Representation of Solutions of Difference Equations with Continuous Time

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Abstract. This paper describes a representation of solutions of the system of nonautonomous functional equations $x(t) = A(t)x(t-1) + B(t)x(p(t))$, in form of series, using the Cauchy matrix of the linear system $y(t) = A(t)y(t-1)$. A representation of analitical solutions of the equation $x(t) = ax(t-1) + bx(pt)$ with constant coefficients is also investigated.

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1. Introduction

Consider the system of difference equations with continuous time

$$x(t) = A(t)x(t-1) + B(t)x(p(t)), \tag{1}$$

where $x(t)$ is an n-dimensional column vector, $A(t) = (a_{ij}(t))$ and $B(t) = (b_{ij}(t))$ are $n \times n$ real matrix functions and $p(t)$ is a nonnegative real function, such that $\lim_{t\to\infty} p(t) = \infty$ and for every $T > t_0$ there exists a $\delta > 0$ such that $p(t) \le t - \delta$ for every $t \in [t_0, T]$.
The purpose of this paper is to obtain a series representation for the solutions of system (1), which can be applied to study the asymptotic behaviour of solutions. The equation with constant coefficients is also investigated because it is important in its own right.
Similar problems were studied for the pantograph differential equation of the form

$$\dot{x}(t) = A(t)x(t) + B(t)x(p(t)).$$

For the differential equation with constant coefficients the well-known Dirichlet series solution is given. The reader interested in this topic can consult Carr and Dyson [2], Fox, Mayers, Ockendon and Taylor [3], Kato and McLeod [4] and Terjéki [7].

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Let t_0 be a positive real number and set

$$t_{-1} = \min\left\{\inf\{p(s) : s \geq t_0\}, t_0 - 1\right\}.$$

By a *solution* of (1) we mean an n-dimensional column vector function x where the components $x_i(t)$, $i = 1, ..., n$, are defined for $t \geq t_{-1}$ and satisfy the system (1) for $t \geq t_0$. For a given vector valued function $\phi(t)$, where the real components ϕ_i, $i = 1, ..., n$ are given on $t_{-1} \leq t < t_0$, the system (1) has a *unique solution* x^ϕ satisfying the initial condition

$$x^\phi(t) = \phi(t) \quad \text{for} \quad t_{-1} \leq t < t_0. \tag{2}$$

Fix a point t such that $t \geq t_0$, and define natural number $k_0(t)$ such that

$$k_0(t) := [t - t_0].$$

Then,

$$t - k_0(t) - 1 < t_0 \quad \text{and} \quad t - k_0(t) \geq t_0$$

and set

$$T_0(t) := \{t - k_0(t) - 1, t - k_0(t), ..., t - 1, t\}.$$

For a given real number t and for a given positive integer n use the notation

$$t^{[n]} = t(t-1)(t-2)...(t-n+1).$$

The *Cauchy matrix* of the initial value problem

$$y(t) = A(t)y(t-1), \quad t \geq t_0, \tag{3}$$

$$y(t) = \phi(t), \quad t_0 - 1 \leq t < t_0 \tag{4}$$

is $W(\tau; t)$, where

$$W(\tau; t) = A(t)A(t-1)...A(\tau+1)$$

for $t \geq t_0$, $\tau \in T_0(t)$, with $W(t; t) = E$ and the n-dimensional unite matrix E.

2. Main Results

First of all we prove a simple but fundamental result.

Theorem 1. *Let $y_0(t)$ denote the solution of the initial value problem (3) and (4) with $\phi(t) \not\equiv 0$ for $t_{-1} \leq t < t_0$, and the sequence $\{y_n(t),\ n = 1, 2, ...\}$ is defined by*

$$y_n(t) = A(t)y_n(t-1) + B(t)y_{n-1}(p(t)), \quad t \geq t_0,$$

$$y_n(t) \equiv 0, \quad t_{-1} \leq t < t_0, \quad n = 1, 2, ...$$

Then

$$x(t) = \sum_{n=0}^{\infty} y_n(t) \tag{5}$$

is a solution of the initial value problem (1) and (2). Moreover, this series is finite on every finite subinterval of $[t_0,\ \infty)$.

