SPECTRAL ANALYSIS OF AN OPERATOR ASSOCIATED WITH LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS AND ITS APPLICATIONS *

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1. INTRODUCTION

In this paper, we treat the (autonomous) linear functional differential equation

$$\dot{x}(t) = L(x_t), \tag{1}$$

where L is a bounded linear operator mapping a uniform fading memory space $\mathcal{B} = \mathcal{B}((-\infty, 0]; \mathbf{C}^n)$ into $\mathbf{C}^n$, and study the admissibility of Eq. (1) for a translation invariant function space $\mathcal{M}$ which consists of functions whose spectrum is contained in a closed set Λ in $\mathbf{R}$. In case of $\Lambda = \mathbf{R}$ or $\Lambda = \{2k\pi/\omega : k \in \mathbf{Z}\}$, the problem for the admissibility is reduced to the one for the existence of bounded solutions, almost periodic solutions or ω-periodic solutions of the equation

$$\dot{x}(t) = L(x_t) + f(t)$$

with the forced function $f(t)$ which is bounded, almost periodic or ω-periodic, and there are many results on the problem (e.g., [1], [3], [5], [7, 8]). In this paper, we study the problem for a general set Λ. Roughly speaking, we solve the problem by determining the spectrum of an operator $D_{\mathcal{M}} - \mathcal{L}_{\mathcal{M}}$ which is associated with Eq. (1).

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2. UNIFORM FADING MEMORY SPACES AND SOME PRELIMINARIES

In this section we explain uniform fading memory spaces which are employed throughout this paper, and give some preliminary results.

Let $\mathbf{C}^n$ be the n-dimensional complex Euclidean space with norm $|\cdot|$. For any interval $J \subset \mathbf{R} := (-\infty, \infty)$, we denote by $C(J; \mathbf{C}^n)$ the space of all continuous functions mapping J into $\mathbf{C}^n$. Moreover, we denote by $\mathrm{BC}(J; \mathbf{C}^n)$ the subspace of $C(J; \mathbf{C}^n)$ which consists of all bounded functions. Clearly $\mathrm{BC}(J; \mathbf{C}^n)$ is a Banach space with the norm $|\cdot|_{\mathrm{BC}(J;\mathbf{C}^n)}$ defined by $|\phi|_{\mathrm{BC}(J;\mathbf{C}^n)} = \sup\{|\phi(t)| : t \in J\}$. If $J = \mathbf{R}^- := (-\infty, 0]$, then we simply write $\mathrm{BC}(J; \mathbf{C}^n)$ and $|\cdot|_{\mathrm{BC}(J;\mathbf{C}^n)}$ as BC and $|\cdot|_{\mathrm{BC}}$, respectively. For any function $x : (-\infty, a) \mapsto \mathbf{C}^n$ and $t < a$, we define a function $x_t : \mathbf{R}^- \mapsto \mathbf{C}^n$ by $x_t(s) = x(t+s)$ for $s \in \mathbf{R}^-$. Let $\mathcal{B} = \mathcal{B}(\mathbf{R}^-; \mathbf{C}^n)$ be a complex linear space of functions mapping $\mathbf{R}^-$ into $\mathbf{C}^n$ with a complete seminorm $|\cdot|_{\mathcal{B}}$. The space $\mathcal{B}$ is assumed to have the following properties:

(A1) There exist a positive constant N and locally bounded functions $K(\cdot)$ and $M(\cdot)$ on $\mathbf{R}^+ := [0, \infty)$ with the property that if $x : (-\infty, a) \mapsto \mathbf{C}^n$ is continuous on $[\sigma, a)$ with $x_\sigma \in \mathcal{B}$ for some $\sigma < a$, then for all $t \in [\sigma, a)$,
(i) $x_t \in \mathcal{B}$,
(ii) x_t is continuous in t (w.r.t. $|\cdot|_{\mathcal{B}}$),
(iii) $N|x(t)| \le |x_t|_{\mathcal{B}} \le K(t-\sigma)\sup_{\sigma \le s \le t}|x(s)| + M(t-\sigma)|x_\sigma|_{\mathcal{B}}$.

(A2) If $\{\phi^k\}$, $\phi^k \in \mathcal{B}$, converges to ϕ uniformly on any compact set in $\mathbf{R}^-$ and if $\{\phi^k\}$ is a Cauchy sequence in $\mathcal{B}$, then $\phi \in \mathcal{B}$ and $\phi^k \to \phi$ in $\mathcal{B}$.

The space $\mathcal{B}$ is called a uniform fading memory space, if it satisfies (A1) and (A2) with $K(\cdot) \equiv K$ (a constant) and $M(\beta) \to 0$ as $\beta \to \infty$ in (A1). A typical one for uniform fading memory spaces is given by the space

$$C_\gamma := C_\gamma(\mathbf{C}^n) = \{\phi \in C(\mathbf{R}^-; \mathbf{C}^n) : \lim_{\theta \to -\infty} |\phi(\theta)|e^{\gamma\theta} = 0\}$$

which is equipped with norm $|\phi|_{C_\gamma} = \sup_{\theta \le 0}|\phi(\theta)|e^{\gamma\theta}$, where γ is a positive constant.

It is known [2, Lemma 3.2] that if $\mathcal{B}$ is a uniform fading memory space, then $\mathrm{BC} \subset \mathcal{B}$ and

$$|\phi|_{\mathcal{B}} \le K|\phi|_{\mathrm{BC}}, \qquad \phi \in \mathrm{BC}. \tag{2}$$

For other properties of uniform fading memory spaces, we refer the reader to the book [4].

We denote by $\mathrm{BUC}(\mathbf{R};\mathbf{C}^n)$ the space of all bounded and uniformly continuous functions mapping $\mathbf{R}$ into $\mathbf{C}^n$. $\mathrm{BUC}(\mathbf{R};\mathbf{C}^n)$ is a Banach space with the supremum norm which will be denoted by $||\cdot||$. The spectrum of a given function $f \in \mathrm{BUC}(\mathbf{R};\mathbf{C}^n)$ is defined as the set

$$sp(f) := \{\xi \in \mathbf{R} : \forall \epsilon > 0 \ \exists u \in L^1(\mathbf{R}),\ supp\ \tilde{u} \subset (\xi - \epsilon, \xi + \epsilon),\ u * f \neq 0\}\ ,$$

where

$$u * f(t) := \int_{-\infty}^{+\infty} u(t-s) f(s) ds \quad ;\ \tilde{u}(s) := \int_{-\infty}^{\infty} e^{-ist} u(t) dt.$$

We collect some main properties of the spectrum of a function, which we will need in the sequel, for the reader's convenience. For the proof we refer the reader to [6], [10-11].

