Stable subharmonic solutions and asymptotic behavior in reaction-diffusion equations*

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Abstract

Time-periodic reaction-diffusion equations can be discussed in the context of discrete-time strongly monotone dynamical systems. It follows from the general theory that typical trajectories approach stable periodic solutions. Among these periodic solutions, there are some that have the same period as the equation, but, possibly, there might be others with larger minimal periods (these are called subharmonic solutions). The problem of existence of stable subharmonic solutions is therefore of fundamental importance in the study of the behavior of solutions. We address this problem for two classes of reaction diffusion equations under Neumann boundary conditions. Namely, we consider spatially inhomogeneous equations, which can have stable subharmonic solutions on any domain, and spatially homogeneous

*This paper is in the final form and no version of it will submitted for publication elsewhere equations, which can have such solutions on some (necessarily non-convex) domains.

1 Introduction

Consider the following parabolic problem

$$\begin{aligned} u_t &= \Delta u + f(u,x,t), && x \in \Omega, \\ \frac{\partial u}{\partial \nu} &= 0, && x \in \partial\Omega, \end{aligned} \tag{1}$$

where $u = u(x,t) \in \mathbb{R}$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $f : \mathbb{R}\times\bar{\Omega}\times\mathbb{R} \to \mathbb{R}$ is a smooth function that is periodic in t with period $\tau > 0$ and ν is the unit outward normal vector field on $\partial\Omega$. We are interested in the existence of linearly stable subharmonic solutions of (1). By a subharmonic solution we mean a solution $p(x,t)$ that is periodic in t with minimal period $k\tau$ for some integer $k > 1$. Such a $p(x,t)$ is said to be linearly stable if the period map (that is, the time-$k\tau$ map) of the linearized problem

$$\begin{aligned} v_t &= \Delta v + f_u(p(x,t),x,t)v, && x \in \Omega, \\ \frac{\partial v}{\partial \nu} &= 0, && x \in \partial\Omega, \end{aligned} \tag{2}$$

has all eigenvalues inside the unit circle in the complex plane.

The existence of stable subharmonic solutions is a fundamental problem in the study of dynamics of (1). As we explain in the next section, most bounded solutions of (1) approach a stable $k\tau$-periodic solution, where k is a positive integer. Absence of stable subharmonic solutions thus implies that most solutions eventually oscillate with the asymptotic period equal to τ, the period of the equation. On the other hand, the existence of linearly stable subharmonic solutions implies that orbits with larger asymptotic periods fill a nonempty open set in the state space.

We present theorems answering the above basic problem in two situations. Theorem 2 in Section 3 asserts that for any bounded domain Ω one can find a nonlinearity $f = f(u,x,t)$ such that (1) has a linearly stable subharmonic solution. This extends earlier results of Takáč [20, 21] and Dancer and Hess [3], where the theorem is proved for special domains.

It is well known that a similar theorem is not true for the class of spatially homogeneous problems

$$\begin{aligned} u_t &= \Delta u + f(u,t), && x \in \Omega, \\ \frac{\partial u}{\partial \nu} &= 0, && x \in \partial\Omega. \end{aligned} \tag{3}$$

Indeed, there are domains Ω such that (3) has no stable subharmonic solutions for any nonlinearity $f(u,t)$. Examples include convex or radially symmetric domains. On the other hand, Theorem 3 in Section 3 shows that there do exist domains and spatially homogeneous nonlinearities such that (3) admits linearly stable subharmonic solutions.

2 Typical behavior of solutions

We choose $X = C(\bar{\Omega})$ as the state space for (1). For any $u_0 \in X$ there exists a unique (local) solution $u := u(\cdot,t;u_0)$ of (1) that satisfies the initial condition $u(\cdot,0) = u_0$. In this section we recall a theorem describing the behavior of typical solutions of (1), that is, solutions emanating from an open and dense set of initial conditions. For a simple formulation we assume the following hypothesis:

$$\limsup_{|u|\to\infty} \frac{f(u,t)}{u} < 0. \tag{4}$$

This dissipativity condition in particular implies that for any $u_0 \in X$ the solution $u(\cdot,t;u_0)$ is defined on $[0,\infty)$ and its orbit $\{u(\cdot,t;u_0) : t \geq 0\}$ is relatively compact in X.

Theorem 1. *Let f be of class C^1, τ-periodic in t, and let (4) be satisfied. Then there exists an open and dense subset $G \subset X$ such that for any $u_0 \in G$ there is a solution $p(x,t)$ of (1) with the following properties:*

(i) $\lim_{t\to\infty} \|u(\cdot,t;u_0) - p(\cdot,t)\|_X = 0$,

(ii) *$p(\cdot,t)$ is $k\tau$-periodic with $k \geq 1$,*

(iii) *$p(\cdot,t)$ is at least linearly neutrally stable.*

Here *at least linearly neutrally stable* means that the eigenvalues of the period map of (2) are contained on or inside the unite circle.

