On some possible extensions of Massera's theorem

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1. Introduction. In this paper we consider ordinary and functional differential equations with T-periodic right hand side and look at some conjectures on proving the existence of a periodic solution. One of the "best" conditions to obtain a periodic solution is the existence of a bounded solution, because it would be then not only a sufficient but also a neccessary condition. There are many authors investigating this condition in the literature. First Massera [8] proved that for one or two dimensional ordinary differential equations if there is a bounded solution of the differential equation and the solutions can be continued for all future times then there is a T-periodic solution. He also proved that a linear ordinary differential system has a T-periodic solution if it has a bounded solution. Yoshizawa [10, Theorem 15.10] proved for ordinary differential systems that if a bounded solution exists with a certain stability property, then there is an mT-periodic solution for some $m > 0$ integer. Chow [4] extended Massera's result for linear functional differential equations with finite delay. The author [7] showed this result for functional differential equations with infinite delay and to a class of integral equations. For other results on infinite delay equations see [3] and [6]. To prove the existence of a T-periodic solution using Liapunov functionals see [5].

In this paper we consider some possible extensions of the above theorems. Our first goal is to show that in most of the cases the existence of a bounded solution is not enough to obtain a periodic solution. We will give examples of ordinary and functional differential equations for which there is a solution for all times but no periodic solution. Although

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some of the examples we give later are examples to the previous conjectures too, this order of presenting them shows the method of constructing such examples.

We also take one step in the direction to prove that if a periodic system has a bounded solution then it also has an almost periodic solution. We prove that every periodic ordinary or functional differential equation is equivalent in some sense to an autonomous system: for every periodic system there is a higher dimensional autonomous system and there is a correspondence between the solutions so that for every bounded and/or almost periodic solution the corresponding solution is also bounded and/or almost periodic. This will help in proving the statement in the first sentence of this paragraph: this means that we only need to consider autonomous systems when proving that statement. And it is obvious that working with autonomous systems is much easier than working with periodic ones.

2. Ordinary differential equations. Looking at Yoshizawa's result the first very natural extension of Massera's theorems is the following

CONJECTURE 1: If an ordinary differential system with T-periodic right hand side has a bounded solution then it also has a T-periodic (or mT-periodic) solution.

Unfortunately this "natural" conjecture is false. When one searches for a counter-example one needs to look for it in the three or more dimensional space. The easiest way to show such an example is when the equation does have a periodic solution but that solution is not T-periodic (or mT-periodic). The example also must use all the three dimensions, i.e., there cannot be a 2 dimensional surface homeomorphic to a plane containing a bounded solution, so that all solutions starting from that surface remain in that surface. One way to fulfill all the above conditions is to have an equation where the solutions are on the cylinder-jackets around the z axis, there is a cylinder which is simply turning around the z axis; the others are moving up or down along the z axis. The following equation is exactly this kind:

$$x'(t) = y(t) \qquad y'(t) = -x(t) \qquad z'(t) = x^2(t) + y^2(t) - 1$$

Clearly, the projection of any solution to the (x, y) plane is a circle, and hence $x^2(t) + y^2(t)$ is constant. Therefore $z'(t)$ is constant, and it is zero only if $x^2(t)+y^2(t) = 1$. So, all periodic solutions are of the form $(\sin(t+\alpha), \cos(t+\alpha), z_0)$, where α and z_0 are constants. Considering this equation to have a 1-periodic right hand side (or even multiplying the right hand side of the equation for z' by a positive 1-periodic function), we have 2π-periodic solutions but not $mT = m$-periodic ones.

One would say next that if one cannot have an mT-periodic solution then one still should be able to prove the existence of a periodic solution with another period.

Conjecture 2: If an ordinary differential system with T-periodic right hand side has a bounded solution then it also has a periodic solution with some period.

One could even sketch a proof for this conjecture. Let us take the bounded solution $x(t, t_0, x_0)$ starting at t_0 from x_0. Then pick some of the points of this solution, preferably the points $x_k := x(t_0 + kT, t_0, x_0)$. Since the sequence $\{x_k\}$ forms a bounded sequence in n dimensional the space, it must have an accumulation point, and starting a solution from that point we hope to get a periodic solution. But the last step will not work.