Proposition 1 *The following statements hold true:*

(i) $sp(e^{i\lambda\cdot}) = \{\lambda\}$ *for* $\lambda \in \mathbf{R}$.
(ii) $sp(e^{i\lambda\cdot} f) = sp(f) + \lambda$ *for* $\lambda \in \mathbf{R}$.
(iii) $sp(\alpha f + \beta g) \subset sp(f) \cup sp(g)$ *for* $\alpha,\ \beta \in \mathbf{C}$.
(iv) $sp(f)$ *is closed. Moreover,* $sp(f)$ *is not empty if* $f \not\equiv 0$.
(v) $sp(f(\cdot + \tau)) = sp(f)$ *for* $\tau \in \mathbf{R}$.
(vi) *If* $f,\ g^k \in \mathrm{BUC}(\mathbf{R};\mathbf{C}^n)$ *with* $sp(g^k) \subset \Lambda$ *for all* $n \in \mathbf{N}$*, and if* $\lim_{k\to\infty} ||g^k - f|| = 0$*, then* $sp(f) \subset \overline{\Lambda}$.
(vii) $sp(\psi * f) \subset sp(f) \cap supp\ \tilde{\psi}$ *for all* $\psi \in L^1(\mathbf{R})$.

In the following we always assume that $\mathcal{B} = \mathcal{B}(\mathbf{R}^-;\mathbf{C}^n)$ is a uniform fading memory space. For any bounded linear functional $L : \mathcal{B} \mapsto \mathbf{C}^n$ we define an operator $\mathcal{L}$ by

$$(\mathcal{L}f)(t) = L(f_t), \qquad t \in \mathbf{R},$$

for $f \in \mathrm{BUC}(\mathbf{R};\mathbf{C}^n)$. It follows from (2) that

$$\begin{aligned} |(\mathcal{L}f)(t) - (\mathcal{L}f)(s)| &\leq ||L|||f_t - f_s|_{\mathcal{B}} \\ &\leq K||L|||f_t - f_s|_{\mathrm{BC}}, \end{aligned}$$

and hence $\mathcal{L}f \in \mathrm{BUC}(\mathbf{R};\mathbf{C}^n)$. Consequently, $\mathcal{L}$ is a bounded linear operator on $\mathrm{BUC}(\mathbf{R};\mathbf{C}^n)$.

For any closed set $\Lambda \subset \mathbf{R}$, we set

$$\Lambda(\mathbf{C}^n) = \{f \in \mathrm{BUC}(\mathbf{R};\mathbf{C}^n) : sp(f) \subset \Lambda\}.$$

From (iii)–(vi) of Proposition 1, we can see that $\Lambda(\mathbf{C}^n)$ is a translation-invariant closed subspace of $\mathrm{BUC}(\mathbf{R};\mathbf{C}^n)$.

Proposition 2 *Let Λ be a closed set in* $\mathbf{R}$. *Then the space $\Lambda(\mathbf{C}^n)$ is invariant under the operator $\mathcal{L}$.*

Proof Let $f \in \mathrm{BUC}(\mathbf{R};\mathbf{C}^n)$. It suffices to establish that $sp(\mathcal{L}f) \subset sp(f)$. Let $\xi \notin sp(f)$. There is an $\epsilon > 0$ with the property that $u * f = 0$ for any $u \in L^1(\mathbf{R})$ such that $supp\ \tilde{u} \subset (\xi - \epsilon, \xi + \epsilon)$. Let v be any element in $L^1(\mathbf{R})$ such that $supp\ \tilde{v} \subset (\xi - \epsilon, \xi + \epsilon)$. Since

$$\begin{aligned} \int_{-\infty}^{\infty} v(t-s) f_s(\theta) ds &= \int_{-\infty}^{\infty} v(t-s) f(s+\theta) ds \\ &= (v * f)(t+\theta) = 0 \end{aligned}$$

for $\theta \leq 0$, (A2) yields that $\int_{-\infty}^{\infty} v(t-s) f_s ds = 0$ in $\mathcal{B}$. Hence

$$\begin{aligned} (v * \mathcal{L}f)(t) &= \int_{-\infty}^{\infty} v(t-s) L(f_s) ds \\ &= L(\int_{-\infty}^{\infty} v(t-s) f_s ds) \\ &= 0, \end{aligned}$$

which shows that $\xi \notin sp(\mathcal{L}f)$.

3. SPECTRUM OF AN OPERATOR ASSOCIATED WITH FUNCTIONAL DIFFERENTIAL EQUATIONS

We consider the linear functional differential equation

$$\dot{x}(t) = L(x_t), \tag{1}$$

where L is a bounded linear operator mapping a uniform fading memory space $\mathcal{B} = \mathcal{B}(\mathbf{R}^-;\mathbf{C}^n)$ into $\mathbf{C}^n$. A translation-invariant space $\mathcal{M} \subset \mathrm{BUC}(\mathbf{R};\mathbf{C}^n)$ is said to be admissible with respect to Eq. (1), if for any $f \in \mathcal{M}$, the equation

$$\dot{x}(t) = L(x_t) + f(t)$$

possesses a unique solution which belongs to $\mathcal{M}$. Let Λ be a closed set in $\mathbf{R}$. An aim in this section is to obtain a condition under which the subspace $\Lambda(\mathbf{C}^n)$ introduced in the previous section is admissible with respect to Eq. (1). To do this, we first introduce the operators $\mathcal{D}_\Lambda$ and $\mathcal{L}_\Lambda$ associated with Eq. (1):

$$\begin{aligned} \mathcal{D}_\Lambda &:= (d/dt)|_{D(\mathcal{D}_\Lambda)} \\ \mathcal{L}_\Lambda &:= \mathcal{L}|_{\Lambda(\mathbf{C}^n)}, \end{aligned}$$

where

$$D(\mathcal{D}_\Lambda) = \{u \in \Lambda(\mathbf{C}^n) : du/dt \in \Lambda(\mathbf{C}^n)\}.$$

Clearly, the admissibility of $\Lambda(\mathbf{C}^n)$ with respect to Eq. (1) is equivalent to the invertibility of the operator $\mathcal{D}_\Lambda - \mathcal{L}_\Lambda$ in $\Lambda(\mathbf{C}^n)$. In fact, we will determine the spectrum $\sigma(\mathcal{D}_\Lambda - \mathcal{L}_\Lambda)$ of $\mathcal{D}_\Lambda - \mathcal{L}_\Lambda$ in Theorem 1, and as a consequence of Theorem 1, we will obtain a condition for $\Lambda(\mathbf{C}^n)$ to be admissible with respect to Eq. (1).