Proof. First we show that the series (5) is absolutely convergent on $[t_0,\ \infty)$. Define

$$M(F,T) := \sup_{t_0 \le t \le T} ||F(t)||$$

for any matrix or vector function F for $T > t_0$ and

$$M(W,T) := \sup_{t_0 \le \tau \le t \le T} ||W(\tau;t)||$$

for the Cauchy matrix $W(\tau;t)$. Since for $t \ge t_0$

$$y_n(t) = \sum_{\tau=t-k_0(t)}^{t} W(\tau;t)B(\tau)y_{n-1}(p(\tau)), \quad n = 1, 2, ...$$

hence, for $T > t_0$ and $t_0 \le t \le T$, the following inequality holds:

$$||y_n(t)|| \le \sum_{\tau=t-k_0(t)}^{t} M(W,t)M(B,t)||y_{n-1}(p(\tau))||.$$

By using mathematical induction we will show that

$$y_n(t) = 0 \quad \text{for} \quad t_0 \le t < t_0 + (n-1)\delta. \tag{6}$$

For $n = 2$ we have

$$||y_2(t)|| \le M(W,T)M(B,T) \sum_{\tau=t-k_0(t)}^{t} ||y_1(p(\tau))||.$$

For $t_0 \le t < t_0 + \delta$ we have $p(t) < t_0$ and $y_1(p(t)) = 0$. Therefore,

$$y_2(t) = 0 \quad \text{for} \quad t_0 \le t < t_0 + \delta.$$

Suppose that statement (6) is valid for $n = k$ and prove it for $n = k + 1$. Then

$$||y_{k+1}(t)|| \le M(W,T)M(B,T) \sum_{\tau=t-k_0(t)}^{t} ||y_k(p(\tau))||.$$

For $t_0 \le t < t_0 + k\delta$ we have $p(t) \le t - \delta < t_0 + (k-1)\delta$ and by the inductional hypothesis $y_k(p(t)) = 0$, and so $y_{k+1}(t) = 0$.

Then, exists a natural number N such that

$$y_m(t) = 0 \quad \text{for all} \quad m \geq N \quad \text{and} \quad t_0 \leq t \leq T.$$

Therefore,

$$x(t) = \sum_{n=0}^{N-1} y_n(t) \quad \text{for} \quad t_0 \leq t \leq T$$

and the convergence is clear. Moreover,

$$x(t) = \sum_{n=0}^{\infty} y_n(t) = y_0(t) + \sum_{n=1}^{\infty} y_n(t) =$$

$$= A(t)y_0(t-1) + \sum_{n=1}^{\infty} A(t)y_n(t-1) + \sum_{n=1}^{\infty} B(t)y_{n-1}(p(t)) =$$

$$= A(t)\sum_{n=0}^{\infty} y_n(t-1) + B(t)\sum_{n=0}^{\infty} y_n(p(t)) =$$

$$= A(t)x(t-1) + B(t)x(p(t)),$$

and the proof is complete.

In the space of vector or matrix functions $f(t)$ let the operators S_p and W^* be defined by

$$S_p f(t) = f(p(t)), \quad W^* f(t) = \sum_{\tau=t-k_0(t)}^{t} W(\tau;t)f(\tau).$$

Then

$$y_n = W^*(BS_p y_{n-1}) = (W^* BS_p)^n y_0, \quad n = 1, 2, ...$$

Therefore Theorem 1 implies the next corollary.

Corollary 1. *The unique solution of the initial value problem (1) and (2) is given by*

$$x(t) = \sum_{n=0}^{\infty}(W^* BS_p)^n W(t-k_0(t)-1;t)\phi(t-k_0(t)-1). \tag{7}$$

In the next result we give conditions garanteeing that series (7) is absolutely and uniformly convergent on the interval $[t_0, \infty)$.