This theorem is a consequence of an abstract result on discrete strongly monotone dynamical systems. See [13, 14] for the proof under a slightly stronger assumption of $f \in C^{1,\alpha}$; [22] contains a different proof for $f \in C^1$. An additional information follows from [6]: under the dissipativity condition (4), the minimal period of the solution p in Theorem 1 is bounded above by a constant independent of $u_0 \in G$ (of course, p itself may depend on u_0).

The theorem says that most solutions are asymptotically periodic, leaving open a possibility for some of these solutions to have large asymptotic periods. The latter can occur only if there exist subharmonic solutions that are at least neutrally linearly stable. The possibility can often be ruled out in specific applications. For example, if (1) is a small perturbation of an autonomous equation no stable subharmonic solutions exist (see [6] for the proof and other examples). The situation may be more complex, however, in different, not too restricted classes of equations. The existence of stable subharmonic solutions is an interesting problem then.

Let us mention for completeness that there is no similar meaningful problem for autonomous equations. If $f = f(u, x)$, then a typical trajectory of (1) converges to an equilibrium. This fact was first proved in [7] invoking the variational structure of the problem. The result has later been extended to a much broader class of differential equations (see [18] for a general abstract theorem and a background on strongly monotone semiflows).

3 Spatially inhomogeneous equations

First examples of stable subharmonic solutions in spatially heterogeneous reaction-diffusion equations were found by Takáč [20, 21] and Dancer and Hess [3]. They gave independent constructions for a specially chosen domain Ω. The following theorem extends their results to any domain.

Theorem 2. *For any integers $N \geq 2$ and $k \geq 1$, and for any bounded domain $\Omega \subset \mathbb{R}^N$ there is a smooth function $f = f(u, x, t)$, τ-periodic in t, such that* (1) *has a linearly stable subharmonic solution of minimal period $k\tau$.*

The proof uses a perturbation argument that can roughly be described as follows. Assume that $f : \mathbb{R}^N \to \mathbb{R}$ is a smooth function, periodic in t, such that for some smooth bounded domain $\Omega_0 \subset \mathbb{R}^N$ the following Dirichlet problem

$$
\begin{aligned}
&u_t = \Delta u + f(u,x,t), && x \in \Omega_0,\\
&u = 0, && x \in \partial\Omega_0,
\end{aligned}
\tag{5}
$$

has a linearly stable periodic solution p_0. Now let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded domain. With an appropriate scaling of Ω_0 we may assume that $\bar{\Omega}_0 \subset \Omega$. Consider the following Neumann problem on Ω

$$
\begin{aligned}
&u_t = \Delta u - \gamma b(x) + f(u,x,t), && x \in \Omega,\\
&\frac{\partial u}{\partial \nu} = 0, && x \in \partial\Omega.
\end{aligned}
\tag{6}
$$

Here $b(x)$ is a smooth function on $\mathbb{R}^N$ such that $b \equiv 0$ on Ω_0 and $b > 0$ on $\mathbb{R}^3 \setminus \bar{\Omega}_0$, and γ is a large positive parameter. Letting $\gamma \to \infty$, (1) with $\Omega = \Omega_0$ turns out to be a "good limit problem" for (6). In particular, for large γ problem (6) has a linearly stable periodic solution p_γ such that $p_\gamma(x,t) \to p_0(x,t)$ for $(x,t) \in \Omega_0 \times [0,\tau]$, the convergence being uniform on compact subsets of $\Omega_0 \times [0,\tau]$. It follows that if p_0 is a subharmonic solution, then so is p_γ. This way we construct equations with linearly stable subharmonic solutions on an arbitrary domain Ω, as soon as we can do it on a particular domain Ω_0. The same idea applies to the Dirichlet problem.

The details of the proof will be given in a forthcoming paper [15]. As a particular problem (5) with linearly stable subharmonic solutions one can use the one given in [21] or an independent one given in [15].

We remark that the idea of using perturbations with localized potentials has been used before in a different context by Prizzi and Rybakowski [17]. See also [12, 11] for a discussion of different localized perturbations.

4 Spatially homogeneous equations

In this section we consider the homogeneous problem (3). It is known that on some domains no stable subharmonic solutions may exist, no matter how the nonlinearity $f = f(u,t)$ is chosen. For example, if Ω is convex, then any periodic solution p of (3) that is at least neutrally linearly stable must be τ-periodic. This follows from a result of Hess [5] which says that p must be spatially homogeneous: $p = p(t)$. Consequently, p solves the ODE $u_t = f(t,u)$, hence it cannot be subharmonic.