Before stating Theorem 1, we prepare some notation. For any $\lambda \in \Lambda$, we define a function $\omega(\lambda) : \mathbf{R}^- \mapsto \mathbf{C} := \mathbf{C}^1$ by

$$[\omega(\lambda)](\theta) = e^{i\lambda\theta}, \quad \theta \in \mathbf{R}^-.$$

Because $\mathcal{B}$ is a uniform fading memory space, it follows that $\omega(\lambda)a \in \mathcal{B}$ for any (column) vector $a \in \mathbf{C}^n$. In particular, we get $\omega(\lambda)e_i \in \mathcal{B}$ for $i = 1, \cdots, n$, where e_i is the element in $\mathbf{C}^n$ whose i-th component is 1 and the other components are 0. We denote by I the $n \times n$ unit matrix, and define an $n \times n$ matrix by

$$(L(\omega(\lambda)e_1), \cdots, L(\omega(\lambda)e_n)) =: L(\omega(\lambda)I).$$

Theorem 1 *Let Λ be a closed subset of $\mathbf{R}$, and let $\mathcal{D}_\Lambda$ and $\mathcal{L}_\Lambda$ be the ones introduced above. Then the following relation holds:*

$$\sigma(\mathcal{D}_\Lambda - \mathcal{L}_\Lambda) = \{\mu \in \mathbf{C} : \det[(i\lambda - \mu)I - L(\omega(\lambda)I)] = 0 \text{ for some } \lambda \in \Lambda\} \ (=: (\widetilde{i\Lambda})).$$

In order to establish the theorem, we need the following result for ordinary differential equations:

Lemma 1 *Let Q be an $n \times n$ matrix such that $\sigma(Q) \subset i\mathbf{R} \backslash i\Lambda$. Then for any $f \in \Lambda(\mathbf{C}^n)$ there is a unique solution x_f in $\Lambda(\mathbf{C}^n)$ of the system of ordinary differential equations*

$$\dot{x}(t) = Qx(t) + f(t).$$

Moreover, the map $f \in \Lambda(\mathbf{C}^n) \mapsto x_f \in \Lambda(\mathbf{C}^n)$ is continuous.

Proof. Without loss of generality, we may assume that Q is a matrix of Jordan canonical form

$$Q = \begin{pmatrix} i\lambda_1 & \delta_1 & & & \mathbf{0} \\ & i\lambda_2 & \delta_2 & & \\ & \cdots & \cdots & & \\ & & & i\lambda_{n-1} & \delta_{n-1} \\ \mathbf{0} & & & & i\lambda_n \end{pmatrix},$$

where $\{\lambda_1, \cdots, \lambda_n\} \cap \Lambda = \emptyset$, and $\delta_k = 0$ or 1 for $k = 1, \cdots, n-1$. The equation for x_n is written as

$$\dot{x}_n(t) = i\lambda_n x_n(t) + f_n(t),$$

where $f_n \in \Lambda(\mathbf{C})$. By setting $z(t) = x_n(t)e^{-i\lambda_n t}$, we get

$$\dot{z}(t) = e^{-i\lambda_n t} f_n(t) =: g(t).$$

It follows that $0 \notin sp(g)$ because of $sp(g) \subset \Lambda - \{\lambda_n\}$. Then, by virtue of [6, Chapter 6, Theorem 3 and its proof] there exists an integrable function ϕ such that $z = \phi * g$ satisfies $\dot{z}(t) = g(t)$ (and hence $x_n(t) = z(t)e^{i\lambda_n t}$ is a solution of the above equation). From (vii) of Proposition 1 it follows that $sp(z) \subset sp(g)$, and hence $sp(x_n) \subset sp(g) + \{\lambda_n\} \subset \Lambda$. If $y(t)$ is another solution in $\Lambda(\mathbf{C})$ of the above equation, then $x_n(t) - y(t) \equiv ae^{i\lambda_n t}$ for some a, and hence $sp(x_n - y) \subset \{\lambda_n\}$. Since $\lambda_n \notin \Lambda$, we must have $x_n - y \equiv 0$. Thus the above equation possesses a unique solution x_n in $\Lambda(\mathbf{C})$, which is represented as the convolution of f_n and an integrable function. For this x_n, let us consider the equation for x_{n-1}

$$\dot{x}_{n-1}(t) = i\lambda_{n-1}x_{n-1}(t) + \delta_{n-1}x_n(t) + f_{n-1}(t).$$

Since the term $\delta_{n-1}x_n(t) + f_{n-1}(t)$ belongs to the space $\Lambda(\mathbf{C})$, the above argument shows that the equation for x_{n-1} possesses a unique solution in $\Lambda(\mathbf{C})$, too. In fact, the solution is represented as

$$\psi_{n-1} * f_{n-1} + \psi_n * f_n$$

for some integrable functions ψ_{n-1} and ψ_n. Continue the procedure to the equations for $x_{n-2}, \cdots,\ x_2$ and x_1, subsequently. Then we conclude that the system possesses a unique solution in $\Lambda(\mathbf{C}^n)$, which is represented as the convolution $Y * f$ for some $n \times n$ matrix-valued integrable function Y.

Proof of Theorem 1. In case where $|\cdot|_{\mathcal{B}}$ is a complete semi-norm of $\mathcal{B}$, one can prove the theorem by considering the quotient space $\mathcal{B}/|\cdot|_{\mathcal{B}}$. In order to avoid some cumbersome notation, we shall establish the theorem in case where $|\cdot|_{\mathcal{B}}$ is a norm and consequently $\mathcal{B}$ is a Banach space.