Consider an equation which has solutions on the surface of a torus so that they are everywhere dense (see for example [9]). This can be achieved by choosing the solutions to be straight lines on the cut-up surface (a rectangle) of the torus so that the tangent of the angle of the line is irrational while the ratio of the rectangle's sides is rational. In this case any point sequence from a solution on the surface of the torus can converge only to a point on the surface, but there is no periodic solution on the surface of the torus.

OK, so this proof will not work but there may be other proofs. The periodic solution might be inside the torus, since no solutions can get out from inside the torus. So, let us get this question settled for good. We show by a modified example of Conjecture 1 that Conjecture 2 is not true. To make the understanding easier, let us write that example in polar coordinates:

$$r'(t) = 0 \qquad \phi'(t) = 1 \qquad z'(t) = r^2(t) - 1$$

We need to get rid of the periodic solutions on the cylinder-jacket $x^2 + y^2 = 1$ $(r = 1)$. The best way to do this is to change the speed of the solutions on this cylinder-jacket, so that the time the solution needs to get back to the initial point is not a rational multiple T. This means changing the equation for ϕ like the following:

$$r'(t) = 0 \qquad \phi'(t) = 1 + 2\sin^2(\pi t) \qquad z'(t) = r^2(t) - 1$$

This made the equation 1-periodic. Under some multiple of one period (say NT) a periodic solution now should turn around the z axis by a degree multiple of 2π, i.e., $\phi(t) - \phi(t_0)$ should be $2k\pi$. Hence we have

$$2k\pi = \phi(t) - \phi(t_0) = \int_{t_0}^{t} 1 + 2\sin^2(\pi t)dt = \int_0^{NT} 1 + 2\sin^2(\pi t)dt = 2NT = 2N$$

which is a contradiction. This proves that there is no periodic solution of this equation. To complete the example, let us rewrite the equation into the (x, y, z) coordinates:

$$x'(t) = (1 + 2\sin^2(\pi t))y(t) \qquad y'(t) = -(1 + 2\sin^2(\pi t))x(t) \qquad z'(t) = x^2(t) + y^2(t) - 1$$

Conjecture 3. One may think that if an autonomous system has a bounded solution, then it has a periodic solution too.

But this, again, is not true. If one thinks of the previous example in which the solutions are everywhere dense on the surface of the torus, then we already have an example, if we say that we consider this system not being defined anywhere else. For an everywhere defined system the example is quite complicated, so we will not formulate it here, but the idea is the following. Consider again the above system and consider this equation in the four dimensional space, so that the system describes the behavior of the solutions' projection to a three dimensional subspace (say the subspace containing the axes x, y, z, transparent to the w axis). Then add one more equation to this system, so that the fourth component of the solution strictly increases, whenever the point is not on the surface of the torus, while on the surface of the torus the fourth component does not change. In this way all periodic solutions must be on the surface of the torus (otherwise the fourth component strictly increases), and there are no periodic solutions there. This completes the example.

3. Functional differential equations. So far all our results were negative. Now lets take a turn and be positive for a while. We now know what cannot be proved, so let us see what can be our goal here. If one looks closely to all of our previous examples, then it can be seen that in all cases we have at least almost periodic solutions. So, let us try to prove this.

Consider the r dimensional system

$$x' = f(t, x_t) \qquad\qquad x_{t_0} = \psi \tag{1}$$

where f is defined on a subset D of $\mathbf{R} \times \mathcal{C}$,

$$\mathcal{C} := \{\phi : [-h, 0] \to \mathbf{R}^r | \phi \text{ is continuous}\},$$

$\psi \in \mathcal{C}$, $h > 0$, and we use the supremum norm ($||\phi|| := \sup_{s\in[-h,0]} |\phi(s)|$) on $\mathcal{C}$. Suppose, that f is continuous and locally Lipschitz in its second variable. Under these conditions an initial value problem has a unique solution which depends continuously on the initial data. As we would like to prove the existence of an almost periodic solution, let us also assume that the right hand side of the equation (1) is periodic in t, i.e., $f(t, \phi) = f(t + T, \phi)$ for all $(t, x) \in D$. We can also safely assume a periodicity in the domain of the definition of f, that is, $(t, \phi) \in D$ if and only if $(t + T, \phi) \in D$. In the following we will always assume these conditions.