Another example is a radially symmetric domain

$$\Omega = \{x \in \mathbb{R}^N : a < |x| < b\}$$

(with $0 < a < b$). It is not difficult to show that any stable periodic solution p must be radially symmetric (see [8, 9, 10, 19] for more general symmetry results of this kind). Hence p solves the one-dimensional problem

$$\begin{aligned} &u_t = u_{rr} + \frac{N-1}{r}u_r + f(u,t), \qquad a < r < b, \\ &u_r(a,t) = 0, \quad u_r(b,t) = 0. \end{aligned} \tag{7}$$

Again, this problem has no subharmonic solutions (see [1]), hence p must have period τ.

On the other hand, the following theorem asserts that on some domains linearly stable subharmonic solutions do occur.

Theorem 3. *For any integers $N \geq 2$ and $k \geq 1$ there exist a domain $\Omega \subset \mathbb{R}^N$ with smooth boundary and a smooth function $f = f(u,t)$, τ-periodic in t, such that* (3) *has a linearly stable subharmonic solution of minimal period $k\tau$.*

We remark that here and likewise in Theorem 2 the condition $N \geq 2$ is necessary. In one space dimension no subharmonic solutions exist, see [1, 2].

The proof of Theorem 3, as given in [16], follows the following scenario.

First, a thin domain in $\mathbb{R}^N$ around a circle is considered. In two dimensions, for example, the domain is given by

$$\Omega_\mu = \{x = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2 : \theta \in [0,2\pi), 1 < r < \mu d(\theta)\},$$

where $\mu > 0$ is a parameter and $d(\theta)$ is a smooth positive 2π-periodic function.

Problem (3) on Ω_μ is compared to the following problem on S^1

$$v_t = \frac{1}{d(\theta)}(d(\theta)v_\theta)_\theta + f(v,t), \quad \theta \in S^1. \tag{8}$$

It can be shown that to any linearly stable periodic solution of (8) there corresponds a linearly stable periodic solution of problem (3) with $\Omega = \Omega_\mu$

and μ sufficiently small. This is one aspect of the relation between the one-dimensional equation and its thin domain approximation (see [4] for a comprehensive discussion of thin domain problems and additional references).

The question of existence of stable subharmonic solutions for (3) is thus reduced to that for (8). The gain is that the latter problem already has some space dependence, although a very special one. In (8) we now choose f to be equal to the bistable nonlinearity

$$f(u,t) = \epsilon^{-2}(v - \varepsilon\beta(t))(1 - v^2),$$

where $\beta(t)$ is a τ periodic function and ϵ is a positive constant. With ϵ very small, the nonlinearity "dominates" over diffusion. This has the effect that solutions that are initially close to 1 or -1 everywhere except for thin transition layers will retain this shape for any t in large time intervals. The dynamics of such solutions is, roughly speaking, governed by the motion of the transition layers. This, in its turn, can be described in terms of ordinary differential equations that reflect the interaction of the asymmetry of the nonlinearity ($\beta(t)$ is chosen close to a piecewise constant function with nonzero values) and the spatial inhomogeneity $d(\theta)$ in the equation. For example, assume a solution $u(\theta,t)$ has two transition layers, one from 1 to -1 near $p(t) \in S^1$, and the other one, from -1 to 1, near $q(t) \in S^1$. The motion of $p(t)$ and $q(t)$ is described by the ODEs

$$\frac{d}{dt}p(t) = g(p(t)) + a(t), \tag{9}$$

$$\frac{d}{dt}q(t) = g(q(t)) - a(t). \tag{10}$$

Here

$$g(\theta) := -d'(\theta)/d(\theta)$$

and $a(t)$ is determined by the speed of the traveling wave solution of the following autonomous equation with artificial time s (and "frozen" time t)

$$v_s = v_{xx} + f(v,t), \quad x \in \mathbb{R}. \tag{11}$$

(The precise correspondence is $c = \epsilon a(t)$ where c is the unique speed of the traveling wave.)

Equations (9), (10) are derived by formal asymptotic analysis but they can effectively be used for rigorous description of solutions of (8). The construction further proceeds as follows. With an appropriate choice of $\beta(t)$

and $d(\theta)$, one finds linearly stable $k\tau$-periodic solutions $p(t), q(t) \in S^1$ of (9), (10), respectively, such that $p(t) \neq q(t)$ for any t. They are then used in a definition of $k\tau$-periodic supersolution and a $k\tau$-periodic subsolution of (8). This yields a stable $k\tau$-periodic solution of (8). (It can also be ensured that $k\tau$ is the minimal period.) With an additional perturbation of the nonlinearity it is finally arranged that the stable $k\tau$-periodic solution perturbs to a linearly stable one.

The details of the proof and a discussion of the related problem with Dirichlet boundary condition can be found in [16].

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