Assume that $\mu \in \mathbf{C}$ satisfies $\det[(i\lambda - \mu)I - L(\omega(\lambda)I] = 0$ for some $\lambda \in \Lambda$. Then there is a nonzero $a \in \mathbf{C}^n$ such that $i\lambda a - L(\omega(\lambda)a) = \mu a$. Set $\phi(t) = e^{i\lambda t}a,\ t \in \mathbf{R}$. Then $\phi \in D(\mathcal{D}_\Lambda)$, and

$$\begin{aligned}
[(\mathcal{D}_\Lambda - \mathcal{L}_\Lambda)\phi](t) &= \dot{\phi}(t) - L(\phi_t) \\
&= i\lambda e^{i\lambda t}a - L(e^{i\lambda t}\omega(\lambda)a) \\
&= e^{i\lambda t}(i\lambda a - L(\omega(\lambda)a)) \\
&= \mu\phi(t),
\end{aligned}$$

or $(\mathcal{D}_\Lambda - \mathcal{L}_\Lambda)\phi = \mu\phi$. Thus $\mu \in P_\sigma(\mathcal{D}_\Lambda - \mathcal{L}_\Lambda)$. Hence $(\tilde{i\Lambda}) \subset \sigma(\mathcal{D}_\Lambda - \mathcal{L}_\Lambda)$.

Next we shall show that $(\tilde{i\Lambda}) \supset \sigma(\mathcal{D}_\Lambda - \mathcal{L}_\Lambda)$. To do this, it is sufficient to prove the claim:

Assertion *If* $\det[(i\lambda - k)I - L(\omega(\lambda)I)] \neq 0 \quad (\forall \lambda \in \Lambda)$, *then* $k \in \rho(\mathcal{D}_\Lambda - \mathcal{L}_\Lambda)$.

To establish the claim, we will show that for each $f \in \Lambda(\mathbf{C}^n)$, the equation

$$\dot{x}(t) = L(x_t) + kx(t) + f(t), \quad t \in \mathbf{R} \tag{3}$$

possesses a unique solution $x_f \in \Lambda(\mathbf{C}^n)$ and that the map $f \in \Lambda(\mathbf{C}^n) \mapsto x_f \in \Lambda(\mathbf{C}^n)$ is continuous. We first treat the homogeneous functional differential equation

$$\dot{x}(t) = L(x_t) + kx(t), \tag{4}$$

and consider the solution semigroup $T(t) : \mathcal{B} \mapsto \mathcal{B},\ t \geq 0$, of Eq. (4) which is defined as

$$T(t)\phi = x_t(\phi), \quad \phi \in \mathcal{B},$$

where $x(\cdot, \phi)$ denotes the solution of (4) through $(0, \phi)$ and x_t is an element in $\mathcal{B}$ defined as $x_t(\theta) = x(t+\theta),\ \theta \leq 0$. Let G be the infinitesimal generator of the solution semigroup $T(t)$. We assert that

$$i\mathbf{R} \cap \sigma(G) = \{i\lambda \in i\mathbf{R} : \det[(i\lambda - k)I - L(\omega(\lambda)I)] = 0\}.$$

Before proving the assertion, we first remark that the constant β introduced in [4, p. 127] satisfies $\beta < 0$ because $\mathcal{B}$ is a uniform fading memory space. In particular, if λ is a real number, then $\mathrm{Re}(i\lambda) = 0 > \beta$, and hence $\omega(\lambda)b \in \mathcal{B}$ for any $b \in \mathbf{C}^n$ by [4, p. 137, Th. 2.4].

Now, let $i\lambda \in i\mathbf{R} \cap \sigma(G)$. Since $i\lambda$ is a normal point of G by [4, p. 141, Th. 2.7], we must have that $i\lambda \in P_\sigma(G)$. Then [4, p. 134, Th. 2.1] implies that there exists a nonzero $b \in \mathbf{C}^n$ such that $i\lambda b - L(\omega(\lambda)b) - kb = 0$, which shows that $i\lambda$ belongs to the set of the right hand side in the assertion. Conversely, assume that $i\lambda$ is an element of the set of the right hand side in the assertion. Then there is a nonzero $a \in \mathbf{C}^n$ such that $i\lambda a = ka + L(\omega(\lambda)a)$. Set $x(t) = e^{i\lambda t}a,\ t \in \mathbf{R}$. Then $x_t = e^{i\lambda t}\omega(\lambda)a$ and

$$\begin{aligned} \dot{x}(t) = i\lambda e^{i\lambda t}a &= e^{i\lambda t}(ka + L(\omega(\lambda)a)) \\ &= kx(t) + L(x_t). \end{aligned}$$

Thus $x(t)$ is a solution of Eq. (4) satisfying $x_0 = \omega(\lambda)a$, and it follows that $T(t)\omega(\lambda)a = T(t)x_0 = x_t = e^{i\lambda t}\omega(\lambda)a$ for $t \geq 0$, which implies that $\omega(\lambda)a \in D(G)$ and $G(\omega(\lambda)a) = i\lambda\omega(\lambda)a$. Thus $i\lambda \in \sigma(G) \cap i\mathbf{R}$, and the assertion is proved.

Now consider the sets $\Sigma_C := \{\lambda \in \sigma(G) : \mathrm{Re}\lambda = 0\}$ and $\Sigma_U := \{\lambda \in \sigma(G) : \mathrm{Re}\lambda > 0\}$. Then the set $\Sigma = \Sigma_C \cup \Sigma_U$ is a finite set [4, p. 144, Prop. 3.2]. Corresponding to the set Σ, we get the decomposition of the space $\mathcal{B}$:

$$\mathcal{B} = S \oplus C \oplus U,$$

where S, C, U are invariant under $T(t)$, the restriction $T(t)|_U$ can be extendable as a group, and there exist positive constants c_1 and α such that

$$\begin{aligned} \|T(t)|_S\| &\le c_1 e^{-\alpha t} \quad (t \ge 0), \\ \|T(t)|_U\| &\le c_1 e^{\alpha t} \quad (t \le 0) \end{aligned}$$