Theorem 2. *Suppose that there exist positive constants M, b and a such that $0 < a < 1$ and there exists a positive scalar function $f(t)$ such that*

$$\sup_{t_{-1} \le \theta \le t_0} \sum_{\tau=\theta}^{\infty} f(\tau) = f_0 < \infty,$$

$$||W(\tau;t)|| \le M a^t, \quad t_0 - 1 \le \tau \le t_0 \le t, \tag{8}$$

$$||W(\tau;t)|| \le M a^{t-\tau}, \quad t_0 \le \tau \le t,$$

$$||B(t)|| \le b + f(t), \quad t \ge t_0,$$

$$M\left(\frac{b}{1-a} + f_0\right) < 1. \tag{9}$$

Then series (7) is absolutely and uniformly convergent for $t \ge t_0$. If, in addition, there exists a positive constant p such that

$$0 < pt \le p(t) \quad \text{for} \quad t \ge t_0, \tag{10}$$

then the solution of the initial value problem (1) and (2) tends to zero, as $t \to \infty$.

Proof. Let p_0 be a real number such that

$$0 \le p_0 < 1 \quad \text{and} \quad p_0 t \le p(t).$$

Introduce the sequence $\{\gamma_n\}$ as follows.

$$\gamma_0 := 1, \quad \gamma_n := M\gamma_{n-1}\left(\frac{b}{1-a^{1-p_0^n}} + f_0\right), \quad n = 1, 2, ...$$

In virtue of (9) it is easy to see that the series

$$\sum_{n=0}^{\infty} \gamma_n$$

is finite. Let

$$y_0(t) = W(t - k_0(t) - 1; t)\phi(t - k_0(t) - 1)$$

and let $\{y_n(t)\}$ be defined as in Theorem 1. We assert that

$$||y_n(t)|| \le M\gamma_n a^{p_0^n t}||\phi||, \quad n = 0, 1, 2, ... \tag{11}$$

Inequality (8) implies assertion (11) for $n = 0$. Suppose that (11) is true for $n - 1$. Then

$$||y_n(t)|| \le \sum_{\tau=t-k_0(t)}^{t} ||W(\tau;t)|| ||B(\tau)|| ||y_{n-1}(p(\tau))||$$

$$\leq \sum_{\tau=t-k_0(t)}^{t} M^2 a^{t-\tau}(b+f(\tau))\gamma_{n-1} a^{p_0^{n-1}p(\tau)}||\phi||$$

$$\leq M^2 a^t \gamma_{n-1}||\phi|| \sum_{\tau=t-k_0(t)}^{t} a^{(p_0^n-1)\tau}(b+f(\tau))$$

$$= M^2 a^t \gamma_{n-1}||\phi|| \left(\sum_{\tau=t-k_0(t)}^{t} b a^{(p_0^n-1)\tau} + \sum_{\tau=t-k_0(t)}^{t} a^{(p_0^n-1)\tau} f(\tau) \right)$$

$$\leq M^2 a^t \gamma_{n-1}||\phi|| \left(\frac{b}{a^{p_0^n-1}-1} a^{(p_0^n-1)\tau}\Big|_{t-k_0(t)}^{t+1} + a^{(p_0^n-1)t} \sum_{\tau=t-k_0(t)}^{t} f(\tau) \right)$$

$$\leq M^2 a^t \gamma_{n-1}||\phi|| \left(\frac{b}{a^{p_0^n-1}-1} a^{(p_0^n-1)(t+1)} + a^{(p_0^n-1)t} f_0 \right)$$

$$= M^2 \gamma_{n-1}||\phi|| a^{p_0^n t} \left(\frac{b a^{p_0^n-1}}{a^{p_0^n-1}-1} + f_0 \right)$$

$$= M^2 \gamma_{n-1}||\phi|| a^{p_0^n t} \left(\frac{b}{1-a^{1-p_0^n}} + f_0 \right)$$