One of the problems we must overcome is that the periodicity condition on the right hand side is somehow "outside" of the equation, does not really fit in the usual initial value problems. First we prove

THEOREM 1. Every r dimensional periodic system has an $r + 2$ dimensional autonomous system, so that for every solution of the first system there is a corresponding solution of the second, and vice versa.

Proof. The corresponding initial value problem to system (1) is defined in the following way: let $y_i(t) = x_i(t)$ for $i = 1, 2, \ldots, r$, $y_{r+1}(t) = \sin(2\pi t/T)$ and $y_{r+2}(t) = \cos(2\pi t/T)$ and hence the vector y satisfies the following system

$$\begin{aligned} y_1' &= f_1(t, x_t) \\ &\cdots \\ y_r' &= f_r(t, x_t) \\ y_{r+1}' &= \frac{2\pi}{T} y_{r+2} \\ y_{r+2}' &= -\frac{2\pi}{T} y_{r+1} \end{aligned} \tag{2}$$

with the extra initial conditions $y_{r+1}(t_0) = \sin(2\pi t_0/T)$ and $y_{r+2}(t_0) = \cos(2\pi t_0/T)$.

First, note that in the case of ordinary differential equations there is a geometrical meaning of the second system. To make the understanding easier, let us assume that the equation is an ordinary differential equation, $r = 1$ and $D = \mathbf{R} \times \mathbf{R}$. Then f is defined in the two dimensional space, and it is T-periodic in its first variable t. Now let us fold this two dimensional space along the t axis to form a cylinder-jacket with circumference T. Put this cylinder-jacket in the three dimensional space so that its axis is the $x = x_1 = y_1$ axis and the other two axes are y_2 and y_3 and you have our equation with the difference that we have a magnification in the variables y_2 and y_3.

Now let us see the exact proof. From the equations and initial conditions for y_{r+1} and y_{r+2} we know that they are exactly of the form as we asked. Hence, another way to look at the above system is that we can consider the functions $y_{r+1}(t)$ and $y_{r+2}(t)$ as a clock which measures the time modulo T: the point described by these functions takes one turn in the two dimensional space during any time interval of length T. They do not interfere with the rest of the system in any way, and the rest of the system does not affect them in any way. From this "clock" we can read the time modulo T; i.e., there is a function $t(y_{r+1}, y_{r+2})$ which gives the time t modulo T. Since system (1) is T-periodic, computing t modulo T is enough to compute the right hand side exactly. By replacing $t(y_{r+1}, y_{r+2})$ in the first r equations in system (2), we get an autonomous system.

Now, for the solution $y(t)$ of the initial value problem (2) there is a solution $x(t)$ of (1) defined by $x_i(t) = y_i(t)$ for $i = 1, 2, \ldots, r$. On the other hand, the construction itself shows that for the solution of (1) there is a corresponding solution of (2). This proves the equivalence of (1) and (2) in this sense and proves the statement of the Theorem.

As we work with bounded, periodic and almost periodic solutions, let us see how these properties are preserved along this equivalence. The proofs of the next four propositions are trivial from the construction.

Proposition 1. The solution of (1) is bounded in the future (or for all times) if and only if the corresponding solution of (2) is bounded in the future (or for all times).

Proposition 2. If the solution of (2) is periodic then the corresponding solution of (1) is also periodic.

Proposition 3. If the solution of (1) is periodic then the corresponding solution of (2) is almost periodic.

Proposition 4. The solution of (1) is almost periodic if and only if the corresponding solution of (2) is almost periodic.

Looking over what we proved so far, we can prove the following

THEOREM 2. The following statements are equivalent:

1. If the system

$$x' = f(t, x_t) \qquad \text{with } f(t, \phi) = f(t + T, \phi) \tag{1}$$

has a solution bounded in the future (or for all times) then it also has an almost periodic solution.

2. If the system

$$Y' = F(Y_t) \tag{2}$$

corresponding to system (1) (as constructed in Theorem 1), has a solution bounded in the future (or for all times) then it also has an almost periodic solution.

3. If an autonomous system

$$z' = g(z_t) \tag{3}$$

has a solution bounded in the future (or for all times) then it also has an almost periodic solution.

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