([4, p. 145, Ths. 3.1, 3.3]). Let Φ be a basis vector in C, and let Ψ be the basis vector associated with Φ. From [4, p. 149, Cor. 3.8] we know that the C-component $u(t)$ of the segment x_t for each solution $x(\cdot)$ of Eq. (3) is given by the relation $u(t) = \langle \Psi, \Pi_C x_t \rangle$ (where Π_C denotes the projection from $\mathcal{B}$ onto C which corresponds to the decomposition of the space $\mathcal{B}$), and $u(t)$ satisfies the ordinary differential equation

$$\dot{u}(t) = Qu(t) - \hat{\Psi}(0^-)f(t), \tag{5}$$

where Q is a matrix such that $\sigma(Q) = \sigma(G) \cap i\mathbf{R}$ and the relation $T(t)\Phi = \Phi e^{tQ}$ holds. Moreover, $\hat{\Psi}$ is the one associated with the Riesz representation of Ψ. Indeed, $\hat{\Psi}$ is a normalized vector-valued function which is of locally bounded variation on $\mathbf{R}^-$ satisfying $\langle \Psi, \phi \rangle = \int_{-\infty}^{0} \phi(\theta) d\hat{\Psi}(\theta)$ for any $\phi \in \mathrm{BC}(\mathbf{R}^-; \mathbf{C}^n)$ with compact support. Observe that $\Sigma_C \subset i\mathbf{R} \setminus i\Lambda$. Indeed, if $\mu \in \Sigma_C$, then $\mu = i\lambda$ for some $\lambda \in \mathbf{R}$, where $\det[(i\lambda - k)I - L(\omega(\lambda)I)] = 0$ by the preceding assertion. Hence we get $\lambda \notin \Lambda$ by the assumption of the claim, and $\mu \in i\mathbf{R} \setminus i\Lambda$, as required. This observation leads to $\sigma(Q) \cap i\Lambda = \oslash$. Since $sp(\hat{\Psi}(0^-)f) \subset \Lambda$, lemma 1 implies that the ordinary differential equation (5) has a unique solution u satisfying $sp(u) \subset \Lambda$ and $\|u\| \le c_2 \|\hat{\Psi}(0^-)f\| \le c_3 \|f\|$ for some constants c_2 and c_3. Consider a function $\xi : \mathbf{R} \mapsto \mathcal{B}$ defined by

$$\xi(t) = \int_{*-\infty}^{t} T^{**}(t-s)\Pi_S^{**}\Gamma f(s) ds + \Phi u(t) - \int_{*t}^{\infty} T^{**}(t-s)\Pi_U^{**}\Gamma f(s) ds,$$

where Γ is the one defined in [4, p. 118] and $\int_*$ denotes the weak-star integration (cf. [4, p. 116]). If $t \ge 0$, then

$$\begin{aligned} & T(t)\xi(\sigma) + \int_{*\sigma}^{t+\sigma} T^{**}(t+\sigma-s)\Gamma f(s) ds \\ = \; & T(t)[\int_{*-\infty}^{\sigma} T^{**}(\sigma-s)\Pi_S^{**}\Gamma f(s) ds + \Phi u(\sigma) - \int_{*\sigma}^{\infty} T^{**}(\sigma-s)\Pi_U^{**}\Gamma f(s) ds] \\ & \qquad + \int_{*\sigma}^{t+\sigma} T^{**}(t+\sigma-s)\Gamma f(s) ds \end{aligned}$$

$$
\begin{aligned}
&= \int_{*-\infty}^{\sigma} T^{**}(t+\sigma-s)\Pi_S^{**}\Gamma f(s)ds + \Phi e^{tQ}u(\sigma) - \int_{*\sigma}^{\infty} T^{**}(t+\sigma-s)\Pi_U^{**}\Gamma f(s)ds \\
&\qquad\qquad + \int_{*\sigma}^{t+\sigma} T^{**}(t+\sigma-s)(\Pi_S^{**}+\Pi_C^{**}+\Pi_U^{**})\Gamma f(s)ds \\
&= \int_{*-\infty}^{t+\sigma} T^{**}(t+\sigma-s)\Pi_S^{**}\Gamma f(s)ds + \Phi[e^{tQ}u(\sigma) + \int_{\sigma}^{t+\sigma} e^{(t+\sigma-s)Q}(-\hat{\Psi}(0^-)f(s))ds] \\
&\quad - \int_{*t+\sigma}^{\infty} T^{**}(t+\sigma-s)\Pi_U^{**}\Gamma f(s)ds \\
&= \int_{*-\infty}^{t+\sigma} T^{**}(t+\sigma-s)\Pi_S^{**}\Gamma f(s)ds + \Phi u(t+\sigma) \\
&\qquad\qquad - \int_{*t+\sigma}^{\infty} T^{**}(t+\sigma-s)\Pi_U^{**}\Gamma f(s)ds \\
&= \xi(t+\sigma),
\end{aligned}
$$

where we used the relation $T^{**}(t)\Pi_C^{**}\Gamma = T^{**}(t)\Phi\langle\Psi,\Gamma\rangle = \Phi e^{tQ}(-\hat{\Psi}(0^-))$. Then [4, p. 121, Th. 2.9] yields that $x(t) := [\xi(t)](0)$ is a solution of (3). Define a $\psi \in \mathcal{B}^* \times \cdots \times \mathcal{B}^*$ (n-copies) by $\langle\psi,\phi\rangle = \phi(0),\ \phi \in \mathcal{B}$. Then

$$
\begin{aligned}
x(t) - \Phi(0)u(t) &= \langle\psi, \xi(t) - \Phi u(t)\rangle \\
&= \langle\psi, \int_{*-\infty}^{t} T^{**}(t-s)\Pi_S^{**}\Gamma f(s)ds - \int_{*t}^{\infty} T^{**}(t-s)\Pi_U^{**}\Gamma f(s)ds\rangle \\
&= \int_{-\infty}^{t} \langle\psi, T^{**}(t-s)\Pi_S^{**}\Gamma\rangle f(s)ds - \int_{t}^{\infty} \langle\psi, T^{**}(t-s)\Pi_U^{**}\Gamma\rangle f(s)ds \\
&= \int_{-\infty}^{\infty} Y(t-s)f(s)ds = Y * f(t),
\end{aligned}
$$

where $Y(\cdot) = \langle\psi, T^{**}(\cdot)\Pi_S^{**}\Gamma\rangle\chi_{[0,\infty)} - \langle\psi, T^{**}(\cdot)\Pi_U^{**}\Gamma\rangle\chi_{(-\infty,0]}$ and it is an $n \times n$ matrix-valued integrable function on $\mathbf{R}$. Then $\sigma(x - \Phi(0)u) \subset \sigma(f) \subset \Lambda$ by (vii) of Proposition 1, and hence $x - \Phi(0)u \in \Lambda(\mathbf{C}^n)$. Thus we get $x \in \Lambda(\mathbf{C}^n)$ because of $sp(u) \subset \Lambda$. Moreover, the map $f \in \Lambda(\mathbf{C}^n) \mapsto x \in \Lambda(\mathbf{C}^n)$ is continuous.