$$= M \gamma_n a^{p_0^n t}||\phi||,$$

and (11) is true for all positive integers n. It means that (7) is absolutely and uniformly convergent on $[t_0, \infty)$ and the first part of the theorem is proved.
If (10) is satisfied then we can choose $p_0 = p$ and for all $\epsilon > 0$ we can find an integer N such that

$$2M \sum_{n=N}^{\infty} \gamma_n < \epsilon.$$

Then

$$||x(t)|| \leq \sum_{n=0}^{\infty} ||y_n(t)|| \leq \sum_{n=0}^{\infty} M \gamma_n a^{p^n t}||\phi|| \leq$$

$$\leq \left(M \sum_{n=N}^{\infty} \gamma_n + M \sum_{n=0}^{N-1} a^{p^{N-1}t} \gamma_n \right) ||\phi|| < \epsilon ||\phi||,$$

if t is so large that

$$2M a^{p^{N-1}t} \sum_{n=0}^{N-1} \gamma_n < \epsilon.$$

This proves the second part of the theorem.

If we apply the above results to the scalar equation with constant coefficients

$$x(t) = ax(t-1) + bx(pt), \tag{12}$$

where a, b, p are real constants such that $0 < a < 1$ and $0 < p < 1$, the form of the functions $y_n(t)$ will be too complicate, not suitable for further investigation. Therefore, to solve Equation (12) by this method, we need a computer. But we can obtain a nice series representation form for the analitical solutions of Equation (12). Of course, it is neccessary for the initial function to be analitical.

Theorem 3. *Let $C_0 \neq 0$ be a given real number. Let a, b, p be real numbers such that $0 < a < 1$, $0 < p < 1$ and $|b| < 1 - a$. Then*

$$x(t) = \sum_{n=0}^{\infty} C_0 b^n \prod_{\ell=1}^{n} (1 - a^{1-p^\ell})^{-1} a^{p^n t}$$

is a series solution of Equation (12) on $[t_0, \infty)$.

Proof. Suppose that a solution of Equation (12) is the series of the form

$$x(t) = \sum_{n=0}^{\infty} C_n \lambda^{p^n t}.$$

Replacing this form in Equation (12) we obtain that

$$\sum_{n=0}^{\infty} C_n \lambda^{p^n t} = a \sum_{n=0}^{\infty} C_n \lambda^{p^n (t-1)} + b \sum_{n=0}^{\infty} C_n \lambda^{p^{n+1} t},$$

and therefore,

$$C_0 \lambda^t + \sum_{n=1}^{\infty} C_n \lambda^{p^n t} = \frac{a}{\lambda} C_0 \lambda^t + \sum_{n=1}^{\infty} \frac{a}{\lambda^{p^n}} C_n \lambda^{p^n t} + \sum_{n=1}^{\infty} b C_{n-1} \lambda^{p^n t}.$$

From the above equality follows that

$$C_0 \left(1 - \frac{a}{\lambda}\right) = 0, \quad \text{so} \quad a = \lambda, \quad C_0 \neq 0.$$

$$C_1 = C_1 a^{1-p} + b C_0,$$

$$C_2 = C_2 a^{1-p^2} + b C_1,$$

$$\ldots$$

$$C_n = C_n a^{1-p^n} + b C_{n-1},$$

$$\ldots$$

Using mathematical induction we obtain that

$$C_n = \frac{b^n C_0}{(1 - a^{1-p})(1 - a^{1-p^2}) \ldots (1 - a^{1-p^n})}, \quad n = 1, 2, \ldots$$

Then the series solution of Equation (12) is of the form

$$x(t) = \sum_{n=0}^{\infty} C_0 b^n \prod_{\ell=1}^{n} (1 - a^{1-p^\ell})^{-1} a^{p^n t}.$$

From the above argumentation follows that the necessary and sufficient condition for the convergence is

$$|b| < 1 - a.$$

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