N. Ronto,* "On the periods of periodic motions in autonomous systems," to appear in *Ukrain. Math. J.*, **52**, No. 12 (2000).

[16] *A. N. Ronto,* "On periodic solutions of autonomous differential functional equations," to appear in *Ukrain. Math. J.*

[17] *A. N. Ronto, M. Rontó, A. M. Samoilenko & S. I. Trofimchuk,* "On periodic solutions of autonomous difference equations," to appear in *Georgian Math. J.*

[18] *A. M. Samoilenko & V. N. Laptinskii,* "On estimates for periodic solutions of periodic solutions of differential equations," *Dokl. Akad. Nauk Ukr. SSR, Ser. A,* No. 1, 30–32 (1982).

[19] *A. M. Samoilenko & N. I . Ronto,* Numerical-Analytic Methods of Investigating Periodic Solutions, Mir Publishers, Moscow (1979).

[20] *E. Trofimchuk,* "Integral operators of the method of periodic successive approximations," *Mat. Fiz. Nelin. Mekh.*, Issue 13 (47), 31–36 (1990).

[21] *J. Vidossich,* "On the structure of periodic solutions of differential equations," *J. Differential Equations,* **21**, No. 2 (1976), 263–278.

[22] *J. A. Yorke*, "Periods of periodic solutions and the Lipschitz constant," *Proc. Amer. Math. Soc.*, **22**, 509–512 (1969).