Finally, we will prove the uniqueness of solutions of (3) in $\Lambda(\mathbf{C}^n)$. Let x be any solution of (3) which belongs to $\Lambda(\mathbf{C}^n)$. By [4, p. 120, Th. 2.8] the $\mathcal{B}$-valued function $\Pi_S x_t$ satisfies the relation

$$
\Pi_S x_t = T(t-\sigma)\Pi_S x_\sigma + \int_{*\sigma}^{t} T^{**}(t-s)\Pi_S^{**}\Gamma f(s)ds
$$

for all $t \geq \sigma > -\infty$. Note that $\sup_{\sigma\in\mathbf{R}} |x_\sigma|_{\mathcal{B}} < \infty$. Therefore, letting $\sigma \to -\infty$ we get

$$
\Pi_S x_t = \int_{*-\infty}^{t} T^{**}(t-s)\Pi_S^{**}\Gamma f(s)ds,
$$

because

$$
\lim_{\sigma\to-\infty} \int_{*\sigma}^{t} T^{**}(t-s)\Pi_S^{**}\Gamma f(s)ds = \int_{*-\infty}^{t} T^{**}(t-s)\Pi_S^{**}\Gamma f(s)ds
$$

converges. Similarly, one gets

$$\Pi_U x_\sigma = -\int_{*\sigma}^{\infty} T^{**}(\sigma - s)\Pi_U^{**}\Gamma f(s)ds.$$

Also, since $\langle\Psi, x_t\rangle$ satisfies Eq. (5) and since $sp(\langle\Psi, x_t\rangle) \subset sp(x) \subset \Lambda$, it follows that $\Pi_C x_t = \Phi\langle\Psi, x_t\rangle = \Phi u(t)$ for all $t \in \mathbf{R}$ by the uniqueness of the solution of (5) in $\Lambda(\mathbf{C}^n)$. Consequently, we have $x_t \equiv \xi(t)$ or $x(t) \equiv [\xi(t)](0)$, which shows the uniqueness of the solution of (3) in $\Lambda(\mathbf{C}^n)$.

Corollary 1 *Suppose that* $\det[i\lambda I - L(\omega(\lambda)I)] \neq 0$ *for all* $\lambda \in \Lambda$. *Then Eq. (1) is admissible for* $\mathcal{M} = \Lambda(\mathbf{C}^n)$.

Proof. The corollary is a direct consequence of Theorem 1, since $0 \notin \sigma(\mathcal{D}_\mathcal{M} - \mathcal{L}_\mathcal{M})$.

Corollary 2 *Let* Λ *be a closed set in* $\mathbf{R}$, *and suppose that* $\det[(i\lambda - k)I - L(\omega(\lambda)I)] \neq 0$ *for all* $\lambda \in \Lambda$. *Then there exists an* $n \times n$ *matrix-valued integrable function* F *such that*

$$[(i\lambda - k)I - L(\omega(\lambda)I)]^{-1} = \tilde{F}(\lambda) := \int_{-\infty}^{\infty} F(t)e^{-i\lambda t}dt \qquad (\forall\lambda \in \Lambda). \tag{6}$$

Furthermore, for any $f \in \Lambda(\mathbf{C}^n)$ *Eq. (3) possesses a unique solution in* $\Lambda(\mathbf{C}^n)$ *which is explicitly given by* $F * f$.

Proof. As seen in the proof of Theorem 1, there exists an $n \times n$ matrix-valued integrable function Y such that $(\mathcal{D}_\mathcal{M} - \mathcal{B}_\mathcal{M} - k)^{-1} f - \Phi(0)u(t) = Y * f$ for all $f \in \mathcal{M} := \Lambda(\mathbf{C}^n)$. Furthermore, as pointed out in the proof of Lemma 1, there exists an integrable matrix-valued function F_1 such that $u = F_1 * f$ is a unique solution of (5) satisfying $sp(u) \subset \Lambda$ for each $f \in \mathcal{M}$. Set $F = Y + \Phi(0)F_1$. Then F is an $n \times n$ matrix-valued integrable function on $\mathbf{R}$, and $F * f$ is is a unique solution in $\mathcal{M}$ of Eq. (3) for each $f \in \mathcal{M}$.

Now we shall prove the relation (6). Let $\lambda \in \Lambda$, and set $x^j(t) = F(t) * e^{i\lambda t}e_j$ for $j = 1, \cdots, n$. We claim that

$$\tilde{F}(\lambda)e_j = \frac{1}{2T}\int_{s-T}^{s+T} x^j(t)e^{-i\lambda t}dt, \qquad j = 1, \cdots, n$$

for all $s \in \mathbf{R}$. Indeed, we get

$$\begin{aligned}
\int_{s-T}^{s+T} x^j(t)e^{-i\lambda t}dt &= \int_{s-T}^{s+T} (\int_{-\infty}^{\infty} F(\tau)e^{i\lambda(t-\tau)}d\tau)e^{-i\lambda t}dt \cdot e_j \\
&= \int_{s-T}^{s+T}\int_{-\infty}^{\infty} F(\tau)e^{-i\lambda\tau}d\tau dt \cdot e_j \\
&= 2T\tilde{F}(\lambda)e_j.
\end{aligned}$$

Since

$$\frac{1}{2T}\int_{-T}^{T} x_t^j(\theta)e^{-i\lambda t}dt = \frac{1}{2T}\int_{-T+\theta}^{T+\theta} x^j(\tau)e^{-i\lambda\tau}d\tau\cdot e^{i\lambda\theta} = [\omega(\lambda)](\theta)\tilde{F}(\lambda)e_j$$

for $\theta \le 0$, (A2) implies that

$$\frac{1}{2T}\int_{-T}^{T} x_t^j e^{-i\lambda t}dt = \omega(\lambda)\tilde{F}(\lambda)e_j, \qquad j = 1,\cdots,n.$$

Then

$$\begin{aligned}\frac{1}{2T}(x^j(T)e^{-i\lambda T} - x^j(-T)e^{i\lambda T}) &= \frac{1}{2T}\int_{-T}^{T}\{-i\lambda x^j(t) + \dot{x}^j(t)\}e^{-i\lambda t}dt \\ &= \frac{1}{2T}\int_{-T}^{T}(-i\lambda x^j(t) + L(x_t^j) + kx^j(t) + e^{i\lambda t}e_j)e^{-i\lambda t}dt \\ &= (k-i\lambda)\tilde{F}(\lambda)e_j + e_j + L(\omega(\lambda)\tilde{F}(\lambda)e_j) \\ &= [(k-i\lambda)I + L(\omega(\lambda)I)]\tilde{F}(\lambda)e_j + e_j.\end{aligned}$$

Letting $T\to\infty$ in the above, we get $0 = [(k-i\lambda)I + L(\omega(\lambda)I)]\tilde{F}(\lambda)e_j + e_j$ for $j = 1,\cdots,n$, or $\tilde{F}(\lambda) = [(i\lambda - k)I - L(\omega(\lambda)I)]^{-1}$, as required.

We denote by $AP(\mathbf{C}^n)$ or AP the set of all almost periodic (continuous) functions $f : \mathbf{R} \mapsto \mathbf{C}^n$. The next result on the admissibility of $\Lambda(\mathbf{C}^n) \cap AP(\mathbf{C}^n)$ with respect to Eq. (1) is a direct consequence of Corollary 2, because $F * f \in AP$ whenever $f \in AP$ and F is integrable.

Corollary 3 *Suppose that* $\det[i\lambda I - L(\omega(\lambda)I)] \neq 0$ *for all* $\lambda \in \Lambda$. *Then Eq. (1) is admissible for* $\Lambda(\mathbf{C}^n) \cap AP(\mathbf{C}^n)$.

The preceding corollary is a result in the non-critical case. In fact, if (1) is a scalar equation (that is, $n = 1$), our result is available even for the crical case.

Corollary 4 *The following statements hold true for Eq. (1) with* $n = 1$*:*

(i) *Let* $f \in AP(\mathbf{C})$ *with discrete spectrum, and assume the following condition:*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \overline{z(s)}f(s)ds = 0 \ \ \textit{for any almost periodic solution } z(t)$$
$$\textit{of Eq. } (1) \textit{ satisfying } sp(z) \subset sp(f).$$

Then the equation $\dot{x}(t) = L(x_t) + f(t)$ *has an almost periodic solution.*

(ii) *Let* $f \in \mathrm{BUC}(\mathbf{R};\mathbf{C})$ *be a periodic function of period* $\tau > 0$*, and assume the*

following condition:

$$\int_0^\tau \overline{z(s)} f(s)ds = 0 \ \ for\ \ any\ \ \tau\text{-periodic solution } z(t) \ \ of\ \ Eq.\ (1).$$

Then the equation $\dot{x}(t) = L(x_t) + f(t)$ has a τ-periodic solution.

Proof. (ii) is a direct consequence of (i). We shall prove (i). To do this, it suffices to show that $i\lambda - L(\omega(\lambda)) \neq 0$ for any $\lambda \in sp(f)$. Suppose that $i\lambda = L(\omega(\lambda))$ for some $\lambda \in sp(f)$, and set $z(t) = e^{i\lambda t}$, $t \in \mathbf{R}$. As seen in the proof of Theorem 1, $z(t)$ is a (periodic) solution of Eq. (1), and moreover $sp(z) \subset sp(f)$. Therefore, by the condition in the statement (i) we get $\lim_{T\to\infty}(1/T)\int_0^T f(s)e^{-i\lambda s}ds = 0$, which shows that λ is not an exponent of $f(t)$. On the other hand, because $sp(f)$ is discrete, any point in $sp(f)$ must be an exponent of $f(t)$. This is a contradiction.

4. APPLICATIONS

As an application, we consider the integro-differential equation

$$\dot{x}(t) = \int_0^\infty [dB(s)]x(t-s), \tag{7}$$

where B is an $n \times n$ matrix-valued function whose components are of bounded variation satisfying

$$\exists \gamma > 0 : \int_0^\infty e^{\gamma s} d|B(s)| < \infty.$$

In order to set up Eq. (7) as an FDE on a uniform fading memory space, we take the space C_γ introduced in Section 2, and define a functional L on C_γ by

$$L(\phi) = \int_0^\infty [dB(s)]\phi(-s), \qquad \phi \in C_\gamma.$$

Then Eq. (7) is rewritten as Eq. (1) with $\mathcal{B} = C_\gamma$, and our previous results are applicable to Eq. (7):

Theorem 2 *Suppose that* $\det[i\lambda I - \int_0^\infty [dB(s)]e^{-i\lambda s}] \neq 0$ *for all $\lambda \in \Lambda$. Then Eq. (7) is admissible for the spaces $\Lambda(\mathbf{C}^n)$ and $\Lambda(\mathbf{C}^n) \cap AP(\mathbf{C}^n)$.*

In fact, there exists an $n \times n$ matrix-valued integrable function F such that

$$[(i\lambda - k)I - \int_0^\infty [dB(s)]e^{-i\lambda s}]^{-1} = \tilde{F}(\lambda) \qquad (\forall \lambda \in \Lambda),$$

and for any $f \in \Lambda(\mathbf{C}^n)$, $F * f$ is a unique solution in $\Lambda(\mathbf{C}^n)$ of the equation

$$\dot{x}(t) = \int_0^\infty [dB(s)]x(t-s) + f(t).$$

Finally, we consider the following integro-differential equation

$$\dot{x}(t) = Ax(t) + \int_0^\infty x(t-s)db(s) \tag{8}$$

in a Banach space X, where A is the infinitesimal generator of an analytic strongly continuous semigroup of linear operators on X, and $b : \mathbf{R}^+ \mapsto \mathbf{C}$ is a function of bounded variation satisfying

$$\exists \gamma > 0 : \int_0^\infty e^{\gamma s} d|b(s)| < \infty.$$

In a similar way for Eq. (7), one can define the operator L on the uniform fading memory space $C_\gamma(X)$.

Now we denote by $\mathrm{BUC}(\mathbf{R}; X), \Lambda(X), AP(X), \mathcal{D}_{\Lambda(X)}, \mathcal{L}_{\Lambda(X)}, \cdots$ the ones corresponding to $\mathrm{BUC}(\mathbf{R}; \mathbf{C}^n), \Lambda(\mathbf{C}^n), AP(\mathbf{C}^n), \mathcal{D}_{\Lambda(\mathbf{C}^n)}, \mathcal{L}_{\Lambda(\mathbf{C}^n)}, \cdots$, and set $\mathcal{M}(\mathbf{C}) = \Lambda(\mathbf{C}) \cap AP(\mathbf{C})$ and $\mathcal{M}(X) = \Lambda(X) \cap AP(X)$. Then $\mathcal{M}(X)$ is a translation invariant closed subspace of $\mathrm{BUC}(\mathbf{R}; X)$, and one can consider the operator $\mathcal{D}_{\mathcal{M}(X)} - \mathcal{L}_{\mathcal{M}(X)}$, together with the operator $\mathcal{D}_{\mathcal{M}(\mathbf{C})} - \mathcal{L}_{\mathcal{M}(\mathbf{C})}$.

Lemma 2 *Under the notation explained above, the following relation holds:*

$$\sigma(\mathcal{D}_{\mathcal{M}(X)} - \mathcal{L}_{\mathcal{M}(X)}) = \sigma(\mathcal{D}_{\mathcal{M}(\mathbf{C})} - \mathcal{L}_{\mathcal{M}(\mathbf{C})})$$

Proof. The inclusion $\sigma(\mathcal{D}_{\mathcal{M}(X)} - \mathcal{L}_{\mathcal{M}(X)}) \subset \sigma(\mathcal{D}_{\mathcal{M}(\mathbf{C})} - \mathcal{L}_{\mathcal{M}(\mathbf{C})})$ is an immediate consequence of Corollary 2 (cf. [9, Lemma 3.6]). We shall establish the converse inclusion. Let $k \in \sigma(\mathcal{D}_{\mathcal{M}(\mathbf{C})} - \mathcal{L}_{\mathcal{M}(\mathbf{C})})$, and assume that $k \notin \sigma(\mathcal{D}_{\mathcal{M}(X)} - \mathcal{L}_{\mathcal{M}(X)})$. It follows from Theorem 1 that $k = i\lambda - \int_0^\infty e^{-i\lambda s} db(s)$ for some $\lambda \in \Lambda$. Let $a \in X$ be any nonzero element, and define a function $f \in \Lambda(X)$ by $f(t) = e^{i\lambda t}a,\ t \in \mathbf{R}$. Then there is a unique solution x in $\mathcal{M}(X)$ of the equation

$$\dot{x}(t) = kx(t) + \int_0^\infty x(t-s)db(s) + f(t). \tag{9}$$

Since $x \in AP(X)$, the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_{-s}^{T-s} x(t)e^{-i\lambda t} dt \ (=: x_\lambda)$$

exists in X uniformly for $s \in \mathbf{R}$. From (9) we get the relation

$$\begin{aligned} [x(T)e^{-i\lambda T} - x(0)]/T &= -(i\lambda/T)\int_0^T x(t)e^{-i\lambda t}dt + (k/T)\int_0^T x(t)e^{-i\lambda t}dt \\ &\quad + (1/T)\int_0^T [\int_0^\infty x(t-s)db(s)]e^{-i\lambda t}dt + a, \end{aligned}$$

and hence letting $T \to \infty$ we get $[-i\lambda + k + \int_0^\infty e^{-i\lambda s} db(s)]x_\lambda + a = 0$, or $a = 0$. This is a contradiction. Hence we must have the inclusion $\sigma(\mathcal{D}_{\mathcal{M}(X)} - \mathcal{L}_{\mathcal{M}(X)}) \supset \sigma(\mathcal{D}_{\mathcal{M}(\mathbf{C})} - \mathcal{L}_{\mathcal{M}(\mathbf{C})})$.

For $\mathcal{M}(X) = \Lambda(X) \cap AP(X)$, we denote by $\mathcal{A}_{\mathcal{M}(\mathcal{X})}$ the operator $f \in \mathcal{M}(X) \mapsto Af(\cdot)$ with $D(\mathcal{A}) = \{f \in \mathcal{M}(X) : f(t) \in D(A),\ Af(\cdot) \in \mathcal{M} \text{ for } \forall t \in \mathbf{R}\}$. For two (unbounded) commuting operators $\mathcal{D}_{\mathcal{M}(X)} - \mathcal{L}_{\mathcal{M}(X)}$ and $\mathcal{A}_{\mathcal{M}(X)}$, it is known (cf. [9, Theorem 2.2]) that

$$\sigma(\overline{\mathcal{D}_{\mathcal{M}(X)} - \mathcal{L}_{\mathcal{M}(X)} - \mathcal{A}_{\mathcal{M}(X)}}) \subset \sigma(\mathcal{D}_{\mathcal{M}(X)} - \mathcal{L}_{\mathcal{M}(X)}) - \sigma(\mathcal{A}_{\mathcal{M}(X)}),$$

here $\overline{(\cdots)}$ denotes the usual closure of the operator. Applying Lemma 2 and this relation, we get the following result on the admissiblity of $\mathcal{M}(X)$ with respect to Eq. (8).

Theorem 3 *Assume that $i\lambda - \int_0^\infty e^{-i\lambda s} db(s) \in \rho(A)$ for all $\lambda \in \Lambda$. Then for any $f \in \Lambda(X) \cap AP(X)$ the equation $\dot{x}(t) = Ax(t) + \int_0^\infty x(t-s)db(s) + f(t)$ has a unique (mild) solution in $\Lambda(X) \cap AP(X)$.*

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